RIGHT-ANGLED ARTIN SUBGROUPS OF ARTIN GROUPS

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Abstract. The Tits Conjecture, proved by Crisp and Paris, states that squares of the standard generators of any Artin group generate an obvious right-angled Artin subgroup. We consider a larger set of elements consisting of all the centers of the irreducible spherical special subgroups of the Artin group, and conjecture that sufficiently large powers of those elements generate an obvious right-angled Artin subgroup. This alleged right-angled Artin subgroup is in some sense as large as possible; its nerve is homeomorphic to the nerve of the ambient Artin group. We verify this conjecture for the class of locally reducible Artin groups, which includes all 2-dimensional Artin groups, and for spherical Artin groups of any type other than E_6, E_7, E_8. We use our results to conclude that certain Artin groups contain hyperbolic surface subgroups, answering questions of Gordon, Long and Reid.

1. Introduction

Suppose (W, S) is a Coxeter system (cf. [4] or [13]). This means that W is a group, S is a distinguished set of generators, and that W has a presentation

$$W := \langle s \in S | s^2 = (st)^{m_{st}} = 1 \rangle ,$$

where $m_{st} \in \{2, 3, \dots\} \cup \{\infty\}$. Given a Coxeter system (W, S) there is an associated *Artin group* A. This group has one generator x_s for each $s \in S$ and the *braid relations*:

$$\underbrace{x_s x_t \ \cdots}_{m_{st} \text{ terms}} = \underbrace{x_t x_s \ \cdots}_{m_{st} \text{ terms}},$$

where both sides of the equation are alternating words in x_s and x_t, and where m_{st} denotes the order of st in W. The Artin group is *right-angled* (and called a RAAG) if $m_{st} \in \{2, \infty\}$.

Consider the subgroup of A generated by the squares x_s^2. These elements are all contained in the *pure Artin group* PA, which is the kernel of the canonical homomorphism $A \to W$ which sends x_s to s. There are obvious commuting relations between the x_s^2, namely if $m_{st} = 2$, then $[x_s^2, x_t^2] = 1$. Crisp and Paris in [12] proved the remarkable fact that these are the only relations between these elements. This verified a conjecture of Tits, who had previously shown that the elements $\{x_s^2\}$ get sent to linearly independent elements in the abelianization of the pure Artin group. In fact, Crisp-Paris showed that the same is true if we replace to 2 with any number $N \geq 2$.

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Tits Conjecture ([12, Thm 1])**.** Let A be an Artin group. For every $N \geq 2$, the subgroup generated by the set $\{x_s^N : s \in S\}$ is a RAAG with presentation

$$\langle x_s^N | [x_s^N, x_t^N] = 1 \text{ if } m_{st} = 2 \rangle$$

This is one of the few theorems known to hold for all Artin groups (e.g. it is not known if all Artin groups are torsion-free, have solvable word problem, ...). The Tits Conjecture had earlier been proved by Appell-Schupp for extra-large Artin groups (where $m_{st} > 3$) [1], by Collins for the braid groups [10], and by Charney for the locally reducible Artin groups (where the associated Coxeter groups have each finite special subgroup a direct product of dihedral groups and $\mathbb{Z}/2$) [7], see also [22], [27] for more partial results.

We will come back to Crisp and Paris' method later in the introduction. Very roughly speaking, they construct a representation from the Artin group into the mapping class group of some surface and show that it is faithful on the alleged RAAG subgroup.

In this paper, we are interested in a conjectural generalization of the Tits Conjecture, which first appeared in [18, Conj 4.9]. This generalization asks for a RAAG subgroup that is as "large" as possible in certain sense. In particular, it contains the RAAG subgroup that Crisp and Paris find and its nerve is homeomorphic to the nerve of the Artin group. We will now explain how there is a natural candidate for this larger RAAG.

Given a Coxeter system (W, S) and $T \subset S$, the subgroup W_T generated by $t \in T$ is called the *special subgroup* corresponding to T. Then (W_T, T) also is a Coxeter system. The subset T is *spherical* if W_T is finite, in this case W_T is a *spherical special subgroup*. The subset T is called *reducible* if it decomposes as $T_1 \cup T_2$, where $m_{tt'} = 2$ for all $t \in T_1$ and $t' \in T_2$, otherwise T is *irreducible* (and we say W_T is as well). If T is reducible with decomposition $T = \cup_{i=1}^n T_i$, then $W_T = W_{T_1} \times \cdots \times W_{T_n}$.

Similarly, the subgroup A_T of A generated by the $\{x_t\}_{t \in T}$ is the Artin group associated to the Coxeter system (W_T, T) [31]. If T is irreducible and spherical, then A_T is a *irreducible, spherical special subgroup* of A. These spherical Artin groups are better understood than general Artin groups. Topologically, the pure Artin group (which in this case is finite index in A) is the fundamental group of an aspherical linear hyperplane arrangement in $\mathbb{C}^n$, which e.g. allows one to compute various cohomological invariants of A [19]. Combinatorially, these groups admit a Garside structure, which e.g. gives an easy to compute normal form [6]. It is also known that the pure spherical Artin groups have an infinite center which is isomorphic to $\mathbb{Z}$ if the Coxeter group is irreducible. This center corresponds to the fundamental group of the fiber after projectiving the hyperplane complement. We denote the generator of this by Δ_T^2 (it is related to the longest element in W_T, see Section 2).

We consider the subgroup of an Artin group A generated by (powers of) the centers of the irreducible, spherical, pure Artin subgroups. There are some obvious commuting relations between these elements, namely if $T \subset U$ or if $m_{ut} = 2$ for all $u \in U$ and $t \in T$, then the corresponding centers Δ_U^2, Δ_T^2 commute. We write $[U, T] = 1$ if $m_{ut} = 2$ for all $u \in U$ and $t \in T$. As in the Tits Conjecture, we ask whether those are the only relations.

Let $\mathcal{S}$ be the set of all irreducible, spherical subsets $T \subseteq S$. Let RA be a RAAG generated by the set $\{z_T\}_{T\in\mathcal{S}}$, with the presentation

$$\mathrm{RA} = \langle z_T \mid [z_T, z_U] = 1 \text{ if } U \subseteq T,\ T \subseteq U,\ \text{ or } [U,T] = 1\rangle. \tag{1.1}$$

By the above, there are homomorphisms $\Phi_N : \mathrm{RA} \to A$ so that $\Phi_N(z_T) = \Delta_T^{2N}$. The map Φ_N is injective if and only if the subgroup of A generated by $\{\Delta_T^{2N}\}_{T\in\mathcal{S}}$ is isomorphic to RA. For a generator x_s, the infinite cyclic subgroup $\langle x_s\rangle$ of A is itself an irreducible, spherical, special Artin subgroup, so the subgroup generated by $\{\Delta_T^{2N}\}_{T\in\mathcal{S}}$ contains an appropriate RAAG subgroup found by Crisp and Paris. It will turn out that the injectivity (when we can verify it) of Φ_N will depend on N, unlike the original Tits Conjecture. Therefore, we propose the following.

Generalized Tits Conjecture. Let A be an Artin group, RA the associated RAAG and $\Phi_N : \mathrm{RA} \to A$ the homomorphism defined above. Then Φ_N is injective for some N.

If we can verify that a specific homomorphism Φ_k is injective, then we say that A satisfies the Generalized Tits Conjecture for $N = k$. The Generalized Tits Conjecture for $N = 1$ was conjectured by Davis, Le, and the second author in [18, Conjecture 4.9]. This turns out to be too optimistic in general, though we can show it for a reasonably large class of Artin groups. In Example 5.8 we show that Φ_1 is not injective for braid groups on at least 8 strands. However, a theorem of Koberda (Theorem 5.3) implies that braid groups satisfy the Generalized Tits Conjecture (for some $N >> 0$).

We claim this is a good generalization of the Tits Conjecture to prove. As evidence, recall that there is a simplicial complex L $(= L(W, S))$, called the *nerve*. Its vertex set is S and a subset $T \subset S$ spans a simplex of L if and only if T is spherical. Davis and Huang showed in [14] that the nerve L' of the RAAG RA defined above is a partial barycentric subdivision of the nerve L. In particular the nerve of the Artin group is homotopy equivalent to the nerve of the alleged RAAG subgroup. This is desirable since many topological properties of this nerve are related algebraic properties of the Artin group. For example, contingent on the $K(\pi, 1)$-conjecture, RA and A will have the same cohomological dimension, their compactly supported cohomology will be nontrivial in the same dimensions, etc. Also, in this sense, the alleged subgroup is as "large" a RAAG subgroup as one can expect to find in A. Davis and Huang in [14] were interested in determining the minimal dimensional manifold for a classifying space BA (see also Le's thesis [30]). These partial subdivisions L' appeared earlier in [16], [17].

1.1. **Results.** We prove the Generalized Tits Conjecture holds for the Charney's class of locally reducible Artin groups. This includes all 2-dimensional Artin groups and Artin groups with $m_{st} \neq 3$ for all $s, t \in S$. The irreducible spherical Artin subgroups correspond to edges of the nerve, so the only new generators that we are considering come from centers of the Artin subgroups generated by $\langle s, t\rangle$ for $m_{st} \geq 3$. We can show the following:

Theorem 1.1. *Artin groups with $m_{st} \neq 3$ for all $s, t \in S$ satisfy the Generalized Tits Conjecture with $N = 1$.*

For general locally reducible Artin groups, we show the following:

Theorem 1.2. *Locally reducible Artin groups satisfy the Generalized Tits Conjecture with $N = 2$.*

We also prove that the Generalized Tits Conjecture holds for large N for the following spherical Artin groups.

Theorem 1.3. *The Generalized Tits Conjecture holds with N sufficiently large for all spherical Artin groups except for those of type E_n (for these it is still open).*

Our methods do not work for these remaining exotic cases. The hardest part of Theorem 1.3 is confirming the conjecture for the Artin groups of type D_n (the conjecture for Artin groups of type B_n also follows from Koberda's result). For technical reasons we have to assume that N is even, though we suspect it works for general large N.

1.2. **Applications.** Here are some immediate applications of our results. The rough moral here is that if an Artin group satisfies the Generalized Tits Conjecture, then its subgroups are as complicated as the subgroups of the corresponding RAAG. In particular, we give a new proof of Wise's result that the spherical Artin group A of type H_3 is incoherent, and can show that A (along with many other Artin groups) contains a closed hyperbolic surface subgroup, answering questions of Gordon-Long-Reid [26]. The advantage of our argument is that the same proof works for both questions; the RAAG subgroup of A has a nerve which is a cone on a pentagon, and it is easy to see this subgroup is incoherent and contains hyperbolic surface subgroups.

1.3. **Outlines of the proofs.** Our methods of proof in the case of spherical Artin group and locally reducible Artin groups are very different. For the locally reducible Artin groups, we use similar methods to [7]. Charney showed that the Deligne complexes of locally reducible Artin groups are CAT(0). She then constructed a cube complex with an action of the predicted RAAG, and showed that it isometrically embeds in the Deligne complex, using arguments from CAT(0) geometry. The crucial case to understand is the dihedral Artin groups A_{2m} (i.e. (W, S) is a dihedral group D_{2m}), since these appear in the links of vertices in the larger Deligne complex. The RAAG that we consider is larger, so we are trying to isometrically embed a larger complex into the Deligne complex. Again, we need to understand the dihedral Artin groups. In this case, our complex looks roughly like a $\mathbb{Z}'s$ worth of Charney's complex, and we need to show that the pieces embed pairwise orthogonally in the Deligne complex of the dihedral Artin group.

