Characterizing the Computable Depth Zero Boolean Algebras

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Outline

Boolean Algebras

Definition

A Boolean algebra *is a structure* $(A; +, \cdot, -, 0, 1)$ *satisfying associativity, commutativity, absorption, distributivity, and complementation:*

1*a*.	$x + (y + z) = (x + y) + z$	1*b*.	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
2*a*.	$x + y = y + x$	2*b*.	$x \cdot y = y \cdot x$
3*a*.	$x + (x \cdot y) = x$	3*b*.	$x \cdot (x + y) = x$
4*a*.	$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$	4*b*.	$x + (y \cdot z) = (x + y) \cdot (x + z)$
5*a*.	$x + (-x) = 1$	5*b*.	$x \cdot (-x) = 0$

Example

The structure $(\mathcal{P}(\omega); \cup, \cap, {}^C, \emptyset, \omega)$ *is a Boolean algebra.*

Linear Orders

Theorem

Given a linear order $\mathcal{L} = (L; <)$ with least element, the set of clopen subsets is a Boolean algebra under union, intersection, and complementation.

Remark

It is usually much easier to specify the structure of a Boolean algebra by describing a linear order that generates it. Note, though, that non-isomorphic linear orders can generate the same Boolean algebra.

Computable Models

Definition

A structure $\mathcal{M} = (M; \cdots)$ is computable if its universe can be identified with a computable subset of ω in such a way that its atomic diagram is computable.

Atoms

Definition

An element x of a Boolean algebra is an atom *(0-atom) if*

$$\forall y(y \leq x \ \rightarrow \ y = 0 \text{ or } x - y = 0).$$

More generally, an element x is an $(n+1)$-atom *if it is not an n-atom but*

$$\forall y(y \leq x \ \rightarrow \ y \text{ or } x - y \text{ is a finite union of } n\text{-atoms}).$$

Example

Let $\mathcal{L} = (\omega + 1; <)$.

- *Then $\{2\}$ and $\{18\}$ are atoms.*
- *Then $\{1, 2\}$ is not an atom as $\{1, 2\} = \{1\} \sqcup \{2\}$.*
- *Then $[0, \omega]$ and $[5, \omega]$ are 1-atoms.*

Cantor-Bendixson Rank

Definition

Given a linear order $\mathcal{L}$, the Cantor-Bendixson derivative *$\mathcal{L}'$ of $\mathcal{L}$ is the sublinear order with universe*

$$L' = L - \{x : x \text{ is isolated in } L\}.$$

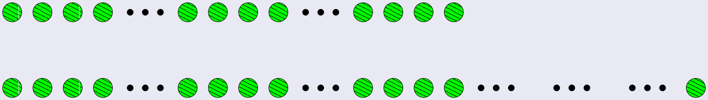
Using transfinite induction, define the α^{th} Cantor-Bendixson derivative of $\mathcal{L}$, denoted $\mathcal{L}^{(\alpha)}$, by

$$\mathcal{L}^{(0)} = \mathcal{L}, \qquad \mathcal{L}^{(\alpha+1)} = \left(\mathcal{L}^{(\alpha)}\right)', \qquad \text{and} \qquad \mathcal{L}^{(\gamma)} = \bigcap_{\alpha<\gamma} \mathcal{L}^{(\alpha)}$$

for limit ordinals γ.

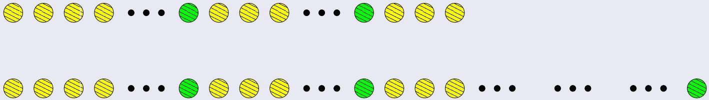
Cantor-Bendixson Example

Example



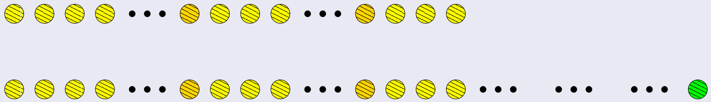
Cantor-Bendixson Example

Example



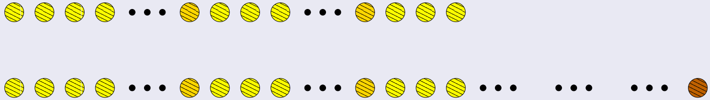
Cantor-Bendixson Example

Example



Cantor-Bendixson Example

Example



The Feiner Hierarchy

Definition

Recall that $\mathbf{0}^{(\omega)} = \{\langle m, n\rangle : m \in \mathbf{0}^{(n)}\}$.

Definition

A set S recursive in $\mathbf{0}^{(\omega)}$ *is said to be* (a, b) *in the Feiner Hierarchy, written* $S \sim (a, b)$, *if there is an index e such that*

1. *The function* $\varphi_e^{\mathbf{0}^{(\omega)}}$ *is total,*
2. *The function* $\varphi_e^{\mathbf{0}^{(\omega)}}$ *is the characteristic function of S, i.e.* $n \in S$ *if and only if* $\varphi_e^{\mathbf{0}^{(\omega)}}(n) = 1$.
3. *The computation* $\varphi_e^{\mathbf{0}^{(\omega)}}(n)$ *uses no part of the oracle above* $\mathbf{0}^{(an+b)}$.

