Characterizing the Computable Structures: Boolean Algebras and Linear Orders

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Abstract

A countable structure (with finite signature) is computable if its universe can be identified with ω in such a way as to make the relations and operations computable functions. In this thesis, I study which Boolean algebras and linear orders are computable.

Making use of Ketonen invariants, I study the Boolean algebras of low Ketonen depth, both classically and effectively. Classically, I give an explicit characterization of the depth zero Boolean algebras; provide continuum many examples of depth one, rank ω Boolean algebras with range $\omega+1$; and provide continuum many examples of depth ω, rank one Boolean algebras. Effectively, I show for sets $S \subseteq \omega+1$ with greatest element, the depth zero Boolean algebras $\mathcal{B}_{u(S)}$ and $\mathcal{B}_{v(S)}$ are computable if and only if $S \setminus \{\omega\}$ is $\Sigma^0_{n \mapsto 2n+3}$ in the Feiner Σ-hierarchy.

Making use of the existing notion of limitwise monotonic functions and the new notion of limit infimum functions, I characterize which shuffle sums of ordinals below $\omega+1$ have computable copies. Additionally, I show that the notions of limitwise monotonic functions relative to $\mathbf{0}'$ and limit infimum functions coincide.

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Chapter 1

Introduction

This work lies at the interface of logic and algebra, focusing on the effective content of algebraic structures. It is motivated by the following general question:

Question 1.1. *What countable algebraic structures are effective? In other words, what countable algebraic structures can, in principle, be implemented on a computer?*

The notion of a computable structure makes this idea precise.

Definition 1.2. *A countable algebraic structure having only finitely many functions and relations is* computable *if its universe can be identified with ω in such a way that the functions and relations become computable operations on ω.*

We address Question 1.1 for two classes of algebraic structures: Boolean algebras (viewed as structures $\mathcal{B} = (B : +, \cdot, -, 0, 1)$) and linear orders (viewed as structures $\mathcal{L} = (L : \prec)$).

1.1 Main Results

For Boolean algebras, we study the class with small Ketonen depth. In addition to various algebraic results, we provide the following classical characterization of the sets S with computable depth zero Boolean algebras.

Theorem 1.3. *For sets $S \subseteq \omega + 1$ with greatest element, the following are equivalent:*

1. *The depth zero Boolean algebra $\mathcal{B}_{u(S)}$ is computable.*
2. *The depth zero Boolean algebra $\mathcal{B}_{v(S)}$ is computable.*
3. *The set $S \setminus \{\omega\}$ is $\Sigma^0_{n \mapsto 2n+3}$ in the Feiner Σ-hierarchy.*

For linear orders, we study the class of shuffle sums of ordinals below $\omega+1$. The main result is the following classical characterization of the sets $S \subseteq \omega + 1$ with computable shuffle sums.

Theorem 1.4. *For sets $S \subseteq \omega + 1$, the following are equivalent:*

1. *The shuffle sum $\sigma(S)$ is computable, i.e., the linear order obtained by interleaving copies of the order types of the ordinals in S is computable.*

2. *The set S is a limit infimum set, i.e., there is a total computable function $g(x,s)$ such that the function $f(x) = \liminf_s g(x,s)$ enumerates S under the convention that $f(x) = \omega$ if $\liminf_s g(x,s) = \infty$.*

3. *The set S is a limitwise monotonic set relative to $\mathbf{0}'$, i.e., there is a total $\mathbf{0}'$-computable function $\tilde{g}(x,t)$ satisfying $\tilde{g}(x,t) \leq \tilde{g}(x,t+1)$ such that the function $\tilde{f}(x) = \lim_t \tilde{g}(x,t)$ enumerates S under the convention that $\tilde{f}(x) = \omega$ if $\lim_s \tilde{g}(x,s) = \infty$.*

Other results discuss the relationship between these sets and the Σ^0_3 sets.

For basic background on computability theory, the reader is referred to [20] or [22]. Although our notation is for the most part standard, we review general notation in Section 1.2, notation specific to linear orders in Section 1.3, and notational conventions in Section 1.4. Chapter 2 is our study of Boolean algebras of small Ketonen depth, and Chapter 3 is our study of shuffle sums of ordinals.

The reader is referred to Section 2.1 and Section 3.1 for introductory material on Boolean algebras and shuffle sums, respectively.

1.2 General Notation

Although the notation used generally conforms to that found in [22], we review the notation that will appear throughout the thesis.

The primary objects we deal with are sets, ordinals, functions, and strings. We will use the symbol S primarily to denote a set of ordinals, with $|S|$ denoting the cardinality of S. We will represent the set of ordinals $\{\beta : \beta < \alpha\}$ by α. We will use f, g, and h, as well as $\tilde{f}$, $\tilde{g}$, and $\tilde{h}$, to denote total functions.

The set of finite binary strings (i.e., strings in the alphabet $\{0,1\}$) will be denoted by $2^{<\omega}$; the set of infinite binary strings will be denoted by 2^ω. The set of binary strings of length k will be denoted by 2^k. The length of a binary string $\tau \in 2^{<\omega}$ will be denoted by $|\tau|$. Concatenation of binary strings $\tau_1, \tau_2 \in 2^{<\omega}$ will be denoted by $\tau_1 {}^\frown \tau_2$. The set of binary strings will be ordered lexicographically. The empty string will be denoted by ε.

The notation $\langle \cdot,\cdot \rangle$ will denote an effective pairing function $\langle \cdot,\cdot \rangle : \omega \times \omega \to \omega$. The symbol $\mathbb{Q}$ will be used to denote the rational numbers, and the symbol $\mathcal{C}$ will be used to denote the Cantor set.

1.3 Notation for Linear Orders

If $\mathcal{L} = (L : \prec)$ is a linear order and $\mathcal{L}_a = (L_a : \prec_a)$ is a linear order for each $a \in L$, then the notation $\sum_{a \in L} \mathcal{L}_a$ represents the *lexicographic sum* of the orders $\mathcal{L}_a$. In particular, it is the linear order with universe $\{(a,b) : a \in L, b \in L_a\}$ under the lexicographic order induced by $\prec$ and $\{\prec_a\}_{a \in L}$.

If $\mathcal{L} = (L : \prec)$ is a linear order, we will use the symbols $-\infty$ and $+\infty$ when denoting intervals in $\mathcal{L}$. In particular, we will write $(-\infty, a)$ and $(a, +\infty)$ to denote the sets $\{z \in L : z \prec a\}$ and $\{z \in L : z \succ a\}$, respectively.

1.4 Notational Conventions

As we will have little need to refer to partial computable functions, we depart from the usual computability-theoretic convention of using lower case Greek letters primarily to denote partial functions. The symbol σ will exclusively denote either a measure or a shuffle sum; the symbol τ will denote a binary string; the symbol φ_e will denote the e^{th} partial computable function.

In many places, we will have a total function serving as an approximation of another function. For example, a common situation will be $f(x) = \lim_s g(x, s)$. We will use the hat symbol to denote the modulus of convergence, i.e., we will denote the least t with the property that $g(x, s) = f(x)$ for all $s \geq t$ by $\hat{s}$.

Chapter 2

Boolean Algebras of Small Ketonen Depth

2.1 Introduction

A fundamental problem in the field of algebra is to classify the isomorphism types existing within a class of structures. If the class of structures is sufficiently well understood classically, computable model theorists are able to ask which of these isomorphism types have effective representations.

For several naturally occurring classes of algebraic structures, the classical characterization is rather straightforward. Vector spaces over the rationals and algebraically closed fields of characteristic zero, for example, are both characterized by a single cardinal number (their dimension or their transcendence degree, respectively). Characterizing the computable vector spaces and algebraically closed fields is straightforward, there being an effective representation if and only if the cardinal invariant is countable.

In this chapter, we restrict our attention to the class of countable Boolean algebras, addressing both of these questions in turn. Ketonen, in [14], found algebraic invariants (Ketonen invariants) that characterize the isomorphism type of a countable Boolean algebra. The main algebraic result below explicitly shows the connection between a depth zero Boolean algebra and the set of ordinals it encodes (Section 2.3.1). Other classical results demonstrate the existence of continuum many depth one, rank ω Boolean algebras with range $\omega + 1$ (Section 2.3.2) and of continuum many depth ω, rank one Boolean algebras (Section 2.3.3).

As one might expect, there is a connection between the complexity of the set of ordinals a depth zero Boolean algebra encodes and whether the Boolean algebra has an effective representation. The main computability-theoretic result of the chapter will show exactly which depth zero, rank ω Boolean algebras are computable (Section 2.4). As the class of sets computable in $\emptyset^{(\omega)}$ is too large for our purposes, we modify the Feiner hierarchy developed in [7] to characterize the sets with computable depth zero, rank ω Boolean algebras. The primary directions of the main result are proved separately (Section 2.5 and Section 2.6).

Before these classical and computability results, we begin with background, notation, and a review of Ketonen invariants (Section 2.2). After, we finish with applications to the $\textsc{Low}_n$ conjecture (Section 2.7) and future directions (Section 2.8).

2.2 Background and Notation

Although a thorough discussion of Ketonen invariants is beyond the scope of this thesis (see [14] for the original paper or [18] for an alternative exposition), we do discuss the background necessary to understand them. Before doing so, we begin with the following important convention.

Convention 2.1. *Throughout, a Boolean algebra will refer exclusively to a countable Boolean algebra. A linear order will refer exclusively to a countable linear order.*

We continue by reviewing Boolean algebras and Ketonen invariants in Section 2.2.1 and defining the Feiner hierarchy in Section 2.2.2.

2.2.1 Boolean Algebras and Ketonen Invariants

We briefly recall that we view Boolean algebras as structures in the signature $\mathcal{B} = (B : +, \cdot, -, 0, 1)$. We use the notation $x \oplus y$ to denote the element $x + y$ with the additional hypothesis that $xy = 0$. As we will frequently have the need to use the symbols 0 and 1 to denote ordinals, we will always refer to the largest and smallest elements of a Boolean algebra as $1_{\mathcal{B}}$ and $0_{\mathcal{B}}$. Certain other elements of Boolean algebras are also of particular importance.

Definition 2.2. *A non-zero element x is an* atom *if the only element strictly below x is the zero element $0_{\mathcal{B}}$.*

An element x is a 0-atom *if it is an atom. An element x is an* α-atom *for $\alpha > 0$ if it cannot be expressed as a finite join of β-atoms for $\beta < \alpha$, but for all y, either xy or $x(-y)$ can be expressed in this form.*

A non-zero element x is atomless *if it bounds no atoms, i.e., there is no atom y with $y \leq x$.*

Several linear orders will also play a prominent role.

Definition 2.3. *Let η denote the order type of the rational numbers, i.e., a countable dense linear order without endpoints.*

Let Fin *denote any order type consisting of at least two, but at most finitely many, points.*

As specifying the construction of a fixed Boolean algebra $\mathcal{B}$ can often be quite cumbersome, we will often describe the construction of a linear order $\mathcal{L}$ whose interval algebra is isomorphic to $\mathcal{B}$.

Definition 2.4. *Let $\mathcal{L}$ be a linear order with least element x_0 with the topology generated by basic open sets $[a, b)$. The* interval algebra of $\mathcal{L}$, *denoted $\mathcal{B}_{\mathcal{L}}$, is the Boolean algebra whose universe is the set of clopen subsets of $\mathcal{L}$. The operations of join, meet, and* *complementation in the Boolean algebra $\mathcal{B}_{\mathcal{L}}$ are given by taking the union, intersection, and complementation of the clopen sets.*

If $\mathcal{B}$ is a Boolean algebra, then $\mathcal{L}$ is a linear order that generates $\mathcal{B}$ *if* $\mathcal{B} \cong \mathcal{B}_{\mathcal{L}}$.

An important fact is that both directions of the Stone Representation Theorem, i.e., the transitions from a Boolean algebra $\mathcal{B}$ to a linear order $\mathcal{L}_{\mathcal{B}}$ that generates it, and from a linear order $\mathcal{L}$ to its interval algebra $\mathcal{B}_{\mathcal{L}}$, are effective. We emphasize that $\mathcal{L}_{\mathcal{B}}$ need not be unique, and, in most cases, is actually far from unique. However, by applying certain algebraic manipulations to any such $\mathcal{L}$, a unique invariant can be obtained. The first of these manipulations is the Cantor-Bendixson derivative.

As we will have need to use the half-open topology with basic open sets $[a, b)$, we alter the standard Cantor-Bendixson derivative slightly.

Definition 2.5. *Let $\mathcal{L}$ be a linear order with least element x_0 under the half-open topology, i.e., the topology with basic open sets $[a, b)$. The* Cantor-Bendixson derivative of $\mathcal{L}$, *denoted $\mathcal{L}'$, is the linear order with universe*

$$\{x_0\} \cup \{x \in \mathcal{L} : x \textit{ is not isolated in } \mathcal{L}\}$$

if $\mathcal{L}$ is infinite and empty universe if $\mathcal{L}$ is finite.

The α^{th} Cantor-Bendixson derivative, *denoted $\mathcal{L}^{(\alpha)}$, is defined recursively by $\mathcal{L}^{(0)} = \mathcal{L}$, $\mathcal{L}^{(\alpha+1)} = (\mathcal{L}^{(\alpha)})'$, and $\mathcal{L}^{(\gamma)} = \cap_{\beta<\gamma}\mathcal{L}^{(\beta)}$ for limit ordinals γ.*

The Cantor-Bendixson rank of $\mathcal{L}$ *is the least ordinal ν such that $\mathcal{L}^{(\nu)} = \mathcal{L}^{(\nu+1)}$ if such an ordinal exists. The* perfect kernel of $\mathcal{L}$ *are the points in $\mathcal{L}^{(\nu)}$.*

For countable linear orders $\mathcal{L}$ with least element under the half-open topology, the Cantor-Bendixson rank exists and is countable. There are two possibilities for $\mathcal{L}^{(\nu)}$: either $\mathcal{L}^{(\nu)} = \emptyset$ or $\mathcal{L}^{(\nu)} = 1 + \eta$ (which we identify with $[0, 1) \cap \mathbb{Q}$). When $\mathcal{L}^{(\nu)} = \emptyset$, the Boolean algebra $\mathcal{B}_{\mathcal{L}}$ is called *superatomic*. These Boolean algebras are well understood both classically (they are exactly the class of α-atoms) and computability-theoretically (computable if and only if $\alpha < \omega_1^{\text{CK}}$), so we restrict our attention to the less understood case when $\mathcal{L}^{(\nu)} = 1 + \eta$. In this case, one approach to describing the algebraic structure of $\mathcal{B}_{\mathcal{L}}$ is through rank functions, measures, and monoid derivatives, as done by Ketonen in [14]. As with Cantor-Bendixson derivatives, we alter the presentation to suit our needs.

Definition 2.6. *The* rank function *for a linear order $\mathcal{L}$ of rank ν is the map $r_{\mathcal{L}}$: $\mathcal{L}^{(\nu)} \to \omega_1$ given by*

$$r(x) = \min\left\{\beta : x \notin \overline{(\mathcal{L}^{(\beta)} \backslash \mathcal{L}^{(\nu)})}\right\},$$

i.e., the minimum ordinal β such that x is not in the closure of $\mathcal{L}^{(\beta)} \backslash \mathcal{L}^{(\nu)}$.

For a point x in the perfect kernel of $\mathcal{L}$, the rank function describes the number of Cantor-Bendixson derivatives required until x is no longer a limit of points not in the perfect kernel. Unfortunately, there may be a disconnect between $\nu(\mathcal{L})$ and the ranks of points in the perfect kernel of $\mathcal{L}$. When they coincide, we call the Boolean algebra uniform.

Definition 2.7. *A countable Boolean algebra $\mathcal{B}$ is* uniform *if*

$$\nu(\mathcal{L}) = \sup_{x \in \mathcal{L}^{(\nu)}} r(x).$$

Restricting one's attention to the uniform Boolean algebras does no harm (classically or effectively) as the following proposition and corollary demonstrate in conjunction with the superatomic Boolean algebras being well understood (classically and effectively).

Proposition 2.8 ([14]). *Every countable Boolean algebra is the disjoint sum of a uniform Boolean algebra and a superatomic Boolean algebra.*

Corollary 2.9. *Every computable Boolean algebra is the disjoint sum of a computable uniform Boolean algebra and a computable superatomic Boolean algebra.*

Proof. Let $\mathcal{B}$ be a computable Boolean algebra. From Proposition 2.8, we obtain a decomposition $\mathcal{B} \cong \mathcal{B}_u \oplus \mathcal{B}_s$ with $\mathcal{B}_u$ uniform and $\mathcal{B}_s$ superatomic. Then both $\mathcal{B}_u$ and $\mathcal{B}_s$ are computable, being intervals in the algebra $\mathcal{B}$. □

Before we can make the transition from rank functions to measures, we introduce the countable free Boolean algebra.

Definition 2.10. *The* countable free Boolean algebra $\mathcal{F}$ *is the (unique) non-trivial countable Boolean algebra having no atoms.*

We view $\mathcal{F}$ as being the Boolean algebra generated by the set of strings $\tau \in 2^{<\omega}$, i.e., elements are finite unions of these cones. We also view $\mathcal{F}$ as being the interval algebra of the linear order $[0, 1) \cap \mathbb{Q}$.

Definition 2.11. *Let $\mathcal{L}$ be a linear order with non-empty perfect kernel, which we identify with $[0, 1) \cap \mathbb{Q}$. The* measure $\sigma = \sigma_r$ *associated to the rank function $r = r_{\mathcal{L}}$ is the map $\sigma : \mathcal{F} \to \omega_1$ given by*

$$\sigma(x) = \sup\{r(p) : p \in x\}$$

for non-zero $x \in \mathcal{F}$ and $\sigma(0_{\mathcal{F}}) = o$, where o is a special symbol satisfying $o < \alpha$ for all ordinals α.

Because of the natural correspondence between $\mathcal{F}$ and $2^{<\omega}$, a measure can be viewed as a map from $2^{<\omega}$ to ω_1.

Remark 2.12. *Any map $\rho : 2^{<\omega} \to \omega_1$ satisfying*

$$\rho(\tau) = \max\{\rho(\tau ^\frown 0), \rho(\tau ^\frown 1)\}$$

for all $\tau \in 2^{<\omega}$ generates a measure $\sigma_\rho : \mathcal{F} \to \omega_1$ by defining $\sigma_\rho(x) = \sup\{\rho(\tau) : \tau \in x\}$.

The map ρ can be viewed as an ordinal labeled complete binary branching tree T_ρ where the label at a node τ in the tree is $\rho(\tau)$.

In order to help prevent confusion, in this chapter we reserve the symbol σ exclusively for measures and the symbol τ exclusively for binary strings. In order to avoid cumbersome language, we make little distinction between ρ, σ_ρ, and T_ρ. We also make little distinction between a measure σ, the Boolean algebra $\mathcal{B}_\sigma$, and any linear order $\mathcal{L}$ that generates $\mathcal{B}_\sigma$. As the context will always make the meaning transparent, there should be no confusion.

We end our discussion of Ketonen invariants by introducing monoid derivatives and depth.

Definition 2.13. *If $\sigma : \mathcal{F} \to \omega_1$ is a measure, define maps $\Delta^\alpha\sigma$ with domain $\mathcal{F}$ for $\alpha < \omega_1$ recursively by setting $\Delta^0\sigma = \sigma$,*

$$\Delta^{\alpha+1}\sigma(x) = \{(\Delta^\alpha\sigma(x_1), \ldots, \Delta^\alpha\sigma(x_n)) : x = x_1 \oplus \cdots \oplus x_n\},$$

and $\Delta^\gamma\sigma(x)$ as the inverse limit of the $\Delta^\beta\sigma(x)$ for $\beta < \gamma$ for limit ordinals γ.

The set $\Delta^\alpha\sigma(1_\mathcal{B})$ is the α^{th} derivative *of $\mathcal{B}_\sigma$.*

Definition 2.14. *The* depth *of a measure $\sigma : \mathcal{F} \to \omega_1$ is the least countable ordinal $\delta = \delta(\sigma)$ such that*

$$\forall x \forall y \left[\Delta^\delta\sigma(x) = \Delta^\delta\sigma(y) \implies \Delta^{\delta+1}\sigma(x) = \Delta^{\delta+1}\sigma(y)\right]. \tag{2.1}$$

A fundamental result by Ketonen is that for every measure σ, the depth of σ is well-defined (i.e., there exists such an ordinal δ). In addition, the derivative $\Delta^{\delta+2}\sigma(1_\mathcal{B})$ characterizes the isomorphism type of $\mathcal{B}_\sigma$. Slightly more is true, namely that $\Delta^{\delta+1}\sigma(1_\mathcal{B})$ characterizes the isomorphism type of $\mathcal{B}_\sigma$ amongst the Boolean algebras with depth at most δ.

