Abstract. The Tits Conjecture, proved by Crisp and Paris, states that squares of the standard generators of any Artin group generate an obvious right-angled Artin subgroup. We consider a larger set of elements consisting of all the centers of the irreducible spherical special subgroups of the Artin group, and conjecture that sufficiently large powers of those elements generate an obvious right-angled Artin subgroup. This alleged right-angled Artin subgroup is, in certain sense, as large as possible, e.g. its nerve is homeomorphic to the nerve of the ambient Artin group. We verify this conjecture for the class of locally reducible Artin groups, which includes all 2-dimensional Artin groups, and for spherical Artin groups of any type other than $E_6, E_7, E_8$. We use our results to conclude that certain Artin groups contain hyperbolic surface subgroups, answering questions of Gordon, Long and Reid.

1. Introduction

Suppose $(W, S)$ is a Coxeter system (cf. [4] or [13]). This means that $W$ is a group, $S$ is a distinguished set of generators, and that $W$ has a presentation

$$W := \langle s \in S | s^2 = (st)^{m_{st}} = 1 \rangle,$$

where $m_{st} \in \{2, 3, \ldots \} \cup \{\infty\}$. Given a Coxeter system $(W, S)$ there is an associated Artin group $A$. This group has one generator $x_s$ for each $s \in S$ and the braid relations:

$$x_s x_t \cdots = x_t x_s \cdots,$$

where both sides of the equation are alternating words in $x_s$ and $x_t$, and where $m_{st}$ denotes the order of $st$ in $W$. The Artin group is right-angled (and called a RAAG) if $m_{st} \in \{2, \infty\}$.

Consider the subgroup of $A$ generated by the squares $x_s^2$. These elements are all contained in the pure Artin group $PA$, which is the kernel of the canonical homomorphism $A \to W$ which sends $x_s$ to $s$. There are obvious commuting relations between the $x_s^2$, namely if $m_{st} = 2$, then $[x_s^2, x_t^2] = 1$. Crisp and Paris in [12] proved the remarkable fact that these are the only relations between these elements. This verified a conjecture of Tits, who had previously shown that the elements $\{x_s^2\}$ get sent to linearly independent elements in the abelianization of the pure Artin group.
**Tits Conjecture.** [12, Thm 1] Let $A$ be an Artin group. Then the subgroup generated by the set $\{x_s^2 : s \in S\}$ is a RAAG with presentation

$$\langle x_s^2 \mid [x_s^2, x_t^2] = 1 \text{ if } m_{st} = 2 \rangle$$

This is one of the few theorems known to hold for all Artin groups (e.g. it is not known if all Artin groups are torsion-free, have solvable word problem, ...). The Tits Conjecture had earlier been proved by Appell-Schupp for extra-large Artin groups (where $m_{st} > 3$) [1], by Collins for the braid groups [10], and by Charney for the locally reducible Artin groups (where the associated Coxeter groups have each finite special subgroup a direct product of dihedral groups and $\mathbb{Z}/2$) [7], see also [22], [27] for more partial results.

We will come back to Crisp and Paris’ method later in the introduction. Very roughly speaking, they construct a representation from the Artin group into the mapping class group of some surface and show that it is faithful on the alleged RAAG subgroup.

In this paper, we are interested in a conjectural generalization of the Tits Conjecture, which first appeared in [18, Conj 4.9]. This generalization asks for a RAAG subgroup that is as “large” as possible in certain sense. In particular, it contains the RAAG subgroup that Crisp and Paris find and its nerve is homeomorphic to the nerve of the Artin group. We will now explain how there is a natural candidate for this larger RAAG.

Given a Coxeter system $(W, S)$ and $T \subset S$, the subgroup $W_T$ generated by $t \in T$ is called the special subgroup corresponding to $T$. Then $(W_T, T)$ also is a Coxeter system. The subset $T$ is spherical if $W_T$ is finite, in this case $W_T$ is a spherical special subgroup. The subset $T$ is called reducible if it decomposes as $T_1 \cup T_2$, where $m_{tt'k} = 2$ for all $t \in T_1$ and $t' \in T_2$, otherwise $T$ is irreducible (and we say $W_T$ is as well). If $T$ is reducible with decomposition $T = \cup_{i=1}^n T_i$, then $W_T = W_{T_1} \times \cdots \times W_{T_n}$.

Similarly, the subgroup $A_T$ of $A$ generated by the $\{x_t^2\}_{t \in T}$ is the Artin group associated to the Coxeter system $(W_T, T)$ [31]. If $T$ is irreducible and spherical, then $A_T$ is a irreducible, spherical special subgroup of $A$. These spherical Artin groups are better understood than general Artin groups. Topologically, the pure Artin group (which in this case is finite index in $A$) is the fundamental group of an aspherical linear hyperplane arrangement in $\mathbb{C}^n$, which e.g. allows one to compute various cohomological invariants of $A$ [19]. Combinatorially, these groups admit a Garside structure, which e.g. gives an easy to compute normal form [6]. It is also known that the pure spherical Artin groups have an infinite center which is isomorphic to $\mathbb{Z}$ if the Coxeter group is irreducible. This center corresponds to the fundamental group of the fiber after projecting the hyperplane complement. We denote the generator of this by $\Delta^2_T$ (it is related to the longest element in $W_T$, see Section 2).

We consider the subgroup of an Artin group $A$ generated by (powers of) the centers of the irreducible, spherical, pure Artin subgroups. There are some obvious commuting relations between these elements, namely if $T \subset U$ or $m_{ut} = 2$ for all
$u \in U$ and $t \in T$, then the corresponding centers $\Delta_2^U, \Delta_2^T$ commute. We write $[U, T] = 1$ if $m_{ut} = 2$ for all $u \in U$ and $t \in T$. As in the Tits Conjecture, we ask whether those are the only relations.

Let $S$ be the set of all irreducible, spherical subsets $T \subseteq S$. Let $RA$ be a RAAG generated by the set $\{s_T\}_{T \in S}$, with the presentation

$$\text{(1.1)} \quad RA = \langle s_T \mid [s_T, s_U] = 1 \text{ if } U \subseteq T, T \subseteq U, \text{ or } [U, T] = 1 \rangle.$$  

By the above, there are homomorphisms $\Phi_N : RA \to A$ so that $\Phi_N(s_T) = \Delta_2^{2N}_T$. The map $\Phi_N$ is injective if and only if the subgroup of $A$ generated by $\{\Delta_2^{2N}_T\}_{T \in S}$ is isomorphic to $RA$. For a generator $x_s$, the infinite cyclic subgroup $\langle x_s \rangle$ of $A$ is itself an irreducible, spherical, special Artin subgroup, so the subgroup generated by $\{\Delta_2^{2N}_T\}_{T \in S}$ contains the RAAG subgroup found by Crisp and Paris. It will turn out that the injectivity (when we can verify it) of $\Phi_N$ will depend on $N$, unlike the original Tits Conjecture. Therefore, we propose the following.

**Generalized Tits Conjecture.** Let $A$ be an Artin group, $RA$ the associated RAAG and $\Phi_N : RA \to A$ the homomorphism defined above. Then $\Phi_N$ is injective for some $N$.

If we can verify that a specific homomorphism $\Phi_k$ is injective, then we say that $A$ satisfies the Generalized Tits Conjecture for $N = k$. The Generalized Tits Conjecture for $N = 1$ was conjectured by Davis, Le, and the second author in [18, Conjecture 4.9]. This turns out to be too optimistic in general, though we can show it for a reasonably large class of Artin groups. In Example 5.8 we show that $\Phi_1$ is not injective for braid groups on at least 8 strands. However, a theorem of Koberda (Theorem 5.3) implies that braid groups satisfy the Generalized Tits Conjecture (for some $N >> 0$).

We claim this is a good generalization of the Tits Conjecture to prove. As evidence, recall that there is a simplicial complex $L (= L(W, S))$, called the nerve. Its vertex set is $S$ and a subset $T \subseteq S$ spans a simplex of $L$ if and only if $T$ is spherical. Davis and Huang showed in [14] that the nerve $L'$ of the RAAG $RA$ defined above is a partial barycentric subdivision of the nerve $L$. In particular the nerve of the Artin group is homotopy equivalent to the nerve of the alleged RAAG subgroup. This is desirable since many topological properties of this nerve are related algebraic properties of the Artin group. For example, contingent on the $K(\pi, 1)$-conjecture, $RA$ and $A$ will have the same cohomological dimension, their compactly supported cohomology will be nontrivial in the same dimensions, etc. Also, in this sense, the alleged subgroup is as “large” a RAAG subgroup as one can expect to find in $A$. Davis and Huang in [14] were interested in determining the minimal dimensional manifold for a classifying space $BA$ (see also Le’s thesis [30]). These partial subdivisions $L'$ appeared earlier in [16], [17].

### 1.1. Results

We prove the Generalized Tits Conjecture holds for the Charney’s class of locally reducible Artin groups. This includes all 2-dimensional Artin groups and Artin groups with $m_{st} \neq 3$ for all $s, t \in S$. The irreducible spherical Artin
subgroups correspond to edges of the nerve, so the only new generators that we are considering come from centers of the Artin subgroups generated by \( (s, t) \) for \( m_{st} \geq 3 \). We can show the following:

**Theorem 1.1.** Artin groups with \( m_{st} \neq 3 \) for all \( s, t \in S \) satisfy the Generalized Tits Conjecture with \( N = 1 \).

For general locally reducible Artin groups, we show the following:

**Theorem 1.2.** Locally reducible Artin groups satisfy the Generalized Tits Conjecture with \( N = 2 \).

We also prove that the Generalized Tits Conjecture holds for large \( N \) for the following spherical Artin groups.

**Theorem 1.3.** The Generalized Tits Conjecture holds with \( N \) sufficiently large for all spherical Artin groups except for those of type \( E_n \) (for these it is still open).

Our methods do not work for these remaining exotic cases. The hardest part of Theorem 1.3 is confirming the conjecture for the Artin groups of type \( D_n \) (the conjecture for Artin groups of type \( B_n \) also follows from Koberda’s result). For technical reasons we have to assume that \( N \) is even, though we suspect it works for general large \( N \).

**Applications.** Here are some immediate applications of our results. The rough moral here is that if an Artin group satisfies the Generalized Tits Conjecture, then its subgroups are as complicated as the subgroups of the corresponding RAAG. In particular, we give a new proof of Wise’s result that the spherical Artin group \( A \) of type \( H_3 \) is incoherent, and can show that \( A \) (along with many other Artin groups) contains a closed hyperbolic surface subgroup, answering questions of Gordon-Long-Reid [26]. The advantage of our argument is that the same proof works for both questions; the RAAG subgroup has a nerve which is a cone on a pentagon, and it is easy to see this subgroup is incoherent and contains hyperbolic surface subgroups.

1.2. **Outlines of the proofs.** Our methods of proof in the case of spherical Artin group and locally reducible Artin groups are very different. For the locally reducible Artin groups, we use similar methods to [7]. Charney showed that the Deligne complexes of locally reducible Artin groups are CAT(0). She then constructed a cube complex with an action of the predicted RAAG, and showed that it isometrically embeds in the Deligne complex, using arguments from CAT(0) geometry. The crucial case to understand is the dihedral Artin groups \( A_{2m} \) (i.e. \((W, S)\) is a dihedral group \( D_{2m} \)), since these appear in the links of vertices in the larger Deligne complex. The RAAG that we consider is larger, so we are trying to isometrically embed a larger complex into the Deligne complex. Again, we need to understand the dihedral Artin groups. In this case, our complex looks roughly like a \( Z'\)'s worth of Charney’s complex, and we need to show that the pieces embed pairwise orthogonally in the Deligne complex of the dihedral Artin group.
For the spherical Artin groups, we follow Crisp and Paris. We start with a representation from the Artin group into the mapping class group of a surface $\Sigma$. For spherical Artin groups of type $A_n - E_n$, these representations are classical and due to Perron and Vannier. The generators of the Artin groups map to Dehn twists around simple closed curves, and powers of the centers of irreducible spherical Artin subgroups map to Dehn twists about the boundary curves of connected subsurfaces of $\Sigma$. Then Koberda’s result implies that high powers of Dehn twists around this collection of curves generates a RAAG subgroup of the mapping class group of $\Sigma$. The reader might suspect that we are done now, and the proof trivially follows from Koberda’s theorem (as we initially thought). However, the boundary of these subsurfaces is not necessarily connected, so each center maps to a product of Dehn twists about disjoint simple closed curves in $\Sigma$. Therefore, we have to study the following question:

**Question 1.4.** Let $RA$ be a RAAG. Let $\{w_i\}$ be a collection of elements of $A$, where each $w_i$ is a product of commuting generators of $A$. Is the subgroup of $A$ generated by $w_i$ the (obvious) RAAG?

This turns out to be subtle, and was also considered in [12] and [29]. Koberda and Crisp-Paris gave different conditions on the $\{w_i\}$ which guaranteed a positive answer to the above question. These conditions fail for the system of curves produced by the Perron-Vannier representation of the Artin groups of type $D_n$. Our main work in this case is to generalize Koberda’s condition to a condition that the curves in this system satisfy. This new condition may be of independent interest.

Unfortunately, our conditions do not work for the RAAG and subwords that arise from the Perron-Vannier representation of the Artin groups of type $E_n$. Even worse, for $E_7$ and $E_8$ we can find words in the alleged RAAG subgroup which are in the kernel of the representation, see Figure 12 in subsection 5.4. One can check by hand that these words correspond to nontrivial elements of the Artin group, so these mapping class group representations are not faithful enough to be used to verify the conjecture, even for spherical Artin groups.

Wajnryb showed that there is no faithful geometric representation of Artin groups of type $E_n$ into Mod($\Sigma$) [40] (a representation is geometric if the generators get mapped to Dehn twists). It is still open whether these Artin groups admit other faithful representations into mapping class groups.

1.3. **Organization of the paper.** In Section 2, we give some background. Section 3 is devoted to proving Theorems 1.1 and 1.2. In Section 4 we study RAAG subgroups of RAAG’s, and in Sections 5 and 6 we use this to prove Theorem 1.3. Section 7 is applications.

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2. Coxeter groups and Artin groups

Let \((W,S)\) be a Coxeter system, and let \(A\) be the corresponding Artin group. As in the Introduction, there is a canonical surjection \(p : A \to W\) which sends \(x_s\) to \(s\). The kernel of \(p\) is called the pure Artin group and denoted \(PA\). It is obviously finite index in \(A\) if and only if \(W\) is a finite Coxeter group. There is also a canonical set-theoretic section \(\sigma : W \to A\) of \(p\) which takes a reduced positive word \(w\) in \(W\) to the same word in \(A\). It follows from Tits’ solution to the word problem for Coxeter groups that this does not depend on the choice of reduced expression for \(w\).