For the spherical Artin groups, we follow Crisp and Paris. We start with a representation from the Artin group into the mapping class group of a surface Σ. For spherical Artin groups of type $A_n - E_n$, these representations are classical and due to Perron and Vannier. The generators of the Artin groups map to Dehn twists around simple closed curves, and powers of the centers of irreducible spherical Artin subgroups map to Dehn twists about the boundary curves of connected subsurfaces of Σ. Then Koberda's result implies that high powers of Dehn twists around this collection of curves generates a RAAG subgroup of the mapping class group of Σ. The reader might suspect that we are done now, and the proof trivially follows from Koberda's theorem (as we initially thought). However, the boundary of these subsurfaces is not necessarily connected, so each center maps to a product of Dehn twists about disjoint simple closed curves in Σ. Therefore, we have to study the following question:

Question 1.4. *Let* RA *be a* RAAG. *Let* $\{w_i\}$ *be a collection of elements of A, where each* w_i *is a product of commuting generators of A. Is the subgroup of A generated by* w_i *the (obvious)* RAAG*?*

This turns out to be subtle, and was also considered in [12] and [29]. Koberda and Crisp-Paris gave different conditions on the $\{w_i\}$ which guaranteed a positive answer to the above question. These conditions fail for the system of curves produced by the Perron-Vannier representation of the Artin groups of type D_n. Our main work in this case is to generalize Koberda's condition to a condition that the curves in this system satisfy. This new condition may be of independent interest.

Unfortunately, our conditions do not work for the RAAG and subwords produced from the Perron-Vannier representation of the Artin groups of type E_n. Even worse, for E_7 and E_8 we can find words in the alleged RAAG subgroup which are in the kernel of the representation, see Subsection 5.4. One can check by hand that these words correspond to nontrivial elements of the Artin group, so these mapping class group representations are not faithful enough to be used to verify the conjecture, even for spherical Artin groups.

1.4. **Organization of the paper.** In Section 2, we give some background on Coxeter and Artin groups. Section 3 is devoted to proving Theorems 1.1 and 1.2. In Section 4 we study RAAG subgroups of RAAG's, and it can be read independently of the rest of the paper. In Sections 5 and 6 we use this to prove Theorem 1.3 first in the small type case (i.e. where all $m_{st} \leq 3$), and then for the remaining cases. Section 7 discusses some applications.

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2. Coxeter groups and Artin groups

Let (W, S) be a Coxeter system, and let A be the corresponding Artin group. There is a canonical surjection $p : A \to W$ which sends x_s to s. The kernel of p is called the *pure Artin group* and denoted PA. It is obviously finite index in A if and only if W is a finite Coxeter group. There is also a canonical set-theoretic section $\sigma : W \to A$ of p which takes a reduced positive word w in W to the same word in A. It follows from Tits' solution to the word problem for Coxeter groups that this does not depend on the choice of reduced expression for w.

2.1. **Coxeter diagrams.** When there are many commuting generators, Coxeter groups and Artin groups can be efficiently described in terms of their *Coxeter diagrams.* Given a Coxeter system (W, S), we consider a graph Γ whose vertex set is the standard generators S, and whose edges correspond to the pairs of generators that do not commute, i.e. there is an edge between s and t if and only if $s \neq t$ and $m_{st} \neq 2$. If $m_{st} \in \{4, 5, 6, \dots\} \cup \{\infty\}$, then the edge between s and t is labeled with m_{st}. Otherwise, i.e. when $m_{st} = 3$, the edge has no label. Given such a graph Γ, we will denote the corresponding Artin group by A_Γ.

Note that if the Coxeter graph Γ has multiple connected components Γ_i, then the Artin group A_Γ splits as the direct product $\prod_i A_{\Gamma_i}$. If the Coxeter graph is connected, we say that A_Γ is an *irreducible* Artin group. We say an Artin group is *small-type* if $m_{st} \in \{2, 3\}$ for all $s, t \in S$. These correspond to Coxeter diagrams which are unlabeled graphs (where there are no loops or multiple edges). The finite Coxeter groups with connected Coxeter diagram were classified by Coxeter, and correspond to the Coxeter graphs in Figure 1. The small type irreducible spherical Artin groups therefore split into two infinite families; type A_n and D_n, and three exotic cases E_6, E_7 and E_8. Each of the other spherical Artin groups injects into a product of small-type spherical Artin groups, see [11] or Section 6.

2.2. **Fundamental elements and Coxeter elements.** Let (W, S) be a Coxeter system with W finite. A *Coxeter element* of W is a product of all the generators of S, in any order, where each generator appears exactly once in the product. Different orderings produce conjugate Coxeter elements (this follows from the fact that if (W, S') is another Coxeter system then S' is conjugate of S). The *Coxeter number* h of (W, S) is the order of a (any) Coxeter element in W.

A *reflection* in W is a conjugate of an element of S. Each finite Coxeter group has a unique *longest element* w_S. This can be characterized as the unique element for which $\ell(sw_S) = \ell(w_S s) < \ell(w_S)$ for all $s \in S$, where $\ell(w)$ is the minimal length of a representative for w. The length of w_S is precisely the number of reflections in W. Conjugation by w_S induces an involution of the Coxeter diagram Γ_S, in the sense that generators are sent to generators and the relations are preserved. It follows from this that w_s is in the center of W if and only if this involution is trivial. This involution happens to be nontrivial if and only if the Coxeter group is of type A_n, D_n with n odd, E_6 or $I_2(p)$ for p odd.

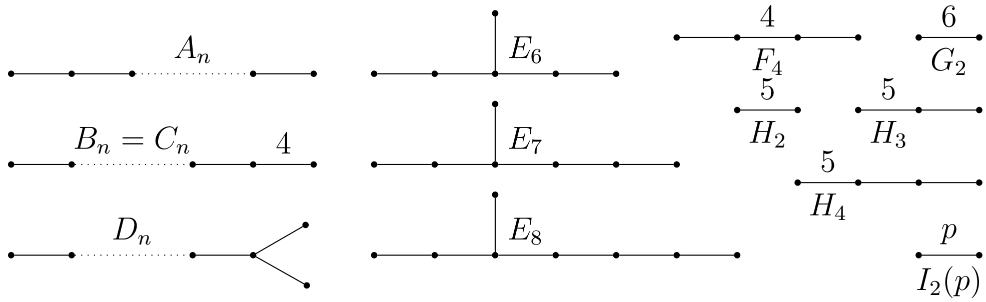

FIGURE 1. Coxeter graphs of the irreducible finite Coxeter groups.

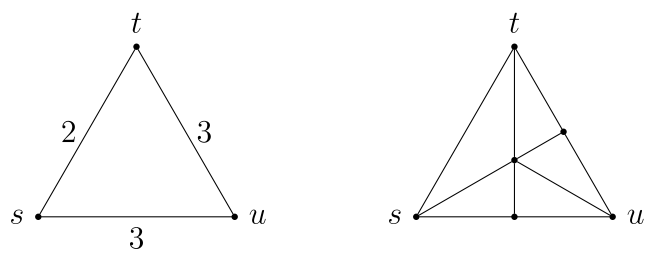

FIGURE 2. The subdivision $\sigma_\oslash$ for the braid group on four strands.

The image of w_S in A under the section $\sigma : W \to A$ will be called the *fundamental element* of A, and we will denote it by Δ. Each spherical Artin group has an infinite cyclic center which is generated by either Δ or Δ^2 (depending as above whether the Coxeter group has a nontrivial center). For simplicity, we will only deal with the squares Δ^2, which generate the center of the pure Artin groups, see [34, Thm 4.7]. We record the following lemma in [6], which will be used in Section 6.

Lemma 2.1. *Let A be a spherical Artin group. We have that $\Delta^2 = \sigma(c)^h$, where c is any Coxeter element and h is the Coxeter number.*

2.3. **Subdivisions.** The *nerve* L of a Coxeter system (W, S) is a simplicial complex L with vertex set S where a subset $T \subset S$ spans a simplex of L if and only if T is spherical. Let σ be a simplex in L corresponding to a spherical subset T, and A_T the corresponding spherical special Artin subgroup of A. Davis and Huang described a partial barycentric subdivision $\sigma_\oslash$ of σ where the vertices of $\sigma_\oslash$ correspond to irreducible subsets of T. For $U \subset T$, we think of the vertex corresponding to A_U as the barycenter of the associated simplex in σ. There are edges between two vertices U and U' if and only if $U \subset U'$, $U' \subset U$, or $[U, U'] = 1$. See Figure 2 for the subdivision corresponding to the braid group on 4 strands.

In terms of the Coxeter diagram Γ, the irreducible spherical Artin subgroups of A_Γ correspond to connected spherical subdiagrams. In this case, there is an edge between two connected subsets U, T of Γ_σ if and only if $U \subset T$, $T \subset U$, or U and T have distance > 2 in Γ.

The subdivision $\sigma_{\oslash}$ can be defined as the flag completion of this graph. Of course, it is not obvious with this definition that this is a subdivision of σ; Davis and Huang provide an alternative description of $\sigma_{\oslash}$ which obviously produces a subdivision of σ, and show that it is a flag complex with the 1-skeleton described above. In either case, these subdivisions $\sigma_{\oslash}$ fit together to give a subdivision $L_{\oslash}$ of L. The simplicial complex $L_{\oslash}$ is the nerve of the RAAG RA described in the introduction.

In the next subsection, we verify that $\Phi_N : \mathrm{RA} \to A$ is injective when restricted the free abelian subgroups corresponding to simplices of the nerve $L_{\oslash}$. This can be seen from looking at the abelianization of PA. We will also need an explicit description of a natural basis for $H_1(PA, \mathbb{Z})$ in the locally reducible case.

2.4. **Abelianization of pure spherical Artin groups.** Each finite Coxeter group W acts on $\mathbb{R}^n$ by linear reflections, where n is the number of elements of S. Complexifying this action gives a group action on $\mathbb{C}^n$ by linear reflections. The complement of the reflecting hyperplanes, denoted by $M(W)$, has $\pi_1(M(W)) = PA$, where A is the associated Artin group.

The Coxeter group acts freely on $M(W)$, and $\pi_1(M(W)/W) = A$. Deligne showed that $M(W)$ is aspherical [19], so in particular $H_1(PA, \mathbb{Z})$ is isomorphic to $H_1(M(W), \mathbb{Z})$. It is easy to see that $H_1(M(W), \mathbb{Z})$ is isomorphic to the free abelian group $\mathbb{Z}^R$, where R is the set of reflections in S, or the set of hyperplanes in this arrangement [34] (the complement deformation retracts to its intersection with S^{2n-1}, which is homeomorphic to $S^{2n-1} - \bigcup_R S^{2n-3}$). Given a spherical subset T of S, let $R_T \subset R$ be the set of reflections in W_T.

Let $\{e_r\}$ be the standard basis of $\mathbb{Z}^R$. For each element $s \in S$, the element x_s^2 in PA corresponds to a loop around the hyperplane corresponding to s in $\mathbb{C}^n$. It turns out that the class of x_s^2 is precisely e_s for each $s \in S$. Any $r \in R$ is a conjugate of an element $s \in S$, i.e. $r = wsw^{-1}$ for some $w \in W$. Let a_w be any element of A that projects to w under $p : A \to W$. Then $a_w x_s^2 a_w^{-1}$ is in PA, and its image in $H_1(PA, \mathbb{Z})$ is precisely e_r.

Lemma 2.2 (Lemma 2.2, [14])**.** *Suppose that T is spherical. Let e_T be the image of Δ_T^2 in $H_1(PA_T, \mathbb{Z})$. Then*

$$e_T = \sum_{t \in R_T} e_t.$$

It follows from this lemma that given any simplex τ in $\sigma_{\oslash}$, the image of the elements in $H_1(PA_T, \mathbb{Z})$ corresponding to the vertices of τ are linearly independent, so in particular these form a free abelian subgroup of PA_T (and hence of PA) of rank $\dim(\tau) - 1$.

Davis and Huang also show that the intersection of these free abelian subgroups is as expected, i.e. if σ and τ are simplices of $L_{\oslash}$, then the subgroups correspond to σ and τ intersect in the subgroup corresponding to $\sigma \cap \tau$. This serves as further motivation for the Generalized Tits Conjecture.