Although we end our discussion of Ketonen invariants here, the reader is referred to [14] or [18] for a more thorough exposition. Ketonen in [14] injects these derivatives into the Ketonen hierarchy and explores which derivatives can be obtained.

The reader is also referred to [8] or [19] for an exposition of Ershov-Tarski invariants. These invariants (which are far simpler) characterize the elementary equivalence classes of the countable Boolean algebras rather than their isomorphism classes, as Ketonen invariants do.

2.2.2 The Feiner Hierarchy and Modifications

In [7], Feiner defined a hierarchy of complexities for certain sets S computable in $\emptyset^{(\omega)} = \{\langle k, m\rangle : m \in \emptyset^{(k)}\}$. Before defining this hierarchy, we define the notation $\emptyset^{(\leq n)}$ for $n \in \omega$. We assume the reader is familiar with fundamental notions of computability theory. The reader is referred to [20] or [22] for such background.

Definition 2.15. *For $n \in \omega$, define $\emptyset^{(\leq n)}$ to be the set*

$$\emptyset^{(\leq n)} = \{\langle k, m\rangle : m \in \emptyset^{(k)}, k \leq n\}.$$

Several observations should quickly be made. The sets $\emptyset^{(n)}$ and $\emptyset^{(\leq n)}$ are Turing equivalent, and in fact Turing equivalent uniformly in n. Also, we have $\emptyset^{(\leq i)} \subseteq \emptyset^{(\leq j)} \subseteq \emptyset^{(\omega)}$ whenever $i \leq j < \omega$. The former property ensures using $\emptyset^{(\leq n)}$ instead of $\emptyset^{(n)}$ is sensical; the latter property simplifies the definition of the Feiner hierarchy, which we now give.

Definition 2.16. *Let $S \subseteq \omega$ be a set computable in $\emptyset^{(\omega)}$. Then S is* (a, b) in the Feiner hierarchy *if there exists an index e such that*

1. *The function $\varphi_e^{\emptyset^{(\omega)}}$ is total and is the characteristic function of S, i.e., $\varphi_e^{\emptyset^{(\omega)}}(n)\downarrow= \chi_S(n)$ for all n.*

2. *The computations $\varphi_e^{\emptyset^{(\leq bn+a)}}(n)$ and $\varphi_e^{\emptyset^{(\omega)}}(n)$ are equal; in particular, neither queries any number $\langle k, m\rangle$ with $k > bn + a$.*

Essentially, a set S is (a, b) in the Feiner hierarchy if membership of n in S can be determined uniformly from the oracle $\emptyset^{(bn+a)}$. Alternately, the second requirement of Definition 2.16 can be viewed as a special kind of restriction on the use of the computation $\varphi_e^{\emptyset^{(\omega)}}(n)$. Whereas the normal restriction is that the oracle cannot be queried above some fixed number, here the restriction is that the oracle cannot be queried above some fixed column (in particular, the column $bn + a$).

For our purposes, we will need membership of n in S to be an existential question over the oracle rather than be computable over the oracle. We therefore introduce a new hierarchy which is a modification of the Feiner hierarchy.

Definition 2.17. *Let $S \subseteq \omega$ be a set computable in $\emptyset^{(\omega)}$. Then S is* $\Sigma^0_{n\mapsto bn+a}$ in the Feiner Σ-hierarchy *if there exists an index e such that*

1. *The set S satisfies $S = W_e^{\emptyset^{(\omega)}}$.*

2. *The computations $\varphi_e^{\emptyset^{(\leq bn+a)}}(n)$ and $\varphi_e^{\emptyset^{(\omega)}}(n)$ are equal; in particular, neither queries any number $\langle k, m\rangle$ with $k > bn + a$.*

Equivalently, a set S is $\Sigma^0_{n\mapsto bn+a}$ in the Feiner Σ-hierarchy if membership of n in S is a Σ^0_{bn+a} question uniformly in n.

In this thesis we will only consider the special case when $a = 3$ and $b = 2$.

2.3 Algebraic Study of Boolean Algebras

As part of his work in [11], Heindorf showed that there are only countably many finite depth, finite rank Boolean algebras. These Boolean algebras have arisen in a variety of contexts: they are exactly the countable Boolean algebras that have a countably categorical weak second-order theory (see [17]); they are exactly the interval algebras of compact zero-dimensional metric spaces of finite type (see [9]); and they are exactly the class of finitary Boolean algebras (see [11]).

We demonstrate that Heindorf's result is optimal in a strong sense. Keeping the depth parameter as small as possible (i.e., zero) and allowing the rank parameter to be as small as possible but not finite (i.e., ω), there are continuum many Boolean algebras, two for each subset $S \subseteq \omega + 1$ with greatest element (one for those S with $|S| = 1$). Keeping the rank parameter as small as possible (i.e., one [zero introduces trivialities]) and allowing the depth parameter to be as small as possible but not finite (i.e., ω), again there are continuum many examples.

In Section 2.3.1, we provide an algebraic characterization of the depth zero measures (of arbitrary rank) and, as a corollary, obtain the existence of continuum many depth zero, rank ω Boolean algebras. Although the image of the depth zero Boolean algebras in the Ketonen hierarchy was previously known (see Proposition 1.18.5 of [18]), the depth zero measures were not explicitly clear. In addition to providing a tractable description of these measures, our work is effective enough to allow us to analyze which depth zero Boolean algebras have effective representations in Section 2.4.

In Section 2.3.2, we construct continuum many depth one, rank ω Boolean algebras with range $\omega+1$. Whereas distinct depth zero measures have the same general structure (with differences only in the range of the measures), these examples have structural differences between distinct measures.

In Section 2.3.3, we finish our algebraic study of Boolean algebras by constructing continuum many depth ω, rank one Boolean algebras. Our technique will be a refinement of a general construction by Pierce in [18].

As preparation, we define certain strings $\tau \in 2^{<\omega}$ to be repeater strings, almost repeater strings, and xor strings.

Definition 2.18. *A string $\tau \in 2^{<\omega}$ is a* repeater string *if the length $|\tau|$ of τ is even and $\tau(2i) = \tau(2i+1)$ for all $i < |\tau|/2$.*

A string $\tau \in 2^{<\omega}$ is an almost repeater string *if τ is a repeater string or of the form $\tau = \tau' \frown 0$ or $\tau = \tau' \frown 1$ for some repeater string τ'.*

A string $\tau \in 2^{<\omega}$ is an xor string *if either $\tau = 01$ or $\tau = 10$.*

2.3.1 Depth Zero Boolean Algebras

A depth zero Boolean algebra $\mathcal{B}$ is almost uniquely characterized by the range S of its measure $\sigma_{\mathcal{B}}$. In fact, all that is needed to make this characterization unique is

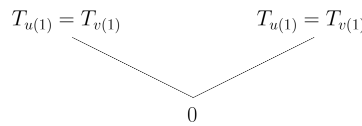


Figure 1: Tree for $T_{u(1)} = T_{v(1)}$

the additional information of whether there are disjoint elements x and y in $\mathcal{B}$ with $\sigma(x) = \max(S) = \sigma(y)$. The intuition for this is relatively straightforward: specifying the measure of an element in a fixed depth zero Boolean algebra dictates the isomorphism type of the element. Hence specifying the range of the measure should dictate the entire Boolean algebra.

We demonstrate the existence of two depth zero Boolean algebras with range S, termed $\mathcal{B}_{u(S)}$ and $\mathcal{B}_{v(S)}$, for each set $S \subseteq \omega_1$ with greatest element. We then show that there are at most two such Boolean algebras, from which the ability to (almost) characterize a depth zero Boolean algebra by the range of its measure follows.

We begin by showing the existence of $\mathcal{B}_{u(S)}$ and $\mathcal{B}_{v(S)}$ in the special case when S is a non-zero ordinal. In the special case when $S = 1$, the Boolean algebras $\mathcal{B}_{u(1)}$ and $\mathcal{B}_{v(1)}$ coincide.

Lemma 2.19. *There is a depth zero measure with range* $1 = \{0\}$.

Proof. Let $\sigma_{u(1)}$ and $\sigma_{v(1)}$ be the measure generated by

$$\sigma(\tau) = 0$$

for $\tau \in 2^{<\omega}$. As for any x, the derivative $\Delta\sigma(x)$ is the set of all finite strings of 0s, the measure σ is depth zero. □

Although the tree $T_{u(1)} = T_{v(1)}$ is quite simple, we include a diagram (see Figure 1) illustrating an alternative recursive definition of the tree.

For larger ordinals α, the measures $T_{u(\alpha+1)}$ and $T_{v(\alpha+1)}$ do not coincide.

Lemma 2.20. *For each non-zero countable ordinal α, there are two depth zero measures with range* $\alpha + 1 = \{0, 1, \ldots, \alpha\}$.

Proof. We show the existence of the depth zero measures $\mathcal{B}_{u(\alpha+1)}$ and $\mathcal{B}_{v(\alpha+1)}$ by induction on α. As preparation we fix, for each countable limit ordinal α, a bijection $f_\alpha : \omega \to \alpha$.

We continue by defining $\mathcal{B}_{u(\alpha+1)}$ and $\mathcal{B}_{v(\alpha+1)}$ when $\alpha = 1$. Let $\sigma_{u(2)}$ be the measure generated by

$$\sigma_{u(2)}(\tau) = \begin{cases} 1 & \text{if } \tau \text{ is an almost repeater string,} \\ 0 & \text{otherwise,} \end{cases}$$

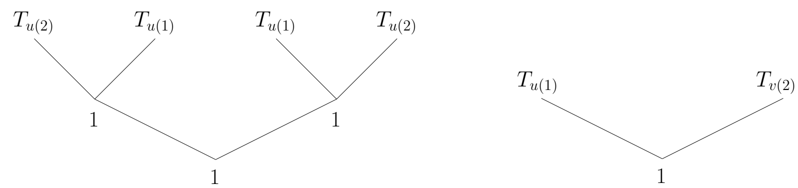


Figure 2: Trees for $T_{u(2)}$ (left) and $T_{v(2)}$ (right)

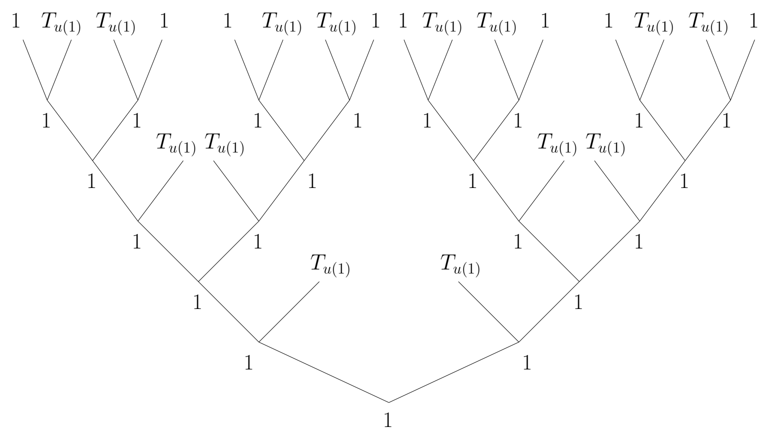


Figure 3: Tree for $T_{u(2)}$ with Additional Nodes Pictured

for $\tau \in 2^{<\omega}$ and let $\sigma_{v(2)}$ be the measure generated by

$$\sigma_{v(2)}(\tau) = \begin{cases} 1 & \text{if } \tau = 1^k \text{ for some integer } k, \\ 0 & \text{otherwise,} \end{cases}$$

for $\tau \in 2^{<\omega}$. In order to assist the reader in understanding $\mathcal{B}_{u(2)}$ and $\mathcal{B}_{v(2)}$, we illustrate the trees $T_{u(2)}$ and $T_{v(2)}$ with an alternative recursive definition (see Figure 2). For the reader's benefit, we also include illustrations of $T_{u(2)}$ and $T_{v(2)}$ with additional nodes pictured (see Figure 3 and Figure 4).

The following observations imply that both $\sigma_{u(2)}$ and $\sigma_{v(2)}$ are depth zero.

$u = 0$: If $\sigma_{u(2)}(x) = 0$, then $\Delta\sigma_{u(2)}(x)$ is the set of all finite strings of 0s.

$u = 1$: If $\sigma_{u(2)}(x) = 1$, then $\Delta\sigma_{u(2)}(x)$ is the set of all finite strings of 0s and 1s with at least one occurrence of 1.

$v = 0$: If $\sigma_{v(2)}(x) = 0$, then $\Delta\sigma_{v(2)}(x)$ is the set of all finite strings of 0s.

$v = 1$: If $\sigma_{v(2)}(x) = 1$, then $\Delta\sigma_{v(2)}(x)$ is the set of all finite strings of 0s and 1s containing exactly one occurrence of 1.

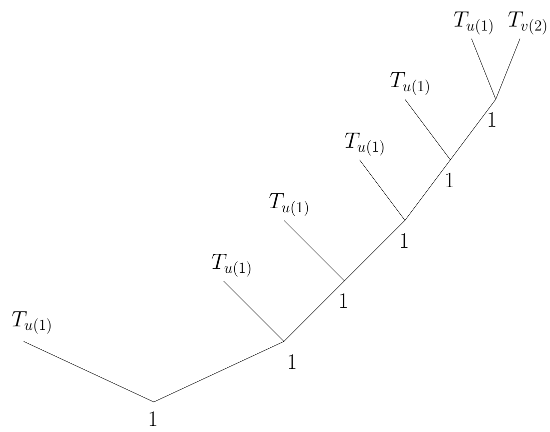


Figure 4: Tree for $T_{v(2)}$ with Additional Nodes Pictured

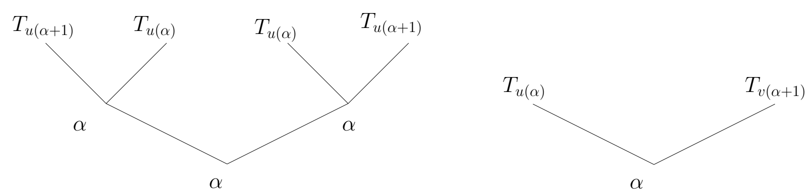


Figure 5: Trees for $T_{u(\alpha+1)}$ (left) and $T_{v(\alpha+1)}$ (right)

We continue by defining $\mathcal{B}_{u(\alpha+1)}$ and $\mathcal{B}_{v(\alpha+1)}$ for successor ordinals α with $\alpha > 1$. Let $\sigma_{u(\alpha+1)}$ be the measure generated by

$$\sigma_{u(\alpha+1)}(\tau) = \begin{cases} \alpha & \text{if } \tau \text{ is an almost repeater string,} \\ \sigma_{u(\alpha)}(\tau_3) & \text{if } \tau = \tau_1 {}^\frown \tau_2 {}^\frown \tau_3 \text{ for some repeater string } \tau_1 \\ & \text{and xor string } \tau_2, \end{cases}$$

for $\tau \in 2^{<\omega}$ and let $\sigma_{v(\alpha+1)}$ be the measure generated by

$$\sigma_{v(\alpha+1)}(\tau) = \begin{cases} \alpha & \text{if } \tau = 1^k \text{ for some integer } k, \\ \sigma_{u(\alpha)}(\tau') & \text{if } \tau = 1^k {}^\frown 0 {}^\frown \tau' \text{ for some integer } k \text{ and string } \tau', \end{cases}$$

for $\tau \in 2^{<\omega}$. Again, we help the reader form a picture of these measures with illustrations of the trees $T_{u(\alpha+1)}$ and $T_{v(\alpha+1)}$ defined recursively (see Figure 5).

The following observations imply that both of these measures are depth zero.

$u = \beta$: If $\sigma_{u(\alpha+1)}(x) = \beta < \alpha$, then $\Delta\sigma_{u(\alpha+1)}(x)$ is the set of all finite strings of ordinals in $\beta + 1$ with at least one occurrence of β.

$u = \alpha$: If $\sigma_{u(\alpha+1)}(x) = \alpha$, then $\Delta\sigma_{u(\alpha+1)}(x)$ is the set of all finite strings of ordinals in $\alpha + 1$ with at least one occurrence of α.

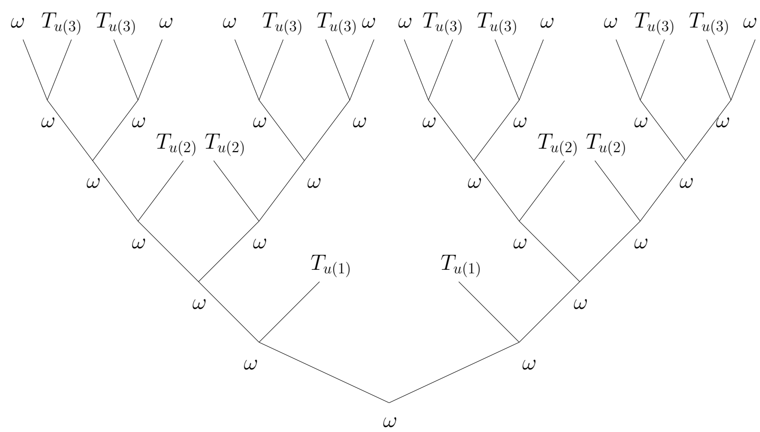


Figure 6: Tree for $T_{u(\omega+1)}$

$v=\beta$: If $\sigma_{v(\alpha+1)}(x)=\beta<\alpha$, then $\Delta\sigma_{v(\alpha+1)}(x)$ is the set of all finite strings of ordinals in $\beta+1$ with at least one occurrence of β.

$v=\alpha$: If $\sigma_{v(\alpha+1)}(x)=\alpha$, then $\Delta\sigma_{v(\alpha+1)}(x)$ is the set of all finite strings of ordinals in $\alpha+1$ containing exactly one occurrence of α.

We finish by defining $\mathcal{B}_{u(\alpha+1)}$ and $\mathcal{B}_{v(\alpha+1)}$ for limit ordinals α. Recalling the fixed bijective functions $f_\alpha:\omega\to\alpha$, let $\sigma_{u(\alpha+1)}$ be the measure generated by

$$\sigma_{u(\alpha+1)}(\tau)=\begin{cases}\alpha & \text{if } \tau \text{ is an almost repeater string,}\\ \sigma_{u(f_\alpha(k))}(\tau_3) & \text{if } \tau=\tau_1{}^\frown\tau_2{}^\frown\tau_3 \text{ for some repeater string } \tau_1 \text{ of}\\ & \text{length } k=\frac{|\tau_1|-2}{2} \text{ and xor string } \tau_2,\end{cases}$$

for $\tau\in 2^{<\omega}$ and let $\sigma_{v(\alpha+1)}$ be the measure generated by

$$\sigma_{v(\alpha+1)}(\tau)=\begin{cases}\alpha & \text{if } \tau=1^k \text{ for some integer } k,\\ \sigma_{u(f_\alpha(k))}(\tau') & \text{if } \tau=1^k{}^\frown 0{}^\frown\tau' \text{ for the integer } k \text{ and string } \tau',\end{cases}$$

for $\tau\in 2^{<\omega}$.

We illustrate the trees $T_{u(\alpha+1)}$ and $T_{v(\alpha+1)}$ for limit ordinals in the special case when α is ω and the bijection $f_\omega:\omega\to\omega$ is given by $n\mapsto n$ (see Figure 6 and Figure 7).

The following observations imply that both of these measures are depth zero.

$u=\beta$: If $\sigma_{u(\alpha+1)}(x)=\beta<\alpha$, then $\Delta\sigma_{u(\alpha+1)}(x)$ is the set of all finite strings of ordinals in $\beta+1$ with at least one occurrence of β.

$u=\alpha$: If $\sigma_{u(\alpha+1)}(x)=\alpha$, then $\Delta\sigma_{u(\alpha+1)}(x)$ is the set of all finite strings of ordinals in $\alpha+1$ with at least one occurrence of α.

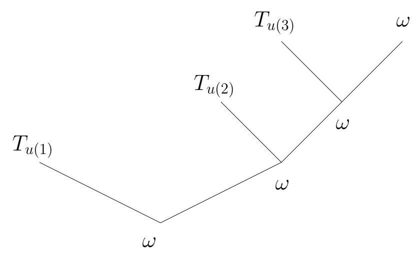


Figure 7: Tree for $T_{v(\omega+1)}$

$v = \beta$: If $\sigma_{v(\alpha+1)}(x) = \beta < \alpha$, then $\Delta\sigma_{v(\alpha+1)}(x)$ is the set of all finite strings of ordinals in $\beta + 1$ with at least one occurrence of β.