2.1. Coxeter diagrams. When there are many commuting generators, Coxeter groups and Artin groups can be efficiently described in terms of their Coxeter diagrams. Given a Coxeter system \((W,S)\), we consider a graph \(\Gamma\) whose vertex set is the standard generators \(S\), and whose edges correspond to the pairs of generators that do not commute, i.e. there is an edge between \(s\) and \(t\) if and only if \(s \neq t\) and \(m_{st} \neq 2\). If \(m_{st} \in \{4,5,6,\ldots\} \cup \{\infty\}\), then the edge between \(s\) and \(t\) is labeled with \(m_{st}\). Otherwise, i.e. when \(m_{st} = 3\), the edge has no label. Given such a graph \(\Gamma\), we will denote the corresponding Artin group by \(A_{\Gamma}\).

Note that if the Coxeter graph \(\Gamma\) has multiple connected components \(\Gamma_i\), then the Artin group \(A_{\Gamma}\) splits as the direct product \(\prod_i A_{\Gamma_i}\). If the Coxeter graph is connected, we say that \(A_{\Gamma}\) is an irreducible Artin group as in the introduction. We say an Artin group is small-type if \(m_{st} \in \{2,3\}\) for all \(s,t \in S\). These correspond to Coxeter diagrams which are unlabeled graphs (where there are no loops or multiple edges). The finite Coxeter groups with connected Coxeter diagram were classified by Coxeter, and correspond to the Coxeter graphs in Figure 1. The small type irreducible spherical Artin groups therefore split into two infinite families; type \(A_n\) and \(D_n\), and three exotic cases \(E_6\), \(E_7\) and \(E_8\). Each of the other spherical Artin groups injects into a product of small-type spherical Artin groups, see [11] or Section 6.

2.2. Fundamental elements and Coxeter elements. Let \((W,S)\) be a Coxeter system with \(W\) finite. A Coxeter element of \(W\) is a product of all the generators of \(S\), in any order, where each generator appears exactly once in the product. Different orderings produce conjugate Coxeter elements (this follows from the fact that if \((W,S')\) is another Coxeter system then \(S'\) is conjugate of \(S\)). The Coxeter number \(h\) of \((W,S)\) is the order of a (any) Coxeter element in \(W\).

A reflection in \(W\) is a conjugate of an element of \(S\). Each finite Coxeter group has a unique longest element \(w_S\). This can be characterized as the unique element for which \(\ell(sw_s) = \ell(w_s) < \ell(w_S)\) for all \(s \in S\), where \(\ell(w)\) is the minimal length of a representative for \(w\). The length of \(w_S\) is precisely the number of reflections in \(W\). Conjugation by \(w_S\) induces an involution of the Coxeter diagram \(\Gamma_S\), in the sense
that generators are sent to generators and the relations are preserved. It follows from this that \( w_s \) is in the center of \( W \) if and only if this involution is trivial. This involution happens to be nontrivial if and only if the Coxeter group is of type \( A_n \), \( D_n \) with \( n \) odd, \( E_6 \) or \( I_2(p) \) for \( p \) odd.

The image of \( w_s \) in \( A \) under the section \( \sigma : W \to A \) will be called the fundamental element of \( A \), and we will denote it by \( \Delta \). Each spherical Artin group has an infinite cyclic center which is generated by either \( \Delta \) or \( \Delta^2 \) (depending as above whether the Coxeter group has a nontrivial center). For simplicity, we will only deal with the squares \( \Delta^2 \), which generate the center of the pure Artin groups, see [34, Thm 4.7]. We record the following lemma in [6], which will be used in Section 6.

**Lemma 2.1.** Let \( A \) be a spherical Artin group. We have that \( \Delta^2 = \sigma(c)h \), where \( c \) is any Coxeter element and \( h \) is the Coxeter number.

### 2.3. Subdivisions.

As in the Introduction, the nerve \( L \) of a Coxeter system \((W,S)\) is a simplicial complex \( L \) with vertex set \( S \) where a subset \( T \subset S \) spans a simplex of \( L \) if and only if \( T \) is spherical. We denote by \( A_L \) the associated Artin group (and say that \( L \) is the nerve of \( A \) as well).

Let \( \sigma \) be a simplex in \( L \) corresponding to a spherical subset \( T \), and \( A_T \) the corresponding spherical special Artin subgroup of \( A_L \). Davis and Huang described a partial barycentric subdivision \( \sigma_\circ \) of \( \sigma \) where the vertices of \( \sigma_\circ \) correspond to irreducible subsets of \( T \). For \( U \subset T \), we think of the vertex corresponding to \( A_U \) as the barycenter of the associated simplex in \( \sigma \). There are edges between two vertices \( U \) and \( U' \) if and only if \( U \subset U' \), \( U' \subset U \), or \( A_U \) commutes with \( A_{U'} \) (this occurs if and only if \( m_{uu'} = 2 \) for all \( u \in U \) and \( u' \in U' \)). See Figure 2 for the subdivision corresponding to the braid group on 4 strands.

In terms of the Coxeter diagram \( \Gamma \), the irreducible spherical Artin subgroups of \( A_\Gamma \) correspond to connected spherical subdiagrams. In this case, there is an edge between two connected subsets \( U,T \) of \( \Gamma_\sigma \) if and only if \( U \subset T \), \( T \subset U \), or \( U \) and \( T \) have distance > 2 in \( \Gamma \).

The subdivision \( \sigma_\circ \) can be defined as the flag completion of this graph. Of course, it is not obvious with this definition that this is a subdivision of \( \sigma \); Davis and Huang provide an alternative description of \( \sigma_\circ \) which obviously produces a subdivision of
Figure 2. The subdivision $\sigma_\otimes$ for the braid group on four strands.

$\sigma$, and show that it is a flag complex with the 1-skeleton described above. In either case, these subdivisions $\sigma_\otimes$ fit together to give a subdivision $L_\otimes$ of $L$. The simplicial complex $L_\otimes$ is the nerve of the RAAG RA described in the introduction.

In the next subsection, we verify that $\Phi_m : RA_{L_\otimes} \to A_L$ is injective on the free abelian subgroups corresponding to simplices of $L_\otimes$. This can be seen from looking at the abelianization of $PA$. We will also need an explicit description of a natural basis for $H_1(PA,\mathbb{Z})$ in the locally reducible case.

2.4. Abelianization of pure Artin groups. Each finite Coxeter group $W$ is a finite reflection group on $\mathbb{R}^n$, where $n$ is the number of elements of $S$. Complexifying this action gives a group action on $\mathbb{C}^n$ by linear reflections. The complement of the reflecting hyperplanes, denoted by $M(W)$, has $\pi_1(M(W)) = PA$, where $A$ is the associated Artin group.

The Coxeter group acts freely on $M(W)$, and $\pi_1(M(W)/W) = A$. Deligne showed that $M(W)$ is aspherical \cite{19}, so in particular $H_1(M(W),\mathbb{Z})$ is isomorphic to $H_1(M(W),\mathbb{Z})$. It is easy to see that $H_1(M(W),\mathbb{Z})$ is isomorphic to the free abelian group $\mathbb{Z}^R$, where $R$ is the set of reflections in $S$, or the set of hyperplanes in this arrangement \cite{34} (the complement deformation retracts to its intersection with $S^{2n-1}$, which is homeomorphic to $S^{2n-1} - \bigcup_R S^{2m-3}$). Given a spherical subset $T$ of $S$, let $R_T \subset R$ be the set of reflections in $W_T$.

Let $\{e_r\}$ be the standard basis of $\mathbb{Z}^R$. For each element $s \in S$, the element $x_s^2$ in $PA$ corresponds to a loop around the hyperplane corresponding to $s$ in $\mathbb{C}^n$. It turns out that the class of $x_s^2$ is precisely $e_s$ for each $s \in S$. Any $r \in R$ is a conjugate of an element $s \in S$, i.e. $r = ws^{-1}$ for some $w \in W$. Let $a_w$ be any element of $A$ that projects to $w$ under $p : A \to W$. Then $a_w x_s^2 a_w^{-1}$ is in $PA$, and its image in $H_1(PA,\mathbb{Z})$ is precisely $e_r$.

Lemma 2.2. Suppose that $T$ is a spherical subset of $S$. Let $e_T$ be the image of $\Delta_T^2$ in $H_1(PA,\mathbb{Z})$. Then

$$e_T = \sum_{t \in R_T} e_t.$$

It follows from this lemma that given any simplex in $\sigma_\otimes$, the image of the elements in $H_1(PA,\mathbb{Z})$ corresponding to the vertices of $\sigma$ are linearly independent, so in particular these form a free abelian subgroup of $PA$ of rank $\dim(\sigma) - 1$. 
Davis and Huang also show that the intersection of these free abelian subgroups is as expected, i.e. if $\sigma$ and $\tau$ are simplices of $L \setminus \emptyset$, then $H_\sigma \cap H_\tau = H_{\sigma \cap \tau}$. This serves as further motivation for the Generalized Tits Conjecture.

**Remark 2.3.** There are similar configurations of free abelian subgroups for affine hyperplane complements in $\mathbb{C}^n$, and as far as we know the analogue to the Generalized Tits Conjecture is also open in this case (see [18, Section 5]). In this case, the relevant simplicial complex $L$ is the geometric realization of the intersection poset of the hyperplane arrangement. Each irreducible central subarrangement has an infinite cyclic center, and these combine to produce standard free abelian subgroups. Again, it is known that these centers are linearly independent vectors in $H_1$ of the arrangement complement. In this case, the relevant partial subdivision of $L$ is the geometric realization of the nested set complex associated to the minimal building set for the arrangement as defined by De Concini and Procesi [20].

**Remark 2.4.** The main impetus for the authors of [18] to consider a generalized Tits Conjecture was the computation of action dimension of RAAG’s in [2]. This is the minimal dimension of a manifold model of the classifying space $B(\text{RA}_L)$. This is obviously monotone, in the sense that if $H$ is a subgroup of $G$, then the action dimension of $G$ is greater than the action dimension of $H$. Therefore, if the Generalized Tits Conjecture was true, the action dimension of the Artin group $A_L$ is larger than the action dimension of $\text{RA}_{L \setminus \emptyset}$, and in many cases this would lead to a complete calculation for $A_L$. On the other hand, combined work of Davis-Huang and Le gave a nearly complete computation for the action dimension of Artin groups (contingent on the $K(\pi, 1)$-conjecture) without using the conjecture (it was enough that the standard free abelian subgroups inject into $A_L$ and intersect as expected) [14], [30].

2.5. **Deligne complex.** The spherical subsets of $S$ form a poset under inclusion. Let $K$ denote the geometric realization of this poset; $K$ is the cone on the barycentric subdivision $bL$ of the nerve, with the cone vertex corresponding to the empty set $\emptyset$. There is another poset of spherical cosets $aA_T$ of $A$, where $a \in A$ and $A_T$ is a spherical special subgroup, again ordered by inclusion. The geometric realization of this poset is called the *modified Deligne complex* [9], and denoted $D(A)$. The Artin group $A$ acts on $D(A)$ by left multiplication, and $K$ is a strict fundamental domain for this action. The analogous construction in the Coxeter group setting (i.e. replace $aA_T$ with $wW_T$) is precisely the Davis complex.

Here is an alternative description of $D(A)$. Given a spherical subset $T$, let $K_{\geq T}$ denote the subcomplex of $K$ spanned by vertices $A_{T'}$ with $T \subset T'$. If $T \neq \emptyset$, then $K_{\geq T}$ is the star in $bL$ of the vertex corresponding to $A_T$, and if $T = \emptyset$, then $K_{\geq T} = bL$. Then

$$D(A) = A \times K / \sim,$$

where $(a_1, x) \sim (a_2, x)$ if and only if $x \in K_{\geq T}$ and $a_1^{-1}a_2 \in A_T$. We identify $K$ with $1 \times K$. If $A$ is itself a spherical Artin group, the poset of spherical cosets has a
maximal element \( A \), so the Deligne complex is a cone. The link of \( A \) is a simplicial complex of dimension \(|S| - 1\), and we denote it by \( B(A) \).

Conjecturally, the modified Deligne complex is contractible for all Artin groups. It follows from Deligne’s work on spherical Artin groups that this would imply the well-known \( K(\pi, 1) \)-conjecture. Motivated by this, Charney and Davis in [9] put two natural piecewise Euclidean metrics on \( D(A) \). Note that \( D(A) \) has a natural cube complex structure, since the cone on the barycentric subdivision of a simplex is combinatorially isomorphic to a cube. The first metric simply makes each cube isometric to a standard Euclidean cube \([0, 1]^n\). It turns out that this metric is CAT(0) if and only if the nerve \( L \) is a flag complex (the induced metric on the link of the vertex \( \emptyset \) is isometric to \( L \) with the all-right spherical metric, so flagness is an obvious necessary condition).

The second metric uses a different cubical metric called the Moussong metric, as it is related to the Moussong metric on the Davis complex, which is CAT(0). Charney and Davis showed that the Moussong metric on \( D(A) \) is CAT(0) for 2-dimensional Artin groups. It is still open whether \( D(A) \) equipped with the Moussong metric is CAT(0) for all Artin groups.

We now describe this metric for Artin groups corresponding to dihedral groups, which is the only important case for this paper. It is slightly more convenient for us to keep the original cellulation of \( D(A) \) as a simplicial complex. See [7] or [9] for details in the general case. The dihedral group \( D_{2m} \) acts by linear reflections on \( \mathbb{R}^2 \) in the standard way, with strict fundamental domain a simplicial cone. Let \( x \) be the unique point in this cone whose distance from the two walls of the cone is 1. The convex hull of the orbit of \( x \) under \( D_{2m} \) is a \( 2m \)-gon with edge lengths 2. We subdivide the \( 2m \)-gon by coning off the orbit of \( x \). The intersection of this convex hull with the simplicial cone is combinatorially isomorphic to \( K \), and we give \( K \) the inherited Euclidean metric. This defines a piecewise Euclidean metric on \( D(A) \), which in this case is CAT(0). This naturally gives a piecewise spherical metric on \( B(A) \), where every edge has length \( \pi/2m \). This turns out to be a CAT(1) metric, which in this case is equivalent to there being no closed geodesics with length \( < 2\pi \).

In the 2-dimensional case, this follows from [1, Lem 6].