Remark 2.3. There are similar configurations of free abelian subgroups for affine hyperplane complements in $\mathbb{C}^n$, and as far as we know the analogue to the Generalized Tits Conjecture is also open in this case (see [18, Section 5]). In this case, the relevant simplicial complex L comes from the intersection poset of the hyperplane arrangement. Each irreducible central subarrangement has an infinite cyclic center, and these combine as above to produce standard free abelian subgroups. Again, it is known that these centers are linearly independent vectors in H_1 of the arrangement complement. In this case, the relevant partial subdivision of L is the geometric realization of the nested set complex associated to the minimal building set for the arrangement as defined by De Concini and Procesi [20].

Remark 2.4. The main impetus for the authors of [18] to consider a generalized Tits Conjecture was the computation of *action dimension* of RAAG's in [2]. This is the minimal dimension of a manifold model of the classifying space $B(\mathrm{RA})$. This is obviously monotone, in the sense that if H is a subgroup of G, then the action dimension of G is greater than the action dimension of H. Therefore, if the Generalized Tits Conjecture was true, the action dimension of the Artin group A is larger than the action dimension of RA, and in many cases this would lead to a complete calculation for A. On the other hand, combined work of Davis-Huang and Le gave a nearly complete computation for the action dimension of Artin groups (contingent on the $K(\pi, 1)$-conjecture) without using the conjecture (it was enough that the standard free abelian subgroups inject into A and intersect as expected) [14], [30].

2.5. **Deligne complex.** The spherical subsets of S form a poset under inclusion. Let K denote the geometric realization of this poset; K is the cone on the barycentric subdivision bL of the nerve, with the cone vertex corresponding to the empty set $\emptyset$. There is another poset of spherical cosets aA_T of A, where $a \in A$ and A_T is a spherical special subgroup, again ordered by inclusion. The geometric realization of this poset is called the *modified Deligne complex* [9], and denoted $D(A)$. The Artin group A acts on $D(A)$ by left multiplication, and K is a strict fundamental domain for this action. The analogous construction in the Coxeter group setting (i.e. replace aA_T with wW_T) is precisely the Davis complex.

Here is an alternative description of $D(A)$. Given a spherical subset T, let $K_{\geq T}$ denote the subcomplex of K spanned by vertices $A_{T'}$ with $T \subset T'$. Then

$$D(A) = A \times K/\sim,$$

where $(a_1, x) \sim (a_2, x)$ if and only if x is in $K_{\geq T}$ and $a_1^{-1}a_2 \in A_T$. We identify K with $1 \times K$. If A is itself a spherical Artin group, the poset of spherical cosets has a maximal element A, so the Deligne complex is a cone with cone point A. The link of A is a simplicial complex of dimension $|S| - 1$, and we denote it by $B(A)$.

Conjecturally, the modified Deligne complex is contractible for all Artin groups. It follows from Deligne's work on spherical Artin groups that this would imply the well-known $K(\pi, 1)$-conjecture. Motivated by this, Charney and Davis in [9] put two natural piecewise Euclidean metrics on $D(A)$. Note that $D(A)$ has a natural cube complex structure, since the cone on the barycentric subdivision of a simplex

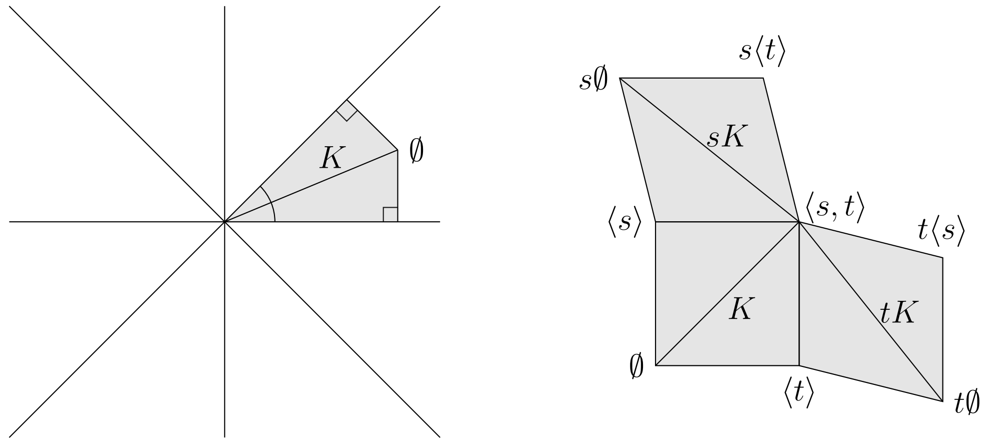

FIGURE 3. The Moussong metric on K for the dihedral Artin group and a small part of the development of the Deligne complex.

is combinatorially isomorphic to a cube. The first metric simply makes each cube isometric to a standard Euclidean cube $[0,1]^n$. It turns out that this metric is CAT(0) if and only if the nerve L is a flag complex (the induced metric on the link of the vertex $\emptyset$ is isometric to L with the all-right spherical metric, so flagness is an obvious necessary condition).

The second metric uses a different cubical metric called the *Moussong metric*, as it is related to the Moussong metric on the Davis complex, which is CAT(0). Charney and Davis showed that the Moussong metric on $D(A)$ is CAT(0) for 2-dimensional Artin groups. It is still open whether $D(A)$ equipped with the Moussong metric is CAT(0) for all Artin groups.

We now describe this metric for Artin groups corresponding to dihedral groups, which is the only important case for this paper. It is slightly more convenient for us to keep the original cellulation of $D(A)$ as a simplicial complex. See [7] or [9] for details in the general case. The dihedral group D_{2m} acts by linear reflections on $\mathbb{R}^2$ in the standard way, with strict fundamental domain a simplicial cone. Let x be the unique point in this cone whose distance from the two walls of the cone is 1. The convex hull of the orbit of x under D_{2m} is a $2m$-gon with edge lengths 2. We subdivide the $2m$-gon by coning off the orbit of x. The intersection of this convex hull with the simplicial cone is combinatorially isomorphic to K, and we give K the inherited Euclidean metric. This defines a piecewise Euclidean metric on $D(A)$, which in this case is CAT(0). This naturally gives a piecewise spherical metric on $B(A)$, where every edge has length $\pi/2m$. This turns out to be a CAT(1) metric, which in this case is equivalent to there being no closed geodesics with length $< 2\pi$. In the 2-dimensional case, this follows from [1, Lem 6].

We now recall the structure of links of vertices in the Deligne complex, again see [9] or [7] for complete details. It suffices to consider vertices in K as this is a strict fundamental domain for the action. Let T be a spherical subset of S, and let v_T be

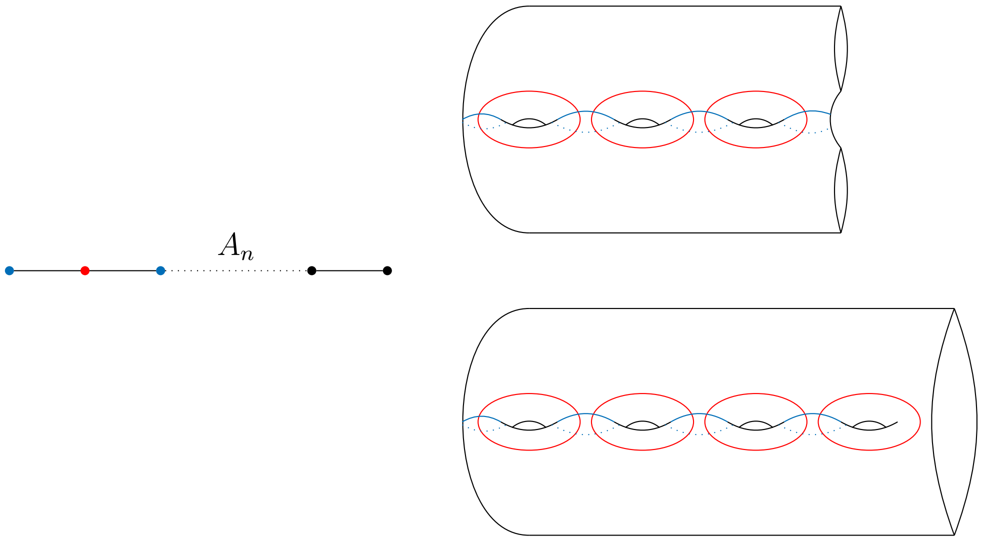

FIGURE 4. The Perron-Vannier representation for the Artin groups of type A_n. If n is even, then the element Δ_S^4 goes to a Dehn twists around the boundary curve. If n is odd, then the element Δ_S^2 goes to multitwist that is a product of single Dehn twists about each of the boundary curves.

the corresponding vertex in K. Then we have the following join decomposition

$$\mathrm{Lk}_{D(A)}(v_T) = \mathrm{Lk}_{K_{\geq T}}(v_T) * B(A_T).$$

The piecewise spherical metric on $\mathrm{Lk}_{D(A)}(v_T)$ is isometric to the orthogonal join of the metrics on $\mathrm{Lk}_{K_{\geq T}}(v_T)$ and $B(A_T)$ [8, Lem 2.2]. Recall the orthogonal join of two piecewise spherical metrics is defined by making each simplex $\sigma * \tau$ isometric to the simplex in $\mathbb{S}^{\dim\sigma+\dim\tau+1}$ spanned by $\sigma \subset S^{\dim\sigma}$ and $\tau \subset S^{\dim\tau}$, where points in σ and τ are all distance $\pi/2$ apart. Charney proves this for the cubical metric on $D(A)$, our links are isometric but have a finer subdivision (which still preserves join structure).

2.6. **Representations of small type Artin groups inside mapping class groups.** Let Σ be an oriented compact surface, possibly with boundary. Let $P = \{P_1, \dots P_n\}$ be a collection of n punctures in the interior of Σ. Let $\mathrm{Homeo}^+(\Sigma, P)$ denote the group of orientation-preserving homeomorphisms of Σ which fix the boundary pointwise, and which preserve P. Let $\mathrm{Homeo}_0^+(\Sigma, P)$ denote the connected component of the identity in $\mathrm{Homeo}^+(\Sigma, P)$. The mapping class group of the pair (Σ, P) is defined to be

$$\mathrm{Mod}(\Sigma, P) = \mathrm{Homeo}^+(\Sigma, P)/\,\mathrm{Homeo}_0^+(\Sigma, P)$$

A *multicurve* is a disjoint union of a finite number of simple, closed, essential curves in Σ. A *multitwist* about a multicurve is the composition of (not necessarily

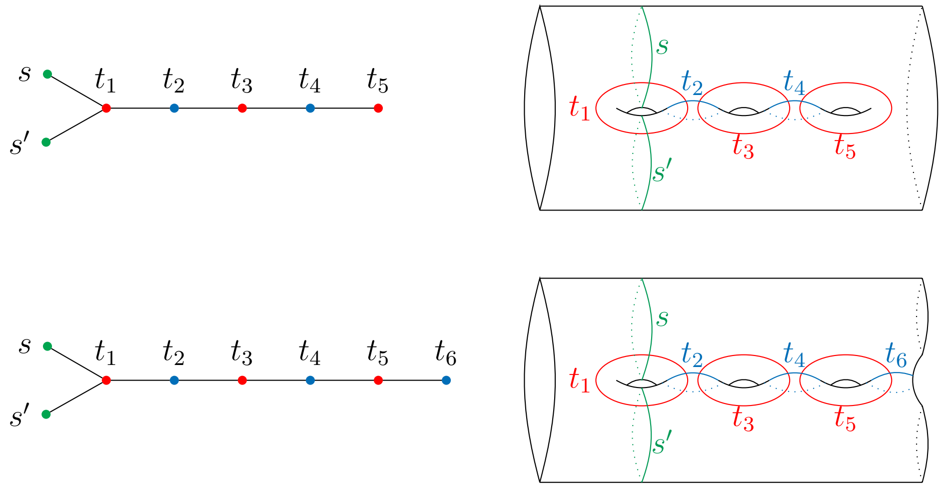

FIGURE 5. The Perron-Vannier representation $A \to Mod(\Sigma)$ for Artin group A of type D_n. For n odd, the element Δ_S^2 gets sent to a product of Dehn twists $\gamma_1\gamma_2^{n-2}$. For n even, the element Δ_S^2 gets sent to a product of Dehn twists $\gamma_1\gamma_2\gamma_3^{\frac{n}{2}-1}$.

the same) powers of Dehn twists about the individual curves. Since the curves are disjoint, the order in which we compose those Dehn twists does not matter.