$v = \alpha$: If $\sigma_{v(\alpha+1)}(x) = \alpha$, then $\Delta\sigma_{v(\alpha+1)}(x)$ is the set of all finite strings of ordinals in $\alpha + 1$ containing exactly one occurrence of α.

We conclude that for each non-zero countable ordinal α, there are two depth zero measures with range $\alpha + 1 = \{0, 1, \ldots, \alpha\}$. □

The transition from successor ordinals to arbitrary subsets of ω_1 with greatest element requires the observation that the order type of any subset of ω_1 with greatest element is a countable successor ordinal.

Proposition 2.21. *For each set $S \subseteq \omega_1$ satisfying $|S| = 1$, there is a depth zero measure with range S. For each set $S \subseteq \omega_1$ with greatest element satisfying $|S| > 1$, there are two distinct depth zero measures with range S.*

Proof. Let $\alpha + 1$ be the order type of S, noting that the order type of S is a successor ordinal as S is a set with a greatest element. Let $g : \alpha + 1 \to S$ be the order preserving bijection that witnesses S has order type $\alpha + 1$.

Let $\sigma_{u(S)}$ be the measure generated by

$$\sigma_{u(S)}(\tau) = g(\sigma_{u(\alpha+1)}(\tau))$$

for $\tau \in 2^{<\omega}$ and let $\sigma_{v(S)}$ be the measure generated by

$$\sigma_{v(S)}(\tau) = g(\sigma_{v(\alpha+1)}(\tau))$$

for $\tau \in 2^{<\omega}$.

The measures $\sigma_{u(S)}$ and $\sigma_{v(S)}$ are depth zero as a consequence of $\sigma_{u(\alpha+1)}$ and $\sigma_{v(\alpha+1)}$ being depth zero. If $|S| = 1$, then $\sigma_{u(S)}$ and $\sigma_{v(S)}$ coincide as $\sigma_{u(1)} = \sigma_{v(1)}$; if $|S| > 1$, then $\sigma_{u(S)}$ and $\sigma_{v(S)}$ are distinct as $\sigma_{u(\alpha+1)} \neq \sigma_{v(\alpha+1)}$ if $\alpha > 0$. □

We continue by demonstrating that for each set $S \subseteq \omega_1$ with greatest element, there are at most two depth zero measures with range S. In order to do so, we use the work of Heindorf. Specifically, we establish that every depth zero Boolean algebra is pseudo-indecomposable and primitive and appeal to a result in [11].

For an element x in a Boolean algebra $\mathcal{B}$, we use the notation $\mathcal{B}_x$ to denote the interval in $\mathcal{B}$ below x, i.e., the Boolean algebra with universe $\{zx : z \in \mathcal{B}\}$.

Definition 2.22. *A Boolean algebra $\mathcal{B}$ is* pseudo-indecomposable *if for every $x \in \mathcal{B}$, either $\mathcal{B} \cong \mathcal{B}_x$ or $\mathcal{B} \cong \mathcal{B}_{-x}$. An element $x \in \mathcal{B}$ is* pseudo-indecomposable *if $\mathcal{B}_x$ is pseudo-indecomposable.*

Definition 2.23. *A Boolean algebra $\mathcal{B}$ is* primitive *if every element in $\mathcal{B}$ is a disjoint union of finitely many pseudo-indecomposable elements.*

Lemma 2.24. *If $\mathcal{B}$ is a depth zero Boolean algebra, then $\mathcal{B}$ is pseudo-indecomposable and primitive.*

Proof. We begin by showing that $\mathcal{B}_x$ is pseudo-indecomposable for an arbitrary element x of a depth zero Boolean algebra $\mathcal{B}$. In order to do so, let z be an arbitrary element of $\mathcal{B}_x$.

As the depth of $\mathcal{B}$ was zero, the depth of $\mathcal{B}_x$ is also zero. Thus if $\sigma_{\mathcal{B}_x}(z) = \sigma_{\mathcal{B}_x}(1_{\mathcal{B}_x})$, then $\mathcal{B}_x \cong (\mathcal{B}_x)_z$ follows immediately from $\mathcal{B}_x$ being depth zero. If instead $\sigma_{\mathcal{B}_x}(z) \neq \sigma_{\mathcal{B}_x}(1_{\mathcal{B}_x})$, then $\sigma_{\mathcal{B}_x}(-z) = \sigma_{\mathcal{B}_x}(1_{\mathcal{B}_x})$ as $\sigma_{\mathcal{B}_x}(1_{\mathcal{B}_x}) = \max\{\sigma_{\mathcal{B}_x}(z), \sigma_{\mathcal{B}_x}(-z)\}$, and so $\mathcal{B}_x \cong (\mathcal{B}_x)_{-z}$ as a consequence of $\mathcal{B}_x$ being depth zero. Hence $\mathcal{B}_x$ is pseudo-indecomposable as either $\mathcal{B}_x \cong (\mathcal{B}_x)_z$ or $\mathcal{B}_x \cong (\mathcal{B}_x)_{-z}$ for every element $z \in \mathcal{B}_x$.

As every $\mathcal{B}_x$ is pseudo-indecomposable, it follows that $\mathcal{B}$ is primitive. The pseudo-indecomposability of $\mathcal{B}$ is precisely the pseudo-indecomposability of $\mathcal{B}_x$ when $x = 1_{\mathcal{B}}$. □

The following lemma of Heindorf's allows us to conclude that there are no other depth zero Boolean algebras besides those described in Proposition 2.21. It is restated using our notation and language.

Lemma 2.25 ([11]). *Let σ be a measure with range S for a pseudo-indecomposable and primitive Boolean algebra $\mathcal{B}$. Then there are at most two possibilities for $\Delta\sigma(1_{\mathcal{B}})$. Moreover one of these possibilities does not exist in the degenerate case when $|S| = 1$.*

It follows that the Boolean algebras given by the measures in Proposition 2.21 are the only depth zero Boolean algebras. We summarize this in the following theorem.

Theorem 2.26. *For each set $S \subseteq \omega_1$ satisfying $|S| = 1$, there is exactly one depth zero Boolean algebra with range S, namely $\mathcal{B}_{u(S)} = \mathcal{B}_{v(S)}$. For each set $S \subseteq \omega_1$ with greatest element satisfying $|S| > 1$, there are exactly two depth zero Boolean algebras with range S, namely $\mathcal{B}_{u(S)}$ and $\mathcal{B}_{v(S)}$.*

Proof. By Proposition 2.21, there are at least this many depth zero Boolean algebras. As a consequence of Lemma 2.25, there are not more than this many, using Lemma 2.24 to obtain the hypotheses of Lemma 2.25. □

The following corollary is then immediate.

Corollary 2.27. *There are continuum many depth zero, rank ω Boolean algebras.*

Rather than appealing to results about pseudo-indecomposable and primitive Boolean algebras, the existence of at most two depth zero measures with range S can be shown directly from the depth zero hypothesis. As Lemma 2.24 implies Proposition 2.28, we keep the proof to a sketch.

Proposition 2.28. *For each set $S \subseteq \omega_1$ with greatest element, there are at most two depth zero measures with range S.*

Proof (Sketch). Let $\alpha = \max(S)$, which we note must exist since S has a greatest element. We consider a depth zero Boolean algebra $\mathcal{B}$ with range S. If $\beta < \alpha$ is in S, there is an element y with $\sigma(y) = \beta$. Then as $\sigma(1_{\mathcal{B}}) = \alpha = \sigma(1_{\mathcal{B}} - y)$ and $(\alpha, \beta) = (\sigma(1_{\mathcal{B}} - y), \sigma(y)) \in \Delta\sigma(1_{\mathcal{B}})$, from the depth zero hypothesis we have that $(\alpha, \beta) \in \Delta\sigma(1_{\mathcal{B}} - y)$. Repeating this argument using $1_{\mathcal{B}} - y$ and any ordinal $\beta' < \alpha$ in S, we conclude that $(\alpha, \beta_1, \ldots, \beta_n) \in \Delta\sigma(1_{\mathcal{B}})$ for any ordinals $\beta_1, \ldots, \beta_n < \alpha$ in S. Moreover, no ordinal δ not in S can appear in an element of $\Delta\sigma(1_{\mathcal{B}})$.

If $(\alpha, \alpha) \in \Delta\sigma(1_{\mathcal{B}})$, then there is a partition $1_{\mathcal{B}} = x_1 \oplus x_2$ with $\sigma(x_1) = \alpha = \sigma(x_2)$. As $\mathcal{B}$ was depth zero and $\sigma(x_1) = \sigma(1_{\mathcal{B}}) = \sigma(x_2)$, we must have $(\alpha, \alpha) \in \Delta\sigma(x_1), \Delta\sigma(x_2)$. Then there are partitions $x_1 = x_{11} \oplus x_{12}$ and $x_2 = x_{21} \oplus x_{22}$ with $\sigma(x_{11}) = \alpha = \sigma(x_{12})$ and $\sigma(x_{21}) = \alpha = \sigma(x_{22})$. Repeating this argument, it follows that $(\alpha, \ldots, \alpha) \in \Delta\sigma(1_{\mathcal{B}})$.

It follows that if $(\alpha, \alpha) \notin \Delta\sigma(1_{\mathcal{B}})$, then $\Delta\sigma(1_{\mathcal{B}})$ consists of the set of all finite sequences of ordinals from S containing exactly one occurrence of α. Otherwise $\Delta\sigma(1_{\mathcal{B}})$ consists of the set of all finite sequences of ordinals from S containing at least one occurrence of α. □

We remark that an algebraic characterization of the depth zero Boolean algebras is not new. Pierce in [18] gave a slightly different classification using a notion called the local refinement property. The explicit characterization given here will be necessary later when we characterize which depth zero Boolean algebras are computable.

2.3.2 Depth One, Rank ω Boolean Algebras

When exhibiting the continuum many depth zero, rank ω Boolean algebras, in some sense all the measures were the same structurally. A monotone function $g : \omega \to \omega$, as in Proposition 2.21, differentiated distinct measures rather than some structural property of the measures.

If the depth parameter is allowed to increase to one, keeping the rank ω, again there are continuum many Boolean algebras. However, unlike in the depth zero case, there are continuum many with range exactly $\omega + 1$.

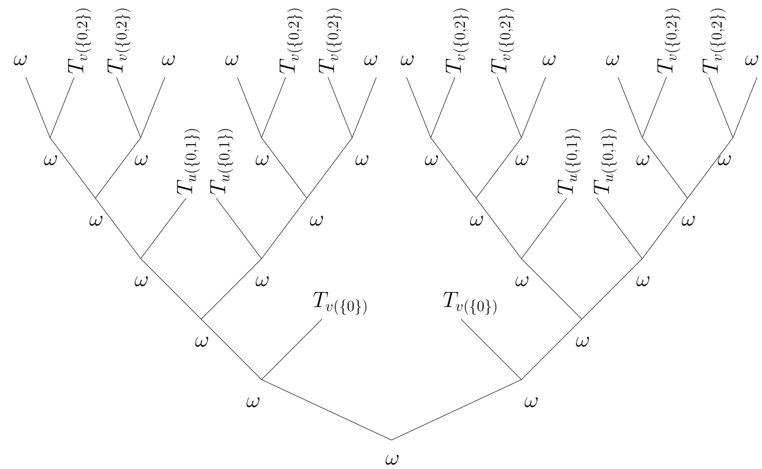


Figure 8: Tree for T^S if $S = \{1, 3, 5, \dots\}$

Proposition 2.29. *There are continuum many depth one, rank ω Boolean algebras with range $\omega + 1$.*

Proof. We demonstrate the existence of continuum many such measures by encoding subsets of the positive integers. For each set S (possibly without greatest element) with $0 \notin S \subseteq \omega$, we define a measure σ^S. The idea is to have the measure $\mathcal{B}_{v(\{0,n\})}$ appear if and only if $n \in S$ and to have the measure $\mathcal{B}_{u(\{0,n\})}$ appear if and only if $n \notin S$. Let σ^S be the measure generated by

$$\sigma^S(\tau) = \begin{cases} \omega & \text{if } \tau \text{ is an almost repeater string,} \\ \sigma_{u(\{0,n\})}(\tau_3) & \text{if } \tau = \tau_1 \frown \tau_2 \frown \tau_3 \text{ for some repeater string } \tau_1 \text{ of length } 2n \\ & \text{and xor string } \tau_2, \text{ and } n \in S, \\ \sigma_{v(\{0,n\})}(\tau_3) & \text{if } \tau = \tau_1 \frown \tau_2 \frown \tau_3 \text{ for some repeater string } \tau_1 \text{ of length } 2n \\ & \text{and xor string } \tau_2, \text{ and } n \notin S, \end{cases}$$

for $\tau \in 2^{<\omega}$. We illustrate the measure σ^S when $S = \{1, 3, 5, \dots\}$ (see Figure 8).

The range of σ^S is easily seen to be $\omega + 1$ as $\sigma^S(0^{2n}10) = n$ and $\sigma(\varepsilon) = \omega$. Before demonstrating the injectivity of the map $S \mapsto \sigma^S$ and that σ^S is depth one, we demonstrate the following claim.

Claim 2.29.1. *If $\sigma^S(x) = n$, then x can be decomposed as $x = y \oplus z$ with $\sigma^S(y) = n$, $\sigma^S(z) < n$, and all sequences of $\Delta\sigma^S(y)$ consisting only of 0's and n's.*

Proof. Let x be such that $\sigma^S(x) = n$. Then x, viewed as a finite sum of elements of $2^{<\omega}$, can be decomposed as the disjoint sum of elements appearing in a copy of $T_{u(\{0,n\})}$ or $T_{v(\{0,n\})}$ and elements not appearing in a copy of $T_{u(\{0,n\})}$ or $T_{v(\{0,n\})}$. This decomposition suffices. □

We continue by demonstrating the injectivity of the map $S \mapsto \sigma^S$ and that σ^S is depth one.

Claim 2.29.2. *The map $S \mapsto \sigma^S$ is injective.*

Proof. We argue that there is an element $x \in \mathcal{B}$ with $\sigma^S(x) = n$ and $\Delta\sigma^S(x)$ containing sequences of arbitrarily many n's if and only if $n \in S$. By Claim 2.29.1, we may write x as the sum of elements y and z with $\sigma^S(y) = n$, $\sigma^S(z) < n$, and $\Delta\sigma^S(y)$ containing sequences of only 0's and n's.

If $n \in S$, then y is part of at least one tree $T_{u(\{0,n\})}$. Hence if $n \in S$, then $\Delta\sigma^S(y)$ (and thus $\Delta\sigma^S(x)$) contains sequences of arbitrarily many n's. If instead $n \notin S$, then y is a sum of trees $T_{v(\{0,n\})}$. Hence if $n \notin S$, then $\Delta\sigma^S(y)$ (and thus $\Delta\sigma^S(x)$) contains sequences of only boundedly many n's.

It follows that if $S_1 \neq S_2$, then $\sigma^{S_1} \ncong \sigma^{S_2}$, so that the map $S \mapsto \sigma^S$ is injective. □

Claim 2.29.3. *For any set S satisfying $|S| \geq 2$, the measure σ^S is depth one.*

Proof. In order to show that σ^S is depth one, we argue that the isomorphism type of an element x is determined by $\Delta\sigma^S(x)$. In order to do so, we use induction on $\sigma^S(x)$. The base case of $\sigma^S(x) = 0$ is trivial, so we consider x with $\sigma^S(x) = n$ for $n > 0$ and assume that the isomorphism type of an element is determined by its first derivative if its measure value is smaller than n.

By Claim 2.29.1, we may write x as the sum of elements y and z with $\sigma^S(y) = n$, $\sigma^S(z) < n$, and $\Delta\sigma^S(y)$ containing sequences of only 0's and n's. The inductive hypothesis gives that z is uniquely determined. As y contains only sequences of 0's and n's, the maximal number of n's occurring in a sequence of $\Delta\sigma^S(y)$ characterizes the isomorphism type of y. It follows that the isomorphism type of x is characterized by $\Delta\sigma^S(x)$.

If $\sigma^S(x) = \omega$, then the isomorphism type of x is clearly determined by $\Delta\sigma^S(x)$.

We note that σ^S is not depth zero if $|S| \geq 2$. For if $n_1, n_2 \in S$ with $n_1 < n_2$, there are elements x_1 and x_2 with $\sigma^S(x_1) = n_1$ and $\sigma^S(x_2) = n_2$. Then $\sigma^S(x_2) = \sigma^S(x_1 + x_2)$, but $\Delta\sigma^S(x_2) \neq \Delta\sigma^S(x_1 + x_2)$, and so σ^S is not depth zero. □

It follows that there is a distinct depth one, rank ω Boolean algebra with range $\omega + 1$ for each set S with $|S| \geq 2$. We conclude there are continuum many such Boolean algebras. □

We remark that the depth one, rank ω Boolean algebras with range $\omega+1$ exhibited in Proposition 2.29 are not exhaustive of this class. It is also possible to code membership of n in S by using $\mathcal{B}_{u(\{0,1,\ldots,n\})}$ and $\mathcal{B}_{v(\{0,1,\ldots,n\})}$, for example.

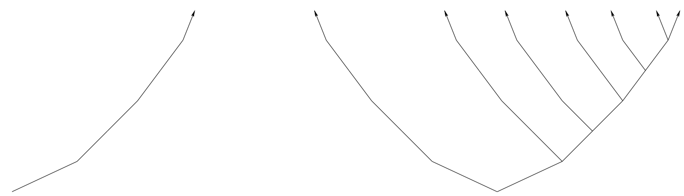


Figure 9: Trees for ς_1 (left) and ς_2 (right). Note that we show the preimage of one rather than the ordinal labels.

2.3.3 Depth ω, Rank One Boolean Algebras

In order to exhibit continuum many depth ω, rank one Boolean algebras, it becomes useful to define a map π from the space of measures to the space of rank one measures. Before doing so, we define an auxiliary measure ς_α for each countable non-limit ordinal α.

Definition 2.30. *For each countable successor ordinal $\alpha+1$, let $X_{\alpha+1}$ be a subset of Cantor space homeomorphic to the ordinal $\omega^\alpha+1$. If $\alpha=0$, we use the (temporary) convention that $\omega^0=0$.*

Let ς_0 be the zero measure. For countable successor ordinals $\alpha+1$, let $\varsigma_{\alpha+1}$ be the measure generated by the characteristic function of $X_{\alpha+1}$.

We illustrate ς_1 and ς_2 by showing the preimage of one (see Figure 9).

Not surprisingly, there is a relationship between α and the depth of $\varsigma_{\alpha+1}$.

Proposition 2.31 (Heindorf in [18]). *The depth of $\varsigma_{\alpha+1}$ is α.*

For an arbitrary measure σ, the measure $\pi(\sigma)$ is, in some sense, a composition of the general structure of depth zero measures interleaved with the measures ς_α for the values of α appearing in the range of σ.

Definition 2.32. *If $\tau\in 2^{<\omega}$ is a repeater string of length $|\tau|$, the string*

$$\tau' = \tau(0)^\frown\tau(2)^\frown\ldots^\frown\tau(k)$$

is a witnesses to τ being a repeater string, *where $k=\dfrac{|\tau|-1}{2}$.*

Definition 2.33. *Fix, for each countable limit ordinal α, a bijection $f_\alpha:\omega\to\alpha$. For a measure σ, define the measure $\pi(\sigma)$ to be the measure generated by*

$$\pi(\sigma)(\tau) = \begin{cases} 1 & \text{if } \tau \text{ is an almost repeater string,} \\ \varsigma_{\sigma(\tau')}(\tau_3) & \text{if } \tau=\tau_1{}^\frown\tau_2{}^\frown\tau_3 \text{ for some repeater string } \tau_1 \\ & \text{witnessed by } \tau', \text{ xor string } \tau_2, \text{ and } \sigma(\tau') \text{ is} \\ & \text{not a limit ordinal,} \\ \varsigma_{f_{\sigma(\tau')}(k)}(\tau_3) & \text{if } \tau=\tau_1{}^\frown\tau_2{}^\frown\tau_3 \text{ for some repeater string } \tau_1 \\ & \text{witnessed by } \tau' \text{ of length } k, \text{ xor string } \tau_2, \\ & \text{and } \sigma(\tau') \text{ is a limit ordinal,} \end{cases}$$

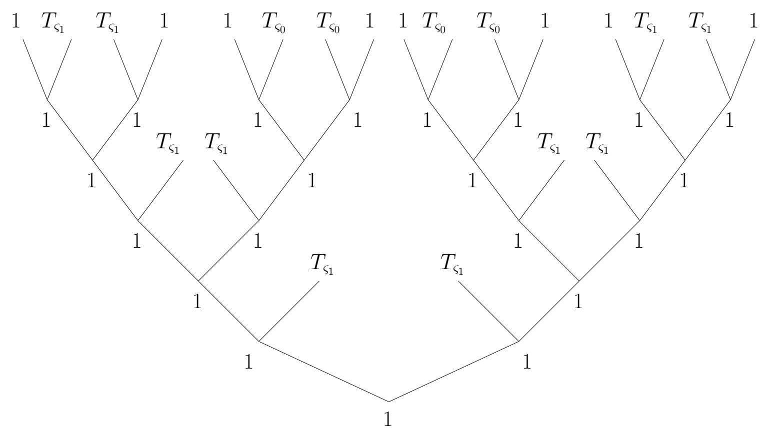


Figure 10: Tree for $\pi(\sigma_{u(2)})$

for $\tau \in 2^{<\omega}$.