We now recall the structure of links of vertices in the Deligne complex, again see [9] or [7] for complete details. It suffices to consider vertices in \( K \) as this is a strict fundamental domain for the action. Let \( T \) be a spherical subset of \( S \), and let \( v_T \) be the corresponding vertex in \( K \). Then we have the following join decomposition

\[
\text{Lk}_{D(A)}(v_T) = \text{Lk}_{K \supseteq T}(v_T) * B(A_T).
\]

The piecewise spherical metric on \( \text{Lk}_{D(A)}(v_T) \) is isometric to the orthogonal join of the metrics on \( \text{Lk}_{K \supseteq T}(v_T) \) and \( B(A_T) \) [Lemma 2.2, [8]], where the orthogonal join of two piecewise spherical metrics is defined by making each simplex \( \sigma * \tau \) isometric to the simplex in \( S^{\dim \sigma + \dim \tau + 1} \) spanned by \( \sigma \subseteq S^{\dim \sigma} \) and \( \tau \subseteq S^{\dim \tau} \), where points in \( \sigma \) and \( \tau \) are all distance \( \pi/2 \) apart. Note that Charney proves this for the cubical
metric on $D(A)$, our links are isometric but $Lk_{K_{≥T}}(v_T)$ and $B(A_T)$ have a finer subdivision.

2.6. Representations of small type Artin groups inside mapping class groups. Let $\Sigma$ be an oriented compact surface, possibly with boundary. Let $P = \{P_1, \ldots, P_n\}$ be a collection of $n$ punctures in the interior of $\Sigma$. Let $\text{Homeo}^+(\Sigma, P)$ denote the group of orientation-preserving homeomorphisms of $\Sigma$ which fix the boundary pointwise, and which preserve $P$. Let $\text{Homeo}_0^+(\Sigma, P)$ denote the connected component of the identity in $\text{Homeo}^+(\Sigma, P)$. The mapping class group of the pair $(\Sigma, P)$ is defined to be

$$\text{Mod}(\Sigma, P) = \text{Homeo}^+(\Sigma, P)/\text{Homeo}_0^+(\Sigma, P)$$

A mult curve is a disjoint union of a finite number of simple, closed, essential curves in $\Sigma$. A multitwist about a mult curve is the composition of (not necessarily the same) powers of Dehn twists about the individual curves. Since the curves are disjoint, the order in which we compose those Dehn twists does not matter.

For the small-type spherical Artin groups, there are classical representations into mapping class groups. This is due to Birman-Hilden for type $A_n$, and Perron, Vannier in general. Crisp and Paris defined similar representations for all small-type Artin groups, however we shall not need this generality. These representations also naturally arise as geometric monodromies of simple singularities of type $\Gamma$.

For $A$ of type $A_n$, we consider the Perron-Vanier representation $A \to \text{Mod}(\Sigma)$ which sends the consecutive generators of $A$ to the Dehn twists around the consecutive curves in Figure 4. The surface $\Sigma$ has genus $\frac{k-1}{2}$ and two boundary components, when $k$ is odd, and $\Sigma$ has genus $\frac{k}{2}$ and one boundary component, when $k$ is even.

If $A$ is an Artin group of type $D_n$, let the standard generators of $A$ be $\{s, s', t_1, \ldots, t_{n-2}\}$ where $s$ and $s'$ are both adjacent to $t_1$ in the Coxeter graph, and $t_i$ and $t_{i+1}$ are adjacent for all $i = 1, \ldots, n-3$. We use the Perron-Vannier representation $A \to \text{Mod}(\Sigma)$,
A

Figure 4. The Perron-Vannier representation for the Artin groups of type $A_n$. If $n$ is even, then the element $\Delta^4_S$ goes to a Dehn twist around the boundary curve. If $n$ is odd, then the element $\Delta^2_S$ goes to a single Dehn twist about the boundary curves.

Figure 5. The Perron-Vannier representation $A \to \text{Mod}(\Sigma)$ for Artin group $A$ of type $D_n$. For $n$ odd, the element $\Delta^2_S$ gets sent to a product of Dehn twists $\gamma_1\gamma_2^{n-2}$. For $n$ even, the element $\Delta^2_S$ gets sent to a product of Dehn twists $\gamma_1\gamma_2^{n\frac{n}{2}-1}$.

which sends the generators to the Dehn twists around curves, as pictured in Figure 5. The surface $\Sigma$ has genus $\frac{n-1}{2}$ and two boundary components when $n$ is odd, and genus $\frac{n-2}{2}$ and three boundary components when $n$ is even.
The Perron-Vannier representation $A \rightarrow \text{Mod}(\Sigma)$ of Artin group $A$ of type $E_n$, where $n = 6, 7, 8$, is illustrated in Figure 6. The surface $\Sigma$ has genus three and one boundary component.

**Remark 2.5.** For any small type spherical Artin group $A$ with the Perron-Vannier representation $\rho : A \rightarrow \text{Mod}(\Sigma)$, the image of the element $\Delta_4$ under $\rho$ is a multitwist about the boundary components of $\partial \Sigma$. It will never matter for us the exact power of each Dehn twist.

3. **Generalized Tits Conjecture for locally reducible Artin groups**

In this section, we show that Artin groups with no 3 labels satisfy the Generalized Tits Conjecture for $N = 1$, and locally reducible Artin groups satisfy the conjecture for $N = 2$. We first record an easy characterization of the locally reducible Artin groups in terms of their Coxeter diagram.

**Lemma 3.1.** Let $\Gamma$ be a Coxeter diagram, and $A$ the associated Artin group. Then $A$ is locally reducible if and only if $\Gamma$ satisfies the following condition:

If two consecutive edges of $\Gamma$ are not contained in a triangle, then their labels satisfy $1/a + 1/b \leq 1/2$.

3.1. **CAT(0) geometry.** We will assume that the reader is comfortable with comparison geometry, particularly in the setting of piecewise Euclidean and piecewise spherical cell complexes, see [7] or [13, Appendix I] for the relevant details. We record some lemmas and definitions that we will need:

**Lemma 3.2.** Let $X$ be a piecewise Euclidean cell complex. Then $X$ is locally $\text{CAT}(0)$ if and only if the induced piecewise spherical metric on the link $\text{Lk}(v, X)$ is $\text{CAT}(1)$ for all vertices $v \in X$. If $X$ is simply connected and locally $\text{CAT}(0)$, then it is $\text{CAT}(0)$.

**Lemma 3.3.** The orthogonal join of two piecewise spherical complexes $L_1$ and $L_2$ is $\text{CAT}(1)$ if and only if $L_1$ and $L_2$ are $\text{CAT}(1)$.

**Definition 3.4.** Let $f : L \rightarrow L'$ be a map between piecewise spherical complexes. We say $f$ is $\pi$-distance preserving if

$$d_L(x_1, x_2) \geq \pi \Rightarrow d_{L'}(f(x_1), f(x_2)) \geq \pi.$$
If $L$ is not connected, then we set $d_L = \infty$ for points in different components. So, $f : L \rightarrow L'$ is $\pi$-distance preserving if it is $\pi$-distance preserving on each component of $L$ and points in different components get mapped $\geq \pi$ apart in $L'$.

**Lemma 3.5.** Suppose $f : X \rightarrow X'$ is a map between two piecewise Euclidean complexes which takes piecewise geodesics to piecewise geodesics. Then $f$ is a locally isometric embedding if and only if the induced maps on all links $\text{Lk}(x, X) \rightarrow \text{Lk}(f(x), X')$ are $\pi$-distance preserving. Furthermore, if $X'$ is CAT(0), then $f$ is an isometric embedding.

**Lemma 3.6.** If $f : L_1 \rightarrow L'_1$ and $g : L_2 \rightarrow L'_2$ are $\pi$-distance preserving, and $L_1 \ast L_2$, $L'_1 \ast L'_2$ are orthogonal joins, then $f \ast g : L_1 \ast L_2 \rightarrow L'_1 \ast L'_2$ is $\pi$-distance preserving.

Charney showed that $D(A)$ being CAT(0) in the 2-dimensional case can be extended to locally reducible Artin groups.

**Theorem 3.7.** [7, Thm 3.2] Let $A$ be a locally reducible Artin group. Then $D(A)$ equipped with the Moussong metric is CAT(0).

### 3.2. Orthogonality in the Deligne complex.

**Definition 3.8.** Let $F_n = \langle x_1, \ldots, x_n \rangle$ be a free group of rank $n$. Then every element $g$ of $F_n$ can be written uniquely as

$$g = x_{i_1}^{n_1} x_{i_2}^{n_2} \ldots x_{i_k}^{n_k}$$

where $x_{i_j} \neq x_{i_{j+1}}$ and $n_i \in \mathbb{Z} - \{0\}$. The *syllable length* of $g$ in this case is equal $k$. If $G$ is a group admitting a surjection $\phi : F_n \rightarrow G$ is a surjection, then for $g \in G$ we define the *syllable length* of $g$ with respect to the generating set $\{\phi(x_1), \ldots, \phi(x_n)\}$, to be the infimum of syllable lengths of words in $\phi^{-1}(g)$.

We will now denote the coset $a\langle \emptyset \rangle$ just by $a$ for each $a \in A$. If $A$ is a dihedral Artin group, then a simplicial path in $B(A)$ between two cosets $g$ and $h$ must contain an even number of vertices. In particular, each pair of consecutive edges connects $g$ to $gt^k$ or $gs^k$. Therefore, we can associate to a path between $g$ and $h$ a word $w$ in the free group $F(s, t)$ so that $gw = h$ in $A$. If this path is embedded, then its length is equal to $\frac{\pi m}{k}$ where $k$ is the syllable length of $w$.

We now record a technical proposition concerning dihedral Artin groups. The pure Artin group in this case is isomorphic to the direct product $F_{m-1} \times \mathbb{Z}$, where $\Delta^2$ is the generator of $\mathbb{Z}$ (in this case, the hyperplane complement is homeomorphic to a $\mathbb{C}^*$-bundle over an $m$-punctured sphere). The proposition states a sort of orthogonality for the action of the pure Artin group on the Deligne complex. Given two elements $g, h$ in the Artin group, we let $d_{B(A)}(g, h)$ denote the distance in $B(A)$ between the vertices $g$ and $h$. 
Proposition 3.9. Let \( A \) be a dihedral Artin group. Let \( g \) be an element of the free subgroup \( \langle s^2, t^2 \rangle \). Then for any \( n \in \mathbb{Z} \) we have that
\[
d_{B(A)}(\Delta^{2n}, g) \geq \pi.
\]
Furthermore, if \( m_{st} > 3 \), then
\[
d_{B(A)}(\Delta^{2n}, g) \geq \pi + \pi/m.
\]
Finally, if \( m_{st} = 3 \) and \( g \) is in the free subgroup \( \langle s^4, t^4 \rangle \), then
\[
d_{B(A)}(\Delta^{4n}, g) \geq \pi + \pi/m.
\]
Proof. Suppose that \( d_{B(A)}(\Delta^{2n}, g) < \pi \), so there is \( a \in A \) of syllable length \( < m \) so that
\[
\Delta^{2n} = ag.
\]
Note that \( a \) is obviously an element of the pure Artin group \( PA \). We now compute the image of \( a \) in \( H_1(PA, \mathbb{Z}) \). Let \( a = bc \), where \( b \) has only odd powers of \( s \) and \( t \), and \( c \) begins with \( s^{2n} \) or \( t^{2n} \). Let \( \pi : A \to W \) be the canonical projection. Since the length of \( \pi(b) \) in \( W \) is \( < 2m \), \( c \) must be nontrivial, otherwise \( a \) would not be in \( PA \). Without loss of generality, suppose that \( c \) starts with \( s^{2n} \). Note that \( c \) must project to \( \text{rev}(b) \) under \( \pi \). In particular, the syllable length of \( b \) is strictly smaller than \( m-1 \).

Thus we can write \( a = bs^{2n}b^{-1}bc' \), where \( bc' \) is in \( PA \) and has strictly smaller syllable length than \( a \). In \( H_1(PA, \mathbb{Z}) \), we have the relation
\[
\overline{a} = bs^{2n}b^{-1} + bc'.
\]

Therefore, by induction on the length of \( a \), we can assume that the image of \( a \) in homology is the sum of images of elements \( b_is^{2n}b_i^{-1} \) and \( b_it^{2n}b_i^{-1} \), where the syllable length of the \( b_i \) is \( \leq \frac{m-2}{2} \) and the \( b_i \) only contain odd powers of \( s \) and \( t \) (or they are trivial). Therefore, the image of \( \Delta^{2n} \) in \( H_1(PA) \) could be written as a sum of elements \( b_is^{2n}b_i^{-1} \) and \( b_it^{2n}b_i^{-1} \) with syllable length of the \( b_i \leq \frac{m-2}{2} \). It is also easy to see that these \( b_i \) can only have syllable length \( \frac{m-2}{2} \) for a single \( i \).

However, this contradicts Lemma 2.2. Recall that \( H_1(PA, \mathbb{Z}) \) has a standard basis given by the hyperplanes in the dihedral arrangement, and the image of \( b_is^{2n}b_i^{-1} \) is precisely the basis vector \( e_r \), where \( r \) is the conjugate \( \pi(b_i)s\pi(b_i)^{-1} \).

In particular, if \( m \) is odd, then any such sum as above misses the hyperplane corresponding to the longest element of \( W \) (which is the conjugate \( (st)^{\frac{m-1}{2}}s(ts)^{\frac{m-1}{2}} \)). If \( m \) is even, there are two hyperplanes corresponding to the conjugates \( (st)^{\frac{m-2}{2}}s(ts)^{\frac{m-2}{2}} \) and \( (ts)^{\frac{m-2}{2}}t(st)^{\frac{m-2}{2}} \) and the elements in our sum miss one of these. We will refer to these as longest hyperplanes. Since \( \Delta^{2n} \) maps in \( H_1(PA, \mathbb{Z}) \) to the sum of all the hyperplanes in the dihedral arrangement, this is a contradiction.

Now, suppose that \( m_{st} > 3 \) and \( d_{B(A)}(\Delta^{2n}, g) = \pi \). We will first do the odd case, so \( m = 2k + 1 \). So, assume that we can write
\[
\Delta^{2n} = ag
\]
where the syllable length of \( a \) is equal to \( m \). In the decomposition \( a = bc \) above, we claim that \( b \) must have syllable length \( k \). Otherwise, we can again write the images
in $H_1(PA, \mathbb{Z})$ as $\bar{a} = bs^{2n}b^{-1} + bc$. If $b$ has syllable length $< k$ then both $bs^{2n}b^{-1}$ and $bc$ will miss the vector in $H_1$ corresponding to the longest hyperplane.

Therefore, without loss of generality we have $a = bc$ and the syllable length of $b$ is $k$. Since $a$ is in the pure Artin group, $c$ is of the form $s^{2n}x$ or $t^{2n}x$ where $x$ projects to $\text{rev}(b)$ (in particular each term of $x$ has odd exponent).