For the small-type spherical Artin groups, there are classical representations into mapping class groups. This is due to Birman-Hilden for type A_n, and Perron-Vannier in general. We will refer to all of them as *Perron-Vannier representations*. Crisp and Paris defined similar representations for all small-type Artin groups, however we shall not need this generality. These representations also naturally arise as geometric monodromies of simple singularities of type Γ [32].

For A of type A_n, the Perron-Vanier representation $A \to \text{Mod}(\Sigma)$ sends the consecutive generators of A to the Dehn twists around the consecutive curves in Figure 4. The surface Σ has genus $\frac{k-1}{2}$ and two boundary components, when k is odd, and Σ has genus $\frac{k}{2}$ and one boundary component, when k is even.

If A has type D_n, let the standard generators of A be $\{s, s', t_1, \ldots, t_{n-2}\}$ where s and s' are both adjacent to t_1 in the Coxeter graph, and t_i and t_{i+1} are adjacent for all $i = 1, \ldots, n-3$. The Perron-Vannier representation $A \to \text{Mod}(\Sigma)$, which sends the generators to the Dehn twists around curves, as pictured in Figure 5. The surface Σ has genus $\frac{n-1}{2}$ and two boundary components when n is odd, and genus $\frac{n-2}{2}$ and three boundary components when n is even.

The Perron-Vannier representation $A \to \text{Mod}(\Sigma)$ of Artin group A of type E_n, where $n = 6, 7, 8$, is illustrated in Figure 6. The surface Σ has

- genus three and one boundary component, if $n = 6$.
- genus three and two boundary components, if $n = 7$.
- genus four and one boundary component, if $n = 8$.

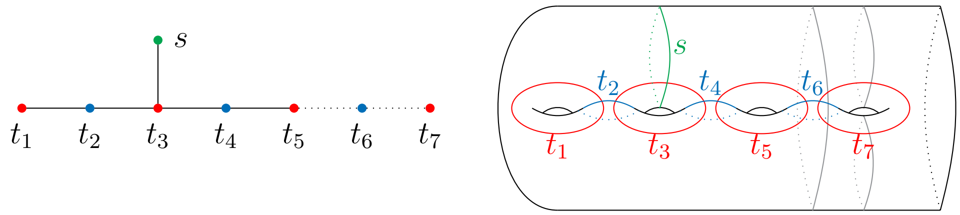

FIGURE 6. The Coxeter diagram for Artin group A of type E_6, E_7, E_8. A representation $A \to \mathrm{Mod}(\Sigma)$. The subsurfaces with gray boundary curves correspond to E_6 and E_7.

Remark 2.5. For any small type spherical Artin group A with the Perron-Vannier representation $\rho : A \to \mathrm{Mod}(\Sigma)$, the image of the element Δ_S^4 under ρ is a multitwist about the boundary components of $\partial\Sigma$. It will never matter for us the exact power of each Dehn twist.

3. Generalized Tits Conjecture for locally reducible Artin groups

In this section, we show that Artin groups with no 3 labels satisfy the Generalized Tits Conjecture for $N = 1$, and locally reducible Artin groups satisfy the conjecture for $N = 2$. We first record an easy characterization of the locally reducible Artin groups in terms of their Coxeter diagram.

Lemma 3.1. *Let Γ be a Coxeter diagram, and A the associated Artin group. Then A is locally reducible if and only if Γ satisfies the following condition:*

If two consecutive edges of Γ are not contained in a triangle, then their labels satisfy $1/a + 1/b \leq 1/2$.

3.1. CAT(0) geometry.

We will assume that the reader is comfortable with comparison geometry, particularly in the setting of piecewise Euclidean and piecewise spherical cell complexes, see [7], [5] or [13, Appendix I] for the relevant details. We record some theorems and definitions that we will need. The first is due to Gromov, proofs can be found in [5].

Theorem 3.2. *Let X be a piecewise Euclidean cell complex. Then X is locally CAT(0) if and only if the induced piecewise spherical metric on the link $\mathrm{Lk}(v, X)$ is CAT(1) for all vertices $v \in X$. If X is simply connected and locally CAT(0), then it is CAT(0).*

Lemma 3.3 (see Appendix of [9])**.** *The orthogonal join of two piecewise spherical complexes L_1 and L_2 is CAT(1) if and only if L_1 and L_2 are CAT(1).*

Definition 3.4. Let $f : L \to L'$ be a map between piecewise spherical complexes. We say f is *π-distance preserving* if

$$d_L(x_1, x_2) \geq \pi \implies d_{L'}(f(x_1), f(x_2)) \geq \pi.$$

If L is not connected, then we set $d_L = \infty$ for points in different components. Thus $f : L \to L'$ is π-distance preserving if it is π-distance preserving on each component of L and points in different components get mapped $\geq \pi$ apart in L'.

Lemma 3.5 ([7, Lem 1.4])**.** *Suppose $f : X \to X'$ is a map between two piecewise Euclidean complexes which takes piecewise geodesics to piecewise geodesics. Then f is a locally isometric embedding if and only if the induced maps on all links $\mathrm{Lk}(x, X) \to \mathrm{Lk}(f(x), X')$ are π-distance preserving. Furthermore, if X' is CAT(0), then f is an isometric embedding.*

Lemma 3.6 (see Appendix of [9])**.** *If $f : L_1 \to L_1'$ and $g : L_2 \to L_2'$ are π-distance preserving, and $L_1 * L_2$, $L_1' * L_2'$ are orthogonal joins, then*

$$f * g : L_1 * L_2 \to L_1' * L_2'$$

is π-distance preserving.

Charney showed that the proof that $D(A)$ is CAT(0) for 2-dimensional Artin groups extends to locally reducible Artin groups.

Theorem 3.7. [7, Thm 3.2] *Let A be a locally reducible Artin group. Then $D(A)$ equipped with the Moussong metric is CAT(0).*

3.2. **Orthogonality in the Deligne complex.**

Definition 3.8. Let $F_n = \langle x_1, \ldots, x_n \rangle$ be a free group of rank n. Then every element g of F_n can be written uniquely as

$$g = x_{i_1}^{n_1} x_{i_2}^{n_2} \ldots x_{i_k}^{i_k}$$

where $x_{i_j} \neq x_{i_{j+1}}$ and $n_i \in \mathbb{Z} - \{0\}$. The *syllable length* of g in this case is equal k. If G is a group admitting a surjection $\phi : F_n \to G$ is a surjection, then for $g \in G$ we define the *syllable length* of g with respect to the generating set $\{\phi(x_1), \ldots, \phi(x_n)\}$, to be the infimum of syllable lengths of words in $\phi^{-1}(g)$.

For convenience, we will denote the generators of A by letters $s, t, \ldots$ rather than $x_s, x_t \ldots$. We will also denote the coset $a\langle \emptyset \rangle$ just by a for each $a \in A$. If A is a dihedral Artin group, then a simplicial path in $B(A)$ between two cosets g and h must contain an even number of vertices. In particular, each pair of consecutive edges connects g to gt^k or gs^k for some $k \in \mathbb{Z}$. Therefore, we can associate to a path in $B(A)$ between g and h a word w in the free group $F(s, t)$ so that $gw = h$ in A. If this path is embedded, then its length is equal to $\frac{\pi}{m}k$ where k is the syllable length of w.

We now record a technical proposition concerning dihedral Artin groups. In this case, the pure Artin group is isomorphic to the direct product $F_{m-1} \times \mathbb{Z}$, where Δ^2 is the generator of $\mathbb{Z}$ (in this case, the hyperplane complement is homeomorphic to a $\mathbb{C}^*$-bundle over an m-punctured sphere). The proposition states a sort of orthogonality for the action of the pure Artin group on the Deligne complex. Given two elements g, h in the Artin group, we let $d_{B(A)}(g, h)$ denote the distance in $B(A)$ between the vertices g and h.

Proposition 3.9. *Let A be a dihedral Artin group. Let g be an element of the free subgroup $\langle s^2, t^2\rangle$. Then for any $n \in \mathbb{Z} - \{0\}$ we have that*

$$d_{B(A)}(\Delta^{2n}, g) \geq \pi.$$

Furthermore, if $m = m_{st} > 3$, then

$$d_{B(A)}(\Delta^{2n}, g) \geq \pi + \pi/m.$$

Finally, if $m = m_{st} = 3$ and g is in the free subgroup $\langle s^4, t^4\rangle$, then

$$d_{B(A)}(\Delta^{4n}, g) \geq \pi + \pi/m.$$

Proof. Suppose that $d_{B(A)}(\Delta^{2n}, g) < \pi$, so there is $a \in A$ of syllable length $< m$ so that

$$\Delta^{2n} = ag.$$

Note that a is obviously an element of the pure Artin group PA, since both g and Δ^{2n} are. We now compute the image $\overline{a}$ of a in $H_1(PA, \mathbb{Z})$. Let $a = bc$, where b has only odd powers of s and t, and c begins with s^{2n} or t^{2n}. Let $\pi : A \to W$ be the canonical projection. Since the length of $\pi(b)$ in W is $< 2m$, c must be nontrivial, otherwise a would not be in PA. Without loss of generality, suppose that c starts with s^{2n}. Note that c must project to $\pi(b)^{-1}$ under π. In particular, the syllable length of b is strictly smaller than $\frac{m-1}{2}$.

Thus we can write $a = bs^{2n}b^{-1}bc'$, where bc' is in PA and has strictly smaller syllable length than a. Since $bs^{2n}b^{-1}$ and bc' are in PA, in $H_1(PA, \mathbb{Z})$ we have the relation between the images

$$\overline{a} = \overline{bs^{2n}b^{-1}} + \overline{bc'}.$$

Therefore, by induction on the syllable length of a, we can assume that $\bar{a} \in H_1(PA, \mathbb{Z})$ is the sum of images of elements $b_i s^{2n} b_i^{-1}$ and $b_i t^{2n} b_i^{-1}$, where the syllable length of the b_i is $\leq \frac{m-2}{2}$ and the b_i only contain odd powers of s and t (or they are trivial). Therefore, the image of Δ^{2n} in $H_1(PA, \mathbb{Z})$ could be written as a sum of elements $b_i s^{2n} b_i^{-1}$ and $b_i t^{2n} b_i^{-1}$ with syllable length of the $b_i \leq \frac{m-2}{2}$. It is also easy to see that these b_i can only have syllable length $\frac{m-2}{2}$ for a single i.

However, this contradicts Lemma 2.2. Recall that $H_1(PA, \mathbb{Z})$ has a standard basis given by the hyperplanes in the dihedral arrangement, and the image of $b_i s^2 b_i^{-1}$ is precisely the basis vector e_r, where r is the conjugate $\pi(b_i)s\pi(b_i)^{-1}$.

In particular, if m is odd, then any such sum as above misses the hyperplane corresponding to the longest element of W, which is a conjugate wsw^{-1} where w has length $\frac{m-1}{2}$. If m is even, there are two hyperplanes corresponding to conjugates wsw^{-1} and $w't(w')^{-1}$ where w and w' have length $\frac{m-2}{2}$, hence the elements in our sum miss one of these. We will refer to these as *longest hyperplanes*. Since Δ^{2n} maps in $H_1(PA, \mathbb{Z})$ to a vector with nontrivial e_r-term for each $r \in R$, this is a contradiction as the image of g obviously misses these longest hyperplanes as well. This completes the proof of the first statement.

Now, suppose that $m_{st} > 3$ and $d_{B(A)}(\Delta^{2n}, g) = \pi$. We first consider the odd case, where $m = 2k+1$. We can write

$$\Delta^{2n} = ag$$

where the syllable length of a is equal to m. In the decomposition $a = bc$ above, we claim that b must have syllable length k. Otherwise, we can again write the images in $H_1(PA, \mathbb{Z})$ as $\overline{a} = \overline{bs^{2n}b^{-1}} + \overline{bc'}$. If b has syllable length $< k$ then both $\overline{bs^{2n}b^{-1}}$ and $\overline{bc'}$ will miss the vector in H_1 corresponding to the longest hyperplane.