We illustrate $\pi(\sigma_{u(2)})$ by showing the location of the ς_α (see Figure 10).

Proposition 2.34. *The measure* $\pi(\sigma)$ *is a measure, i.e., it satisfies*

$$\pi(\sigma)(\tau) = \max\{\pi(\sigma)(\tau ^\frown 0), \pi(\sigma)(\tau ^\frown 1)\}$$

for all $\tau \in 2^{<\omega}$.

The map $\sigma \to \pi(\sigma)$ *is well-defined on homeomorphism types, i.e., if* $\sigma_1 \cong \sigma_2$*, then* $\pi(\sigma_1) \cong \pi(\sigma_2)$.

The map $\sigma \to \pi(\sigma)$ *is injective.*

Proof. The map $\pi(\sigma) : 2^{<\omega} \to \omega_1$ satisfies $\pi(\sigma)(\tau) = \max\{\pi(\sigma)(\tau ^\frown 0), \pi(\sigma)(\tau ^\frown 1)\}$ as the ς_α are measures and $\varsigma_\alpha(\varepsilon) = 1$. If $\sigma_1 \cong \sigma_2$, then any witnessing homeomorphism induces a homeomorphism witnessing $\pi(\sigma_1) \cong \pi(\sigma_2)$. Thus π is well defined. Conversely if $\pi(\sigma_1) \cong \pi(\sigma_2)$, then the witnessing homeomorphism induces a homeomorphism witnessing $\sigma_1 \cong \sigma_2$. Thus π is injective. □

The measure $\pi(\sigma)$ is directly related to the Boolean algebra $\mathcal{B}_\sigma$. Every Boolean algebra can be realized as a closed subset of Cantor space. The measure $\pi(\sigma)$ is the characteristic function of one such subset.

Although the map π is defined on the space of all measures, we apply it only to the depth zero Boolean algebras $\mathcal{B}_{u(S)}$ with $S \subseteq \omega + 1$. In this case, the map π does not increase the depth beyond ω.

Proposition 2.35. *For any set* $S \subseteq \omega + 1$ *with greatest element, the measure* $\pi(\sigma_{u(S)})$ *has depth at most* ω.

Proof. In order to show that $\pi(\sigma_{u(S)})$ has depth at most ω, we argue that non-isomorphic elements have unequal ω^{th} derivatives. We therefore suppose that $x \ncong y$ and consider the cases when neither, one, or both of x and y contain a perfect kernel of rank one points above themselves.

If neither x nor y contain a perfect kernel of rank one points above themselves, then their homeomorphism type is given by a countable ordinal below ω^ω. By Proposition 2.31, they will have unequal ω^{th} derivatives.

If exactly one of x or y contains a perfect kernel of rank one points above themselves (without loss of generality x), then the homemorphism type of y is given by a countable ordinal below ω^ω. Again by Proposition 2.31, they will have unequal ω^{th} derivatives.

If both x and y contain a perfect kernel of rank one points above themselves, then x and y correspond to the image of elements x' and y' in $\sigma_{u(S)}$ under π. Since π is injective, we must have that $x' \ncong y'$. Since $\sigma_{u(S)}$ is depth zero, we must have $\sigma_{u(S)}(x') \neq \sigma_{u(S)}(y')$. It then follows from Proposition 2.31 that the elements $x = \pi(x')$ and $y = \pi(y')$ will have unequal ω^{th} derivatives. □

We obtain the following corollary, making use of Corollary 2.27.

Corollary 2.36. *There are continuum many depth ω, rank one measures.*

We note that the above is a refinement of Corollary 1.11.3 in [18] where Pierce produces continuum many rank one Boolean algebras. Following the notation in [18], in the above we have that V_k is the set of paths in 2^ω with rank at least k for the measure $\sigma_{v(\omega+1)}$. The strictly increasing map $\alpha : \omega \to \omega$ corresponding to a set $S = \{a_0 < a_1 < a_2 < \dots\}$ is $\alpha(n) = a_n$.

2.4 Computable Characterization

Having characterized the algebraic structure of the depth zero Boolean algebras in Section 2.3.1, we turn our attention to characterizing those which have effective representations. We begin by relating the complexity of a measure σ and its derivative $\Delta\sigma(1_{\mathcal{B}})$.

In order to be able to code the finite sequences in $\Delta\sigma(1_{\mathcal{B}})$ as integers, we fix an ordinal $\lambda < \omega_1^{\text{CK}}$ and an ordinal notation for λ, which we denote by ℓ. An ordinal α less than λ is then associated with its unique ordinal notation a satisfying $a <_{\mathcal{O}} \ell$. Having identified ordinals with their ordinal notations, finite sequences of ordinals less than λ can be coded using the standard encoding of finite sequences of integers. The reader is referred to [3] for background on ordinal notations.

In order to be able to maintain uniformity, the following important convention will be assumed for the rest of the paper.

Convention 2.37. *Fix an ordinal $\lambda < \omega_1^{\text{CK}}$. From here on, an ordinal will refer exclusively to an ordinal less than λ. A Boolean algebra will refer to a Boolean algebra of rank less than λ. A measure will refer to a measure of rank less than λ.*

The set $\Delta\sigma(1_{\mathcal{B}})$ is then a set of integers. In order to prevent cumbersome language, we abuse notation and view the first derivative as containing sequences $(\alpha_1, \ldots, \alpha_n)$ of ordinals rather than integers coding sequences of integers, each of which in turn is coding an ordinal.

With these conventions in place, we relate the complexity of a measure σ and its derivative $\Delta\sigma(1_{\mathcal{B}})$.

Proposition 2.38. *If σ is computable, then $\Delta\sigma(1_{\mathcal{B}})$ is computably enumerable. Moreover, there is an effective procedure which, given an index for a measure σ, yields an index for $\Delta\sigma(1_{\mathcal{B}})$.*

Proof. As there is an effective enumeration $\{(x_1, \ldots, x_{m(n)})\}_{n\in\omega}$ of all the disjoint partitions of $1_{\mathcal{F}}$, the set $\Delta\sigma(1_{\mathcal{B}}) = \{(\sigma(x_1), \ldots, \sigma(x_{m(n)}))\}_{n\in\omega}$ is computably enumerable if σ is computable. □

We cannot hope for the converse of Proposition 2.38 to be true, as in general the first derivative $\Delta\sigma(1_{\mathcal{B}})$ does not dictate the measure σ. Modifying the statement to reflect this ambiguity in σ, the converse does hold.

Proposition 2.39. *If $\Delta\sigma(1_{\mathcal{B}})$ is computably enumerable, then there is a computable measure $\hat{\sigma}$ satisfying $\Delta\sigma(1_{\mathcal{B}}) = \Delta\hat{\sigma}(1_{\mathcal{B}})$. Moreover, there is an effective procedure which, given an index $\Delta\sigma(1_{\mathcal{B}})$, yields an index for such a measure $\hat{\sigma}$.*

In addition, the depth of the computable measure $\hat{\sigma}$ produced can be partially controlled: if $\Delta\sigma(1_{\mathcal{B}})$ is the first derivative of a depth zero measure, then the $\hat{\sigma}$ produced will be depth zero.

Proof. We describe an effective procedure to define a measure $\hat{\sigma}$ in ω many stages from $\Delta\sigma(1_{\mathcal{B}})$ with the intention that $\Delta\sigma(1_{\mathcal{B}}) = \Delta\hat{\sigma}(1_{\mathcal{B}})$. In order to guarantee that $\hat{\sigma}$ is total, and thus computable, at stage k we will ensure that $\hat{\sigma}$ is defined on all strings τ with $|\tau| \leq k$.

The construction of $\hat{\sigma}$ is an interplay between trying to split enough (have all elements of $\Delta\sigma(1_{\mathcal{B}})$ belong to $\Delta\hat{\sigma}(1_{\mathcal{B}})$), not split too much (have all elements of $\Delta\hat{\sigma}(1_{\mathcal{B}})$ belong to $\Delta\sigma(1_{\mathcal{B}})$), and split densely enough (have $\hat{\sigma}$ be depth zero, if possible).

In order to ensure that we split enough, we formalize a set of requirements Φ_j for $j \in \omega$.

> *Requirement* Φ_j: If $(\alpha_0, \ldots, \alpha_n)$ is the j^{th} element of $\Delta\sigma(1_{\mathcal{B}})$, then there are disjoint $x_0, \ldots, x_n$ such that $\hat{\sigma}(x_i) = \alpha_i$ for $i \leq n$.

The construction will work to satisfy the Φ_j in increasing order, noting that once Φ_j is satisfied, it can never be injured.

In order to ensure that we do not split too much, at every stage k we ensure that $(\hat{\sigma}(\tau))_{\tau\in 2^k}$ is an element of $\Delta\sigma(1_{\mathcal{B}})$.

In order to ensure that we split densely enough, at every stage k and for every ordinal α, we rank the α-priority of the strings $\tau \in 2^{k-1}$ as follows (with lexicographic order breaking ties).

- If $\hat{\sigma}(\tau) < \alpha$, then τ has no α-priority.
- If $\hat{\sigma}(\tau) = \alpha$, then the α-priority of τ is $|\tau| - |\tau'|$, where τ' is the longest substring of τ such that $\hat{\sigma}(\tau' \frown 0) = \alpha = \hat{\sigma}(\tau' \frown 1)$. If no such substring τ' exists, then the α-priority of τ is $|\tau|$.
- If $\hat{\sigma}(\tau) > \alpha$, then the α-priority of τ is $|\tau| - |\tau'|$, where τ' is the longest substring of τ such that τ' has an extension τ'' with $\hat{\sigma}(\tau'') = \alpha$. If no such substring τ' exists, then the α-priority of τ is $|\tau|$.

In order to track the construction, we maintain a dynamic parameter μ specifying the minimal ordinal in the range of σ seen thus far. Its value may change multiple times during a single stage.

Construction: At stage 0, search for the (unique) sequence in $\Delta\sigma(1_\mathcal{B})$ consisting of exactly one ordinal (i.e., the sequence $(\sigma(1_\mathcal{B}))$). Define $\hat{\sigma}(\varepsilon) = \sigma(1_\mathcal{B})$. Set μ to be the minimum ordinal seen in the range of σ while searching $\Delta\sigma(1_\mathcal{B})$ for the value of $\sigma(1_\mathcal{B})$.

At stage k for $k > 0$, let H_k be the sequence of ordinals (with multiplicity) $(\hat{\sigma}(\tau))_{\tau \in 2^{k-1}}$. Let $j = j_k$ be minimal such that the requirement Φ_j has not yet been satisfied. A requirement Φ_j is satisfied at stage k if H_k is an extension of the j^{th} element of $\Delta\sigma(1_\mathcal{B})$. Decrease the value of μ appropriately if a smaller ordinal was seen in the range of σ while determining j.

Let Z_k be the sequence of ordinals (with multiplicity) appearing in the j^{th} element of $\Delta\sigma(1_\mathcal{B})$ but not in H_k. For each $\alpha \in Z_k$, let τ_α be the string τ in 2^{k-1} with highest α-priority. As there must be a string $\tau \in 2^{k-1}$ with $\hat{\sigma}(\tau) = \sigma(1_\mathcal{B}) \geq \alpha$, there must be a string with some α-priority, so this is well-defined.

We define $\hat{\sigma}(\tau \frown 0)$ and $\hat{\sigma}(\tau \frown 1)$ for all $\tau \in 2^{k-1}$ in two steps. For those strings τ satisfying $\tau = \tau_\alpha$ for some $\alpha \in Z_k$, we define $\hat{\sigma}(\tau \frown 0) = \hat{\sigma}(\tau)$ and $\hat{\sigma}(\tau \frown 1) = \max\{\alpha \in Z_k : \tau = \tau_\alpha\}$. For those strings τ satisfying $\tau \neq \tau_\alpha$ for all $\alpha \in Z_k$, we define $\hat{\sigma}(\tau \frown 0) = \hat{\sigma}(\tau)$. We then search for a sequence ζ in $\Delta\sigma(1_\mathcal{B})$ extending the sequence $P_k = (\hat{\sigma}(\tau) : \hat{\sigma}(\tau) \text{ defined})_{\tau \in 2^k}$ by the requisite number of appearances of μ. If while searching for such a sequence ζ we find a new minimal ordinal in the range of σ, we update μ immediately. When such a sequence ζ is found, define $\hat{\sigma}(\tau \frown 1) = \mu$ for all those τ satisfying $\tau \neq \tau_\alpha$ for all $\alpha \in Z_k$.

Verification: It remains to argue that $\hat{\sigma}$ is a total (and thus computable) measure, that $\Delta\sigma(1_\mathcal{B}) = \Delta\hat{\sigma}(1_\mathcal{B})$, and that $\hat{\sigma}$ is depth zero if possible. We begin with a claim showing that $\hat{\sigma}$ is a measure satisfying $\Delta\hat{\sigma}(1_\mathcal{B}) \subseteq \Delta\sigma(1_\mathcal{B})$.

Claim 2.39.1. *For every integer k, the map $\hat{\sigma}$ is defined on all $\tau \in 2^{k-1}$, satisfies $\hat{\sigma}(\tau) = \max\{\hat{\sigma}(\tau^\frown 0), \hat{\sigma}(\tau^\frown 1)\}$ for all $\tau \in 2^{k-1}$, and the sequence $(\hat{\sigma}(\tau))_{\tau\in 2^k}$ is an element of $\Delta\sigma(1_{\mathcal{B}})$.*

Proof. We prove the claim by induction on k. When $k = 0$, we have $\hat{\sigma}(\varepsilon) = \sigma(1_{\mathcal{B}})$, noting that the search for the sequence in $\Delta\sigma(1_{\mathcal{B}})$ containing exactly one ordinal will terminate.

Assuming the claim for all $m < k$, we show the claim for k. The search for a sequence ζ will terminate. By the inductive hypothesis, we have that $H_k \in \Delta\sigma(1_{\mathcal{B}})$. The extension of H_k to P_k will preserve this inclusion as P_k is a simultaneous refinement of H_k and the j^{th} sequence in $\Delta\sigma(1_{\mathcal{B}})$. If the current value of μ reflects the true minimum of the range of σ, then the desired sequence ζ must exist and be a refinement of P_k. If the current value of μ is incorrect, we will either find such a sequence ζ before the correct value of μ is found, or we will find the correct value of μ and later find such a sequence ζ.

It follows that $\hat{\sigma}$ is defined on 2^k, and, by construction, therefore satisfies $\hat{\sigma}(\tau) = \max\{\hat{\sigma}(\tau^\frown 0), \hat{\sigma}(\tau^\frown 1)\}$. The choice of ζ guarantees that $(\hat{\sigma}(\tau))_{\tau\in 2^k} \in \Delta\sigma(1_{\mathcal{B}})$. □

It follows from the claim that $\hat{\sigma}$ is a total (and thus computable) measure and that $\Delta\hat{\sigma}(1_{\mathcal{B}}) \subseteq \Delta\sigma(1_{\mathcal{B}})$. To show the reverse inclusion, it suffices to establish that every requirement Φ_j is eventually satisfied. For if the $x_0, \ldots, x_n$ witnessing that Φ_j is satisfied do not sum to $1_{\mathcal{B}}$, by changing one x_i with $\hat{\sigma}(x_i) = \sigma(1_{\mathcal{B}})$ to $1_{\mathcal{B}} - x_0 - \cdots - x_{i-1} - x_{i+1} - \cdots - x_n$, we have that $(\alpha_0, \ldots, \alpha_n) \in \Delta\hat{\sigma}(1_{\mathcal{B}})$. In order to show that Φ_j is satisfied, it suffices to show $Z_{k+1} \subsetneq Z_k$ if $j_{k+1} = j_k$. Let $\alpha = \max Z_k$. Then the string $\tau \in 2^{k-1}$ with maximal α-priority will have an extension with measure value α, namely $\tau^\frown 1$. Thus α will appear in Z_{k+1} with multiplicity at least one less than in Z_k. We note that the above is justified as there must be at least one string $\tau \in 2^{k-1}$ with some α-priority as a string $\tau \in 2^{k-1}$ with $\hat{\sigma}(\tau) = \sigma(1_{\mathcal{B}}) \geq \alpha$ must exist. We conclude $\Delta\sigma(1_{\mathcal{B}}) \subseteq \Delta\hat{\sigma}(1_{\mathcal{B}})$, and thus $\Delta\sigma(1_{\mathcal{B}}) = \Delta\hat{\sigma}(1_{\mathcal{B}})$.

We finish by demonstrating $\hat{\sigma}$ is depth zero, if possible. The following claim will be the backbone of this argument.

Claim 2.39.2. *If $\hat{\sigma}(\tau) = \alpha$ and there are arbitrarily many disjoint nodes in σ with measure value $\beta \leq \alpha$, then there are arbitrarily many nodes in $\hat{\sigma}$ above τ with measure value β.*

Proof. If $\beta = \min S$, then β will appear above every node with measure value α, so we assume that $\beta > \min S$. After the stage where $\mu = \min S$, the only new instances of β appearing in the tree (either above a node with measure greater than β or a node of measure β splitting) are as a result of β belonging to some Z_k. By hypothesis, there will be requirements Φ_j necessitating arbitrarily many disjoint elements with measure value β. It follows that β will appear above every τ with $\hat{\sigma}(\tau) = \beta$ as a consequence of the β-priorities. □

If $\Delta\sigma(1_{\mathcal{B}})$ is the derivative of a depth zero measure $\sigma_{u(S)}$, then every $\alpha \in S$ and τ with $\hat{\sigma}(\tau) = \alpha$ satisfies the hypothesis of the claim. It follows that if if $\hat{\sigma}(\tau_1) = \alpha = \hat{\sigma}(\tau_2)$, then $\Delta\hat{\sigma}(\tau_1) = \Delta\hat{\sigma}(\tau_2)$, from which we conclude that $\hat{\sigma}$ is depth zero.

If $\Delta\sigma(1_{\mathcal{B}})$ is the derivative of a depth zero measure $\sigma_{v(S)}$, then every $\alpha \in S$ with $\alpha < \max S$ and τ with $\hat{\sigma}(\tau) = \alpha$ satisfies the hypothesis of the claim. It follows that if $\hat{\sigma}(\tau_1) = \alpha = \hat{\sigma}(\tau_2)$, then $\Delta\hat{\sigma}(\tau_1) = \Delta\hat{\sigma}(\tau_2)$. On the other hand if $\alpha = \max S$ and τ_1 and τ_2 satisfy $\hat{\sigma}(\tau_1) = \alpha = \hat{\sigma}(\tau_2)$, then the proof of the claim is valid for $\beta < \alpha$. We therefore conclude that $\hat{\sigma}$ is depth zero.

This completes the proof of Proposition 2.39. □

As $\Delta\sigma(1_{\mathcal{B}}) = \Delta\hat{\sigma}(1_{\mathcal{B}})$ implies $\sigma \cong \hat{\sigma}$ if both σ and $\hat{\sigma}$ are depth zero, we obtain the following corollary after observing the uniformity present in the proofs of Proposition 2.38 and Proposition 2.39.

Corollary 2.40. *If $\mathcal{B}_\sigma$ is a depth zero Boolean algebra, then σ is computable (i.e., there is a computable measure $\hat{\sigma}$ with $\sigma \cong \hat{\sigma}$) if and only if $\Delta\sigma(1_{\mathcal{B}})$ is computably enumerable. Moreover, there is a procedure that, given an index for a depth zero measure σ, gives an index for $\Delta\sigma(1_{\mathcal{B}})$, and vice versa.*

In addition to there being a relationship between the complexity of σ and $\Delta\sigma(1_{\mathcal{B}})$, there is a relationship between their complexity and the complexity of $\mathcal{B}_\sigma$. Although much more is true, we begin by observing that if σ (of arbitrary depth) is computable, then $\mathcal{B}_\sigma$ is also computable.

Proposition 2.41. *If the measure σ is computable, then the Boolean algebra $\mathcal{B}_\sigma$ is computable.*

Proof. The relationship between σ and $\mathcal{B}_\sigma$ as described in the background is effective. □

It might seem reasonable to conjecture that the converse is also true, namely that a depth zero measure σ is computable if and only if the Boolean algebra $\mathcal{B}_\sigma$. However this is far from true, as the following theorem demonstrates.