Therefore, without loss of generality we can rewrite $a = bs^{2n}b^{-1}a'$. By the above, $a' = b'c'$, where $b'$ has all odd exponents, has syllable length $k - 1$, and where $c' = s^{2n}x'$ or $t^{2n}x'$ as above. Repeating this argument gives us that in $H_1(PA, \mathbb{Z})$,

$$\bar{a} = e_{r_1} + e_{r_2} + \cdots + e_{r_{k+1}}$$

where the length of the reflections $r_i$ is strictly decreasing. If $m_{st} > 3$, then there are two hyperplanes of length $1 < \ell(r) < m$, and hence the image $\bar{a}$ misses one of those.

If $m = 2k$ is even, the proof essentially extends. The point here is that there are two longest hyperplanes and the image $\bar{a}$ will again miss one of these. More precisely, suppose that $a = bc$ as above. In order to hit one of the longest hyperplanes, $b$ must have syllable length $k - 1$. This implies that $c = s^{2n}x$ or $c = t^{2n}x$, and $x$ projects to $\text{rev}(b)$. In particular, we must have that $x = yt^{2n}$ or $x = ys^{2n}$ where $y$ projects to $\text{rev}(b)$. Therefore, we can push the last syllable into $g$. Since we now have a word $a'$ of length $< m$, the proof in the first case rules out this possibility.

We now prove the last statement. We will now assume that $m_{st} = 3$ (i.e. $A$ is the Artin group on 3-strands). We need the following lemma:

**Lemma 3.10.** If $n > 0$ we have that

$$\Delta^{2n} = st^{2n} s t^2 s^2 \cdots = ts^{2n} t s^2 t^2 \cdots$$

$$\Delta^{-2n} = s^{-1} t^{-2n} s^{-1} t^{-2} s^2 \cdots = t^{-1} s^{-2n} t^{-1} s t^{-2} \cdots$$

**Proof.** We only prove the first equalities for both $\Delta^{2n}$ and $\Delta^{-2n}$; the same argument with $s$ and $t$ switched will give the second. Since $\Delta^2 = st^2 st^2 = t^2 st^2 s$ we have by induction that for $n > 1$

$$\Delta^{2n} = \underbrace{st^{2n-2} t^2 s^2}_{2n-3 \text{ terms}} \cdots \underbrace{(t^2 st^2 s)}_{2n-3 \text{ terms}} = \underbrace{st^{2n-2} (t^2 st^2 s)}_{2n-3 \text{ terms}} s t^2 s^2 \cdots$$

$$= \underbrace{st^{2n} st^2 s^2}_{2n-1 \text{ terms}} t^2 s^2 \cdots$$

Similarly, $\Delta^{-2} = s^{-1} t^{-2} s^{-1} t^{-2} = t^{-2} s^{-1} t^{-2} s^{-1}$, so by induction for $n > 1$ we have

$$\Delta^{-2n} = \underbrace{s^{-1} t^{-2n+2} s^{-1} t^{-2} s^2}_{2n-3 \text{ terms}} \cdots \underbrace{t^{-2} s^{-1} t^{-2} s^{-1}}_{2n-3 \text{ terms}} = \underbrace{s^{-1} t^{-2n} s^{-1} t^{-2} s^2}_{2n-1 \text{ terms}} \cdots$$

$\square$
Now, suppose that $\Delta^{4n} = ag$ where $a$ has syllable length 3 and $g \in \langle s^4, t^4 \rangle$. Then without loss of generality we can assume that $a = s^k t^{4n} s^l$ where $k$ and $l$ are odd integers.

By conjugating with an even power of $s$, we can assume that $k = 1$. Then we have

$$st^{4n} st^2 s^2 t^2 s^2 \cdots = st^{4n} s^l g^k$$

so in particular $t^2 s^2 \cdots = s^{l-1} g s^l$, where $l - 1$ and $k - 1$ are even.

Since the original Tits Conjecture holds for $A$, we must have that $t^2 s^2 \cdots$ is equivalent to $s^{l-1} g s^l$ in the free group on $s$ and $t$. But the powers of $s$ and $t$ in $g$ are powers of 4, which is a contradiction. \hfill $\square$

3.3. Proof. Let $A$ be a locally reducible Artin group and $RA$ the associated RAAG. Let $K$ be the fundamental domain for the action of $A$ on its Deligne complex $D(A)$ with the metric as described in Section 2.5. Let $\hat{A}$ denote the RAAG subgroup that is in the original Tits Conjecture (i.e. the generators of $\hat{A}$ correspond to generators of $A$). In \cite{7} Charney defined a complex

$$\hat{D}(A) = \hat{A} \times K / \sim,$$

where $(a_1, x) \sim (a_2, x)$ if and only if $x \in K_{\geq T}$ and $a_1^{-1} a_2 \in \hat{A}_T$. This is not the Deligne complex for $\hat{A}$, since $\hat{A}_T$ may not be spherical. We define

$$Y(A) = RA \times K / \sim,$$

where $(a_1, x) \sim (a_2, x)$ if and only if $x \in K_{\geq T}$ and $a_1^{-1} a_2 \in RA_T$.

The Moussong metric on $K$ induces a piecewise Euclidean metric on $Y(A)$, and the homomorphisms $\Phi_{2n} : RA \to A$ define $\Phi_{2n} : Y(A) \to D(A)$ which sends $s \times K$ isomorphically onto $\Phi_{2n}(s) \times K$.

If $A$ is spherical, then both $\hat{D}(A)$ and $Y(A)$ have a cone point, labelled by $A$. We let $\hat{B}(A)$ be the link of this cone point in $Y(A)$, and $\hat{C}(A)$ the link of the cone point in $\hat{D}(A)$. Note that $\hat{C}(A)$ is naturally a subcomplex of $\hat{B}(A)$. If $A$ is dihedral, then we denote the generators of $RA$ by $\{s, t, \Delta\}$.

Lemma 3.11. If $A$ is a dihedral Artin group, then $\hat{B}(A)$ is the disjoint union $\cup_{i \in \mathbb{Z}} \Delta_i \, \hat{C}(A)$.

Proof. The adjacent vertices in $\hat{B}(A)$ to $w(s)$ are $w\emptyset$ and $ws\emptyset$, (and similarly for $w(s)$). Since $w = g \Delta^i$ and $ws = gs \Delta^i$ for some $g \in F_2$, the power $\Delta^i$ is the same for $w$ and $ws$. Therefore, any vertices that can be connected by a path in $\hat{B}(A)$ have the same power of $\Delta$. The $\Delta$-action on $\hat{B}(A)$ identifies each connected component of $\hat{B}(A)$ with $\hat{C}(A)$. \hfill $\square$
Proposition 3.12. Let $A$ be a totally reducible, finite type Artin group with $m_{st} \neq 3$ for each factor. Then the induced map $\Phi_2 : \text{Lk}(A,Y(A)) \to \text{Lk}(A,D(A))$ is $\pi$-distance preserving. For any totally reducible finite type Artin group, the induced map $\Phi_4 : \text{Lk}(A,Y(A)) \to \text{Lk}(A,D(A))$ is $\pi$-distance preserving.  

**Proof.** We have that $A$ and $RA$ decompose as  

$$A = A_1 \times A_2 \times \cdots \times A_k; RA = RA_1 \times RA_2 \times \cdots \times RA_k$$  

where each $A_i$ is an irreducible spherical subgroup of rank 2 or $\mathbb{Z}$. Therefore, both $B(A)$ and $\hat{B}(A)$ decompose as orthogonal joins  

$$B(A) = B(A_1) * B(A_2) * \cdots * B(A_k); \hat{B}(A) = \hat{B}(A_1) * \hat{B}(A_2) * \cdots * \hat{B}(A_k),$$  

so it suffices to check $\pi$-distance preserving for each $A_i$. If $A = \mathbb{Z}$ this is obvious, so suppose that $A$ is a dihedral Artin group. 

By [7, Lem 4.1], the induced map $\Phi_{2k} : \hat{C}(A) \to B(A)$ is $\pi$-distance preserving. Since the map $\Phi$ is equivariant, this implies that the induced map is distance preserving on each component $(\Delta^2)^k \hat{C}(A)$. By Lemma 3.11 it suffices to verify that for $x, y$ in two different copies of $\hat{D}(A)$ in $Y(A)$, their images $\phi_*(x), \phi_*(y)$ have distance at least $\pi$ in $\hat{D}(A)$. Let $x$ lie in an edge of $\Delta^{2n_1} \hat{C}(A)$ and $y$ lie in an edge an edge of $\Delta^{2n_2} \hat{C}(A)$, where $g_1, g_2 \in \hat{A}$, and $n_1 \neq n_2$. Then $x$ and $y$ are within distance $\pi/2m$ from vertices $\Delta^{2n_1}g_1$ and $\Delta^{2n_2}g_2$. By Proposition 3.9  

$$d_{B(A)}(\Delta^{2n_1}g_1, \Delta^{2n_2}g_2) = d_{\hat{B}(A)}(\Delta^{2(n_2-n_1)}, g_1g_2^{-1}) \geq \pi + \pi/m.$$  

This implies that the images of $x$ and $y$ are $\geq \pi$ apart. 

**Proposition 3.13.** The map $\Phi_4 : RA \to A$ is injective for every locally reducible Artin group. If $m_{st} \neq 3$ for all $s, t \in S$, then $\Phi_2 : RA \to A$ is injective.  

**Proof.** We will prove that the induced map $\hat{\Phi} : \hat{D}(A) \to D(A)$ is an isometric embedding. By Lemma 3.5 $\Phi : Y(A) \to D(A)$ is an isometric embedding provided that the map $\text{Lk}(x,Y(A)) \to \text{Lk}(\hat{\phi}(x), D(A))$ induced by $\hat{\phi}$ is $\pi$-distance preserving for every $x \in \hat{D}(A)$. We only check this where $x$ is a vertex, essentially the same argument works for all $x$. Since $\hat{\Phi}$ is equivariant, it suffices to check vertices of $K$. For $T$ a spherical subset and $v_T \in Y(A) \cap K$, the link of $v_T$ decomposes as $\text{Lk}(x,K_{\geq T}) \times \hat{B}(A_T)$. Since the link of $v_T$ in $D(A)$ decomposes as $\text{Lk}(x,K_{\geq T}) \times B(A_T)$ and the map between links decomposes as $\text{Id} \times \Phi_2$ the result follows from Proposition 3.12. 

3.4. **Intersections with special subgroups.** Finally, we use a coning trick to show that for any locally reducible Artin group $A$ and special subgroup $A_T$, we have that our RAAG subgroup for $A$ intersects $A_T$ in the RAAG subgroup for $T$. Charney used CAT(0) geometry to prove this for the RAAG provided by the original Tits Conjecture [7, Thm 5.2]. 

Let $RA$ be the associated RAAG for $A$, and let $\Phi_4 : RA \to A$ be the monomorphism. Suppose that there was a reduced word $w$ in $RA$ so that $w \notin RA_T$ but
Φ(\(w\)) ∈ \(A_T\). Define a larger Artin group \(\tilde{A}\) by "coning" off \(T\), i.e. introduce a new generator \(s\) so that \(m_{st} = 2\) for all \(t \in T\) and \(m_{su} = \infty\) otherwise. Then \(\tilde{A}\) is a locally reducible Artin group, so we know the Generalized Tits Conjecture for \(\tilde{A}\). Note that the RAAG for \(\tilde{A}\) is just the RAAG for \(A\) with the RAAG for \(A_T\) coned off. Now, by assumption we have that \([s, w] = 1\), which contradicts the RAAG for \(\tilde{A}\) injecting into \(\tilde{A}\), as the centralizer of \(s\) in that RAAG is the RAAG subgroup \(RA_T \times \langle s \rangle\).

**Remark 3.14.** The same argument shows that the RAAG subgroup that Crisp and Paris find intersects each special Artin group in the expected way. If we knew the Generalized Tits Conjecture for all Artin groups, then we would know this intersection property as well.

### 4. RAAG subgroups of RAAGs

In this section, we study whether a subgroup of a RAAG generated by words which are powers of commuting elements is the "obvious" RAAG. In Sections 5 and 6 we apply these ideas to the generalized Tits Problem, but we hope that this will be of independent interest.

In general, this is a delicate question; the subgroup may be a RAAG but a different one than expected, or may not be a RAAG at all. See [28] and [29] for a detailed analysis and many (positive and negative) examples. Our main goal in this section is Theorem 4.10, which generalizes a condition on the words given in [29] called Property PP (short for ping-pong).

**4.1. Koberda's Property PP.** Let \(L\) be a flag complex and \(RA_L\) the corresponding RAAG. For each simplex \(\sigma \in L\), let \(w_\sigma\) be a (possibly trivial) word with all positive powers (or all negative powers) of the generators corresponding to vertices in \(\sigma\) (in particular, \(w_\sigma\) has a nontrivial power of each generator in \(\sigma^{(0)}\)).

The collection \(\{w_\sigma\}_{\sigma \in L}\) determines a flag complex \(L'\), whose vertex set is \(\{\sigma \in L : w_\sigma\) is nontrivial\}, and whose simplices correspond to collections \(\sigma_1, \ldots, \sigma_k\) where \(w_{\sigma_1}, \ldots, w_{\sigma_k}\) pairwise commute in \(RA_L\). Of course, we have that \(w_\sigma\) and \(w_\tau\) commute if any only if \(\sigma\) and \(\tau\) span a simplex in \(L\).

**Definition 4.1.** We say that the collection \(\{w_\sigma\}_{\sigma \in L}\) satisfies Property PP if there is an injective simplicial map \(p : L' \to L\), so that \(p(L')\) is a full subcomplex of \(L\) and the vertex of \(L'\) corresponding to \(w_\sigma\) maps to a vertex of \(\sigma\) for each \(\sigma \in L\).

One can think of this map \(p\) as choosing a representative vertex in \(L\) for each \(w_\sigma\). The requirement that \(p(L')\) is a full subcomplex ensures that if two words \(w_\sigma\) and \(w_{\sigma'}\) do not commute, then their representative vertices do not commute. The following proposition of Koberda follows quickly from the normal form for RAAG’s.

**Proposition 4.2 (Koberda).** Let \(w_\sigma\) have Property PP. Then the map \(f : RA_{L'} \to RA_L\) sending the generators \(\sigma\) of \(RA_{L'}\) to \(w_\sigma\), is injective. Therefore, the subgroup generated by \(\langle w_\sigma \rangle\) is isomorphic to \(RA_{L'}\).
Example 4.3. The following example is taken from [29], where it is attributed to M. Casals. Let $R_{A_L} = F_2 \times F_2$ where the first $F_2$ is generated by $a, c$ and the second generated by $b, d$. Consider the subgroup $H_n \leq A_L$ generated by $a^n, d^n$ and $(bc)^n$ for $n \in \mathbb{N}$. It turns out that $H_n$ is not isomorphic to any right-angled Artin group. We will not provide the full proof of this, the key point is that in this group there is the relation
\[ [a^n, (bc)^n d^n (bc)^{-n}] = 1. \]
Note that this collection of words does not satisfy property PP, in this case $L'$ is the disjoint union of an edge and a point, and hence there is no injective map from $L' \to L$ where the image is a full subcomplex.