Therefore, without loss of generality we have $a = bc$ and the syllable length of b is k. Since a is in the pure Artin group, c is of the form $s^{2n}x$ or $t^{2n}x$ where x projects to $\pi(b)^{-1}$ (in particular each term of x has odd exponent).

Therefore, without loss of generality we can rewrite $a = bs^{2n}b^{-1}a'$. By the above, $a' = b'c'$, where b' has all odd exponents, has syllable length $k-1$, and where $c' = s^{2n}x'$ or $t^{2n}x'$ as above. Repeating this argument gives us that in $H_1(PA, \mathbb{Z})$,

$$\overline{a} = e_{r_1} + e_{r_2} + \cdots + e_{r_{k+1}}$$

where the length of the reflections r_i is strictly decreasing. If $m_{st} > 3$, then there are two hyperplanes of length $1 < \ell(r) < m$, and hence the image $\overline{a}$ misses one of those.

If $m = 2k$ is even, the proof essentially extends. The point here is that there are two longest hyperplanes and the image $\overline{a}$ will again miss one of these. More precisely, suppose that $a = bc$ as above. In order to hit one of the longest hyperplanes, b must have syllable length $k-1$. This implies that $c = s^{2n}x$ or $c = t^{2n}x$, and x projects to $\pi(b)^{-1}$. In particular, we must have that $x = yt^{2m}$ or $x = ys^{2m}$ where y projects to $\pi(b)^{-1}$. Therefore, we can push the last syllable into g. Since we now have a word a' of length $< m$, the proof in the first case rules out this possibility.

We now prove the last statement. We will now assume that $m_{st} = 3$, i.e. A is the braid group on 3-strands. We need the following lemma:

Lemma 3.10. *If $n > 0$ we have that*

$$\Delta^{2n} = st^{2n}s \underbrace{t^2s^2 \cdots}_{2n-1 \text{ terms}} = ts^{2n}t \underbrace{s^2t^2 \cdots}_{2n-1 \text{ terms}}$$

$$\Delta^{-2n} = s^{-1}t^{-2n}s^{-1} \underbrace{t^{-2}s^{-2} \cdots}_{2n-1 \text{ terms}} = t^{-1}s^{-2n}t^{-1} \underbrace{s^{-2}t^{-2} \cdots}_{2n-1 \text{ terms}}$$

Proof. We only prove the first equalities for both Δ^{2n} and Δ^{-2n}; the same argument with s and t switched will give the second. Since $\Delta^2 = st^2st^2 = t^2st^2s$ we have by induction that for $n > 1$

$$\Delta^{2n} = st^{2n-2}s \underbrace{t^2s^2 \cdots}_{2n-3 \text{ terms}} (t^2st^2s) = st^{2n-2}(t^2st^2s)s \underbrace{t^2s^2 \cdots}_{2n-3 \text{ terms}}$$

$$= st^{2n}st^2s^2 \underbrace{t^2s^2 \cdots}_{2n-1 \text{ terms}}$$

Similarly, $\Delta^{-2} = s^{-1}t^{-2}s^{-1}t^{-2} = t^{-2}s^{-1}t^{-2}s^{-1}$, so by induction for $n > 1$we have

$$\Delta^{-2n} = s^{-1}t^{-2n+2}s^{-1}\underbrace{t^{-2}s^{-2}\ \cdots}_{2n-3\text{ terms}}t^{-2}s^{-1}t^{-2}s^{-1} = s^{-1}t^{-2n}s^{-1}\underbrace{t^{-2}s^{-2}\ \cdots}_{2n-1\text{ terms}}$$

□

Now, suppose that $\Delta^{4n} = ag$ where a has syllable length 3 and $g \in \langle s^4, t^4\rangle$. Then without loss of generality we can assume that $a = s^k t^{4n} s^l$ where k and l are odd integers. We'll assume that $n > 0$, a similar argument works for $n < 0$.

By conjugating with an even power of s, we can assume that $k = 1$. Then we have

$$st^{4n}st^2s^2\underbrace{t^2s^2\ \cdots}_{2n-1\text{ terms}} = st^{4n}s^lgs^{k-1}$$

so in particular $\underbrace{t^2s^2\ \cdots}_{2n+1\text{ terms}} = s^{l-1}gs^{k-1}$, where $l-1$ and $k-1$ are even.

Since the original Tits Conjecture holds for A, we must have that $\underbrace{t^2s^2\ \cdots}_{2n+1\text{ terms}}$ is equivalent to $s^{l-1}gs^{k-1}$ in the free group on s and t. But the powers of s and t in g are powers of 4, which is a contradiction. □

3.3. **Proof of Generalized Tits Conjecture in the locally reducible case.** Let A be a locally reducible Artin group and RA the associated RAAG. Let K be the fundamental domain for the action of A on its Deligne complex $D(A)$ with the metric as described in Section 2.5. Let $\hat{A}$ denote the RAAG subgroup that is in the original Tits Conjecture (i.e. the generators of $\hat{A}$ correspond to generators of A). In [7] Charney defined a complex

$$\widehat{D}(A) = \widehat{A} \times K/\sim,$$

where $(a_1, x) \sim (a_2, x)$ if and only if $x \in K_{\geq T}$ and $a_1^{-1}a_2 \in \widehat{A}_T$. This is not the Deligne complex for $\widehat{A}$, since $\widehat{A}_T$ may not be spherical. We define

$$\widehat{\widehat{D}}(A) = RA \times K/\sim,$$

where $(a_1, x) \sim (a_2, x)$ if and only if $x \in K_{\geq T}$ and $a_1^{-1}a_2 \in RA_T$.

The Moussong metric on K induces a piecewise Euclidean metric on $\widehat{\widehat{D}}(A)$, and the homomorphisms Φ_N : RA $\to A$ define an induced map $\widehat{\widehat{\Phi}}_N : \widehat{\widehat{D}}(A) \to D(A)$ which sends $s \times K$ isomorphically onto $\Phi_N(s) \times K$.

If A is spherical, then both $\widehat{D}(A)$ and $\widehat{\widehat{D}}(A)$ have a cone point, labeled by A. We let $\widehat{\widehat{B}}(A)$ be the link of this cone point in $\widehat{\widehat{D}}(A)$, and $\widehat{B}(A)$ the link of the cone point in $\widehat{D}(A)$. Note that $\widehat{B}(A)$ is naturally a subcomplex of $\widehat{\widehat{B}}(A)$. If A is dihedral, then we denote the generators of RA by $\{z_s, z_t, z_{\{s,t\}}\}$. In particular, $\Phi_N(z_{\{s,t\}}) = \Delta^{2N}$.

Lemma 3.11. *If A is a dihedral Artin group, then $\widehat{\widehat{B}}(A)$ is the disjoint union*

$$\widehat{\widehat{B}}(A) = \bigsqcup_{i\in\mathbb{Z}} z^i_{\{s,t\}}\widehat{B}(A).$$

Proof. The adjacent vertices in $\widehat{\widehat{B}}(A)$ to $w\langle s\rangle$ are $w\emptyset$ and $wz_s\emptyset$, (and similarly for $w\langle t\rangle$). Since $w = gz^i_{\{s,t\}}$ and $wz_s = gz_sz^i_{\{s,t\}}$ for some $g \in F_2$, the power $z^i_{\{s,t\}}$ is the same for w and ws. Therefore, any vertices that can be connected by a path in $\widehat{\widehat{B}}(A)$ have the same power of $z_{\{s,t\}}$. The $z_{\{s,t\}}$-action on $\widehat{\widehat{B}}(A)$ identifies each connected component of $\widehat{\widehat{B}}(A)$ with $\widehat{B}(A)$. □

Proposition 3.12. *Let A be a totally reducible, finite type Artin group with $m_{st} \neq 3$ for each factor. Then the induced map $\Phi_1 : \mathrm{Lk}(A, \widehat{\widehat{D}}(A)) \to \mathrm{Lk}(A, D(A))$ is π-distance preserving. For any totally reducible finite type Artin group, the induced map $\Phi_2 : \mathrm{Lk}(A, \widehat{\widehat{D}}(A)) \to \mathrm{Lk}(A, D(A))$ is π-distance preserving.*

Proof. In this proof $N = 1$ or 2, depending on whether there are s, t with $m_{st} = 3$. We have that A and RA decompose as

$$A = A_1 \times A_2 \times \cdots \times A_k; \quad \mathrm{RA} = \mathrm{RA}_1 \times \mathrm{RA}_2 \times \cdots \times \mathrm{RA}_k$$

where each A_i is an irreducible spherical subgroup of rank 2 or $\mathbb{Z}$. Therefore, both $B(A)$ and $\widehat{\widehat{B}}(A)$ decompose as orthogonal joins

$$B(A) = B(A_1) * B(A_2) * \cdots * B(A_k); \quad \widehat{\widehat{B}}(A) = \widehat{\widehat{B}}(A_1) * \widehat{\widehat{B}}(A_2) * \cdots * \widehat{\widehat{B}}(A_k),$$

so by Lemma 3.6 it suffices to check π-distance preserving for each A_i. If $A = \mathbb{Z}$ this is obvious, so suppose that A is a dihedral Artin group.

By [7, Lem 4.1], the induced map $\widehat{\Phi}_N : \widehat{B}(A) \to B(A)$ is π-distance preserving. Since the map Φ_N is equivariant, this implies that the induced map is π-distance preserving on each component $(\Delta^{2N})^i\widehat{B}(A)$. By Lemma 3.11, it suffices to verify that for x, y in two different copies of $\widehat{D}(A)$ in $\widehat{\widehat{D}}(A)$, their images $\widehat{\widehat{\Phi}}_N(x), \widehat{\widehat{\Phi}}_N(y)$ have distance at least π in $D(A)$. Let x lie in an edge of $\Delta^{2n_1}\widehat{B}(A)$ and y lie in an edge of $\Delta^{2n_2}\widehat{B}(A)$, where $g_1, g_2 \in \widehat{A}$, and $n_1 \neq n_2$. Then x and y are within distance $\pi/2m$ from vertices $\Delta^{2n_1}g_1$ and $\Delta^{2n_2}g_2$. By Proposition 3.9,

$$d_{B(A)}(\Delta^{2n_2}g_2, \Delta^{2n_1}g_1) = d_{B(A)}(\Delta^{2(n_2-n_1)}, g_1g_2^{-1}) \geq \pi + \pi/m.$$

This implies that the images of x and y are $\geq \pi$ apart. □

Theorem 3.13. *The map $\Phi_2 : RA \to A$ is injective for every locally reducible Artin group. If $m_{st} \neq 3$ for all $s, t \in S$, then $\Phi_1 : RA \to A$ is injective.*

Proof. In this proof $N = 1$ or 2, depending on whether there are s, t with $m_{st} = 3$. We prove that the induced map $\widehat{\widehat{\Phi}}_N : \widehat{\widehat{D}}(A) \to D(A)$ is an isometric embedding. By Lemma 3.5, $\widehat{\widehat{\Phi}}_N : \widehat{\widehat{D}}(A) \to D(A)$ is an isometric embedding provided that the map $\mathrm{Lk}(x, \widehat{\widehat{D}}(A)) \to \mathrm{Lk}(\widehat{\widehat{\Phi}}_N(x), D(A))$ induced by $\widehat{\widehat{\Phi}}_N$ is π-distance preserving for every $x \in \widehat{\widehat{D}}(A)$. We only check this where x is a vertex, essentially the same argument works for all x. Since $\widehat{\widehat{\Phi}}_N$ is equivariant, it suffices to check vertices of K. For T a spherical subset and $v_T \in \widehat{\widehat{D}}(A) \cap K$, the link of v_T decomposes as $\mathrm{Lk}(x, K_{\geq T}) \times \widehat{\widehat{B}}(A_T)$. Since the link of v_T in $D(A)$ decomposes as $\mathrm{Lk}(x, K_{\geq T}) \times B(A_T)$ and the map between links decomposes as $\mathrm{Id} \times \Phi_N$ the result follows from Proposition 3.12. □

3.4. **Intersections with special subgroups.** Finally, we use a coning trick to show that for any locally reducible Artin group A and special subgroup A_T, we have that our RAAG subgroup for A intersects A_T in the RAAG subgroup for T. Charney used CAT(0) geometry to prove this for the RAAG provided by the original Tits Conjecture [7, Thm 5.2]. Since $\Phi_2 : \mathrm{RA} \to A$ injective, by Theorem 3.13, we write RA_T for the image of RA_T under Φ_2.