Theorem 2.42. *Let $S \subseteq \omega + 1$ be a set with greatest element. Then the following are equivalent:*

1. *The Boolean algebra $\mathcal{B}_{u(S)}$ is computable.*
2. *The Boolean algebra $\mathcal{B}_{v(S)}$ is computable.*
3. *The set $S \setminus \{\omega\}$ is $\Sigma^0_{n \mapsto 2n+3}$ in the Feiner Σ-hierarchy.*

The proof of Theorem 2.42 is lengthy, and is thus split into Section 2.5 and Section 2.6 where we show (1), (2) $\implies$ (3) and (3) $\implies$ (1), (2), respectively.

2.5 Proof of Theorem 2.42 $(1),(2) \implies (3)$

In order to show that $S \setminus \{\omega\}$ is $\Sigma^0_{n \mapsto 2n+3}$ in the Feiner Σ-hierarchy if $\mathcal{B}_{u(S)}$ or $\mathcal{B}_{v(S)}$ is computable, we give, uniformly for each $n \in \omega$, a sentence φ_n of complexity Σ^0_{2n+3}. The sentence φ_n will be constructed so that the depth zero Boolean algebras $\mathcal{B}_{u(S)}$ and $\mathcal{B}_{v(S)}$ satisfy φ_n if and only if $n \in S$.

We start by noting the complexity of various well-known arithmetical formulas (see [3], for example). There is a Π^0_2 formula atomless(x) saying whether x is atomless. There is also, for each computable ordinal α, a computable $\Pi^0_{2\alpha+1}$ formula atom$_\alpha(x)$ saying whether x is an α-atom.

Using these, we define the sentences φ_n and argue that the depth zero Boolean algebras $\mathcal{B}_{u(S)}$ and $\mathcal{B}_{v(S)}$ satisfy φ_n if and only if $n \in S$.

Definition 2.43. *Define φ_0 to be the sentence*

$$\varphi_0 := \exists x \, \mathrm{atomless}(x)$$

and define φ_n for $n > 0$ to be the sentence

$$\varphi_n := \exists x \big[\forall y \leq x \, [\neg \, \mathrm{atom}_n(y)] \;\; \textit{and} \; \forall k \, \exists x_1, \ldots, x_k \\ (x_i \text{ disjoint } \;\textit{and}\; x_i < x \;\textit{and}\; \mathrm{atom}_{n-1}(x_i)) \big]$$

Proposition 2.44. *The depth zero Boolean algebras $\mathcal{B}_{u(S)}$ and $\mathcal{B}_{v(S)}$ satisfy φ_n if and only if $n \in S$.*

Proof. We consider the case when $n = 0$ separately from the case when $n > 0$. If $n = 0$, then $\mathcal{B}_{u(S)}$ and $\mathcal{B}_{v(S)}$ satisfy φ_0 if and only if they have an atomless element, which happens if and only if $0 \in S$.

If $n > 0$, then $\mathcal{B}_{u(S)}$ and $\mathcal{B}_{v(S)}$ satisfy φ_n if and only if they have an element bounding infinitely many $(n-1)$-atoms but no n-atoms. An element witnessing this must be part of the perfect kernel not bounding any n-atoms. For if it were not part of the perfect kernel, then it would be superatomic, an impossibility. The measure of any such element is thus n, which happens if and only if $n \in S$. □

We finish by showing that $(1),(2) \implies (3)$.

Theorem 2.45. *If $\mathcal{B}_{u(S)}$ is computable, then $S \setminus \{\omega\}$ is $\Sigma^0_{n \mapsto 2n+3}$ in the Feiner Σ-hierarchy. If $\mathcal{B}_{v(S)}$ is computable, then $S \setminus \{\omega\}$ is $\Sigma^0_{n \mapsto 2n+3}$ in the Feiner Σ-hierarchy.*

Proof. We begin by analyzing the quantifier complexity of the φ_n. Since atomless(x) has a Π^0_2 representation, the sentence φ_0 has quantifier complexity Σ^0_3. Since atom$_n(x)$ has a Π^0_{2n+1} representation, the sentence φ_n has quantifier complexity Σ^0_{2n+3} for $n > 0$.

As a consequence of Proposition 2.44, it follows that if $\mathcal{B}_{u(S)}$ (respectively $\mathcal{B}_{v(S)}$) is computable, then S is $\Sigma^0_{n \mapsto 2n+3}$ in the Feiner Σ-hierarchy. □

2.6 Proof of Theorem 2.42 (3) $\implies$ (1), (2)

In order to show that $\mathcal{B}_{u(S)}$ and $\mathcal{B}_{v(S)}$ are computable if $S\setminus\{\omega\}$ is $\Sigma^0_{n\mapsto 2n+3}$ in the Feiner Σ-hierarchy, we construct computable copies of them from an index e witnessing that $S\setminus\{\omega\}$ is $\Sigma^0_{n\mapsto 2n+3}$ in the Feiner Σ-hierarchy. Before doing so, we prove a lemma which will be iterated in the construction of $\mathcal{B}_{u(S)}$ and $\mathcal{B}_{v(S)}$. The lemma we prove is a modification of the following well-known theorem. We cite folklore as it appears numerous times in the literature with various attributions.

Theorem 2.46 (Folklore). *There is a procedure, uniform in α and in a $\Delta^0_{2\alpha+1}$ index for the atomic diagram $D(\mathcal{A})$ of a linear order $\mathcal{A}$ with distinguished least element, which yields a Δ^0_1 linear order $\mathcal{L}$ such that $\mathcal{L}\cong\omega^\alpha\cdot\mathcal{A}$.*

For depth zero Boolean algebras with $S\subseteq\omega+1$, it suffices to consider only finite ordinals α. Thurber's argument in [23] therefore serves as an outline to prove the main technical lemma needed.

Lemma 2.47. *There is a procedure, uniform in a Δ^0_3 index for the atomic diagram $D(\mathcal{A})$ of a linear order $\mathcal{A}=(A:\prec)=(\{a_0,a_1,\dots\}:\prec)$ with distinguished least element a_0 and an index for a Σ^0_3 predicate $\exists n\forall u\exists vR(n,u,v)$, which yields a Δ^0_1 linear order $\mathcal{L}$ such that $\mathcal{L}\cong\sum_{a\in A}\mathcal{L}_a$, where $\mathcal{L}_{a_n}\cong 1+\eta+\omega$ if $\forall u\exists vR(n,u,v)$ and $\mathcal{L}_{a_n}\cong\omega$ otherwise.*

Proof. By hypothesis, we have a Δ^0_3 function h such that $h(n)$ codes the atomic diagram of $\mathcal{A}$ restricted to $\{a_0,a_1,a_2,\dots,a_n\}$. By the Limit Lemma, we also have a Δ^0_2 function $g(n,s)$ and a Δ^0_1 function $f(n,s,k)$ (uniformly from an index for h) such that

$$\lim_s g(n,s)=h(n)\qquad\text{and}\qquad\lim_k f(n,s,k)=g(n,s).$$

We impose the following *constraints* on the approximations $g(n,s)$ and $f(n,s,k)$ without any loss of uniformity.

1. The approximations $g(n,s)$ say that a_0 is the least element of $\mathcal{A}$ for all n and s.
2. The approximations $f(n,s,k)$ say that a_0 is the least element of $\mathcal{A}$ for all n, s, and k.
3. The approximations $g(n,s)$ satisfy $g(m,s)\subset g(n,s)$ for all s and $m<n$, i.e., the linear order specified by $g(n,s)$ extends the linear order specified by $g(m,s)$.
4. The approximations $f(n,s,k)$ satisfy $f(m,s,k)\subset f(n,s,k)$ for all s, k, and $m<n$, i.e., the linear order specified by $f(n,s,k)$ extends the linear order specified by $f(m,s,k)$.

Using f we will try to build, for each point a_n in $\mathcal{A}$, a linear order $\mathcal{L}_n$ at the location in $\mathcal{L}$ where we believe a_n to be. The linear order $\mathcal{L}_n$ (termed a *block*) will consist of three parts (termed *segments*): a *singleton segment*, a *dynamic segment* to the right of the singleton segment, and a *discrete segment* to the right of the dynamic segment.

The singleton segment will start as a single point and never have additional points added to it.

The dynamic segment will start as a single point and, in the limit, have order type η or Fin. Whenever a new witness is found for a_n, i.e., a v is discovered such that $R(n, u, v)$ for the least u with no such pre-existing v, the dynamic segment is densified. More specifically, new elements are added at either end of the dynamic segment and between every pair of already existing elements in the dynamic segment. In order to help track for which u witnesses v have been found for n, parameters u_n and v_n are used.

The discrete segment will also start as a single point but, in the limit, have order type ω. At each stage, a new element is added to the right end of the discrete segment. The discrete segment will also serve as a garbage collection for unwanted blocks that were mistakenly created as a consequence of misapproximations of $h(n)$ by $g(n, s)$ or $f(n, s, k)$.

At a given stage k, we will act on behalf of a pair $(n, s) = (n_k, s_k)$ for some $n \leq s \leq k$ using the approximation $f(n, s, k)$. Several possibilities exist, which we describe informally. Our approximation $f(n, s, k)$ may suggest new work, in which case we begin a new block $\mathcal{L}_n$. Our approximation $f(n, s, k)$ may agree with previous work, in which case we simply expand all linear orders $\mathcal{L}_m$ for $m \leq n$. Our approximation $f(n, s, k)$ may disagree with the previous guess about the location of a_n, in which case we *attach* each $\mathcal{L}_m$ for every $m \geq n$ to its predecessor by associating its points with its predecessor's discrete segment. Depending on certain conditions, elaborated on later, we begin building a new instantiation of $\mathcal{L}_n$ at the new location of a_n or *detach* a previously attached instantiation of $\mathcal{L}_n$.

Since we may build a block $\mathcal{L}_n$ for a_n at a wrong location relative to $\mathcal{L}_m$ for $m < n$ because of a misapproximation, multiple instantiations of $\mathcal{L}_n$ for a_n may be started. Ones believed to be incorrectly placed will be attached to their predecessor as suggested above. In order to help determine whether a previously attached version should be detached, each instantiation of a block $\mathcal{L}_n$ is tagged with a tuple $(\tilde{n}, \tilde{s}, \tilde{k})$. The value of $\tilde{s}$ will be the block's *approximation priority*, which will be the value of s_k when the block is begun. The value of $\tilde{k}$ will be the block's *chronological priority*, which will be the value of k when the block is begun. If the chronological priorities of blocks $\mathcal{L}_n$ and $\mathcal{L}_{n'}$ are $\tilde{k}$ and $\tilde{k}'$, we say that $\mathcal{L}_n$ is *chronologically older* than $\mathcal{L}_{n'}$ if $\tilde{k} < \tilde{k}'$.

In order to help track the construction and aid its success, an auxiliary function $r(n, s, k)$ and parameters n_k and s_k are used. The partial function $r(n, s, k)$ describes the approximation when we last acted on behalf of the pair (n, s). More specifically, the value of $r(n, s, k)$ is $f(n, s, k')$, where k' is the last stage when we acted on behalf of (n, s). The parameters n_k and s_k specify the pair $(n, s) = (n_k, s_k)$ acted on behalf of at stage k.

Construction: The construction involves ω many stages and builds a linear order $\mathcal{L}$ with universe $\{b_0, b_1, b_2, \dots\}$. At stage 0, we fix b_0 as the least element of $\mathcal{L}$, tag the block with the tuple $(\tilde{n}, \tilde{s}, \tilde{k}) = (0, 0, 0)$, and commit ourselves to never putting anything before it. We also set the parameters n_0 and s_0 to 0 and put $r(0, 0, 0) = f(0, 0, 0)$.

At the end of each stage $k-1$ for $k \geq 1$, we assume that various blocks have been started and tagged with tuples $(\tilde{n}, \tilde{s}, \tilde{k})$ with $\tilde{n} \leq \tilde{s} \leq \tilde{k}$, that at stage $k-1$ we acted on behalf of the pair (n_{k-1}, s_{k-1}), and that we have defined a partial function $r(n, s, k-1)$ for $n \leq s \leq k-1$ which describes our last action for the pair (n, s).

Each stage $k > 0$ proceeds in three substages: defining the values of the parameters n_k and s_k, acting on behalf of the pair (n_k, s_k), and enlarging all active blocks.

Substage 0 defines the values of the parameters n_k and s_k in the following manner. Let s_k be the least $s \leq s_{k-1}$ such that $f(n, s, k) \neq r(n, s, k-1)$ for some $n \leq s$ if such an s exists; otherwise let $s_k = s_{k-1} + 1$. Then let n_k be the least $n \leq s_k$ such that $f(n, s_k, k) \neq r(n, s_k, k-1)$ if such an n exists; otherwise let $n_k = s_k$.

Substage 1 acts on behalf of the pair (n_k, s_k) and defines more of the function $r(n, s, k)$ in the following manner. Our action depends on which of the following scenarios occurs: we haven't yet begun building a block L_{n_k}; we haven't yet acted for (n_k, s_k), but what we've done so far for (n_k, s) for $s < s_k$ seems correct; we agree with what we've done so far for (n_k, s_k); or we think the block built for (n_k, s_k) is at the wrong place.

1. *Scenario: We haven't yet begun building a block $\mathcal{L}_{n_k}$.* More precisely, the function $r(n, s, k)$ is undefined at n_k for all values of s and k.

   Action: We start a new block $\mathcal{L}_{n_k}$ for a_{n_k}, tag $\mathcal{L}_{n_k}$ with (n_k, s_k, k), initialize the parameters u_{n_k} and v_{n_k}, and update the function $r(n, s, k)$ appropriately. More precisely:

   We begin the block $\mathcal{L}_{n_k}$ in the place indicated by $f(n_k, s_k, k)$ relative to the blocks $\mathcal{L}_m$ for $m < n_k$. In particular, we insert a new element as the singleton segment of $\mathcal{L}_{n_k}$, a new element as the dynamic segment of $\mathcal{L}_{n_k}$, and a new element as the discrete segment of $\mathcal{L}_{n_k}$.

   We tag the block $\mathcal{L}_{n_k}$ with the tuple $(\tilde{n}, \tilde{s}, \tilde{k}) = (n_k, s_k, k)$, thus giving it approximation priority s_k and chronological priority k. These will never change. We initialize the parameters u_{n_k} and v_{n_k} (which are specific to this instantiation of $\mathcal{L}_{n_k}$) to zero. We also set $r(n_k, s_k, k) = f(n_k, s_k, k)$ and set $r(n, s, k) = r(n, s, k-1)$ for all $n < n_k$ and $s \leq s_k$.

2. *Scenario: We haven't yet acted for (n_k, s_k), but what we've done so far for (n_k, s) for $s < s_k$ seems correct.* More precisely, the function $r(n, s, k)$ is undefined at n_k

and s_k for all values of k, but $f(n_k, s_k, k) = r(n_k, s', k-1)$ for all s' with $\tilde{s} \leq s' < s_k$, where $\tilde{s}$ is the approximation priority of the active instantiation of the block $\mathcal{L}_{n_k}$.

Action: We update the function $r(n, s, k)$ appropriately. More precisely:

We set $r(n_k, s_k, k) = f(n_k, s_k, k)$ and set $r(n, s, k) = r(n, s, k-1)$ for all $n \leq n_k$ and $s \leq s_k$ (excepting $n = n_k$ and $s = s_k$).

3. *Scenario: We agree with what we've done so far for* (n_k, s_k). More precisely, $f(n_k, s_k, k) = r(n_k, s_k, k-1)$.

   Action: We update the function $r(n, s, k)$ appropriately. More precisely:

   We set $r(n, s, k) = r(n, s, k-1)$ for all $n \leq n_k$ and $s \leq s_k$.

4. *Scenario: We think the block built for* (n_k, s_k) *is at the wrong place.* More precisely, $f(n_k, s_k, k) \neq r(n_k, s_k, k-1)$.

   Action: We attempt to correct our previous "mistake" by attaching the blocks thought wrongly built to their predecessor. We then build $\mathcal{L}_{n_k}$ at the new location, either by starting a new instantiation of $\mathcal{L}_{n_k}$ or detaching a previously attached instantiation. We then update the function $r(n, s, k)$ appropriately. More precisely:

   Each block $\mathcal{L}_m$ with $m \geq n_k$ is attached to the block $\mathcal{L}_{\ell(m)}$ immediately to its left, beginning with $\mathcal{L}_{n_k}$ and counting upwards. We note the block $\mathcal{L}_{\ell(m)}$ must exist for each m as a result of Constraint 2. The attachment is done by associating all the points in $\mathcal{L}_m$ with the discrete segment of the linear order $\mathcal{L}_{\ell(m)}$. The points from $\mathcal{L}_m$ retain the tuple with which they were tagged at their creation.

   We illustrate an attachment involving several blocks (see Figure 11).

   The instantiation of the block $\mathcal{L}_{n_k}$ at the location given by $f(n_k, s_k, k)$ is either built from scratch or possibly detached from the block $\mathcal{L}_m$ immediately to the left of this location. The block is detached only if:

   (a) It was previously started at some stage k' at this location relative to a_ℓ for $\ell < n_k$, tagged with a tuple $(\tilde{n}, \tilde{s}, \tilde{k})$ satisfying $\tilde{n} = n_k$ and $\tilde{s} \leq s_k$, and attached to $\mathcal{L}_m$ at some stage k'' with $k' < k'' < k$,

   (b) There is no t with $\tilde{s} < t < s_k$ such that $f(n_k, s_k, k) \neq f(n_k, t, k)$.

   (c) Detaching the block would result in no chronologically older block being detached.

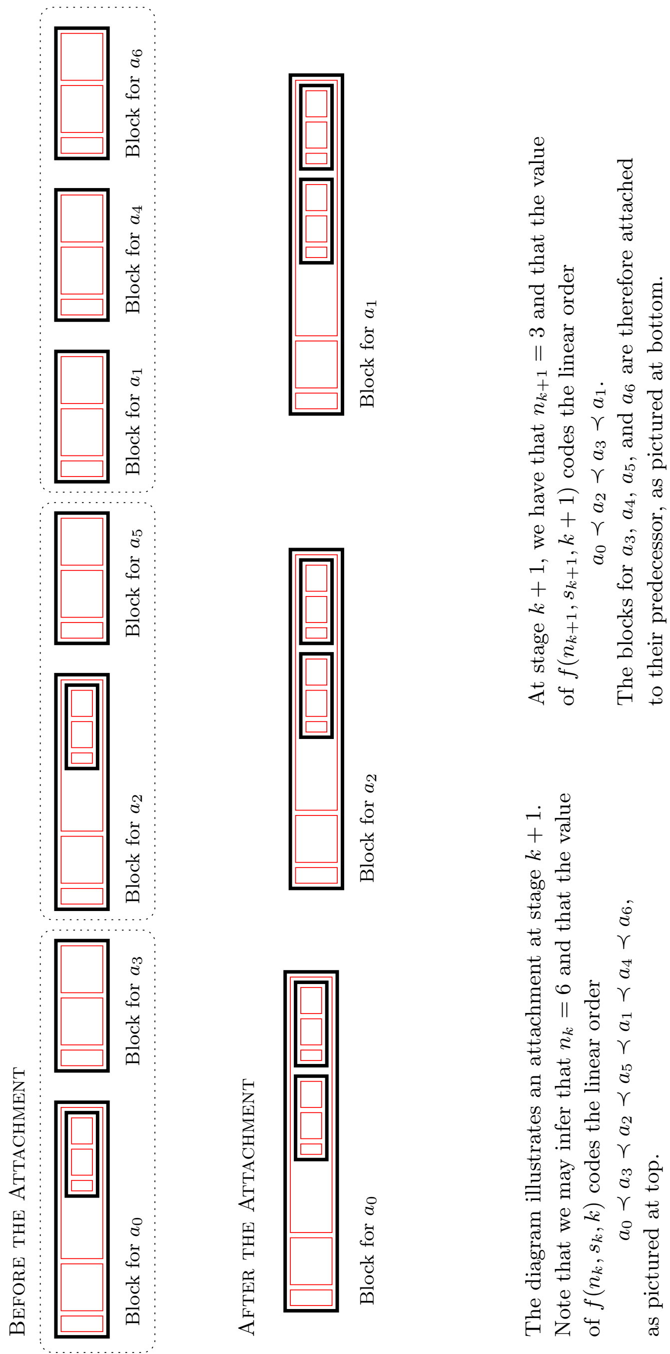


Figure 11: Block Attachment

The detachment is done by splitting off all the elements in the block $\mathcal{L}_m$ that were attached to L_m at the stage k'' or were added to the block $\mathcal{L}_m$ to the right of these elements.