4.2. Generalization of Property PP. We start with a motivating example.

Example 4.4. Let $L$ be a path with 4 vertices $a, b, c, d$, and let $R_{A_L}$ be the corresponding RAAG. Consider the subgroup $H$ generated by $\{a, d, bc\}$. By the same reasoning as Example 4.3, this collection of words do not satisfy property PP (in this case $L'$ is 3 points and there is no injective map from 3 points to $L$ with image a full subcomplex). However, we claim that the subgroup $H$ generated by these words is still isomorphic to the free group $F_3$.

To see this, note that $R_{A_L}$ splits as the amalgamated product
\[ R_{A_L} = \langle a, b, c \rangle \ast \langle b, c \rangle \langle b, c, d \rangle \]
and $F_3$ decomposes as $F_2 \ast_{\mathbb{Z}} F_2$. We can use Property PP to say that the subgroups generated by $\langle a, bc \rangle$ and $\langle bc, d \rangle$ inside $\langle a, b, c \rangle$ and $\langle b, c, d \rangle$ respectively are both $F_2$. Furthermore, each of these subgroups intersects $\langle b, c \rangle$ in $\langle bc \rangle$ (this is not completely obvious and generalizing this is the majority of our work below). Therefore, we can apply the following lemma of Serre:

Proposition 4.5. Let $G_i$ be a collection of groups with common subgroup $A$ and let $\ast_A G_i$ denote the amalgamated product. Let $H_i \leq G_i$ be subgroups and suppose the intersection $B = H_i \cap A$ is independent of $i$. Then the natural homomorphism $\ast_B H_i \to \ast_A G_i$ is injective.

So, in the above example, we get an injection from $F_3 \cong F_2 \ast_{\mathbb{Z}} F_2 \to \langle a, bc, d \rangle$, which is obviously an isomorphism. We shall also need the following lemma about subgroups of amalgamated products, which follows immediately from the normal form for amalgamated products.

Lemma 4.6. Let $G = A \ast_C B$ be an amalgamated product, and $H = D \ast_F E$ a subgroup of $G$ so that $D \subset A, E \subset B$ and $D \cap E = F$. If $K$ is a subgroup of $G$ contained in $A$, then $K \cap H = K \cap D$.

For general RAAG's, we suppose the following: our nerve $L$ decomposes as $L = L_1 \cup_{L_0} L_2$, where each $L_i$ is a full subcomplex. We consider a collection of words $\{w_\sigma\}_{\sigma \in L}$ where, as before, $w_\sigma$ is a (possibly trivial) word with all positive powers (or all negative powers) of the generators corresponding to vertices in $\sigma$. We assume that each of the collections $\{w_\sigma\}_{\sigma \in L_1}, \{w_\sigma\}_{\sigma \in L_2}$ satisfies Property PP in $R_{A_{L_1}}, R_{A_{L_2}}$.
The collection subcomplex. Let $L$ be a flag complex and function $p : L' \to L$. We say that the collection $\{w_\sigma\}_{\sigma \in L}$ avoids $L_0$ if the following two conditions are satisfied:

1. If $\sigma \notin L_0$, then $p(\sigma) \notin L_0$.
2. If $\sigma \notin L_0$, then $w_\sigma$ is trivial for any other $\tau$ containing $p(\sigma)$.

The next lemma guarantees that if $\{w_\sigma\}_{\sigma \in L}$ satisfies property $PP$ and avoid $L_0$, then the intersection of the RAAG subgroup generated by the $\{w_\sigma\}$ intersects the special subgroup $A_{L_0}$ as expected.

**Lemma 4.8.** Let $L$ be a flag complex, $RA_L$ the RAAG on $L$, and let $L_0$ be a full subcomplex. Let $\{w_\sigma\}_{\sigma \in L}$ be words satisfying Property $PP$ avoiding $L_0$. Let $RA_{L'}$ be the RAAG subgroup of $A_L$ generated by the $\{w_\sigma\}_{\sigma \in L}$, and $RA_{L_0'}$ the RAAG subgroup generated by $\{w_\sigma\}_{\sigma \in L_0}$. Then $RA_{L'} \cap RA_{L_0} = RA_{L_0'}$.

**Proof.** Note that we are obviously done if $w_\sigma$ is trivial for each $\sigma$ not contained in $L_0$. We assume that this is not the case. We induct on the number of vertices of $L$.

We first prove the case where $L$ is a simplex, so that $RA_L$ is free abelian. Note that $RA_L$ splits as a product $RA_{L_0} \times RA_{L-L_0}$. We claim that $p$ is injective while restricted to the subgroup generated by $\{w_\sigma\}_{\sigma \notin L_0}$. Indeed, by Condition 2 of Definition 4.7 each $w_\sigma$ with $\sigma \notin L_0$ has a non-trivial coordinate in $RA_L$ such that all other $w_\tau$ have this coordinate zero. It follows that $RA_L \cap RA_{L_0} = RA_{L_0'}$.

We now consider the general case. Let $\mu$ be the largest standard factor of $L$, i.e. the largest dimensional simplex so that $L = \mu \ast K$. In particular, $K$ is not a cone. Choose $\sigma$ not contained in $L_0$ with non-trivial $w_\sigma$. We consider different cases.

First, suppose there exists such $\sigma$ with $p(w_\sigma) \in K$. We denote $p(w_\sigma)$ by $v$. By Condition 1 of Definition 4.7 $v$ is not in $L_0$. Since $K$ has no non-trivial standard factors, $St(v)$ is a proper subcomplex of $L$. Thus we have a nontrivial decomposition

$$RA_L = RA_{L-v'} \ast RA_{Lk(v)} RA_{St(v)}.$$

The collection $\{w_\sigma\}_{\sigma \in L-v}$ satisfies property $PP$ and avoids $L_0$. Let $RA_{L-v'}$ be the RAAG subgroup generated by $\{w_\sigma\}_{\sigma \in L-v}$. By induction, $RA_{L-v'} \cap RA_{L_0} = RA_{L_0'}$. Since $\{w_\sigma\}_{\sigma \in L}$ satisfies property $PP$, if $\tau$ is contained in $St(v)$, then $w_\tau$ commutes with $w_\sigma$, since $v$ commutes with each vertex in $w_\tau$, and hence commutes with $p(w_\tau)$. By Condition 2 $w_\sigma$ is the only non-trivial word such with $v \in \sigma$. Thus the words $\{w_\sigma\}_{\sigma \in St(v)}$ satisfy Property $PP$ and avoid $Lk(v)$. Let $RA_{St(v)}$ denote the RAAG subgroup generated by $\{w_\sigma\}_{\sigma \in St(v)}$. Again by induction, we have that $RA_{St(v)}' \cap RA_{Lk(v)} = RA_{Lk(v)}'$. Therefore, we have that

$$RA_{L'} = RA_{L-v'}' \ast RA_{Lk(v)}' RA_{St(v)}'.$$
and Lemma 4.6, we have that $RA_{L'} \cap RA_{L_0} = RA_{(L-v)'} \cap RA_{L_0} = RA_{L'_0}$.

Now suppose that for every $\sigma$ not contained in $L_0$ with nontrivial $w_\sigma$, we have $p(w_\sigma) \notin K$. Take any such $\sigma$. We have $p(w_\sigma) \in \mu$. We first consider the case where $\sigma$ is not entirely contained in $\mu$. In this case, since each nontrivial $w_\sigma$ must commute with $w_\mu$, the subgroup $RA_{L'}$ is contained in $RA_{St(\sigma)}$. The star $St(\sigma)$ decomposes as a join $(\mu * (\sigma \cap K)) * (St_K(\sigma \cap K))$. By the choice of $\mu$, this join is properly contained in $L$, and in particular has fewer vertices than $L$. By induction we are done.

Finally, if all $\sigma$ not contained in $L_0$ with nontrivial $w_\sigma$, are entirely contained in $\mu$, then the subgroup $RA_{L'}$ splits as a product of a free abelian group generated by $\{w_\sigma\}_{\sigma \notin L_0}$ and $RA_{L'_0}$. The intersection of the free abelian group generated by $\{w_\sigma\}_{\sigma \notin L_0}$ with $RA_{L_0}$ is trivial, by Condition 2 of Definition 4.7. The conclusion follows. □

Remark 4.9. Without the second condition in Definition 4.7, Lemma 4.8 is false even for free abelian groups. For example, if $L$ is a simplex with vertices $s,t,u$, then the words $stu$ and $st$ avoid $u$, but the subgroup they generate contains $u$. On the other hand, we expect this second condition can be significantly weakened.

This lemma combined with Serre’s proposition implies the following theorem:

**Theorem 4.10.** Let $L$ be a flag complex and suppose that $L = L_1 \cup L_0 L_2$ where each $L_i$ is a full subcomplex. Let $\{w_\sigma\}_{\sigma \in L}$ be words so that each subcollection $\{w_\sigma\}_{\sigma \in L_i}$ satisfies property PP and avoids $L_0$. Then the subgroup generated by the $\{w_\sigma\}_{\sigma \in L}$ is the RAAG based on $L'$.

**Proof.** By Property PP, we know that the subgroup $RA_{L'}$ of $RA_{L_i}$ generated by $\{w_\sigma\}_{\sigma \in L_i}$ is a RAAG. By Lemma 4.8, we know that

$$RA_{L'_1} \cap RA_{L_0} = RA_{L'_2} \cap RA_{L_0} = RA_{L'_0}.$$  

By Proposition 4.5, $RA_{L'} = RA_{L'_1} *_{RA_{L_0}} RA_{L'_2}$ injects into $RA_L = RA_{L_1} *_{RA_{L_0}} RA_{L_2}$. □

Remark 4.11. The words $b,d,ac$ in Example 4.3 do not satisfy generalized PP for any decomposition of $L$. For example, if we take $L_1$ the full subcomplex containing $a,b,c$ and $L_2$ the full subcomplex containing $b,c,d$, then for the edge $ac$ we have to choose the vertex $c$ in property PP, which is in $L_1 \cap L_2$. Note that the subgroup generated by $ac$ and $b$ contains $cbc^{-1}$, and hence intersects the subgroup $\langle b, c \rangle$ outside of $\langle b \rangle$.

5. Generalized Tits Conjecture and small type spherical Artin groups

In this section, we describe how Generalized Tits Conjecture holds for small-type spherical Artin groups of type $A_n$ and $D_n$. The main work is type $D_n$, where we require the generalized Property PP of the previous section.
Let $A$ be any small-type spherical Artin group with the standard generating set $S$, and the Perron-Vannier representation $A \to \text{Mod}(\Sigma)$. Let RA be the associated RAAG with the presentation $1.1$. Let us record the following observation.

**Lemma 5.1.** If the composition $RA \xrightarrow{\Phi_N} A \to \text{Mod}(\Sigma)$ is an embedding, then $A$ satisfies the Generalized Tits Conjecture with $N$.

For every irreducible subset $T \subset S$, let $A_T \to \text{Mod}(\Sigma_T)$ be its Perron-Vannier representation of the special subgroup $A_T$. If $T = \{s\}$, by $\Sigma_T$ as well as $\partial\Sigma_T$ we mean the single curve in $\Sigma$ corresponding to $s$. The surface $\Sigma_T$ can be embedded in $\Sigma$ and that embedding induces a homomorphism $\text{Mod}(\Sigma_T) \to \text{Mod}(\Sigma)$ which makes the following diagram commute

$$
\begin{array}{ccc}
A_T & \longrightarrow & \text{Mod}(\Sigma_T) \\
\downarrow & & \downarrow \\
A & \longrightarrow & \text{Mod}(\Sigma).
\end{array}
$$

Note that $\partial\Sigma_T$ is a multicurve in $\Sigma$. By Remark 2.5, $\Delta^1_T$ is sent to a multitwist around the boundary components of $\partial\Sigma_T$ in $\text{Mod}(\Sigma)$. We summarize the above discussion in the following lemma.

**Lemma 5.2.** Let $A$ be a small type spherical Artin group and let $N \geq 2$. Then each of the generators $\{s_T\}_T$ of the RAAG RA is mapped to a multitwist in $\text{Mod}(\Sigma)$ via $\rho \circ \Phi_N$.

The following theorem of Koberda is our main tool for proving the generalized Tits conjecture for the spherical Artin groups.

**Theorem 5.3 (29).** Let $f_i$ be a finite collection of nontrivial powers of Dehn twists around simple closed curves so that the subgroup $\langle f_i, f_j \rangle$ is not cyclic for all $i, j$. Then there is an $M > 0$ so that the powers $f_i^M$ generate a RAAG subgroup of $\text{Mod}(\Sigma)$.

The flag complex that is the nerve of the RAAG generated by those powers of Dehn twists has the collection of the curves as its vertex set, where a subcollection of curves spans a simplex if and only if they can be realized as pairwise disjoint curves. The analogous result does not hold for multitwists instead of Dehn twists about single curves. Indeed, it is easy to find surfaces and multicurves so that the RAAG generated by Dehn twists about the individual curves is $F_2 \times F_2$, and the words given by the multicurves are as in Example 4.3.

Coming back to the surface $\Sigma$ from the Perron-Vannier representation of $A$, let $C$ denote the collection of all the curves in the support of the multitwists $\{\rho \circ \Phi_N(s_T)\}_T$. By Theorem 5.3, high powers of Dehn twists about the curves in $C$ generate a RAAG RA$_C$. Therefore, we have an induced homomorphism $RA \to RA_C \subset \text{Mod}(\Sigma)$, where each generator of RA goes to a product of commuting generators of RA$_C$. In order to prove that $A$ satisfies the Generalized Tits Conjecture with $N$, it suffices to show
that \( RA \to RA_C \) is injective. In the case of \( A \) of type \( A_n \) and \( D_n \), we show it using Property PP and generalized Property PP from Section 4, respectively. To verify Property PP, we pick one of the boundary components \( p(T) \) in \( \partial \Sigma_T \) for each irreducible \( T \) such that the curves \( p(T), p(T') \) intersect if and only if \( [\Delta^2_T, \Delta^2_{T'}] = 1 \) (if and only if \( T \subset T', T' \subset T \), or \( [T, T'] = 1 \)). For details on how we verify generalized Property PP, see subsection on type \( D_n \).

We now rephrase Property PP in terms of curves on surfaces in the context of RAAG subgroups of mapping class groups.

**Lemma 5.4.** Let \( \hat{C} \) a collection of multicurves in a surface \( \Sigma \) with and let \( C \) be the collection of all the connected component of elements of \( \hat{C} \). Suppose there exists a function \( p : \hat{C} \to C \) such that
- \( p(\alpha) \in \alpha \) and
- \( p(\alpha), p(\beta) \) intersect if and only if some connected component of \( \alpha \) intersect some connected component of \( \beta \).