Proposition 3.14. *Let A be a locally reducible Artin group. Then* $\mathrm{RA} \cap A_T = \mathrm{RA}_T$.

Proof. Suppose that there was a reduced word w in RA so that $w \notin \mathrm{RA}_T$ but $w \in A_T$. Define a larger Artin group $\widetilde{A}$ by "coning" off T, i.e. introduce a new generator s so that $m_{st} = 2$ for all $t \in T$ and $m_{su} = \infty$ otherwise. Then $\widetilde{A}$ is a locally reducible Artin group, so we know the Generalized Tits Conjecture for $\widetilde{A}$. Note that the RAAG for $\widetilde{A}$ is just the RAAG for A with the RAAG for A_T coned off. Now, by assumption we have that $[s, w] = 1$, which contradicts the RAAG for $\widetilde{A}$ injecting into $\widetilde{A}$, as the centralizer of s in that RAAG is the RAAG subgroup $\mathrm{RA}_T \times \langle s \rangle$. □

Remark 3.15. The same argument shows that the RAAG subgroup that Crisp and Paris find intersects each special Artin group in the expected way. If we knew the Generalized Tits Conjecture for all Artin groups, then we would know this intersection property as well.

4. RAAG subgroups of RAAGs

In this section, we study whether a subgroup of a RAAG generated by words which are powers of commuting elements is the "obvious" RAAG. In Sections 5 and 6 we apply these ideas to the Generalized Tits Conjecture, but we hope that this will be of independent interest.

In general, this is a delicate question; the subgroup may be a RAAG but a different one than expected, or may not be a RAAG at all. See [28] and [29] for a detailed analysis and many (positive and negative) examples. Our main goal in this section is Theorem 4.10, which generalizes a condition on the words given in [29] called *Property PP* (short for ping-pong).

4.1. **Koberda's Property PP.** Let L be a flag complex and RA_L the corresponding RAAG. For each simplex $\sigma \in L$, let w_σ be a (possibly trivial) word with all positive powers (or all negative powers) of the generators corresponding to vertices in σ (in particular, if w_σ is nontrivial it has a nontrivial power of each generator in $\sigma^{(0)}$).

The collection $\{w_\sigma\}_{\sigma \in L}$ determines a flag complex L'. The vertex set of L' is $\{\sigma \in L : w_\sigma \text{ is nontrivial}\}$ (the reader can imagine the vertex at the barycenter of σ), and the simplices correspond to collections $\sigma_1, \dots, \sigma_k$ where $w_{\sigma_1}, \dots w_{\sigma_k}$ pairwise commute in RA_L. Of course, we have that w_σ and w_τ commute if any only if σ and τ span a simplex in L.

Definition 4.1. We say that the collection $\{w_\sigma\}_{\sigma\in L}$ satisfies *Property PP* if there is an injective simplicial map $p : L' \to L$, so that $p(L')$ is a full subcomplex of L and the vertex of L' corresponding to w_σ maps to a vertex of σ for each $\sigma \in L$.

One can think of this map p as choosing a representative vertex in L for each w_σ. The requirement that $p(L')$ is a full subcomplex ensures that if two words w_σ and $w_{\sigma'}$ do not commute, then their representative vertices do not commute. The following proposition of Koberda follows quickly from the normal form for RAAG's.

Proposition 4.2 ([29, Lem 5.2])**.** *Let w_σ have Property PP. Then the map $f : \mathrm{RA}_{L'} \to \mathrm{RA}_L$ sending the generators σ of $\mathrm{RA}_{L'}$ to w_σ, is injective. Therefore, the subgroup generated by $\langle w_\sigma\rangle$ is isomorphic to $\mathrm{RA}_{L'}$.*

Example 4.3. The following example is taken from [29], where it is attributed to M. Casals. Let $\mathrm{RA}_L = F_2 \times F_2$ where the first F_2 is generated by a, c and the second generated by b, d. Consider the subgroup $H_n < \mathrm{RA}_L$ generated by a^n, d^n and $(bc)^n$ for $n \in \mathbb{N}$. It turns out that H_n is not isomorphic to any right-angled Artin group. We will not provide the full proof of this, the key point is that in this group there is the relation

$$[a^n, (bc)^n d^n (bc)^{-n}] = 1.$$

Note that this collection of words does not satisfy property PP, in this case L' is the disjoint union of an edge and a point, and hence there is no injective map from $L' \to L$ where the image is a full subcomplex.

4.2. **Generalization of Property PP.** We start with a motivating example.

Example 4.4. Let L be a path with 4 vertices a, b, c, d, and let RA_L be the corresponding RAAG. Consider the subgroup H generated by $\{a, d, bc\}$. By the same reasoning as Example 4.3, this collection of words do not satisfy property PP (in this case L' is 3 points and there is no injective map from 3 points to L with image a full subcomplex). However, we claim that the subgroup H generated by these words is still isomorphic to the free group F_3.

To see this, note that RA_L splits as the amalgamated product

$$\mathrm{RA}_L = \langle a, b, c\rangle *_{\langle b,c\rangle} \langle b, c, d\rangle$$

and F_3 decomposes as $F_2 *_{\mathbb{Z}} F_2$. We can use Property PP to say that the subgroups generated by $\langle a, bc\rangle$ and $\langle bc, d\rangle$ inside $\langle a, b, c\rangle$ and $\langle b, c, d\rangle$ respectively are both F_2. Furthermore, each of these subgroups intersects $\langle b, c\rangle$ in $\langle bc\rangle$ (this is not completely obvious and generalizing this is the majority of our work below). Therefore, we can apply the following lemma of Serre:

Proposition 4.5. *Let G_i be a collection of groups with common subgroup A and let $\ast_A, G_i$ denote the amalgamated product. Let $H_i \subset G_i$ be subgroups and suppose the intersection $B = H_i \cap A$ is independent of i. Then the natural homomorphism $\ast_B\ H_i \to \ast_A\ G_i$ is injective.*

So, in the above example, we get an injection from $F_3 \cong F_2 *_{\mathbb{Z}} F_2 \to \langle a, bc, d\rangle$, which is obviously an isomorphism. We shall also need the following lemma about subgroups of amalgamated products, which follows immediately from the normal form for amalgamated products.

Lemma 4.6. *Let $G = A *_C B$ be an amalgamated product, and $H = D *_F E$ a subgroup of G so that $D \subset A$, $E \subset B$ and $D \cap E = F$. If K is a subgroup of G contained in A, then $K \cap H = K \cap D$.*

For general RAAG's, we suppose the following: our nerve L decomposes as $L = L_1 \cup_{L_0} L_2$, where each L_i is a full subcomplex. We consider a collection of words $\{w_\sigma\}_{\sigma \in L}$ where, as before, w_σ is a (possibly trivial) word with all positive powers (or all negative powers) of the generators corresponding to vertices in σ. We assume that each of the collections $\{w_\sigma\}_{\sigma\in L_1}$, $\{w_\sigma\}_{\sigma\in L_2}$ satisfies Property PP in RA_{L_1}, RA_{L_2} respectively. Note that the functions in property PP for each L_i do not have to agree on the words for L_0 (if they do it is easy to see that the words already satisfy property PP). However, we have to make some additional assumptions, to conclude that $\mathrm{RA}_{L'}$ embeds in RA_L.

Definition 4.7. Let L be a flag complex and L_0 a full subcomplex. Suppose we have a collection of words $\{w_\sigma\}_{\sigma\in L}$ satisfying property PP, and L' is the associated flag complex and function $p : L' \to L$. We say that the collection $\{w_\sigma\}_{\sigma\in L}$ *avoids* L_0 if the following two conditions are satisfied:

(1) If $\sigma \notin L_0$, then $p(\sigma) \notin L_0$.
(2) If $\sigma \notin L_0$, then w_τ is trivial for any other τ containing $p(\sigma)$.

The next lemma guarantees that if $\{w_\sigma\}_{\sigma\in L}$ satisfies property PP and avoid L_0, then the intersection of the RAAG subgroup generated by the $\{w_\sigma\}$ intersects the special subgroup A_{L_0} as expected.

Lemma 4.8. *Let L be a flag complex,* RA_L *the* RAAG *on L, and let L_0 be a full subcomplex. Let $\{w_\sigma\}_{\sigma\in L}$ be words satisfying Property PP avoiding L_0. Let* $\mathrm{RA}_{L'}$ *be the* RAAG *subgroup of* RA_L *generated by the $\{w_\sigma\}_{\sigma\in L}$, and* $\mathrm{RA}_{L'_0}$ *the* RAAG *subgroup generated by $\{w_\sigma\}_{\sigma\in L_0}$. Then* $\mathrm{RA}_{L'} \cap \mathrm{RA}_{L_0} = \mathrm{RA}_{L'_0}$.

Proof. Note that we are obviously done if w_σ is trivial for each σ not contained in L_0. We assume that this is not the case. We induct on the number of vertices of L.

We first prove the case where L is a simplex, so that RA_L is free abelian. Note that RA_L splits as a product $\mathrm{RA}_{L_0} \times \mathrm{RA}_{L-L_0}$. We claim that p is injective while restricted to the subgroup generated by $\{w_\sigma\}_{\sigma\notin L_0}$. Indeed, by Condition 2 of Definition 4.7, each w_σ with $\sigma \notin L_0$ has a non-trivial coordinate in RA_L such that all other w_τ have this coordinate zero. It follows that $\mathrm{RA}_L \cap \mathrm{RA}_{L_0} = RA_{L'_0}$.

We now consider the general case. Let μ be the largest standard factor of L, i.e. the largest dimensional simplex so that $L = \mu * K$. In particular, K is not a cone. Choose σ not contained in L_0 with nontrivial w_σ. We consider different cases.

First, suppose there exists such σ with $p(w_\sigma) \in K$. We denote $p(w_\sigma)$ by v. By Condition 1 of Definition 4.7, v is not in L_0. Since K has no non-trivial standard factors, $\mathrm{St}(v)$ is a proper subcomplex of L. Thus we have a nontrivial decomposition

$$\mathrm{RA}_L = \mathrm{RA}_{L-v} *_{\mathrm{RA}_{\mathrm{Lk}(v)}} \mathrm{RA}_{\mathrm{St}(v)} .$$

The collection $\{w_\sigma\}_{\sigma \in L-v}$ satisfies property PP and avoids L_0. Let $\mathrm{RA}_{(L-v)'}$ be the RAAG subgroup generated by $\{w_\sigma\}_{\sigma\in L-v}$. By induction, $\mathrm{RA}_{(L-v)'} \cap \mathrm{RA}_{L_0} = \mathrm{RA}_{L_0'}$. Since $\{w_\sigma\}_{\sigma\in L}$ satisfies property PP, if τ is contained in $\mathrm{St}(v)$, then w_τ commutes with w_σ, since v commutes with each vertex in w_τ, and hence commutes with $p(w_\tau)$. By Condition 2 w_σ is the only non-trivial word such with $v \in \sigma$. Thus the words $\{w_\sigma\}_{\sigma\in \mathrm{St}(v)}$ satisfy Property PP and avoid $\mathrm{Lk}(v)$. Let $\mathrm{RA}_{\mathrm{St}(v)'}$ denote the RAAG subgroup generated by $\{w_\sigma\}_{\sigma\in\mathrm{St}(v)}$. Again by induction, we have that $\mathrm{RA}_{\mathrm{St}(v)'} \cap \mathrm{RA}_{\mathrm{Lk}(v)} = \mathrm{RA}_{\mathrm{Lk}(v)'}$. Therefore, we have that

$$\mathrm{RA}_{L'} = \mathrm{RA}_{(L-v)'} *_{\mathrm{RA}_{\mathrm{Lk}(v)'}} \mathrm{RA}_{\mathrm{St}(v)'}$$

and Lemma 4.6, we have that $\mathrm{RA}_{L'} \cap \mathrm{RA}_{L_0} = \mathrm{RA}_{(L-v)'} \cap \mathrm{RA}_{L_0} = \mathrm{RA}_{L_0'}$.