We illustrate a detachment in which all the necessary conditions are met (see Figure 12).

If no detachment occurs, we begin a new instantiation of the block $\mathcal{L}_{n_k}$ at the location given by $f(n_k, s_k, k)$. In particular, we insert a new element as the singleton element segment of $\mathcal{L}_{n_k}$, a new element as the dynamic segment of $\mathcal{L}_{n_k}$, and a new element as the discrete segment of $\mathcal{L}_{n_k}$. We also initialize new parameters u_{n_k} and v_{n_k} (specific to this instantiation of $\mathcal{L}_{n_k}$) to zero.

We set $r(n_k, s_k, k) = f(n_k, s_k, k)$ and set $r(n, s, k) = r(n, s, k-1)$ for all $n \leq n_k$ and $s \leq s_k$ (except $n = n_k$ and $s = s_k$).

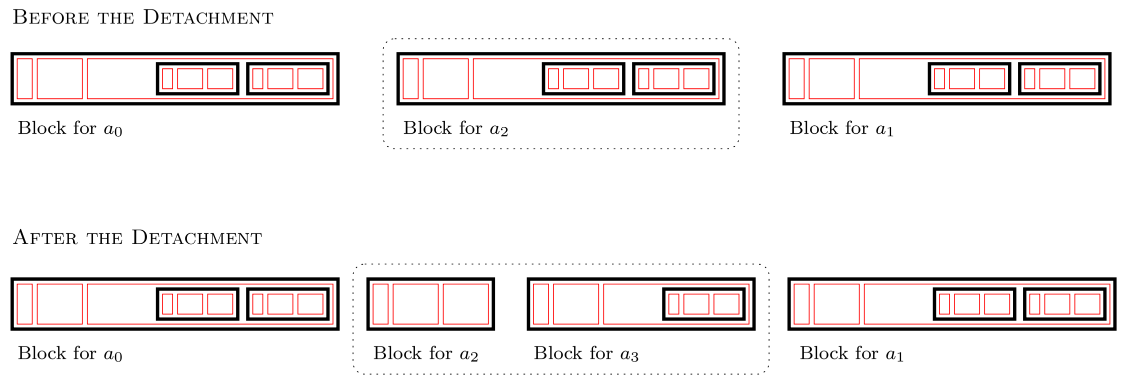
Substage 2 enlarges all blocks $\mathcal{L}_n$ for $n \leq n_k$ in the following manner. The dynamic segment is densified depending on whether $R(n, u_n, v_n)$ holds or not. If $R(n, u_n, v_n)$ holds, then u_n is incremented, v_n is reset to 0, and the dynamic segment is densified by adding new elements at either end and between every pair of already existing elements of the dynamic segment. If $R(n, u_n, v_n)$ fails to hold, we leave u_n unchanged, increment v_n, and do not modify the dynamic segment. Independent of whether $R(n, u_n, v_n)$ holds, a new element is added at the right end of the discrete segment.

This completes the description of the construction.

Verification: In order to be sure that our construction yields $\sum_{a \in A} \mathcal{L}_a$, we verify that a block $\mathcal{L}_n$ of the correct order type is built at the correct location relative to blocks $\mathcal{L}_m$ for $m \leq n$. As blocks will be absorbed and unabsorbed into other blocks through the processes of attachment and detachment, the *owner* of an instantiation of a block $\mathcal{L}_n$ at stage k is the block $\mathcal{L}_{n'}$ such that $\mathcal{L}_n$ is part of the discrete segment of $\mathcal{L}_{n'}$ at stage k (allowing the possibility of $n = n'$). A stage k where $h(n) \subseteq f(n_k, s_k, k)$ is said to be a *true stage for n*.

We begin by establishing the following claims.

- All wrongly started pieces are eventually found to be wrongly started and permanently attached.
- For each n, a block $\mathcal{L}_n$ is built for a_n at the correct location relative to a_ℓ for $\ell < n$ that is detached again after each time it is attached.
- For each n, the block $\mathcal{L}_n$ built for a_n at the correct location relative to a_ℓ for $\ell < n$ is of the correct isomorphism type.



The diagram illustrates a detachment at stage $k + 1$. We assume that the left block inside the ω segment of the block for a_2 was built on behalf of a_3 at some prior stage and that it satisfies the requirements for detachment. In particular, we assume that the block detached with a_3 is of a chronologically younger age then the block for a_3.

The detachment leaves the blocks for a_0 and a_1 untouched. It separates the pre-existing block for a_3 from the discrete segment of a_2, incorporating the other block attached to a_2 at the beginning of the detachment into the discrete segment of the detached a_3 block.

Figure 12: Block Detachment

Roughly speaking, the first guarantees that we have no extra elements in the linear order $\mathcal{L}$ while the second and third together guarantee that we have at least the appropriate elements in the linear order $\mathcal{L}$. Before formalizing and proving these claims, we show the following combinatorial claims about the construction.

Claim 2.47.1. *For a fixed s, there are only finitely many k such that $s_k = s$. Thus, for a fixed s, there are only finitely many k such that $s_k \leq s$.*

Proof. We show there are only finitely many k such that $s_k = s$ by induction on s. For $s = 0$, we have $s_k = 0$ if and only if $k = 0$ as there is only one possible atomic diagram for the linear order consisting only of a_0. Consequently the value of $f(0,0,k)$ is constant in k, so $r(0,0,k)$ will be constant in k, and thus the condition $f(0,0,k) \neq r(0,0,k-1)$ will never be met.

Assuming $s_k = s$ for only finitely many k, we argue $s_k = s+1$ for only finitely many k. If $s_k = s+1$, either $s_{k-1} = s$ or $f(n, s+1, k) \neq r(n, s+1, k-1)$ for some $n \leq s+1$. By the inductive hypothesis, the former can happen only finitely often. Fixing n, let $\hat{k}$ be least such that $f(n, s+1, k) = g(n, s+1)$ for all $k > \hat{k}$. For $k \geq \hat{k}$, we will have the inequality $f(n, s+1, k) \neq r(n, s+1, k-1)$ for at most one k; if we witness this inequality at stage k, we will set $r(n, s+1, k) = f(n, s+1, k)$ and both $f(n, s+1, \cdot)$ and $r(n, s+1, \cdot)$ will be constant and equal thereafter. Thus the latter condition can also happen only finitely often.

The first statement in the claim then follows by induction, from which the second statement in the claim immediately follows. □

Claim 2.47.2. *For each n, there are infinitely many true stages for n.*

Proof. We begin by noting that if k is a true stage for n, then k is a true stage for all $m < n$ by Constraint 4. Also as $n_k \leq k$ for all k, a stage k can only be a true stage for finitely many n. We therefore need only show that for each n, there is a stage k such that k is a true stage for n.

In order to do so, fix n and let $\hat{s}$ be least such that $g(n,s) = h(n)$ for all $s \geq \hat{s}$. Let $\hat{k}$ be least such that $f(n, \hat{s}, k) = g(n, \hat{s})$ for all $k \geq \hat{k}$. Finally, let s' be least greater than $\hat{s}$ such that $f(n, s', \cdot)$ has not yet converged to $g(n,s)$ by stage $\hat{k}$. If no such s' exists, then the claim follows as any stage k with $n_k \geq n$ and $s_k \geq \hat{s}$ is then a true stage for n.

Let $\hat{k}'$ be least such that $f(n, s', k) = g(n, s')$ for all $k \geq \hat{k}'$. At stage $\hat{k}'$, we will have $n_{\hat{k}'} \leq n$ and $s_{\hat{k}'} \leq s'$. If $n_{\hat{k}'} = n$ and $s_{\hat{k}'} = s'$, then $\hat{k}'$ will be a true stage for n. Otherwise, the first stage $k > \hat{k}'$ with $n_k \geq n$ and $s_k \geq s'$ will be a true stage for n. □

We continue by proving the three claims.

Claim 2.47.3. *If a block $\mathcal{L}_n$ for a_n is started at the wrong location relative to $\mathcal{L}_m$ for $m < n$, there is a stage k at which $\mathcal{L}_n$ is attached and never detached afterwards.*

Proof. We prove the claim by using induction on the chronological priority of the wrongly started block $\mathcal{L}_n$. The oldest chronological block, the block $\mathcal{L}_0$ created at stage 0, is never found to be wrongly started by Constraint 2. Proceeding with the induction, we assume that $n > 0$ and that the claim is valid for all wrongly started blocks with stronger chronological priority.

A block $\mathcal{L}_n$ for a_n can be wrongly started for several reasons: if $g(n,s)$ was wrong but we guessed correctly for it, i.e., $f(n,s,k) = g(n,s) \neq h(n)$; if $g(n,s)$ was correct but we guessed incorrectly for it, i.e., $f(n,s,k) \neq g(n,s) = h(n)$; and if $g(n,s)$ was wrong and we guessed incorrectly for it, i.e., $f(n,s,k) \neq g(n,s) \neq h(n) \neq f(n,s,k)$.

In all cases, there is an $\hat{s}$ such that $g(n,s) = h(n)$ for all $s \geq \hat{s}$ and a $\hat{k}$ such that $f(n,\hat{s},k) = g(n,\hat{s}) = h(n)$ for all $k \geq \hat{k}$. The strategy in Scenario 4 implies that we will not detach this block for the sake of a pair (n,s) for any $s \geq \hat{s}$ after stage $\hat{k}$. Moreover, as $s_k < \hat{s}$ for only finitely many k by Claim 2.47.1, this block will be detached only finitely often for (n,s) for $s < \hat{s}$.

Moreover, as we are assuming the claim for all blocks of stronger chronological priority, each of these blocks will be detached wrongly only finitely often as a consequence of their own actions. Thus $\mathcal{L}_n$ will eventually be attached and never detached afterwards.

We note that if $f(n,s,k) \neq g(n,s) \neq h(n) = f(n,s,k)$ for some n, s, and k, then the approximation errors seem to "cancel" each other. However, at some later stage, $f(n,s,k)$ will converge to $g(n,s)$, and shortly afterward the wrongly started block will be permanently attached to its predecessor. □

Claim 2.47.4. *For each n, there is a block $\mathcal{L}_n$ built for a_n at the correct location relative to $\mathcal{L}_m$ for $m < n$ that is eventually detached after every time it is attached.*

Proof. We establish the claim ignoring the possibly detrimental effects of chronological priorities. By this we mean that every such block will be eventually detached if Detachment Condition (c) is not required. After doing so, we argue that the claim holds after considering chronological priorities.

Fixing n, let $\hat{s}$ be least such that $g(n,s) = h(n)$ for all $s \geq \hat{s}$. Let $\hat{k}$ be least such that $f(n,\hat{s},k) = g(n,\hat{s}) = h(n)$ for all $k \geq \hat{k}$. Consider any stage $\ell \geq \hat{k}$ such that the following criterion are met:

1. For all stages $k \geq \ell$, $s_k \geq \hat{s}$.

2. The stage ℓ is a true stage for n.

We argue that the block $\mathcal{L}_n$ existing for a_n at stage ℓ is eventually detached after every time it is attached, ignoring the possibly detrimental effects of chronological priorities. We therefore assume that this block is later attached on behalf of some pair (n', s') with $n' \leq n$ and $s' > \hat{s}$. We note these inequalities on n' and s' must be satisfied as a result of Constraint 4 and the hypotheses on ℓ.

As this block was built at the correct location relative to a_m for $m < n$, the attachment must have been a result of $f(n', s', \cdot)$ misapproximating $g(n', s')$. We note that the attachment could not have been a result of $g(n', \cdot)$ misapproximating $h(n')$ as a consequence of Constraint 3.

At some point after the attachment on behalf of the pair (n', s'), there will be a stage k such that the pair (n_k, s_k) satisfies $n_k = n$ and $f(n, s, k) = g(n, s) = h(n)$ for all $\hat{s} \leq s \leq s_k$. At this stage k, this block will be detached (ignoring chronological priorities) as the other conditions required for a block detachment are necessarily met. We conclude that, ignoring the possibly detrimental effects of chronological priorities, the claim holds.

We continue by showing, using induction on n, that the claim holds after considering the possibly detrimental effects of chronological priorities. Assuming the claim for all $m < n$, we show it is true for n. We may assume that the block $\mathcal{L}_n$ discussed will eventually have an older block permanently attached to it, else we are done with the claim. Consequently, we will create a new instantiation of the block $\mathcal{L}_n$ as the older block will prevent a detachment. We may assume that this instantiation exists at a stage ℓ' later than ℓ such that all true blocks $\mathcal{L}_m$ for $m < n$ have been created and ℓ' is a true stage for n. We note that such a stage must exist as there are infinitely many true stages for n and by the inductive hypothesis.

We argue that the instantiation of the block $\mathcal{L}_n$ existing at stage ℓ' satisfies the claim. In order to do so, we need only show that it will never have a chronologically older block attached to it. As all the true blocks $\mathcal{L}_m$ have been created for $m < n$ and ℓ' is a true stage, the instantiations of the blocks for $m < n$ at stage ℓ' must be the true instantiations. It follows that no chronologically older incorrect blocks will ever be attached to $\mathcal{L}_n$ (at a true stage).

By induction, we conclude that the claim holds after considering the possibly detrimental effects of chronological priorities. □

Claim 2.47.5. *For the block $\mathcal{L}_n$ at the correct location that is eventually detached after every time it is attached, the order type is $1 + \eta + \omega$ if $\forall u \exists v R(n, u, v)$ and ω otherwise.*

Proof. Claim 2.47.4 established that there is a block $\mathcal{L}_n$ at the correct location that is detached again after each time it is attached, and so this instantiation is active infinitely often. Thus there are infinitely many stages at which work is done for the dynamic segment and the discrete segment. If $\forall u \exists v R(n, u, v)$, then for every u a new witness v will eventually be found. Each time one is, the dynamic segment is densified, from which it follows that the order type η is built. If instead $\exists u \forall v \neg R(n, u, v)$, then the dynamic segment is densified only finitely often, from which it follows that the order type FIN is built. The existence of infinitely many stages at which work is done for the discrete segment is enough for the discrete segment to have order type ω in the limit.

Additionally, the actions of $\mathcal{L}_m$ for $m \neq n$ do not interfere with the order type of $\mathcal{L}_n$. If a block $\mathcal{L}_m$ is attached to $\mathcal{L}_n$ at some point permanently, the finite amount of $\mathcal{L}_m$ built to that stage is successfully incorporated into the discrete segment of $\mathcal{L}_n$. If a block $\mathcal{L}_m$ is attached and detached infinitely often to $\mathcal{L}_n$, it is not part of the block $\mathcal{L}_n$. Moreover, at least each time it is detached, the discrete segment of $\mathcal{L}_n$ is extended. As no block $\mathcal{L}_m$ interferes with the singleton element segment or the dynamic segment of $\mathcal{L}_n$, the $\mathcal{L}_m$ for $m \neq n$ do not interfere with the order type of the block $\mathcal{L}_n$.

The claim follows, observing that $1 + \textsc{Fin} + \omega = \omega$. □

Combined, we argue that the claims guarantee that $\mathcal{L} \cong \sum_{a \in A} \mathcal{L}_a$. Claim 2.47.3 guarantees that any wrongly started block is eventually permanently attached. Claim 2.47.4 guarantees that a block $\mathcal{L}_n$ is built for a_n at the correct location, and Claim 2.47.5 guarantees that this block is of the correct order type.

This completes the proof of Lemma 2.47. □

Noting that the lemma relativizes, we obtain the following corollary.

Corollary 2.48. *Uniformly in*

1. *an integer k,*
2. *a Δ^0_{2k+3} index for the atomic diagram $D(\mathcal{A})$ of a linear order $\mathcal{A} = (A : \prec) = (\{a_0, a_1, \dots\} : \prec)$ with distinguished least element a_0, and*
3. *an index for a Σ^0_{2k+3} predicate $\exists n \forall u \exists v R(n, u, v)$ where $R(n, u, v)$ is Δ^0_{2k+1},*

there is an index for a Δ^0_{2k+1} linear order $\mathcal{L}$ such that $\mathcal{L} \cong \sum_{a \in A} \mathcal{L}_a$, where $\mathcal{L}_{a_n} = 1 + \eta + \omega$ if $\forall u \exists v R(n, u, v)$ and $\mathcal{L}_{a_n} = \omega$ otherwise.

Before continuing, we make several remarks and observations about the preceding lemma and corollary. Although Thurber's argument serves as an outline for the above proof, the argument offered in [23] seems to be unclear: it fails to utilize chronological priorities, and as such allows ω^* chains of wrongly started blocks to exist.

By altering the Σ^0_3 predicate in Lemma 2.47 (or the Σ^0_{2k+3} predicate in Corollary 2.48), we can partially control how many copies of η are built.

Remark 2.49. *From an index of a Σ^0_3 predicate $\exists n \forall u \exists v R(n, u, v)$, we can uniformly obtain an index of a Σ^0_3 predicate $\exists n \forall u \exists v R'(n, u, v)$ such that if there is an n satisfying $\forall u \exists v R(n, u, v)$, then cofinitely many n satisfy $\forall u \exists v R'(n, u, v)$ (see [22]).*

Thus in Corollary 2.48 we may assume that either none or cofinitely many of the $\mathcal{L}_n$ are $1 + \eta + \omega$.

Therefore, under the above remark, the effect of moving from a Δ^0_{2k+3} linear order to a Δ^0_{2k+1} linear order as in Corollary 2.48 is that every point is replaced by a copy of ω, possibly with dense intervals η inserted after all but finitely many of the first points of the copies of ω.

We continue by giving the basic construction that, when iterated, will form the crux of the construction of $\mathcal{B}_{u(S)}$ and $\mathcal{B}_{v(S)}$.

Lemma 2.50. *Uniformly in*

1. *an integer* n*,*
2. Σ^0_{2k+3} *predicates* R_k *for* $0 \leq k < n$ *specifying membership of* k *in* S*, and*
3. *a* Π^0_{2n+2} *predicate* R_n *specifying membership of* n *in* S*,*

there is an index for

(a) *the algebra* $\mathcal{B}_{u(S\restriction(n+1))}$ *if* $n \in S$ *via the* Π^0_{2n+2} *predicate* R_n*, or*

(b) *the algebra* $\mathcal{B}_{u(S\restriction(k+1))} \oplus \mathcal{B}_{\ell\cdot\omega^n}$ *for some integers* $k < n$ *and* ℓ*, or the algebra* $\mathcal{B}_{\ell\cdot\omega^n}$ *for some integer* ℓ *if* $n \notin S$ *via the* Π^0_{2n+2} *predicate* R_n*.*

Proof. Fix n and an index e uniformly giving the predicates R_k. We note that membership of k in S for $0 \leq k < n$ can be viewed as a Σ^0_3 question relative to Δ^0_{2k+1} and membership of n in S can be viewed as a Π^0_2 question relative to Δ^0_{2n+1}. In order to construct an appropriate Boolean algebra, we begin by building a Δ^0_{2n+1} linear order using the relativized Π^0_2 predicate. Using Corollary 2.48, we will iteratively build Δ^0_{2n-1}, Δ^0_{2n-3}, ..., Δ^0_3, and Δ^0_1 linear orders. The interval algebra of the Δ^0_1 linear order so produced will be the desired Boolean algebra.

Construction: Using the Π^0_2 predicate R_n relative to Δ^0_{2n+1}, we build a Δ^0_{2n+1} linear order that is $1+\eta$ if the predicate R_n holds and Fin otherwise.

Assuming we have a Δ^0_{2k+3} linear order for some k with $0 \leq k < n$, we describe a uniform procedure to obtain a Δ^0_{2k+1} linear order. We view the Δ^0_{2k+3} linear order as a Δ^0_3 linear order relative to Δ^0_{2k+1} and view the predicate R_k as a Σ^0_3 predicate relative to Δ^0_{2k+1}. The Δ^0_{2k+1} linear order is then the linear order produced by Corollary 2.48 relativized to Δ^0_{2k+1}.

This completes our description of the construction.

Verification: We begin by letting $\lfloor k \rfloor$ denote the greatest integer not greater than k that is a member of S. We proceed by establishing the following claim by induction on k for $k < n$.