Let \( \{ T_\alpha : \alpha \in \hat{C} \} \) be a collection of multitwists about multicurves in \( \hat{C} \). Then the exists \( M > 0 \) such that \( \langle T_\alpha^M : \alpha \in \hat{C} \rangle \) is a RAAG.

**Proof.** By Theorem 5.3 the group generated by sufficiently large powers of Dehn twists around \( C \) is a RAAG. The assumption about existence of function \( p \) is just reformulation of Property PP from Definition 4.1. The conclusion follows from Proposition 4.2. \( \square \)

**5.1. Intersections of boundary curves.** Let \( A \) be a small-type spherical Artin group, and let \( A \to \text{Mod}(\Sigma) \) be the Perron-Vannier representation. Suppose that \( T \) and \( T' \) are two irreducible spherical subsets of \( S \), and let \( \Sigma_T \) and \( \Sigma_{T'} \) be the associated subsurfaces of \( \Sigma \). If \( \Delta^2_T \) and \( \Delta^2_{T'} \) commute, then each of the boundary curves of \( \Sigma_T \) and \( \Sigma_{T'} \) are necessarily disjoint. If they do not commute, then there are different possibilities for the possible intersections between the curves. We shall record a few lemmas that we will need later.

**Lemma 5.5.** Let \( A \) be of type \( A_n \), and suppose that \( T \) and \( T' \) are irreducible subsets such that \( \Delta^2_T \) and \( \Delta^2_{T'} \) do not commute. If \( \Sigma_T \) has one boundary component up to homotopy (so \( |T| = 1 \) or \( |T| \) is even), then the boundary curve of \( \Sigma_T \) has nontrivial intersection number with every boundary curve of \( A_T \).

**Proof.** Note that \( \partial \Sigma_T \) is fixed up to homotopy by the hyperelliptic involution. If \( A_T \) has two boundary components, then these are permuted up to homotopy by the hyperelliptic involution. Therefore, we only have to show that \( \partial \Sigma_T \) has nontrivial intersection number with a single boundary component of \( \partial \Sigma_{T'} \). Since the Perron-Vannier representation is injective in this case \( [33] \), this follows from the fact that \( [\Delta^2_T, \Delta^2_{T'}] \neq 1 \) in \( A_n \).

**Lemma 5.6.** Suppose \( \Sigma_{T'} \) and \( \Sigma_T \) both have two boundary components, i.e. both \( |T| \) and \( |T'| \) are odd and \( \neq 1 \). Then
Each boundary component of $\Sigma_T$ intersects at least one boundary component of $\Sigma_{T'}$, and vice versa.

If $|T - T'|$ is odd, then each boundary component of $\Sigma_T$ intersects each boundary component of $\Sigma_{T'}$.

Proof. For the first statement, again by injectivity of the Perron-Vannier representation we must have that at least one component of $\partial \Sigma_T$ intersects a component of $\partial \Sigma_{T'}$. Since both components are permuted by the hyperelliptic involution, we get the other intersection.

We will prove the second statement in the Alternative proof of Proposition 5.9 below, once we have developed more notation.

There is a similar lemma for the images of curves in the Perron-Vannier representation $A \to \text{Mod}(\Sigma)$ for Artin groups $A$ of type $D_n$. Note that $\Sigma$ has a unique boundary curve which is contained in the boundary of $\Sigma_T$ for any irreducible subset $T \subset S$ of type $D_m$ for $m < n$. We will call this the outer component, and the other boundary components inner components. We shall see that the Dehn twist around the outer component never factors into any of our calculations.

Lemma 5.7. Let $A$ be of type $D_n$, and suppose that $T$ and $T'$ are irreducible subsets such that $\Delta^2_T$ and $\Delta^2_{T'}$ do not commute.

1. If $A_T$ has type $D_m$ and $m = 2k + 1$ for $k > 0$, then the inner boundary component of $\partial \Sigma_T$ intersects each curve in $\partial \Sigma_{T'}$, if $s, s' \notin T'$, and intersects one boundary curve of $\Sigma_{T'}$ if one of $s$ or $s' \in T'$.

2. If $A_T$ has type $D_m$ and $m = 2k$ for $k > 1$, then if $s, s' \notin T'$ each inner boundary component either intersect all curves in $\partial \Sigma_{T'}$ or there are two pairs of components that intersect. If $s$ or $s' \in T'$, there is one non-central component of $\partial \Sigma_T$ which has nontrivial intersection with $\partial \Sigma_{T'}$. Furthermore, one component intersects all $\partial \Sigma_{T'}$ for $T'$ that contain $s$, and one component intersects all $\partial \Sigma_{T'}$ for $T'$ that contain $s'$.

3. If $s \in T$ and $s' \in T'$, then the components of $\partial \Sigma_T$ and $\partial \Sigma_{T'}$ which are not contained in $\partial \Sigma_{D_m}$ for any $m$ always intersect.

Proof. The proof of the first two items is similar to Lemma 5.5. The non-central component(s) of $\partial \Sigma$ is fixed by the hyperelliptic involution. If $\partial \Sigma_T$, has two components then if $s, s' \notin T'$ then these are permuted by the hyperelliptic involution. If $s$ or $s' \in T'$, then one of the curves of $\partial \Sigma_{T'}$ is contained in $\partial \Sigma_{D_m}$ for $m > n$, and hence misses the non-central component(s) of $\partial \Sigma_{D_m}$. The other curve therefore intersects $\partial \Sigma_{D_m}$. For the last statement of the second item, note that each of the boundary components of $\partial \Sigma_{D_m}$ is a boundary component of $\partial \Sigma_{A_n}$ and $\partial \Sigma_{A'_{m}}$. Therefore, they are disjoint from $\partial \Sigma_{A_n}$ and $\partial \Sigma_{A'_{m}}$ respectively for $m > n$.

For the last item, note that both these curves have nontrivial intersection number with $\gamma_s$ and $\gamma'_s$ respectively. The curves $\gamma_s$ and $\gamma'_s$ and the outer boundary component are the boundary of a pair of pants. If our curves had trivial intersection number,
Figure 7. Dehn twists around the grey curves in this diagram are images of $\Delta^2_T$ for certain braid subgroups of the braid group on 8 strands. The red and blue curves arise via the lantern relation, and obviously commute in the braid group. It is easy to check that the words in the lantern relation which produce these curves do not commute in the RAAG on the $\Delta^2_T$, and hence the induced homomorphism from this RAAG to the braid group is not injective.

then we could homotope one of them to have trivial intersection number with both $\gamma_s$ and $\gamma_s'$.

5.2. Type $A_n$. Suppose that $A$ is a spherical Artin group of type $A_n$, i.e., $A$ is the braid group on $n+1$-strands. Then $A$ is the mapping class group of the $n+1$-punctured disc. We give the punctures an arbitrary labeling $\{1, 2, \ldots, n+1\}$. The squares of the standard generators of $A$ correspond to the Dehn twists about the simple closed curves around two consecutive punctures $\{i, i+1\}$. Any irreducible subset $T$ of $S$ corresponds to a subset $I \subset \{1, \ldots, n+1\}$ of consecutive numbers, where $|I| = |T| + 1$. The center of the pure braid group $PB_T$ corresponds to a Dehn twist $T_I$ around punctures in $I$, see Figure 7. Therefore, the Generalized Tits Conjecture for $N = 1$ asks if this collection of Dehn twists about these simple closed curves generates a RAAG subgroup of the braid group. For simplicity, we denote the generators $s_T$ of RA by $s_I$.

Example 5.8. The braid groups on $\geq 8$ strands do not satisfy the Generalized Tits Conjecture with $N = 1$. The map $\Phi_1$ is not injective, as there are deeper relations between the $\Delta^2_T$. See Figure 7. By the lantern relation (see e.g. [23, Prop 5.1]), the Dehn twist $T_r$ about the red curve in Figure 7 satisfies

$$T_r = T_{3456}^{-1}T_{1234}^{-1}T_{12}T_{34}T_{56}T_{123456}. $$

Similarly, the Dehn twist $T_b$ about the blue curve satisfies

$$T_b = T_{5678}^{-1}T_{3456}^{-1}T_{34}T_{56}T_{78}T_{345678}. $$

However, the elements $s_{3456}^{-1}s_{1234}^{-1}s_{34}^{-1}s_{56}^{-1}s_{123456}^{-1}$ and $s_{5678}^{-1}s_{3456}^{-1}s_{34}^{-1}s_{56}^{-1}s_{78}^{-1}s_{345678}^{-1}$ do not commute in RA. Indeed, it suffices to consider their images under the retraction $RA \to \langle s_{123456}, s_{345678} \rangle \simeq F_2$. 


As we have noted, Theorem 5.3 implies that high powers of these elements generate a RAAG, i.e. we get the following

**Proposition 5.9.** The Generalized Tits Conjecture holds for all spherical Artin groups of type $A_n$.

We shall also show how the conjecture follows from using the Perron-Vannier representation and property PP (this will serve as a warmup for the other cases).

**Alternative proof of Proposition 5.9.** By Lemma 5.4, it suffices to show that the multicurves that arise in the Perron-Vannier representation satisfy the condition in the statement of the lemma. Let $S = \{t_1, \ldots, t_n\}$ be the standard set of generators.

Note that if $n$ is odd, then $\Sigma - \{t_1, \ldots, t_3, \ldots, t_n\}$ has two connected components. Similarly, if $n$ is even, then $\Sigma - \{t_1, \ldots, t_3, \ldots, t_{n-1}, k\}$ has two connected components, where $k$ is an arc with both endpoints in $\partial \Sigma$ and which intersect only $t_n$ among the curves in $S$. In either case, we pick a connected component, and denote it by $\Sigma_+$. Similarly, let $\Sigma_{\#}$ be the connected component that does not contain $\partial \Sigma$ of respectively $\Sigma - \bigcup \{t_2, t_4, \ldots, t_{n-1}, \ell, \ell'\}$ in case of odd $n$, and of $\Sigma - \bigcup \{t_2, t_4, \ldots, t_n, k\}$ in case of even $n$. Here, $\ell, \ell'$ denote arcs with the endpoint in two connected components of $\partial \Sigma$ where $\ell$ intersects only $t_1$, and $\ell'$ intersects only $t_n$ among curves in $S$.

The arc $k$ has both endpoint in $\partial \Sigma$ and intersects only $t_1$ among curves in $S$.

Let $T = \{t_i, \ldots, t_j\}$. If $|T|$ is even, then $\Sigma_T$ has a unique boundary component, denoted by $t_{ij}$. Otherwise, if $|T|$ is odd, then we denote the two boundary components of $\Sigma_T$ by $\{t_{ij}, t'_{ij}\}$. If $T = \{t_i, \ldots, t_j\}$ and $|T|$ is even, we denote the unique boundary curve of $\Sigma_T$ by $t_{ij}$. When $|T|$ is odd, then we denote the boundary curves of $\Sigma_T$ by $\{t_{ij}, t'_{ij}\}$. If additionally, $i$ is odd, then again exactly one of these curves is contained in $\Sigma_+$.

Similarly, if $i$ is even, then we assume that $t_{ij}$ is contained in $\Sigma_{\#}$. For each irreducible set $T = \{t_i, \ldots, t_j\}$, we set $p(T) = t_{ij}$, if $i > 1$. See Figure 8.

We claim that our choice function $p$ satisfies Lemma 5.4. By the first part of Lemma 5.5, we only need to worry about irreducible subsets of odd cardinality. Suppose we have two subsets $T = \{t_i, \ldots, t_j\}$ and $T' = \{t_{i'}, \ldots, t_{j'}\}$ of odd cardinality which do not commute, i.e. $[s_T, s_{T'}] \neq 1$ in RA.

We claim that if $i$ is odd and $i'$ is even or vice versa, then each curve in $\partial \Sigma_T$ intersects each curve in $\partial \Sigma_{T'}$, (this is the second part of Lemma 5.7). To see this, note that it suffices to assume that $T = \{1, 2, \ldots, j\}$ and $T' = \{k, \ldots, l\}$. Since $j$ is odd, $\partial \Sigma_T$ has two components, and both components of $\partial \Sigma_{T'}$ intersect the curve $t_{k-1}$ exactly once (and miss all other $t_i$ for $i \leq j$). Therefore, each component of $\Sigma_{T'}$ intersects both components of $\Sigma_T - \{t_1, t_3, \ldots, t_{k-1}, \ldots, t_j\}$, and therefore must intersect both components of $\partial \Sigma_T$.

If $i$ and $i'$ are both odd, then $p(T)$ cannot intersect the curve of $\partial \Sigma_{T'}$ which is contained in $\Sigma - \Sigma_+$, and by the second part of Lemma 5.5, $p(T)$ intersect $p(T')$. Similarly, if $i$ and $i'$ are both even, then $p(T)$ and $p(T')$ intersect.

We do not know if $N = 2$ suffices for the braid group. In fact, Runnels and Seo have independently and recently shown an effective version of Koberda’s result [35],


Figure 8. Curves $s_2, s_3, s'_3, \ldots, s_{2k}, s_{2k+1}, s'_{2k+1} \ldots$ and curves $t_{i:j}, t'_{i:j}$ in the surface $\Sigma$ of type $A_n$. The thick curves are an example of a choice of the representatives for property PP.

For the collections of Dehn twists that arise in the braid group case, it turns out that $N = 17$ works, and we suspect there is a not very large $N$ that works for all Artin groups.

5.3. Type $D_n$. We now consider an Artin group $A$ of type $D_n$ with the standard generating set $S = \{s, s', t_1, \ldots, t_n\}$, as in Figure 5.

Theorem 5.10. Generalized Tits Conjecture holds for all spherical Artin groups of type $D_n$.

Proof. There are four families of irreducible subsets of $S$ with at least two elements.

1. If $T = \{s, s', t_1, \ldots, t_j\}$ where $j \leq n - 2$, then $A_T$ has type $D_{j+2}$.
2. If $T = \{t_i, \ldots, t_j\}$ where $1 \leq i < j \leq n - 2$, then $A_T$ has type $A_{j-i+1}$.
3. If $T = \{s, t_1, \ldots, t_j\}$ where $1 \leq j \leq n - 2$, then $A_T$ has type $A_{j+1}$.
4. If $T = \{s', t_1, \ldots, t_j\}$ where $1 \leq j \leq n - 2$, then $A_T$ has type $A_{j+1}$.

We consider the Perron-Vannier representation $\rho : A \to \text{Mod}(\Sigma)$ (see Section 2.6 and Figure 5). By Lemma 5.2, the generator $s_T$ of $RA$ is sent to a multitwist about the boundary of a subsurface $\Sigma_T$.