Now suppose that for every σ not contained in L_0 with nontrivial w_σ, we have $p(w_\sigma) \notin K$. Take any such σ. We have $p(w_\sigma) \in \mu$. We first consider the case where σ is not entirely contained in μ. In this case, since each nontrivial w_τ must commute with w_σ, the subgroup $\mathrm{RA}_{L'}$ is contained in $\mathrm{RA}_{\mathrm{St}(\sigma)}$. The star $\mathrm{St}(\sigma)$ decomposes as a join $(\mu * (\sigma \cap K)) * (St_K(\sigma \cap K)$. By the choice of μ, this join is properly contained in L, and in particular has fewer vertices than L. By induction we are done.

Finally, if all σ not contained in L_0 with nontrivial w_σ, are entirely contained in μ, then the subgroup $\mathrm{RA}_{L'}$ splits as as a product of a free abelian group generated by $\{w_\sigma\}_{\sigma\notin L_0}$ and $\mathrm{RA}_{L_0'}$. The intersection of the free abelian group generated by $\{w_\sigma\}_{\sigma\notin L_0}$ with RA_{L_0} is trivial, by Condition 2 of Definition 4.7. The conclusion follows. ∎

Remark 4.9. Without the second condition in Definition 4.7, Lemma 4.8 is false even for free abelian groups. For example, if L is a simplex with vertices s, t and u, then the words stu and st avoid u, but the subgroup they generate contains u. On the other hand, we expect this second condition can be weakened to only requiring the words w_σ to not be equal outside of L_0.

This lemma combined with Serre's proposition implies the following theorem:

Theorem 4.10. *Let L be a flag complex and suppose that $L = L_1 \cup_{L_0} L_2$ where each L_i is a full subcomplex. Let $\{w_\sigma\}_{\sigma\in L}$ be words so that each subcollection $\{w_\sigma\}_{\sigma\in L_i}$ satisfies property PP and avoids L_0. Then the subgroup generated by the $\{w_\sigma\}_{\sigma\in L}$ is the* RAAG *based on L'.*

Proof. By Property PP, we know that the subgroup $\mathrm{RA}_{L_i'}$ of RA_{L_i} generated by $\{w_\sigma\}_{\sigma\in L_i}$ is a RAAG. By Lemma 4.8, we know that

$$\mathrm{RA}_{L_1'} \cap \mathrm{RA}_{L_0} = \mathrm{RA}_{L_2'} \cap \mathrm{RA}_{L_0} = \mathrm{RA}_{L_0'} .$$

By Proposition 4.5, $\mathrm{RA}_{L'} = \mathrm{RA}_{L_1'} *_{\mathrm{RA}_{L_0'}} \mathrm{RA}_{L_2'}$ injects into $\mathrm{RA}_L = \mathrm{RA}_{L_1} *_{\mathrm{RA}_{L_0}} \mathrm{RA}_{L_2}$. ∎

Remark 4.11. The words b, d, ac in Example 4.3 do not satisfy generalized PP for any decomposition of L. For example, if we take L_1 the full subcomplex containing a, b, c and L_2 the full subcomplex containing b, c, d, then for the edge ac we have to choose the vertex c in property PP, which is in $L_1 \cap L_2$. Note that the subgroup generated by ac and b contains cbc^{-1}, and hence intersects the subgroup $\langle b, c\rangle$ outside of $\langle b\rangle$.

5. Generalized Tits Conjecture and small type spherical Artin groups

In this section, we describe how the Generalized Tits Conjecture holds for small-type spherical Artin groups of type A_n and D_n. The main work is type D_n, where we require the generalized Property PP of the previous section.

Let A be any small-type spherical Artin group with the standard generating set S, and the Perron-Vannier representation $A \to \mathrm{Mod}(\Sigma)$. Let RA be the associated RAAG with the presentation 1.1. Let us record the following observation.

Lemma 5.1. *If the composition* $\mathrm{RA} \xrightarrow{\Phi_k} A \to \mathrm{Mod}(\Sigma)$ *is injective, then A satisfies the Generalized Tits Conjecture for $N = k$.*

For every irreducible subset $T \subset S$, let $A_T \to \mathrm{Mod}(\Sigma_T)$ be its Perron-Vannier representation of the special subgroup A_T. If $T = \{s\}$, by Σ_T as well as $\partial\Sigma_T$ we mean the single curve in Σ corresponding to s. The surface Σ_T can be embedded in Σ and that embedding induces a homomorphism $\mathrm{Mod}(\Sigma_T) \to \mathrm{Mod}(\Sigma)$ which makes the following diagram commute:

$$\begin{CD} A_T @>>> \mathrm{Mod}(\Sigma_T) \\ @VVV @VVV \\ A @>>> \mathrm{Mod}(\Sigma) \end{CD}$$

Note that $\partial\Sigma_T$ is a multicurve in Σ. By Remark 2.5, Δ_T^4 is sent to a multitwist around the boundary components of $\partial\Sigma_T$ in $\mathrm{Mod}(\Sigma)$. We summarize the above discussion in the following lemma.

Lemma 5.2. *Let A be a small type spherical Artin group and let $N \geq 2$. Then each of the generators $\{z_T\}_T$ of the* RAAG RA *is mapped to a multitwist in* $\mathrm{Mod}(\Sigma)$ *via* $\rho \circ \Phi_N$.

The following theorem of Koberda is our main tool for proving the generalized Tits conjecture for the spherical Artin groups.

Theorem 5.3 ([29])**.** *Let f_i be a finite collection of nontrivial powers of Dehn twists around simple closed curves so that the subgroup $\langle f_i, f_j\rangle$ is not cyclic for all i, j. Then there is an $M > 0$ so that the powers f_i^m generate a* RAAG *subgroup of* $\mathrm{Mod}(\Sigma)$ *for all $m \geq M$.*

The flag complex that is the nerve of the RAAG generated by those powers of Dehn twists has the collection of the curves as its vertex set, where a subcollection of curves spans a simplex if and only if they can be realized as pairwise disjoint curves. The analogous result does not hold for multitwists instead of Dehn twists about single curves. Indeed, it is easy to find surfaces and multicurves so that the RAAG generated by Dehn twists about the individual curves is $F_2 \times F_2$, and the words given by the multicurves are as in Example 4.3.

Coming back to the surface Σ from the Perron-Vannier representation of A, let $\mathcal{C}$ denote the collection of all the curves in the support of the multitwists $\{\rho\circ\Phi_N(z_T)\}_T$. By Theorem 5.3, high powers of Dehn twists about the curves in $\mathcal{C}$ generate a RAAG $\mathrm{RA}_{\mathcal{C}}$. Therefore, we have an induced homomorphism $\mathrm{RA} \to \mathrm{RA}_{\mathcal{C}} \subset \mathrm{Mod}(\Sigma)$, where each generator of RA goes to a product of commuting generators of $\mathrm{RA}_{\mathcal{C}}$. In order to prove that A satisfies the Generalized Tits Conjecture with N, it suffices to show that $\mathrm{RA} \to \mathrm{RA}_{\mathcal{C}}$ is injective. In the case of A of type A_n and D_n, we show it using Property PP and generalized Property PP from Section 4, respectively. To verify Property PP, we pick one of the boundary components $p(T)$ in $\partial\Sigma_T$ for each irreducible T such that the curves $p(T), p(T')$ intersect if and only if $[\Delta_T^2, \Delta_T^2] = 1$ (if and only if $T \subseteq T'$, $T' \subseteq T$, or $[T, T'] = 1$). For details on how we verify generalized Property PP, see subsection on type D_n.

We now rephrase Property PP in terms of curves on surfaces in the context of RAAG subgroups of mapping class groups.

Lemma 5.4. *Let $\hat{\mathcal{C}}$ a collection of multicurves in a surface Σ with and let $\mathcal{C}$ be the collection of all the connected component of elements of $\hat{\mathcal{C}}$. Suppose there exists a function $p : \hat{\mathcal{C}} \to \mathcal{C}$ such that*

- *$p(\alpha) \in \alpha$ and*
- *$p(\alpha), p(\beta)$ intersect if and only if some connected component of α intersect some connected component of β.*

Let $\{T_\alpha : \alpha \in \hat{\mathcal{C}}\}$ be a collection of multitwists about multicurves in $\hat{\mathcal{C}}$. Then the exists $M > 0$ such that $\langle T_\alpha^M : \alpha \in \hat{\mathcal{C}}\ \rangle$ is a RAAG.

Proof. By Theorem 5.3 the group generated by sufficiently large powers of Dehn twists around $\mathcal{C}$ is a RAAG. The assumption about existence of function p is just reformulation of Property PP from Definition 4.1. The conclusion follows from Proposition 4.2. □

5.1. **Intersections of boundary curves.** Let A be a small-type spherical Artin group, and let $A \to \mathrm{Mod}(\Sigma)$ be the Perron-Vannier representation. Suppose that T and T' are two irreducible spherical subsets of S, and let Σ_T and $\Sigma_{T'}$ be the associated subsurfaces of Σ. If Δ_T^2 and $\Delta_{T'}^2$ commute, then each of the boundary curves of Σ_T and $\Sigma_{T'}$ are necessarily disjoint. If they do not commute, then there are different possibilities for the possible intersections between the curves. We shall record a few lemmas that we will need later.

Lemma 5.5. *Let A be of type A_n, and suppose that T and T' are irreducible subsets such that Δ_T^2 and $\Delta_{T'}^2$ do not commute. If Σ_T has one boundary component up to homotopy (so $|T| = 1$ or $|T|$ is even), then the boundary curve of Σ_T has nontrivial intersection number with every boundary curve of $A_{T'}$.*

Proof. Note that $\partial\Sigma_T$ is fixed up to homotopy by the hyperelliptic involution. If $A_{T'}$ has two boundary components, then these are permuted up to homotopy by the hyperelliptic involution. Therefore, we only have to show that $\partial\Sigma_T$ has nontrivial intersection number with a single boundary component of $\partial\Sigma_{T'}$. Since the Perron-Vannier representation is injective in this case [33], this follows from the fact that $[\Delta_T^m, \Delta_{T'}^n] \neq 1$ in A_n.

□

Lemma 5.6. *Suppose $\Sigma_{T'}$ and Σ_T both have two boundary components, i.e. both $|T|$ and $|T'|$ are odd and $\neq 1$. Then*

- *Each boundary component of Σ_T intersects at least one boundary component of Σ_T', and vice versa.*
- *If $|T - T'|$ is odd, then each boundary component of Σ_T intersects each boundary component of $\Sigma_{T'}$.*

Proof. For the first statement, again by injectivity of the Perron-Vannier representation we must have that at least one component of $\partial\Sigma_T$ intersects a component of $\partial\Sigma_{T'}$. Since both components are permuted by the hyperelliptic involution, we get the other intersection.

We will prove the second statement in the Alternative proof of Proposition 5.9 below, once we have developed more notation.

□

There is a similar lemma for the images of curves in the Perron-Vannier representation $A \to \mathrm{Mod}(\Sigma)$ for Artin groups A of type D_n. Note that Σ has a unique boundary curve which is contained in the boundary of Σ_T for any irreducible subset $T \subset S$ of type D_m for $m < n$. We will call this the *outer component*, and the other boundary components *inner components*. We shall see that the Dehn twist around the outer component never factors into any of our calculations.