Measures

Definition

A measure is a map σ from the clopen subsets of 2^ω into ω_1 induced by a map $\tilde{\sigma} : 2^{<\omega} \to \omega_1$ satisfying

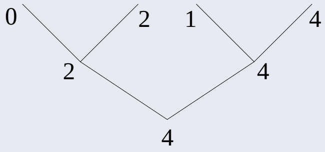
$$\tilde{\sigma}(\tau) = \max\left\{\tilde{\sigma}(\tau^\frown 0), \tilde{\sigma}(\tau^\frown 1)\right\} \qquad \text{for all } \tau \in 2^{<\omega}$$

where

$$\sigma(\tau) = \max\{\tilde{\sigma}(\tau_0), \ldots, \tilde{\sigma}(\tau_n) : \tau = \tau_0 \sqcup \cdots \sqcup \tau_n\}.$$

Example

The following map is (induces) a measure.



Measures and Boolean Algebras

Theorem (Ketonen)

There is a natural bijection between (isomorphism types) of the countable uniform Boolean algebras and (homeomorphism types) of measures.

Proof.

Move from countable uniform Boolean algebras to linear orders to rank functions to measures.

□

Depth Zero Boolean Algebras

Definition

Given a measure σ and an element x, define

$$\Delta\sigma(x) = \{(\sigma(x_0), \ldots, \sigma(x_n)) : x = x_0 \sqcup \cdots \sqcup x_n\}.$$

More generally, define

$$\Delta^{\alpha+1}\sigma(x) = \{(\Delta^{\alpha}\sigma(x_0), \ldots, \Delta^{\alpha}\sigma(x_n)) : x = x_0 \sqcup \cdots \sqcup x_n\}.$$

Definition

A measure σ is depth zero if

$$\sigma(x) = \sigma(y) \implies \Delta\sigma(x) = \Delta\sigma(y).$$

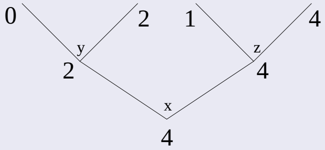
Remark

Note that the reverse implication always holds.

Depth Example

Example

Note that for the measure



we have

- $\Delta\sigma(x)$ *is the set of all strings of* 0, 1, 2, *and* 4 *containing at least one* 4
- $\Delta\sigma(y)$ *is the set of all strings of* 0 *and* 2 *containing at least one* 2
- $\Delta\sigma(z)$ *is the set of all strings of* 1, *and* 4 *containing at least one* 4

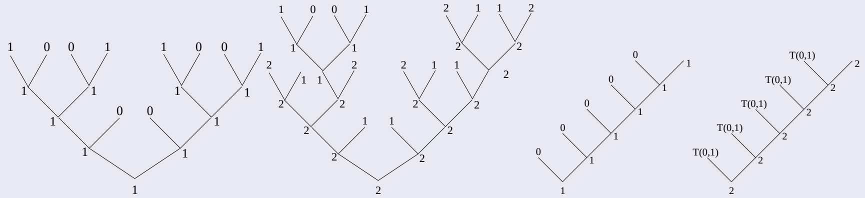
In particular, σ *is not depth zero since* $\sigma(x) = \sigma(z)$ *but* $\Delta\sigma(x) \neq \Delta\sigma(z)$.

Proposition (K)

Given a bounded set $S \subset \omega_1$ with maximal element, there are exactly two depth zero measures with range S.

Proof.

For example, with $S = \{0, 1\}$ and $S = \{0, 1, 2\}$, the Boolean algebras are given by the measures



□

Results (I)

Theorem (K)

If $S \sim (2,0)$, then B_S is computable.

Proof.

Begin with the measure $\sigma_{\omega+1}$. When it appears that an integer n isn't in S, attempt to build an n-atom (e.g. the linear order $w^n + 1$) rather than an $(n-1)$-atom (e.g. the linear order $w^{n-1} + 1$) at the appropriate place. Carefully modifying the proof of the following theorem,

Theorem (Thurber)

There is a procedure, uniform in n and a Δ^0_{2n+1} index for the open diagram of a linear order $\mathcal{A}$ with distinguished first element, which builds a Δ^0_1 copy of the linear order $\omega^n \cdot \mathcal{A}$.

it is possible to relativize and nest the construction controlled by a Σ^0_2 predicate that builds either FIN or $\omega + 1$ to yield B_S.

Results (II)

Theorem (K)

If B_S is computable, then $S \sim (2,3)$.

Proof.

A number $n \geq 1$ is in S if and only if

$$B_S \models \exists x \forall m \exists x_1, \ldots, x_m \left[\left(\bigwedge_{i=1}^{m} x_i \leq x \right) \wedge \left(\bigwedge_{i \neq j}^{m} x_i x_j = 0 \right) \wedge \left(\bigwedge_{i=1}^{m} \mathrm{At}_{n-1}(x_i) \right) \right] \bigwedge \forall y \left[y \leq x \implies \neg \mathrm{At}_n(y) \right]$$

Since the formula $\mathrm{At}_n(x)$ stating that x is an n-atom is expressible as an infinitary Π^0_{2n+1} formula, counting quantifiers the above is Σ^0_{2n+3}. □

Current and Future Work...

Conjecture

With $S \subseteq \omega + 1$, the depth zero Boolean algebras B_S are computable if and only if $S \sim (2, 3)$.

Conjecture

More generally with $S \subset \omega_1^{\mathrm{CK}}$, the depth zero Boolean algebras are computable if and only if $S \sim (2, 3)$.

Other future projects include analyzing both the finite depth Boolean algebras and the rank one, depth ω Boolean algebras.

Definition (The Low-n Problem)

The Low-n *Problem is the statement that every* Low-n *Boolean algebra is computable.*

Remark

If an iff statement can be achieved, then the Low-n *Problem would be solved (positively) for the depth zero Boolean algebras.*

Theorem

The Low-n *Problem is true for* $n = 1$ *(Downey and Jockusch),* $n = 2$ *(Thurber), and* $n = 4$ *(Knight and Stob).*

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