We pick a connected component of, respectively, $\Sigma_+$ of $\Sigma - \{t_2, t_4, \ldots, t_{n-2}, k\}$ when $n$ is even, and of $\Sigma - \{t_2, t_4, \ldots, t_{n-3}, k, \ell\}$ when $n$ is odd. In both cases $k$ denotes an arc with both endpoints in the outer boundary component of $\Sigma$ that intersects $t_1$ and no other curves in $S$. When $n$ is odd, then $\ell$ denotes an arc with both endpoints in the unique inner boundary component of $\Sigma$, that intersects $t_{n-2}$ and no other curves in $S$. We denote the chosen component by $\Sigma_+$. We can also pick a connected component $\Sigma_#$ of, respectively, $\Sigma - \{t_1, t_3, \ldots, t_{n-3}, k_1, k_2, k_3\}$ when $n$ is even, and of $\Sigma - \{t_1, t_3, \ldots, t_{n-2}, k_1, k_2\}$ when $n$ is odd. By $k_1, k_2, k_3$ we denote arc with endpoints in distinct connected components of $\partial \Sigma$ that intersect only respectively $s, s', t_{n-2}$ (the last one only when $n$ is even) among curves of $S$.

Let us now analyze what those multicurves are for the four families of irreducible special subgroups. Let $s_0$ denote the outer boundary curve of $\partial \Sigma$.

For each irreducible set $T = \{s, s', t_1, \ldots, t_j\}$, the multicurve $\partial \Sigma_T$ is of the form $\{s_0, s_{j+1}, s'_{j+1}\}$ when $|T|$ is even, and of the form $\{s_0, s_j\}$ when $|T|$ is odd. We also
set $s_1 = s$ and $s'_1 = s'$. See the left side of Figure 9. Without loss of generality, we can assume that all the curves $s_j$ are contained in $\Sigma_+$. Now consider an irreducible subset $T = \{t_i, \ldots, t_j\}$ where $1 \leq i < j \leq n - 2$. If $|T|$ is even, let $t_{i,j}$ be the unique boundary curve of the subsurface $\Sigma_T$. If $|T|$ is odd, let $t_{i,j}, t'_{i,j}$ be the two boundary curves of the subsurface containing $t_i, \ldots, t_j$. See the right side of Figure 9. If $i$ is even, we assume that $t_{i,j}$ lies in $\Sigma_+$, and if $i$ is odd, we assume that $t_{i,j}$ is contained in $\Sigma_\#$.

Finally consider the irreducible subset $T = \{s, t_1, \ldots, t_j\}$. When $j$ is odd, we denote the unique boundary component of $\Sigma_T$ by $r_{j+1}$. When $j$ is even, the boundary of the subsurface $\partial \Sigma_T$ has two connected components, one is $s'_{j+1}$, and the other is denoted by $r_{j+1}$. See Figure 10. We define curves $r'_2, r'_3, \ldots$ analogously, so that $\partial \Sigma_T = \{r'_{j+1}\}$ or $\{s'_{j+1}, r'_{j+1}\}$ for $T = \{s, t_1, \ldots, t_j\}$.

Let $C = \{s_j, s'_j\} \cup \{t_j\} \cup \{t_{i,j}, t'_{i,j}\} \cup \{r_j, r'_j\}$ be the collection of all the curves that arise as boundary curves of subsurfaces $\Sigma_T$ for irreducible subsets $T \subset S$. By Theorem 5.3, the subgroup generated by sufficiently large powers of Dehn twists around curves in $C$ is a RAAG, which we denote by $\text{RA}_C$. Note that the words $w_\sigma$ corresponding to centers of irreducible spherical Artin subgroups, regarded as elements of $\text{RA}_C$ do not satisfy property PP, see Figure 11.

We now show that generalized property PP implies the conjecture for Artin groups of type $D_n$. Let $C_1 = C - \{r'_k\}_{k \geq 2}$, $C_2 = C - \{r_k\}_{k \geq 2}$, and $C_0 = C_1 \cap C_2 = C - \{r_k, r'_k\}_{k \geq 2}$.

Let $L$ be the flag complex on the vertex set $C$ defining $\text{RA}_C$, and let $L_0, L_1, L_2$ be the full subcomplexes of $L$ on the sets $C_0, C_1, C_2$ respectively. Since the curves $r_i$ and...
$r_j'$ intersect for every $i, j \geq 2$, the complex $L$ decomposes as $L_1 \cup_{L_0} L_2$. Therefore we have a splitting of the RA$\mathcal{C}$ as $A_{L_1} *_{A_{L_0}} A_{L_2}$ where $A_{L_i}$ is a RAAG with nerve $L_i$. Also, the RAAG RA associated with $A$ splits as $RA = RA_{L_1} *_{RA_{L_0}} RA_{L_2}$ where

- $RA_{L_1}$ omits all the generators corresponding to centers of the irreducible special subgroups from family (4), i.e. $A_T$ with $T = \{s', t_1, \ldots, t_k\}$ and $k \geq 1$,
- $RA_{L_2}$ omits generators corresponding to centers of the irreducible special subgroups from family (3), i.e. $A_T$ with $T = \{s, t_1, \ldots, t_k\}$ and $k \geq 1$, and
- $RA_{L_0}$ omits all the generators corresponding to centers of the irreducible special subgroups from families (3) and (4).

We claim that each $L_i$ satisfies Property PP avoiding $L_0$. We only verify it for $L_1$, the proof for $L_2$ is analogous. For each irreducible special subgroup $A_T$ (which corresponds to a simplex in $L$ spanned by the boundary curves of its corresponding subsurface $\Sigma_T$) from families (1), (2) and (3), we need to make a choice of a boundary curve of $\Sigma_T$. For a subgroup from family (3) of type $A_k$ we choose the curve $r_k$. Note that these subgroups are exactly the ones corresponding to simplices $\sigma$ with nontrivial $w_\sigma$ such that $\sigma \notin L_0$. We thus see that $p(\sigma) \notin L_0$ for such simplices $\sigma$. Also, $w_\tau$ is trivial for any other simplex $\tau$ containing $p(\sigma)$, because $\Sigma_T$ is the unique subsurface corresponding to an irreducible special subgroup whose boundary contains $r_k$.

By Lemma 5.7 for each irreducible special subgroup $A_T$ of type $D_k$ (i.e. from family (1)), there is a unique boundary curve of $\Sigma_T$ which intersects $\partial \Sigma_U$ for $U$ in family (3). Indeed, it is curve $s_{k-1}$, see Figure 9. Finally, for irreducible subset in family (2), i.e. $T = \{t_i, \ldots, t_j\}$, we pick the curve $t_{i:j}$, i.e. a curve in $\partial \Sigma_T$ that intersects $r_k$ for every $i \leq k \leq j$, which is unique unless $i,j$ are both odd (in which case either choice works).

We now claim that these choices satisfy the Property PP relative to $L_0$. For each simplex in $L_1 - L_0$, we have chosen curves not contained in $L_0$. Also, for each edge in $L_1 - L_0$, we have chosen a simple closed curve that is not a boundary curve of any subsurface corresponding to a different irreducible Artin subgroup. Therefore, condition (2) is satisfied. By Theorem 4.10, the conclusion follows.

5.4. Types $E_6$, $E_7$ and $E_8$. We now show that any homomorphism $A_{E_7} \to \text{Mod}(\Sigma)$ which sends generators to Dehn twists has nontrivial kernel which intersects the image of $\Phi_m$. Wajnryb has previously shown that there is no injective homomorphism from the Artin groups of type $E_6, E_7, E_8$ to any mapping class group which maps generators to Dehn twists [40].

The Dehn twists in Figure 12 show an element in the kernel of the Perron-Vannier representation $\rho : A \to \text{Mod}(\Sigma)$ where $A$ is of type $E_7$. A specific element in the kernel of $\rho$ is the commutator

$$[\Delta_T^{2m} \Delta_U^{2m} s^{2m} \Delta_U^{-2m} \Delta_T^{-2m} v^m, \Delta_V^{2m}]$$
These curves show the failure of property PP for the images of centers of $D_n$. The red and orange curves are boundary components of subsurfaces $\Sigma_{A_5}$ and $\Sigma_{A_5}'$. The green curves are the boundary components of the subsurface $\Sigma_{D_4}$. Since the other boundary component misses $\gamma$ and $\gamma'$ respectively, we are forced to choose these. However, this implies that we cannot correctly choose a boundary component of $\partial \Sigma_{D_4}$.

High powers of Dehn twists around the multicurves above do not generate the obvious RAAG subgroup. The figure on the left shows the corresponding centers of irreducible Artin subgroups. To see this, start with the Dehn twist around $s$, then conjugate by the red multitwist, then conjugate by the green multitwist. This element commutes with the blue multitwist.

where $T = \{t_1, t_2, t_3, t_4, s\}$, $U = \{t_2, t_3, t_4, t_5, t_6\}$ and $V = \{t_2, t_3, t_4, t_5, t_6, s\}$. The nerve of the subgroup of RA on the generators corresponding to $T, \{s\}, V, U$ is a path on 4 vertices. It is easy to verify that the corresponding commutator in RA is nontrivial. The commutator above is also nontrivial in $A$. This can be verified by computing the Deligne’s normal form for its positive representatives. We have performed the computation in GAP 3 with the package CHEVIE [39] [24]. Note that the above does not imply that the Generalized Tits Conjecture does not hold for Artin group of type $E_7$ (or $E_8$). It only shows that our strategy, using the Perron-Vannier representation, does not work in that case.

Remark 5.11. The multicurves produced by the Perron-Vannier representation of the Artin group of type $E_6$ do not satisfy generalized property PP, but we cannot find a word in RA as above in the kernel of the representation.
6. Generalized Tits Conjecture for other spherical Artin groups

In this section, we show the Generalized Tits Conjecture holds for all spherical Artin groups which are not small type. The main tool is a folding trick due to Crisp and Paris (see [12, Section 6]), which embeds any of these Artin groups into small type spherical Artin groups.

6.1. Folding homomorphisms. In this section it is more convenient to work with Coxeter diagrams rather than nerves. So, suppose that $A_\Gamma$ is an Artin group with a connected Coxeter diagram $\Gamma$ that has no $\infty$-labels. Crisp and Paris define a *folding homomorphism* $\Psi : A_\Gamma \to A_{\tilde{\Gamma}}$ where $A_{\tilde{\Gamma}}$ is small-type. Here is the construction: let $N = \text{lcm}\{m_{st} - 1|s \neq t \in S\}$. For each vertex $s \in S$, let $I(s)$ be a set with $N$ elements. For $m \geq 3$, let $\Gamma(m)$ be the Coxeter diagram for $A_{m-1} \times A_{m-1}$, i.e. $\Gamma(m)$ is a disjoint union of two copies of the Coxeter graph of type $A_{m-1}$. See Figure 13.

Let $\tilde{\Gamma}$ be a Coxeter diagram so that:

- The vertex set of $\tilde{\Gamma}$ is the disjoint union of the sets $I(s)$.
- If there is no edge in $\Gamma$ between $s$ and $t$, there is no edge between vertices in $I(s)$ and the vertices of $I(t)$. In particular, there are no edges between any two vertices in $I(s)$.
- If $m_{st} \geq 3$, then the subgraph of $\tilde{\Gamma}$ spanned by $I(s) \cup I(t)$ is isomorphic to $Nm_{st} - 1$ copies of $\Gamma(m_{st})$.

It is easy to see that such a $\tilde{\Gamma}$ can always be constructed, though it will not be unique. There is a map of graphs $\tilde{\Gamma} \to \Gamma$ sending every vertex in $I(s)$ to $s$. Crisp and Paris show that the map $\Psi : A_\Gamma \to A_{\tilde{\Gamma}}$ which sends a generator $s$ to the product $\prod_{s \in I(s)} s_i$ extends to a homomorphism $[12, \text{Prop 13}]$. Crisp showed that this homomorphism is injective when restricted to the Artin monoid $[11]$, and it follows that it is injective when $A_\Gamma$ is spherical. It is still open whether $\Psi$ is injective in general.

We now verify that $\Psi$ has some additional properties. For each $T \subset S$, let $\tilde{T}$ denote the preimage of $T$ under $\tilde{\Gamma} \to \Gamma$, i.e. $\tilde{T}$ contains the vertices in $I(s)$ for each $s \in T$.

**Lemma 6.1.** Let $A_\Gamma$ be an Artin group with connected Coxeter diagram $\Gamma$ and no $\infty$-labels. Let $\tilde{\Gamma}$ as above, and let $\Psi : A_\Gamma \to A_{\tilde{\Gamma}}$ be the above homomorphism. Then $\Psi$ satisfies the following:

- For every spherical Artin subgroup $A_T \subset A_\Gamma$, $A_{\tilde{T}}$ is spherical.
• If $A_T$ is an irreducible spherical subgroup of $A_Γ$, then
\[
Ψ(Δ^2_T) = \prod_{T_i \subseteq \tilde{T}} Δ^2_{T_i}
\]
where the $\{T_i\}$ are the irreducible components of $\tilde{T}$.

Proof. If $A_T$ is small type, then the Coxeter diagram of $A_\tilde{T}$ is a disjoint union of finitely many copies of Coxeter diagram for $A_T$, and hence $A_\tilde{T}$ is a direct product of copies of $A_T$. In this case, both statements follow easily. Suppose $σ$ is not small type. If $A_T$ is irreducible, the Coxeter subdiagram $Γ_T$ for $A_T$ is a line, with exactly one edge $e$ labeled with number greater than 3. Let $Γ^e_T$ be a connected component of the Coxeter diagram for $Γ_T$. Then $Γ^e_T$ is obtained from $Γ_T$ by replacing $e$ with a copy of $Γ(m)$ and then attaching copies of $T - e$. In each case we check directly that this produces a spherical Artin group.

For the second statement, by the definition of $Ψ$, a Coxeter element for $W_Γ$ maps to a Coxeter element for $W_Γ$. The Coxeter number is preserved in the above identifications (see Lemma 6.2 below), in that if $Γ^e_T$ is connected component of $Γ_T$, then the Coxeter number of $A_T$ is the same as the number for $A_Γ$). This immediately implies the second statement by Lemma 2.1. □

We now record the following specific cases as a lemma (these were previously tabulated in \([11]\)).

Lemma 6.2. Let $A_Γ$ be a spherical Artin group as above, and let $A_Γ$, be an Artin group in the image of $Ψ$ where $Γ_e$ is a connected component of $Γ_Γ$. Let $h$ be the Coxeter number of the Coxeter system of $A_Γ$. Then the Coxeter number of the Coxeter system of $A_Γ_e$ is $h$. Moreover:

1. If $A_Γ = B_n$, then $A_Γ$ has type $D_{n+1}$ or $A_{2n-1}$, and $h = 2n$.
2. If $A_Γ = I_2(p)$, then $A_Γ$ has type $A_{p-1}$ and $h = p$.
3. If $A_Γ = H_3$, then $A_Γ$ has type $D_6$ and $h = 10$.
4. If $A_Γ = H_4$, then $A_Γ_e$ has type $E_8$ and $h = 30$.
5. If $A_Γ = F_4$, then $A_Γ$ has type $E_6$ and $h = 12$.