Claim 2.50.1. *If* $k \in S$*, then cofinitely many of the points appearing in the* Δ^0_{2k+3} *linear order become a copy of a suborder in the* Δ^0_1 *linear order whose interval algebra is* $\mathcal{B}_{u(S\restriction(k+1))} \oplus \mathcal{B}_{\omega^{k+1}}$*. The remaining points appearing in the* Δ^0_{2k+3} *linear order become a copy of a suborder in the* Δ^0_1 *linear order whose interval algebra is* $\mathcal{B}_{u(S\restriction(\lfloor k' \rfloor+1))} \oplus \mathcal{B}_{\omega^{k+1}}$ *for some integer* $k' < k$*.*

If $k \notin S$*, then all of the points appearing in the* Δ^0_{2k+3} *linear order become a copy of a suborder in the* Δ^0_1 *linear order whose interval algebra is* $\mathcal{B}_{u(S\restriction(\lfloor k \rfloor+1))} \oplus \mathcal{B}_{\omega^{k+1}}$*.*

Proof. If $k = 0$, then the claim follows from Lemma 2.47 and Remark 2.49.

Assuming the claim for all $m < k$, we show that it holds for k. By Corollary 2.48 and Remark 2.49, the points in the Δ^0_{2k+3} linear order will become copies of $\omega^{k-\lfloor k \rfloor}$ in the $\Delta^0_{2\lfloor k \rfloor+3}$ linear order.

If $\lfloor k \rfloor < k$, by induction, cofinitely many of the $\Delta^0_{2\lfloor k \rfloor+3}$ points will have a suborder whose interval algebra is $\mathcal{B}_{u(S\restriction(\lfloor k \rfloor+1))} \oplus \mathcal{B}_{\omega^{\lfloor k \rfloor+1}}$ built for them, while the remaining finitely many will have a suborder whose interval algebra is $\mathcal{B}_{u(S\restriction(\lfloor k' \rfloor+1))} \oplus \mathcal{B}_{\omega^{\lfloor k \rfloor+1}}$ for some integer $k' < k$ built for them. Thus each point of the Δ^0_{2k+3} linear order has a suborder in the Δ^0_1 linear order whose interval algebra is $\mathcal{B}_{u(S\restriction(\lfloor k \rfloor+1))} \oplus \mathcal{B}_{\omega^{k+1}}$ built for it.

If $\lfloor k \rfloor = k$, by Corollary 2.48, cofinitely many of the points in the Δ^0_{2k+3} linear order will become copies of $1 + \eta + \omega$ in the Δ^0_{2k+1} linear order, while the remaining finitely many will become ω in the Δ^0_{2k+1} linear order. Making use of the inductive hypothesis, it follows that each point in the Δ^0_{2k+3} linear order has a suborder in the Δ^0_1 linear order whose interval algebra is $\mathcal{B}_{u(S\restriction(k+1))} \oplus \mathcal{B}_{\omega^{k+1}}$ built for them. □

By Claim 2.50.1, at least cofinitely many of the points in the Δ^0_{2n+1} linear order has a suborder in the Δ^0_1 linear order whose interval algebra is $\mathcal{B}_{u(S\restriction n)} \oplus \mathcal{B}_{\omega^n}$ built for them. If $n \in S$, then the Δ^0_{2n+1} linear order is $1+\eta$, so the Δ^0_1 linear order will have $\mathcal{B}_{u(S\restriction(n+1))}$ as its interval algebra. If $n \notin S$, then the Δ^0_{2n+1} linear order is Fin, so the Δ^0_1 linear order will have $\mathcal{B}_{u(S\restriction n)} \oplus \mathcal{B}_{\ell\cdot\omega^n}$ as its interval algebra. □

By repeating the construction in Lemma 2.50 for increasing n, we are able to build computable copies of $\mathcal{B}_{u(S)}$ and $\mathcal{B}_{v(S)}$ if $S \setminus \{\omega\}$ is $\Sigma^0_{n\mapsto 2n+3}$ in the Feiner Σ- hierarchy.

Theorem 2.51. *If $S \subseteq \omega+1$ is a set with greatest element such that $S \setminus \{\omega\}$ is $\Sigma^0_{n\mapsto 2n+3}$ in the Feiner Σ-hierarchy, then $\mathcal{B}_{u(S)}$ ($\mathcal{B}_{v(S)}$, respectively) is computable.*

Proof. Let $S \subseteq \omega + 1$ be a set with greatest element such that $S \setminus \{\omega\}$ is $\Sigma^0_{n\mapsto 2n+3}$ in the Feiner Σ-hierarchy and let the index e witness this. We build a computable linear order $\mathcal{L}_{u(S)}$ ($\mathcal{L}_{v(S)}$, respectively) by iterating Lemma 2.50 for increasing n. We note that we may assume that S is infinite, else $\mathcal{B}_{u(S)}$ ($\mathcal{B}_{v(S)}$, respectively) is computable by Proposition 2.41.

The linear order $\mathcal{L}_{u(S)}$ ($\mathcal{L}_{v(S)}$, respectively) is constructed by building a linear order of the form $\sum_{\tau\in 2^{<\omega}} \mathcal{L}_\tau$. The linear order $\mathcal{L}_\tau$ depends on the value of $\sigma_{u(\omega+1)}(\tau)$ ($\sigma_{v(\omega+1)}(\tau)$, respectively) and the set S.

Construction: We build a linear order $\sum_{\tau\in 2^{<\omega}} \mathcal{L}_\tau$, where $\mathcal{L}_\tau$ is as follows.

Value ω: If $\sigma_{u(\omega+1)}(\tau) = \omega$ ($\sigma_{v(\omega+1)}(\tau) = \omega$, respectively) and $|\tau| = \ell$, the linear order $\mathcal{L}_\tau$ is ω^ℓ.

Value 0: If $\sigma_{u(\omega+1)}(\tau) = 0$ ($\sigma_{v(\omega+1)}(\tau) = 0$, respectively), the linear order $\mathcal{L}_\tau$ is the empty linear order.

Value n: If $\sigma_{u(\omega+1)}(\tau) = n$ ($\sigma_{v(\omega+1)}(\tau) = n$, respectively) with $0 < n < \omega$ and $|\tau| = \ell$, we use Lemma 2.50 to build $\mathcal{L}_\tau$. More specifically, the Π^0_{2n+2} predicate is the ℓ^{th} column of the Σ^0_{2n+3} predicate giving membership of n in S.

This completes our description of the construction.

Verification: We verify that the interval algebra of $\sum_{\tau\in 2^{<\omega}} \mathcal{L}_\tau$ is $\mathcal{B}_{u(S)}$ ($\mathcal{B}_{v(S)}$, respectively). As $\mathcal{L}_\tau = \omega^\ell$ if $\sigma_{u(\omega+1)}(\tau) = \omega$ ($\sigma_{v(\omega+1)}(\tau) = \omega$, respectively) and $|\tau| = \ell$, the placement of the rank ω points is correct.

Above each rank ω point in the measure $\sigma_{u(\omega+1)}$ ($\sigma_{v(\omega+1)}$, respectively) are rank n points for arbitrarily large n. For $n \in S$, a copy of $\mathcal{B}_{u(S\restriction(n+1))}$ will appear above cofinitely many of the rank n points by Lemma 2.50. As S was assumed to be infinite, the placement of the rank n points is correct.

It remains to verify that the points built for numbers $n \notin S$ and for the finitely many exceptional rank n points for $n \in S$ do not disturb the isomorphism type of the interval algebra. In either case, by Lemma 2.50, such interval algebras will either be superatomic of small rank or $\mathcal{B}_{u(S\restriction(k+1))}$ for some integer k. Because of the placement of such points, in the former case they can be thought of as contributing to the rank ω points; in the latter case they can be thought of as contributing to the rank k points.

We conclude that the interval algebra of $\sum_{\tau\in 2^{<\omega}} \mathcal{L}_\tau$ is $\mathcal{B}_{u(S)}$ ($\mathcal{B}_{v(S)}$, respectively). □

2.7 Applications to the Low$_n$ Conjecture

As a corollary to Theorem 2.42, we obtain partial results on the Low$_n$ Conjecture. For a fixed n, let Low$_n$ be the statement that every Boolean algebra with a presentation computable in a Low$_n$ degree is computable.

Conjecture 2.52. *For every n, the statement* Low$_n$ *holds.*

Jockusch showed Low$_1$ in [5]; Thurber showed Low$_2$ in [24]; and Knight and Stob showed Low$_4$ in [16]. Although progress is being made towards resolving the Low$_n$ conjecture for $n \geq 5$ (see [10], for example), the Low$_n$ conjecture remains open for $n \geq 5$.

As a corollary to Theorem 2.42, we resolve the conjecture for depth zero, rank ω Boolean algebras.

Corollary 2.53. *If a depth zero, rank ω Boolean algebra $\mathcal{B}$ has a* Low$_k$ *presentation for some k, it has a computable presentation.*

Proof. Fix a Low$_k$ set A so that $\mathcal{B}$ has an A-computable presentation. Relativizing Theorem 2.45 to A, we have that $n \in S$ if and only if a Σ^A_{2n+3} predicate holds.

For $n > k$, we have that $\Sigma^A_{2n+3} = \Sigma^0_{2n+3}$ as $A^{2n+3} = (A^{(k)})^{(2n-k+3)} = (\emptyset^{(k)})^{(2n-k+3)} = \emptyset^{(2n+3)}$. Thus from the finite information of $S \restriction k$, we can build the Boolean algebra $\mathcal{B}$. □

2.8 Future Directions

Although we have characterized the computable depth zero, rank ω Boolean algebras, we leave open the characterization for higher rank depth zero Boolean algebras. A generalization of the Feiner Σ-hierarchy seems necessary, which we state under Convention 2.37 where we fixed an ordinal $\lambda < \omega_1^{\mathrm{CK}}$ and an ordinal notation ℓ for λ.

Definition 2.54. *For an ordinal $\alpha < \lambda$ (i.e., $a <_{\mathcal{O}} \ell$ with $\alpha = |a|_{\mathcal{O}}$), define $\emptyset^{(\leq\alpha)}$ to be the set*

$$\emptyset^{(\leq\alpha)} = \{\langle k, m\rangle : m \in \emptyset^{(|k|_{\mathcal{O}})}, k \leq_{\mathcal{O}} a\},$$

where $|k|_{\mathcal{O}}$ denotes the ordinal for which k is a notation.

Definition 2.55. *Let $S \subseteq \lambda$ be a set computable in $\emptyset^{(\lambda)}$. Then S is* $\Sigma^0_{\alpha\mapsto b\cdot\alpha+a}$ in the Feiner Σ-hierarchy *if there exists an index e such that*

1. *The set S satisfies $S = W_e^{\emptyset^{(\lambda)}}$.*
2. *The computations $\varphi_e^{\emptyset^{(\leq b\cdot\alpha+a)}}(n)$ and $\varphi_e^{\emptyset^{(\lambda)}}(n)$ are equal where $\alpha = |n|_{\mathcal{O}}$; in particular, neither queries any number $\langle k, m\rangle$ with $|k|_{\mathcal{O}} > b\cdot\alpha + a$.*

We conjecture that the straightforward generalization of Theorem 2.42 is true, namely:

Conjecture 2.56. *Let $S \subseteq \lambda$ be a set with greatest element. Then the following are equivalent:*

1. *The Boolean algebra $\mathcal{B}_{u(S)}$ is computable.*
2. *The Boolean algebra $\mathcal{B}_{v(S)}$ is computable.*
3. *The set S is $\Sigma^0_{\alpha\mapsto 2\cdot\alpha+3}$ in the Feiner Σ-hierarchy.*

An analog of Lemma 2.47 for limit ordinals should be all that is necessary to prove Conjecture 2.56. Unfortunately, the proof of Lemma 2.47 does not generalize to limit ordinals. Instead it is likely that the method of infinite games, as in [1], will be needed to prove the analog of Lemma 2.47 for limit ordinals.

In addition to a characteration of the computable depth zero Boolean algebras of higher rank, we leave open the characterization of the computable Boolean algebras of higher depth. Although the unrestricted question may still be intractable, certain subclasses seem both natural and tractable. For example, a characterization of the computable Boolean algebras of the form $\pi(\sigma)$, or even $\pi(\sigma_{u(S)})$ and $\pi(\sigma_{v(S)})$, would be desirable.

Examining the boundary conditions around ω_1^{CK} is also an important question. As the linear order $\omega_1^{\mathrm{CK}} \cdot (1 + \eta)$ has a computable copy, it follows that the depth zero Boolean algebra $\mathcal{B}_{u(\{\omega_1^{\mathrm{CK}}\})} = \mathcal{B}_{v(\{\omega_1^{\mathrm{CK}}\})}$ is computable. Are there other computable depth zero measures with ω_1^{CK} in the range? Are there computable depth zero measures with both ω_1^{CK} and ordinals cofinal in ω_1^{CK} in the range?

Chapter 3

Shuffle Sums of Ordinals

3.1 Introduction

A countable linear order is said to be computable if its universe can be identified with ω in such a way that the order is a computable relation on $\omega \times \omega$. The class of computable linear orders has been studied extensively; see [6] for an overview. In this chapter we discuss the class of linear orders that are the shuffle sums of ordinals.

Definition 3.1. *The* shuffle sum *of a countable set of linear orders $S = \{\mathcal{L}_i\}_{i\in\omega}$, denoted $\sigma(S)$, is the (unique) linear order obtained by partitioning the rationals $\mathbb{Q}$ into dense sets $\{Q_i\}_{i\in\omega}$ and replacing each rational of Q_i by the linear order $\mathcal{L}_i$.*

Equivalently, the shuffle sum of $S = \{\mathcal{L}_i\}_{i\in\omega}$ is the linear order obtained by interleaving copies of each $\mathcal{L}_i$ densely and unboundedly amongst each other. Shuffle sums can also be defined in terms of lexicographic sums as per the following remark.

Remark 3.2. *The shuffle sum of a set $S = \{\mathcal{L}_i\}_{i\in\omega}$ can also be defined as the (unique) linear order $\sigma(S) = \sum_{a\in\mathbb{Q}} \mathcal{L}_a$, where $\mathcal{L}_a$ is the linear order $\mathcal{L}_i$ if $a \in Q_i$.*

The class of shuffle sums of ordinals has yielded various results in computable model theory. In [2], the authors use shuffle sums to produce, for each computable ordinal $\alpha \geq 2$, a linear order $\mathcal{A}_\alpha$ such that $\mathcal{A}_\alpha$ has αth jump degree but not βth jump degree for any $\beta < \alpha$. In [12], shuffle sums of ordinals are used to exhibit a linear order with both a computable model and a prime model, but no computable prime model.

In this chapter, we characterize which shuffle sums of the finite order types and the order type ω are computable. In order to do so, we need the following notions.

Definition 3.3. *A set $S \subseteq \omega + 1$ is a* limit infimum set*, written* LimInf set*, if there is a total computable function $g : \omega \times \omega \to \omega$ such that the function $f : \omega \to \omega$ given by*

$$f(x) = \liminf_s g(x, s)$$

enumerates S under the convention that $f(x) = \omega$ if $\liminf_s g(x, s) = \infty$. We say that g is a LimInf witnessing function *for S.*

Definition 3.4. *A set $S \subseteq \omega + 1$ is a* limitwise monotonic set relative to a degree $\mathbf{a}$, *written* LimMon($\mathbf{a}$) *set, if there is a total* $\mathbf{a}$*-computable function* $\tilde{g} : \omega \times \omega \to \omega$ *satisfying* $\tilde{g}(x,t) \leq \tilde{g}(x,t+1)$ *for all x and t such that the function* $\tilde{f} : \omega \to \omega$ *given by*

$$\tilde{f}(x) = \lim_t \tilde{g}(x,t)$$

enumerates S under the convention that $\tilde{f}(x) = \omega$ if $\lim_t \tilde{g}(x,t) = \infty$. We say that $\tilde{g}$ is a LimMon($\mathbf{0}'$) witnessing function for S.

Although the notion of LimInf sets is new, LimMon($\mathbf{0}'$) sets have been previously studied. Limitwise monotonic functions were first introduced and relativized in [13] and further studied in [4], [12] and [15]. Our definition departs slightly from the literature where $\lim_t \tilde{g}(x,t)$ is required to be finite. With the exception of the conclusion, we will only have need to consider limitwise monotonic sets relative to the degree $\mathbf{a} = \mathbf{0}'$.

Blurring the distinction between an ordinal α and the linear order of order type α (which we will do throughout the chapter), we note that $\sigma(S) = \sigma(S \cup \{0\})$ for any set S of linear orders. In order to avoid complications in several of the proofs, we assume the following conventions.

Convention 3.5. *Any set S of ordinals is assumed to not contain 0. Any set S of linear orders is assumed to not contain the empty linear order.*

Any LimInf *witnessing function $g(x,s)$ is assumed to satisfy $g(x,s) \neq 0$ for all x and s. Any* LimMon($\mathbf{0}'$) *witnessing function $\tilde{g}(x,t)$ is assumed to satisfy $\tilde{g}(x,t) \neq 0$ for all x and t.*

The following facts justify that all the results in this chapter are correct as stated, without needing to invoke Convention 3.5.

Fact 3.6. *If S is a Σ^0_3 set, then $S \backslash \{0\}$ is a Σ^0_3 set. If S is a Σ^0_3 set, then $S \cup \{0\}$ is a Σ^0_3 set.*

Fact 3.7. *If S is a* LimInf *set, then $S \backslash \{0\}$ is a* LimInf *set. If S is a* LimMon($\mathbf{0}'$) *set, then $S \backslash \{0\}$ is a* LimMon($\mathbf{0}'$) *set.*

If S is a LimInf *set, then $S \cup \{0\}$ is a* LimInf *set. If S is a* LimMon($\mathbf{0}'$) *set, then $S \cup \{0\}$ is a* LimMon($\mathbf{0}'$) *set.*

Fact 3.8. *If S is a* LimInf *set not containing 0, then there is a* LimInf *witnessing function g for S satisfying $g(x,s) \neq 0$ for all x and s.*

If S is a LimMon($\mathbf{0}'$) *set not containing 0, then there is a* LimMon($\mathbf{0}'$) *witnessing function g for S satisfying $\tilde{g}(x,t) \neq 0$ for all x and t.*

As these facts are all straightforward, the proofs are omitted. Having introduced all the relevant notions, we are now in a position to state the main results of the chapter. The first result is in computable model theory. In particular, it provides a necessary and sufficient condition for the shuffle sum $\sigma(S)$ to be computable in terms of the new computability-theoretic notion of LimInf sets.

Theorem 3.9. *For sets $S \subseteq \omega + 1$, the shuffle sum $\sigma(S)$ is computable if and only if S is a* LimInf *set.*

The next result is in classical computability theory. It provides an alternate characterization of the LimInf sets, showing their equivalence with the pre-existing notion of $\text{LimMon}(\mathbf{0}')$ sets.

Theorem 3.10. *A set $S \subseteq \omega+1$ is a* LimInf *set if and only if S is a* $\text{LimMon}(\mathbf{0}')$ *set.*

In Section 3.2 we prove Theorem 3.9, and in Section 3.3 we prove Theorem 3.10. In Section 3.4 we discuss the relationship between the LimInf and $\text{LimMon}(\mathbf{0}')$ sets and the Σ^0_3 sets, making use of previous work in [4] and [15]. We note that in [4] the authors show that for sets $S \subseteq \omega$, if the shuffle sum $\sigma(S)$ is computable, then S is a $\text{LimMon}(\mathbf{0}')$ set.

3.2 Proof of Theorem 3.9

We prove Theorem 3.9 by proving the forwards and backwards directions separately, making each a proposition.

Proposition 3.11. *If $S \subseteq \omega + 1$ is a* LimInf *set, then the shuffle sum $\sigma(S)$ is computable.*

Proof. Let $g(x, s)$ be a LimInf witnessing function for S. Fix a uniformly computable partition of the rationals $\mathbb{Q}$ into dense sets $\{Q_x\}_{x\in\omega}$ with $Q_x = \{q_{x,y}\}_{y\in\omega}$. We build a computable copy of $\sigma(S)$ in ω many stages s using $g(x, s)$.

The basic idea is to build the finite linear order $g(x, s+1)$ at a rational $q_{x,y}$ at stage $s + 1$. If $g(x, s + 1)$ is larger than $g(x, s)$, then the appropriate number of points are added to the linear order already built for $q_{x,y}$. If $g(x, s+1)$ is smaller than $g(x, s)$, then the extra points already built for $q_{x,y}$ are no longer associated with $q_{x,y}$; instead they eventually become associated with some other rational at a later stage. In order to track whether a point is currently associated with some rational $q_{x,y}$, the states *associated* and *unassociated* will be used.

Construction: At each stage s we build a computable linear order $\mathcal{L}_s$ such that $\mathcal{L}_s \subseteq \mathcal{L}_{s+1}$ for all s. With $\mathcal{L} = \bigcup_s \mathcal{L}_s$, we aim for $\mathcal{L} \cong \sigma(S)$. At stage 0 we begin with the empty linear order, i.e., $\mathcal{L}_0$ is the empty linear order. At stage $s + 1$ we work on behalf of all rationals $q_{x,y}$ with $x, y < s$. This work is done in s^2 substages, with a substage devoted to each such rational $q_{x,y}$ (in lexicographic order). Fixing a rational $q_{x,y}$ with $x, y < s$, we compare the value of $g(x, s + 1)$ and $g(x, s)$; our action is determined by which is greater and whether or not work has already been done for the rational $q_{x,y}$.