Let $A_Γ$ be an Artin group with no $∞$-labels and connected $Γ$. The RAAG with presentation \([11]\) associated with $A_Γ$ is denoted by $RA_Γ$, and the RAAG associated with $A_Γ$ is denoted by $RA_Γ$. We denote the homomorphism $RA_Γ \to A_Γ$ from the statement of the Generalized Tits Conjecture by $Π_N$.

We consider the composition $Ψ \circ Φ_N : RA_Γ \to A_Γ$. Our aim is to verify the Generalized Tits Conjecture for $A_Γ$, i.e. to show that $Φ_N$ is injective for some $N$. Let $F : RA_Γ \to RA_Γ$. If $T$ is an irreducible, special subset, then $\tilde{T}$ will generally not be irreducible. For a generator $s_Γ$ of $RA_Γ$ its image $F(s_Γ)$ is the product $\prod_{T_i \subseteq \tilde{T}} s_Γ$. For each $N$, we have the following equality $Φ_N \circ F = Ψ \circ Φ_N$. Crisp and Paris show the following
Lemma 6.3. The homomorphism $F$ is injective. Therefore, if $\Phi_N$ is injective, then $\Phi_N$ is injective.

Proof. Note that if $T, U \subseteq S$ are distinct irreducible subsets, then $s_{\tilde{T}_i} \neq s_{\tilde{U}_j}$ for all $i$ and $j$. Furthermore, if $s_T$ and $s_U$ do not commute in $R\Gamma$, then for every $s_{\tilde{T}_i}$ there exists a $s_{\tilde{T}_i}$ that does not commute with $s_{\tilde{T}_i}$ in $R\Gamma$. To see this, note that if two spherical subsets $T$ and $U$ do not commute, there is $t \in T - U$ and $u \in U - T$ with $m_{tu} \neq 2$. Any component $\tilde{T}_i$ of $\tilde{T}$ contains a vertex $\tilde{t}$ of $I(t)$. There exists a vertex $\tilde{u} \in I(u)$ with $m_{\tilde{t}\tilde{u}} \neq 2$. Therefore, the component of $\tilde{U}$ containing $\tilde{u}$ does not commute with $\tilde{T}_i$. By the normal form for RAAGs this implies $F$ is injective, as it takes a reduced word in $R\Gamma$ to a reduced word in $R\Gamma$. □

6.2. Generalized Tits Conjecture for $B_n, F_4, H_3, H_4, I_2(p)$. We now finish the remaining spherical cases. By Lemma 6.2 Artin groups of types $B_n$, $H_3$, and $I_2(p)$ embed via the folding homomorphisms in Artin groups of types $A_m$ and $D_m$, for which we already know that the conjecture holds by Proposition 5.9 and Theorem 5.10. By Lemma 6.1 and Lemma 6.3 we get the following.

Corollary 6.4. The Generalized Tits Conjecture holds for spherical Artin group of type $B_n$, $H_3$, and $I_2(p)$.

In the case of $F_4$ and $H_4$, we do not know the conjecture for $E_6$ and $E_8$ respectively. However, in each of those cases, we can still show that the generalized Tits conjecture holds by considering $R\Gamma \rightarrow A\Gamma \leftarrow A\tilde{\Gamma} \rightarrow \text{Mod}(\Sigma)$.

where $A\Gamma$ is of type $F_4$ (respectively $H_4$), $R\Gamma$ its associated RAAG, and $A\tilde{\Gamma}$ is of type $E_6$ (respectively $E_8$) with its Perron-Vannier representation in a mapping class group (see Section 2.6).

Theorem 6.5. The Generalized Tits Conjecture holds for the spherical Artin group of type $F_4$.

Proof. Let $S = \{s, t, u, v\}$ be the standard generators of $A\Gamma$ of type $F_4$ where $m_{st} = m_{uv} = 3, m_{tu} = 4$. Consider the homomorphism $\Psi : A\Gamma \rightarrow A\tilde{\Gamma}_1$, where $A\Gamma$ has type $E_8$, and where $\Psi$ is the composition of the folding homomorphism $\Psi$ with the projection $A\tilde{\Gamma}_1 \rightarrow A\tilde{\Gamma}_1$, where $\tilde{\Gamma}_1$ is a connected component of $\tilde{\Gamma}$. See Figure 15. We also consider the Perron-Vannier representation $\rho : A\tilde{\Gamma}_1 \rightarrow \text{Mod}(\Sigma)$, as discussed in Section 2.6. The images of the elements $\Delta_4^2$ for irreducible subsets $T \subseteq S$ are powers of Dehn twists around curves in $\Sigma$, pictured in Figure 14. Let $C$ denote the collection of all these curves. Again, by Theorem 5.3 sufficiently large powers $k$ of Dehn twists around single curves in $C$ generate a RAAG. By Lemma 5.4 to show that $A$ satisfies the Generalized Tits Conjecture, we need to make a choice $p(T)$ of a curve in $\partial \Sigma_T$ for each of the irreducible subsets $T$ of $S$. For $T = \{u\}$ or $\{v\}$, the choice is
unique. It remains to make choices for \( \{s\}, \{t\}, \{s,t\}, \{t,u\}, \{s,t,u\}, \{t,u,v\} \). There is a unique curve in \( \Sigma_{t,u} \) that intersects the curve \( p(\{v\}) \), we choose that curve for \( p(\{t,u\}) \). Note that that curve intersects both boundary components of \( \Sigma_{\{s\}} \), both boundary components of \( \Sigma_{\{s,t\}} \) and the unique boundary component of \( \Sigma_{\{u,v\}} \). For each choice of a curve in \( \partial \Sigma_{\{s\}} \), there is a unique intersecting curve in \( \partial \Sigma_{\{t\}} \), and there is a unique curve in \( \partial \Sigma_{\{s,t\}} \) that is the boundary of a subsurface containing those two curves. We make such a consistent choice of \( p(\{s\}), p(\{t\}), p(\{s,t\}) \). Then there is a unique choice of \( p(\{t,u,v\}) \) that intersects \( p(\{s\}) \) and both components of \( \Sigma_{\{s,t,u\}} \). Finally, any choice of \( p(\{s,t,u\}) \) works. \( \square \)

**Theorem 6.6.** The Generalized Tits Conjecture holds for the spherical Artin group of type \( H_4 \).

*Proof.* Let \( S = \{s, t, u, v\} \) be the standard generators of \( A_\Gamma \) of type \( H_4 \) where \( m_{st} = 5 \) and \( m_{tu} = m_{uv} = 3 \). Consider the restriction of a folding homomorphism to a single component of Coxeter diagram. We get a homomorphism \( \Phi : A_{H_4} \to A_{E_8} \) as pictured in the left of Figure 15. We also consider the Perron-Vannier representation \( A_{E_6} \to \text{Mod}(\Sigma) \), as discussed in Section 2.6. The images of the (squares of the?) original generators, and of the elements \( \Delta^2_T \) for other irreducible subsets \( T \subseteq S \) are Dehn twists around curves in two copies of \( \Sigma \) in Figure 14. Let \( C \) denote the collection of all these curves. As in previously considered cases, by Theorem 5.3 sufficiently large powers \( k \) of Dehn twists around single curves in \( C \) generate a RAAG. We need to verify that Property PP is satisfied for \( \{\Phi(\Delta^{2k}_T)\} \).
Figure 15. The restriction of folding homomorphism for $A_\Gamma$ of type $E_8$ to one connected component. The thick curves are the ones we choose for the standard generators in verifying Property PP. The gray and purple curves are the curves we choose for $\{t,u,v\}$ and $\{s,t,u\}$ respectively. The black, brown, and orange curves are the unique curves for the irreducible rank 2 subsets.

Specifically, we need to make choices for the following irreducible subsets of $S$: $\{s\}$, $\{t\}$, $\{s,t\}$, $\{t,u\}$, $\{s,t,u\}$, $\{t,u,v\}$. There is a unique curve in $\Sigma_{t,u}$ that intersects the curve corresponding to $v$, we choose that curve for $p(\{t,u\})$. Note that that curve intersects both boundary components of $\Sigma_{\{s\}}$, both boundary components of $\Sigma_{\{s,t\}}$ and the unique boundary component of $\Sigma_{\{u,v\}}$. For each choice of a curve in $\partial \Sigma_{\{s\}}$, there is a unique intersecting curve in $\partial \Sigma_{\{t\}}$, and there is a unique curve in $\partial \Sigma_{\{s,t\}}$ that is the composition of those two curves. We make such a consistent choice of $p(\{s\}), p(\{t\}), p(\{s,t\})$. That also forces a choice of $p(\{t,u,v\})$. Finally, any choice of $p(\{s,t,u\})$ works. □

7. Applications

If the Generalized Tits Conjecture holds in full generality, one immediate application is that the subgroups of Artin groups are as complicated as subgroups of right-angled Artin groups. The latter are currently more well understood. In this section, we give a few applications of this idea.

7.1. Incoherence. Recall that a group $G$ is coherent if every finitely generated subgroup of $G$ is finitely presented. Droms had showed that the right-angled Artin group $A_L$ was coherent if and only if $L$ was a chordal graph [21]. Gordon showed that if the Artin group of type $H_3$ was incoherent, there was a similar classification of coherent Artin groups. Wise answered this in the affirmative in 2013.

Theorem 7.1 ([41]). The Artin group of type $H_3$ is incoherent.

Since $A$ of type $H_3$ satisfies the generalized Tits conjecture, we can give an alternative proof of Wise’s theorem. In this case, the nerve $L_\varnothing$ of the RAAG subgroup RA in $A$ is the cone on a pentagon. The RAAG based on a pentagon is well known to be incoherent, for example the Bestvina-Brady subgroup of the RAAG (the kernel of the map to $\mathbb{Z}$ which sends every generator to 1) is finitely generated and not finitely presented.
7.2. Hyperbolic surface subgroups. In [26], Gordon, Long, and Reid studied which Coxeter groups and Artin groups contained hyperbolic surface subgroups. They showed that all finite type Artin groups except types $A_1$, $I_2(m)$, and $H_3$ contained these subgroups. The first two classes do not contain such subgroups (More generally, any Artin group where the nerve $L$ is a tree does not contain a hyperbolic surface subgroup), and type $H_3$ was left as an open question. It follows from the Generalized Tits Conjecture that the answer to their question is yes.

**Corollary 7.2.** The Artin group of type $H_3$ contain a hyperbolic surface subgroup.

*Proof.* The proof is the same as above. The RAAG based on a pentagon is commensurable to a right-angled Coxeter group which has the pentagon as a full subcomplex [15]. It follows that the Coxeter group contains a hyperbolic surface subgroup, hence so does the RAAG based on a pentagon, and hence so does the type $H_3$ Artin group. □

In [26], they also considered Artin groups of Euclidean type, and asked specifically if the Artin group of type $\tilde{B}_2$ or type $\tilde{G}_2$ contain a hyperbolic surface subgroup. The nerves of these Artin groups are triangles with labels $(2, 3, 6)$ and $(2, 4, 4)$ respectively. In unpublished work, Sang-hyun Kim can show the existence of such a subgroup in the $(2, 4, 4)$ case, but we believe the $(2, 3, 6)$-case was still open. In either case, these groups are locally reducible. Therefore, the answer to this question is also yes, and follows as above from the Generalized Tits Conjecture.

**Corollary 7.3.** The Artin groups of types $\tilde{B}_2$ and $\tilde{G}_2$ contain hyperbolic surface subgroups.

At this point, we do not have a complete characterization of which Artin groups contain hyperbolic surface subgroups. This is still open in the right-angled case. On the other hand, these should be easier to construct outside of the right-angled case. For example, if the nerve $L$ is a 4-cycle, then the RAAG is $F_2 \times F_2$, which does not contain such a subgroup. On the other hand, if any edge has a label $> 2$ then the Artin group is locally reducible and the associated RAAG subgroup has nerve a pentagon.

7.3. Subgroups of type $F_n$ and not $F_{n+1}$. Spherical Artin groups are generally too high-dimensional for coherence to be an interesting question. A more interesting question is when a spherical Artin group contains a subgroup which is type $F_n$ but not type $F_{n+1}$ where $n + 2$ is the cohomological dimension (recall a group is type $F_n$ if it admits a classifying space with finite $n$-skeleton). For example, the type $H_3$ Artin group has cohomological dimension 3, so coherence is an interesting question. It follows again from the generalized Tits Conjecture that Artin groups of type $H_4$ or $F_4$ have subgroups which are $F_2$ but not $F_3$.

In [42], Zaremsky showed that the pure braid groups $PB_n$ contained subgroups $N$ so that $N$ was type $F_{m-3}$ but not $F_{m-2}$ for $3 \leq m \leq n$. The existence of these subgroups again follows from the fact that the generalized Tits conjecture holds
for the braid groups (on the other hand Zaremsky’s examples are normal and have $PB_n/N \cong \mathbb{Z}$; our subgroups have neither of these properties). In this case, $L_\omega$ is the cone on a flag triangulation of $S^{n-1}$. The Bestvina-Brady subgroup of this RAAG is type $F_{m-3}$ but not type $F_{m-2}$. To get subgroups that are $F_{m-3}$ but not $F_{m-2}$ for $3 \leq m < n$, one can instead map $A_{L_\omega}$ to $\mathbb{Z}$ and send some generators to 0.

Since the generalized Tits conjecture holds for type $D_n$, we have an analogous theorem with the same proof as above.

**Theorem 7.4.** The Artin group of type $D_n$ contains subgroups $N$ so that $N$ is type $F_{m-3}$ but not $F_{m-2}$ for $3 \leq m \leq n$.

Of course, the Artin group of type $D_n$ contains the braid group $A_{n-1}$, so the only improvement on what Zaremsky’s theorem provides is $n = m$.

8. Questions

We end the paper with some open questions which

**Question 8.1.** Does the generalized Tits conjecture hold for spherical Artin groups of type $E_n$?

If one can show the conjecture holds for all spherical Artin groups, a natural next step is the Artin groups of FC type.

**Question 8.2.** Does the conjecture hold for all Artin groups of FC type?

It would be very interesting to know some geometric properties of these subgroups:

**Question 8.3.** Each standard free abelian subgroup quasi-isometrically embeds into the Artin group $A$. When the generalized Tits conjecture is true, when are the entire RAAG subgroups quasi-isometrically embedded? Does a quasi-isometry between two Artin groups coarsely preserve these RAAG subgroups?

**References**