If $g(x, s + 1) > g(x, s)$ and work has already been done for $q_{x,y}$, then we insert the appropriate number of new points (namely $g(x, s + 1) - g(x, s)$) at the right end of the linear order built at $q_{x,y}$ and give these inserted points the state associated.

If $g(x, s+1) < g(x, s)$ and work has already been done for $q_{x,y}$, then we split off the appropriate number of points (namely $g(x, s) - g(x, s+1)$) from the right end of the linear order built at $q_{x,y}$. The points split off have their state switched to unassociated and receive a priority amongst all points unassociated based first on the stage at which they became unassociated (lower stage, higher priority) and then their position in the linear order (further left, higher priority).

If no work has been done for $q_{x,y}$, then we insert the linear order $g(x, s+1)$ at $q_{x,y}$. In particular, we note whether or not there are any unassociated points greater than the greatest associated point to the left of $q_{x,y}$ and less than the least associated point to the right of $q_{x,y}$. If there are such unassociated points, we use the one with highest priority for the first point of the linear order built at $q_{x,y}$ and insert the appropriate number of new points (namely $g(x, s+1) - 1$) immediately to the right of this first point. If there are no such unassociated points, we insert the appropriate number of new points (namely $g(x, s+1)$) at $q_{x,y}$. All points inserted at $q_{x,y}$ are given the state associated, including the previously unassociated point if one was used. This completes the construction.

Verification: As the construction is computable, it suffices to show that $\mathcal{L} \cong \sigma(S)$. In order to demonstrate this equality, we verify the following two claims. The first implies that no extra points are built, and the second implies that enough points are built.

Claim 3.11.1. *Every point is unassociated for at most finitely many stages.*

Proof. When a point changes its state to unassociated, there are at most finitely many unassociated points with higher priority. Moreover, as priority is determined first by stage, no later point will receive a higher priority.

As a consequence of the density of the rationals and that only those rationals $q_{x,y}$ with $x, y < s$ have had work done for them by stage s, at some later stage the point will meet the criterion for becoming the first point built for some rational which is having work done for it for the first time. When the point does meet this criterion, it will never again become unassociated as by convention $g(x, s) > 0$ for all s, and thus it will never be split off. Thus each point becomes unassociated at most once and eventually becomes associated permanently at some later stage. □

Claim 3.11.2. *In $\mathcal{L}$, the linear order $f(x) = \liminf_s g(x, s)$ is built at the rational $q_{x,y}$.*

Proof. Since $f(x) = \liminf_s g(x, s)$, there is a stage $\hat{s}$ such that $g(x, s) \geq f(x)$ for all $s \geq \hat{s}$. As a result, the rational $q_{x,y}$ will have at least $f(x)$ points built at it at every stage $s \geq \hat{s}$. On the other hand, no other points will remain permanently associated with $q_{x,y}$ as infinitely often the value of $g(x, s)$ will drop to $f(x)$, causing all other points to be split off from $q_{x,y}$.

As the rationals are dense, eventually the points split off will be separated from the $f(x)$ points permanently associated with $q_{x,y}$. Thus the linear order $f(x)$ is built at the rational $q_{x,y}$ in $\mathcal{L}$. □

It follows from the first claim that every point of the linear order eventually becomes associated permanently with some rational $q_{x,y}$. As each $q_{x,y}$ has the correct linear order built at it by the second claim, we conclude that $\mathcal{L} = \sigma(S)$.

$\square$

Before demonstrating the converse, we introduce some vocabulary and notation which will simplify the language in its proof.

Definition 3.12. *A* maximal block *in a linear order is a collection of points maximal with respect to the property that for every pair of points a and b in the collection, the interval $[a, b]$ is finite.*

The block size *of an element x, denoted BlockSize(x), is the number of points in the (unique) maximal block containing x.*

Definition 3.13. *If $A = \{a_x\}_{x\in\omega}$ is an enumeration of a linear order $\mathcal{A} = (A : \prec)$, define $|(a_i, a_j)|_s$ to be the number of points strictly between a_i and a_j amongst the first s points in the enumeration, i.e., the cardinality of the set $\{k : a_i \prec a_k \prec a_j, k < s\}$.*

Proposition 3.14. *If the shuffle sum $\sigma(S)$ is computable with $S \subseteq \omega + 1$, then S is a* LimInf *set.*

Proof. Assume $\sigma(S)$ is computable and let $\mathcal{A} = (A : \prec)$ be a computable presentation of $\sigma(S)$ with universe $A = \{a_x\}_{x\in\omega}$. In order to show that S is a LimInf set, we define a LimInf witnessing function $g : \omega \times \omega \to \omega$ for S.

The idea will be to define auxiliary functions $\ell(x, s)$ and $r(x, s)$ that guess the number of points to the left and right of x in its maximal block. The difficulty is that all linear orders of a fixed finite cardinality are isomorphic. This obstacle is resolved by believing the left and right boundaries of the maximal block are determined by the most recently enumerated point on the left and on the right. Because of the dense nature of the maximal blocks, infinitely often $\ell(x, s)$ and $r(x, s)$ will be correct.

From the functions $\ell(x, s)$ and $r(x, s)$, we define the function $g(x, s)$. The idea will be to add $\ell(x, s)$ and $r(x, s)$ to obtain the value of $g(x, s)$, but we cannot do so directly as $\ell(x, s)$ and $r(x, s)$ may never be at their correct values simultaneously.

Construction: Before defining $g(x, s)$, we first define auxiliary functions $\ell(x, s) : \omega\times\omega \to \omega$ and $r(x, s) : \omega \times \omega \to \omega$ by

$$\ell(x, s) = |(a_i, a_x)|_s \qquad \text{and} \qquad r(x, s) = |(a_x, a_j)|_s$$

where i is the greatest index less than s such that $a_i \prec a_x$ and j is the greatest index less than s such that $a_x \prec a_j$. If no such index i exists, define $\ell(x, s) = |(-\infty, a_x)|_s$. Similarly, if no such index j exists, define $r(x, s) = |(a_x, +\infty)|_s$.

Fixing x and s, let v be the most recent time before s such that $r(x, \cdot)$ took the value $r(x, s)$. More formally, we define $v = v_{x,s}$ to be the greatest integer u less than s such that $\ell(x,u)=\ell(x,s)$ if one exists; otherwise we define $v=v_{x,s}$ to be s. We then define $g(x,s)$ by

$$g(x,s)=\ell(x,s)+1+\min_{z\in[v,s]} r(x,z).$$

Verification: Since $\mathcal{A}$ is a computable presentation of $\sigma(S)$, it is clear that $\ell(x,s)$ and $r(x,s)$ are computable, from which it follows that $g(x,s)$ is computable. We claim that the range of $f(x)=\liminf_s g(x,s)$ is exactly S, which we will show by demonstrating that $\liminf_s g(x,s)=\text{BlockSize}(a_x)$. Fixing x, we consider the cases when $\text{BlockSize}(a_x)$ is finite and infinite separately.

Claim 3.14.1. *If BlockSize(a_x) is finite, then* $\liminf_s g(x,s)=$ *BlockSize*(a_x).

Proof. If $\text{BlockSize}(a_x)=n$, then there is an $\hat{s}$ such that $\{a_0,\dots,a_{\hat{s}}\}$ includes all of the elements of the maximal block of a_x. Moreover, we may assume that at stage $\hat{s}$, the points a_i and a_j (as in the definition of $\ell(x,s)$ and $r(x,s)$) are not part of the maximal block of a_x.

Denote the elements in a_x's maximal block by $\{a_{x_1}<\cdots<a_x=a_{x_k}<\cdots<a_{x_n}\}$. Then $\{a_{x_1},\dots,a_{x_n}\}\subseteq\{a_0,\dots,a_{\hat{s}}\}$. Note that for any $s>\hat{s}$, we have $\ell(x,s)\geq k-1$ and $r(x,s)\geq n-k$.

When a new element is enumerated directly to the left of a_{x_1}, we have $\ell(x,s)=k-1$; similarly, when a new element is enumerated directly to the right of a_{x_n}, we have $r(x,s)=n-k$. Because of the dense nature of shuffle sums, such points will be enumerated infinitely often. Thus $\liminf_s \ell(x,s)=k-1$ and $\liminf_s r(x,s)=n-k$, from which it follows that $\liminf_s g(x,s)=(k-1)+1+(n-k)=n$. □

Claim 3.14.2. *If BlockSize(a_x) is infinite, then* $\liminf_s g(x,s)=$ *BlockSize*(a_x).

Proof. If $\text{BlockSize}(a_x)=\infty$, then a_x belongs to a maximal block of order type ω. For every k, there is an $\hat{s}=\hat{s}_k$ such that $\{a_0,\dots,a_{\hat{s}}\}$ includes the k points immediately to the right of a_x in $\sigma(S)$. Moreover, we may assume that at stage $\hat{s}$, the point a_j (as in the definition of $r(x,s)$) is not part of the maximal block of a_x. Then $r(x,s)\geq k$ for all $s>\hat{s}$. Since there is such a stage $\hat{s}=\hat{s}_k$ for every k, it follows that $\lim_s r(x,s)=\infty$. We conclude that $\liminf_s g(x,s)=\infty$. □

As a consequence of $f(x)=\liminf_s g(x,s)=\text{BlockSize}(a_x)$ for all x, we conclude that $g(x,s)$ is a LimInf witnessing function for S.

□

3.3 Proof of Theorem 3.10

We prove Theorem 3.10, again by proving the forwards and backwards directions separately as separate propositions.

Proposition 3.15. *If $S \subseteq \omega + 1$ is a* LIMINF *set, then S is a* LIMMON($\mathbf{0}'$) *set.*

Proof. Let $g(x,s)$ be a LIMINF witnessing function for S. Define a function $\tilde{g} : \omega \times \omega \to \omega$ by setting $\tilde{g}(x,t)$ equal to the largest number n such that $g(x,s) \geq n$ for all $s \geq t$. Note that $\tilde{g}(x,t)$ is total, increasing in t, and computable in $\mathbf{0}'$. Moreover $\lim_t \tilde{g}(x,t) = \liminf_s g(x,s)$, so that the range of $\tilde{f}(x) = \lim_t \tilde{g}(x,t)$ is the same as the range of $f(x) = \liminf_s g(x,s)$. It follows that $\tilde{g}(x,t)$ is a LIMMON($\mathbf{0}'$) witnessing function for S. □

Proposition 3.16. *If $S \subseteq \omega + 1$ is a* LIMMON($\mathbf{0}'$) *set, then S is a* LIMINF *set.*

Proof. Let $\tilde{g}(x,t)$ be a LIMMON($\mathbf{0}'$) witnessing function for S. By the Limit Lemma, there is a total computable function $\tilde{h} : \omega \times \omega \times \omega \to \omega$ such that $\lim_k \tilde{h}(x,t,k) = \tilde{g}(x,t)$. Fixing x, for each s we define a natural number t_s by recursion. Let $t_0 = 0$ and let t_s for $s > 0$ be the least t not greater than t_{s-1} such that $\tilde{h}(x,t,s) \neq \tilde{h}(x,t,s-1)$ if such a t exists, and otherwise let t_s be $t_{s-1} + 1$.

We define a function $g : \omega \times \omega \to \omega$ by

$$g(x,s) = \max\left\{\tilde{h}(x,i,s) : 0 \leq i \leq t_s\right\}.$$

As $g(x,s)$ is clearly total and computable, it suffices to show that $\liminf_s g(x,s) = \lim_t \tilde{g}(x,t)$ for all x. We begin with a combinatorial claim about the sequence $\{t_s\}_{s\in\omega}$.

Claim 3.16.1. *For every t, there are at most finitely many s with $t_s = t$. In particular, for every t, there are at most finitely many s with $t_s \leq t$.*

Proof. We prove the claim by induction on t. For $t = 0$, we have $t_s = 0$ when $s = 0$ and when $\tilde{h}(x,0,s) \neq \tilde{h}(x,0,s-1)$. Since $\lim_k \tilde{h}(x,0,k)$ exists, the latter condition occurs at most finitely often. Thus $t_s = 0$ for only finitely many s.

For $t+1$, we have $t_s = t+1$ possibly when $\tilde{h}(x,t+1,s) \neq \tilde{h}(x,t+1,s-1)$ and possibly when $t_{s-1} = t$. Since $\lim_k \tilde{h}(x,t+1,k)$ exists, the former condition happens at most finitely often. The inductive hypothesis assures that the latter condition happens at most finitely often. Thus $t_s = t+1$ for only finitely many s. □

In order to show that $\liminf_s g(x,s) \geq \lim_t \tilde{g}(x,t)$, we argue that for every t, there is an $\hat{s}$ such that $g(x,s) \geq \tilde{g}(x,t)$ for all $s \geq \hat{s}$. Fixing t, let $\hat{s}$ be such that $t_s \geq t$ for all $s \geq \hat{s}$, which is possible by the claim. As $\lim_k \tilde{h}(x,t,k)$ exists, we may assume that $\hat{s}$ satisfies $\tilde{h}(x,t,s) = \tilde{g}(x,t)$ for all $s \geq \hat{s}$. Then for $s \geq \hat{s}$ we have

$$\begin{aligned} g(x,s) &= \max\left\{\tilde{h}(x,i,s) : 0 \leq i \leq t_s\right\} \\ &\geq \max\left\{\tilde{h}(x,i,s) : 0 \leq i \leq t\right\} \\ &\geq \tilde{h}(x,t,s) \quad = \quad \tilde{g}(x,t). \end{aligned}$$

Thus for every t there is an $\hat{s}$ such that $g(x,s) \geq \tilde{g}(x,t)$ for all $s \geq \hat{s}$, from which the inequality $\liminf_s g(x,s) \geq \lim_t \tilde{g}(x,t)$ follows.

In order to show that $\liminf_s g(x,s) \leq \tilde{g}(x,t)$, we argue that for every t there is an s such that $g(x,s) = \tilde{g}(x,t)$ (with $s \neq s'$ if $t \neq t'$). Fixing t, let $\hat{s}$ be minimal such that $\tilde{h}(x,i,s) = \tilde{g}(x,i)$ for all $i \leq t$ and $s \geq \hat{s}$. Let s be the least number greater than or equal to $\hat{s}$ such that $t_s = t$, which is possible since $\hat{s}$ was chosen to satisfy $\tilde{h}(x,i,\hat{s}) = \tilde{g}(x,i) \neq \tilde{h}(x,i,\hat{s}-1)$ for some $i \leq t$. Then

$$\begin{aligned} g(x,s) &= \max\left\{\tilde{h}(x,i,s) : 0 \leq i \leq t_s\right\} \\ &= \max\left\{\tilde{h}(x,i,s) : 0 \leq i \leq t\right\} \\ &= \max\left\{\tilde{g}(x,i) : 0 \leq i \leq t\right\} \\ &= \tilde{g}(x,t). \end{aligned}$$

Moreover, the value of s will be distinct for distinct values of t as s satisfies $t_s = t$. Thus for every t there is an s such that $g(x,s) = \tilde{g}(x,t)$, with $s \neq s'$ if $t \neq t'$. The inequality $\liminf_s g(x,s) \leq \lim_t \tilde{g}(x,t)$ then follows.

We conclude that $\liminf_s g(x,s) = \lim_t \tilde{g}(x,t)$ for all x, so that $g(x,s)$ is a LimInf witnessing function for S. □

3.4 LimInf and LimMon($\mathbf{0}'$) Sets

With the characterization of the computable shuffle sums of subsets $S \subseteq \omega + 1$ in terms of LimInf and LimMon($\mathbf{0}'$) sets completed, it is natural to ask which subsets of $\omega + 1$ are LimInf and LimMon($\mathbf{0}'$) sets. We note that a LimInf set (and thus a LimMon($\mathbf{0}'$) set) can be no more complicated than a Σ^0_3 set. For if $g(x,s)$ is a LimInf witnessing function for S, then

$$\begin{aligned} n \in S \quad &\text{iff} \quad \exists x\,[\liminf_s g(x,s) = n] \\ &\text{iff} \quad \exists x\big[\exists \hat{s}\forall s > \hat{s}\,[g(x,s) \geq n] \ \&\ \forall s \exists s' > s\,[g(x,s') = n]\big]. \end{aligned}$$

As the last predicate is Σ^0_3, membership in S cannot be more complicated than Σ^0_3. We state this as a proposition.

Proposition 3.17. *If S is a* LIMINF *and* LIMMON($\mathbf{0}'$) *set, then S is a Σ^0_3 set.*

We next show that for sets S with $\omega \in S$, the LIMINF sets (and thus LIMMON($\mathbf{0}'$) sets) coincide exactly with the Σ^0_3 sets.

Proposition 3.18. *If $S \subseteq \omega$ is a Σ^0_3 set, then $S \cup \{\omega\}$ is a* LIMINF *and* LIMMON($\mathbf{0}'$) *set.*

Proof. Let S be a Σ^0_3 set witnessed by the predicate $\exists m \exists^\infty s\, R(n, m, s)$, where R is a computable relation. Define a function $g : \omega \times \omega \to \omega$ by

$$g(x, s) = g(\langle n, m \rangle, s) = \begin{cases} n & \text{if } R(n, m, s), \\ s & \text{otherwise.} \end{cases}$$

Note that $g(x, s)$ is computable as R is computable.

If $n \in S$, we have $\exists m \exists^\infty s\, R(n, m, s)$. Letting m_0 witness this, we have $g(\langle n, m_0 \rangle, s) = n$ for infinitely many s. As s will be less than n only a finite number of times, it follows that $\liminf_s g(\langle n, m_0 \rangle, s) = n$. Thus n is in the range of $f(x) = \liminf_s g(x, s)$.

If instead $n \notin S$, we have $\forall m \exists^{<\infty} s\, R(n, m, s)$. For any $x = \langle n, m \rangle$, it follows that $g(x, s) = n$ for only finitely many s. Thus $\liminf_s g(x, s) = \infty$, and so ω is in the range of $f(x)$ and n is not in the range of $f(x)$.

In the extreme case when $S = \omega$, we can (non-uniformly) arrange to have the range of $f(x) = \liminf_s g(x, s)$ be $\omega \cup \{\omega\}$ if ω would otherwise not be in the range. □

It follows immediately from Theorem 3.9 and Proposition 3.18 that $\sigma(S \cup \{\omega\})$ is computable for every Σ^0_3 set S, a result shown in [2]. However $\sigma(S)$ is not computable for every Σ^0_3 set S, a corollary of our results and the following result found in [4] (which is a relativization of a result in [15]).

Proposition 3.19 ([4]). *There is a Σ^0_3 set S that is not a* LIMMON($\mathbf{0}'$) *set. Moreover, the set S can be made to be a d.c.e. set.*

3.5 Conclusion

We conclude our discussion of shuffle sums of ordinals by asking several questions.

Question 3.20. *For which subsets $S \subseteq \omega_1^{\mathrm{CK}} + 1$ is $\sigma(S)$ computable?*

We note that Question 3.20 will probably require significantly more work. For example, if S has a computable shuffle sum and is known to satisfy $\omega^2 < \alpha < \omega^2 + \omega$ for all $\alpha \in S$, then S must be a LIMINF and LIMMON($\mathbf{0}'$) set as a consequence of the proof of Proposition 3.14. On the other hand, if S has a computable shuffle sum and is known to satisfy $\omega^2 < \alpha < \omega^3$ for all $\alpha \in S$, there is no reason to believe that S must be a LIMINF and LIMMON($\mathbf{0}'$) set.

In addition, if $S \subseteq \omega + 1$ is a $\textsc{LimMon}(\emptyset^{(3)})$ set, then $\sigma(\omega \cdot S) = \sigma(\{\omega \cdot \alpha : \alpha \in S\})$ is computable as a consequence of the following theorem (see [1] or [23], for example) and the fact that $\sigma(\omega \cdot S) = \omega \cdot \sigma(S)$.

Theorem 3.21. *If $\mathcal{L}$ is a Δ^0_3 linear order, then there is a computable copy of $\omega \cdot \mathcal{L}$.*

In addition to exploring shuffle sums of subsets of larger ordinals, it is natural to further investigate which sets are $\textsc{LimInf}$ and $\textsc{LimMon}(\mathbf{0}')$ sets.

Question 3.22. *Which subsets $S \subseteq \omega + 1$ are Σ^0_3 sets but not $\textsc{LimInf}$ and $\textsc{LimMon}(\mathbf{0}')$ sets?*

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