Lemma 5.7. *Let A be of type D_n, and suppose that T and T' are irreducible subsets such that Δ_T^2 and $\Delta_{T'}^2$ do not commute.*

(1) If A_T has type D_m and $m = 2k + 1$ for $k > 0$, then the inner boundary component of $\partial\Sigma_T$ intersects each curve in $\partial\Sigma_{T'}$ if $s, s' \notin T'$, and intersects one boundary curve of $\Sigma_{T'}$ if one of s or $s' \in T'$.

(2) If A_T has type D_m and $m = 2k$ for $k > 1$, then if $s, s' \notin T'$ each inner boundary component either intersect all curves in $\partial\Sigma_{T'}$ or there are two pairs of components that intersect. If s or $s' \in T'$, there is one non-central component of $\partial\Sigma_T$ which has nontrivial intersection with $\partial\Sigma_{T'}$. Furthermore, one component intersects all $\partial\Sigma_{T'}$ for T' that contain s, and one component intersects all $\partial\Sigma_{T'}$ for T' that contain s'.

(3) *If $s \in T$ and $s' \in T'$, then the components of $\partial\Sigma_T$ and $\partial\Sigma_{T'}$ which are not contained in $\partial\Sigma_{D_m}$ for any m always intersect.*

Proof. The proof of the first two items is similar to Lemma 5.5. The non-central component(s) of $\partial\Sigma$ is fixed by the hyperelliptic involution. If $\partial\Sigma_{T'}$ has two components then if $s, s' \notin T'$ then these are permuted by the hyperelliptic involution. If s or $s' \in T'$, then one of the curves of $\partial\Sigma_{T'}$ is contained in $\partial\Sigma_{D_m}$ for $m > n$, and hence misses the non-central component(s) of $\partial\Sigma_{D_n}$. The other curve therefore intersects $\partial\Sigma_{D_n}$ For the last statement of the second item, note that each of the boundary components of $\partial\Sigma_{D_n}$ is a boundary component of $\partial\Sigma_{A_n}$ and $\partial\Sigma_{A'_n}$. Therefore, they are disjoint from $\partial\Sigma_{A_m}$ and $\partial\Sigma_{A'_m}$ respectively for $m > n$.

For the last item, note that both these curves have nontrivial intersection number with γ_s and γ'_s respectively. The curves γ_s and $\gamma_{s'}$ and the outer boundary component are the boundary of a pair of pants. If our curves had trivial intersection number, then we could homotope one of them to have trivial intersection number with both γ_s and γ'_s. □

5.2. **Type A_n.** Suppose that A is a spherical Artin group of type A_n, i.e, . A is the braid group on $n+1$-strands. Then A is the mapping class group of the $n+1$-punctured disc. We give the punctures an arbitrary labeling $\{1, 2, \dots, n+1\}$. The squares of the standard generators of A correspond to the Dehn twists about the simple closed curves around two consecutive punctures $\{i, i+1\}$. Any irreducible subset T of S corresponds to a subset $I \subset \{1, \dots, n+1\}$ of consecutive numbers, where $|I| = |T| + 1$. The center of the pure braid group PB_T corresponds to a Dehn twist T_I around punctures in I, see Figure 7. Therefore, the Generalized Tits Conjecture for $N = 1$ asks if this collection of Dehn twists about these simple closed curves generates a RAAG subgroup of the braid group. For simplicity, we denote the generators z_T of RA by z_I.

Example 5.8. The braid groups on ≥ 8 strands do not satisfy the Generalized Tits Conjecture with $N = 1$. The map Φ_1 is not injective, as there are deeper relations between the Δ_T^2. See Figure 7. By the lantern relation (see e.g. [23, Prop 5.1]), the Dehn twist T_r about the red curve in Figure 7 satisfies

$$T_r = T_{3456}^{-1} T_{1234}^{-1} T_{12} T_{34} T_{56} T_{123456}.$$

Similarly, the Dehn twist T_b about the blue curve satisfies

$$T_b = T_{5678}^{-1} T_{3456}^{-1} T_{34} T_{56} T_{78} T_{345678}.$$

However, the elements $z_{3456}^{-1} z_{1234}^{-1} z_{12} z_{34} z_{56} z_{123456}$ and $z_{5678}^{-1} z_{3456}^{-1} z_{34} z_{56} z_{78} z_{345678}$ do not commute in RA. Indeed, it suffices to consider their images under the retraction $\mathrm{RA} \to \langle z_{123456}, z_{345678} \rangle \simeq F_2$

As we have noted, Theorem 5.3 implies that high powers of these elements generate a RAAG, i.e. we get the following

Proposition 5.9. *The Generalized Tits Conjecture holds for all spherical Artin groups of type A_n.*

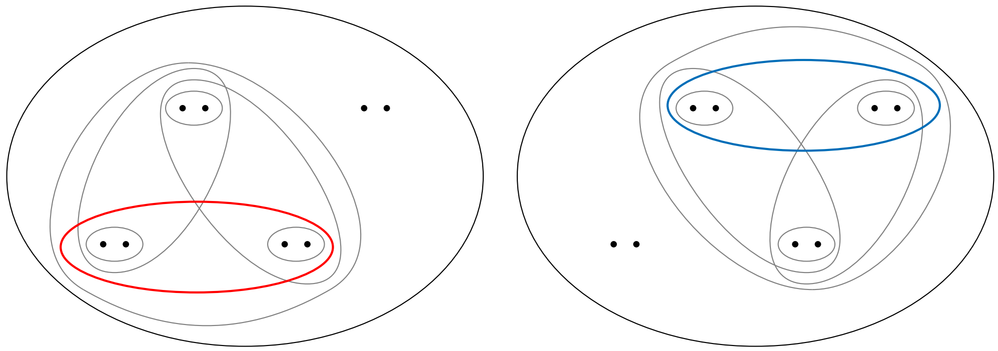

FIGURE 7. Dehn twists around the grey curves in this diagram are images of Δ_T^2 for certain braid subgroups of the braid group on 8 strands. The red and blue curves arise via the lantern relation, and obviously commute in the braid group. It is easy to check that the words in the lantern relation which produce these curves do not commute in the RAAG on the Δ_T^2, and hence the induced homomorphism from this RAAG to the braid group is not injective.

We shall also show how the conjecture follows from using the Perron-Vannier representation and property PP (this will serve as a warmup for the other cases).

Alternative proof of Proposition 5.9. By Lemma 5.4, it suffices to show that the multicurves that arise in the Perron-Vannier representation satisfy the condition in the statement of the lemma. Let $S = \{t_1, \dots t_n\}$ be the standard set of generators.

Note that if n is odd, then the $\Sigma - \{t_1, \dots t_3, \dots t_n\}$ has two connected components. Similarly, if n is even, then $\Sigma - \{t_1, \dots t_3, \dots t_{n-1}, k\}$ has two connected components, where k is an arc with both endpoints in $\partial\Sigma$ and which intersect only t_n among the curves in S. In either case, we pick a connected component, and denote it by Σ_+. Similarly, let $\Sigma_\#$ be the connected component that does not contain $\partial\Sigma$ of respectively $\Sigma - \bigcup\{t_2, t_4, \dots t_{n-1}, \ell, \ell'\}$ in case of odd n, and of $\Sigma - \bigcup\{t_2, t_4, \dots t_n, k\}$ in case of even n. Here, ℓ, ℓ' denote arcs with the endpoint in two connected components of $\partial\Sigma$ where ℓ intersects only t_1, and ℓ' intersects only t_n among curves in S. The arc k has both endpoint in $\partial\Sigma$ and intersects only t_1 among curves in S.

Let $T = \{t_i, \dots, t_j\}$. If $|T|$ is even, then Σ_T has a unique boundary component, denoted by $t_{i:j}$. Otherwise, if $|T|$ is odd, then we denote the two boundary component of Σ_T by $\{t_{i:j}, t'_{i:j}\}$.

If $T = \{t_i, \dots, t_j\}$ and $|T|$ is even, we denote the unique boundary curve of Σ_T by $t_{i:j}$. When $|T|$ is odd, then we denote the boundary curves of Σ_T by $\{t_{i:j}, t'_{i:j}\}$. If additionally, i is odd, then again exactly one of these curves is contained in Σ_+.

Similarly, if i is even, then we assume that $t_{i:j}$ is contained in $\Sigma_\#$. For each irreducible set $T = \{t_i, \dots, t_j\}$, we set $p(T) = t_{i:j}$, if $i > 1$. See Figure 8.

We claim that our choice function p satisfies Lemma 5.4. By the first part of Lemma 5.5, we only need to worry about irreducible subsets of odd cardinality.

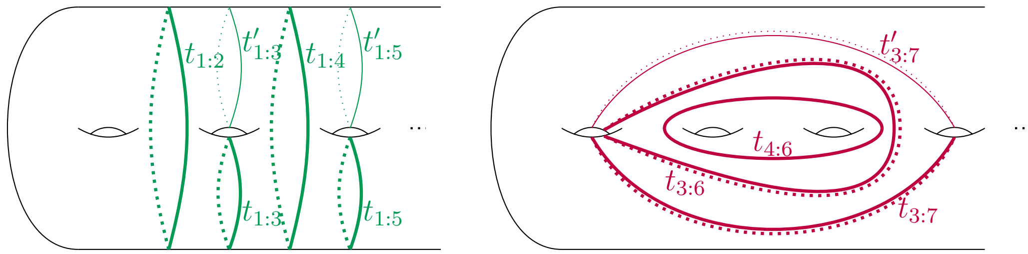

FIGURE 8. Curves $s_2, s_3, s'_3, \dots s_{2k}, s_{2k+1}, s'_{2k+1} \dots$ and curves $t_{i:j}, t'_{i:j}$ in the surface Σ of type A_n. The thick curves are an example of a choice of the representatives for property PP.

Suppose we have two subsets $T = \{t_i, \dots, t_j\}$ and $T' = \{t_{i'}, \dots, t_{j'}\}$ of odd cardinality which do not commute, i.e. $[z_T, z_{T'}] \neq 1$ in RA.

We claim that if i is odd and i' is even or vice versa, then each curve in $\partial\Sigma_T$ intersects each curve in $\partial\Sigma_{T'}$, (this is the second part of Lemma 5.7). To see this, note that it suffices to assume that $T = \{1, 2, \dots, j\}$ and $T' = \{k, \dots, l\}$. Since j is odd, $\partial\Sigma_T$ has two components, and both components of $\partial\Sigma_{T'}$ intersect the curve t_{k-1} exactly once (and miss all other t_i for $i \leq j$). Therefore, each component of $\partial\Sigma_{T'}$ intersects both components of $\Sigma_T - \{t_1, t_3, \dots, t_{k-1}, \dots t_j\}$, and therefore must intersect both components of $\partial\Sigma_T$.

If i and i' are both odd, then $p(T)$ cannot intersect the curve of $\partial\Sigma_{T'}$ which is contained in $\Sigma - \Sigma_+$, and by the second part of Lemma 5.5, $p(T)$ intersects $p(T')$. Similarly, if i and i' are both even, then $p(T)$ and $p(T')$ intersect. □

We do not know if $N = 2$ suffices for the braid group. In fact, Runnels and Seo have independently and recently shown an effective version of Koberda's result [35], [37]. For the collections of Dehn twists that arise in the braid group case, it turns out that $N = 17$ works, and we suspect there is a not very large N that works for all Artin groups.

5.3. **Type D_n.** We know consider an Artin group A of type D_n with the standard generating set $S = \{s, s', t_1, \dots t_n\}$, as in Figure 5.

Theorem 5.10. *Generalized Tits Conjecture holds for all spherical Artin groups of type D_n.*

Proof. There are four families of irreducible subsets of S with at least two elements.

(1) If $T = \{s, s', t_1, \dots, t_j\}$ where $j \leq n-2$, then A_T has type D_{j+2}.
(2) If $T = \{t_i, \dots, t_j\}$ where $1 \leq i < j \leq n-2$, then A_T has type A_{j-i+1}.
(3) If $T = \{s, t_1, \dots, t_j\}$ where $1 \leq j \leq n-2$, then A_T has type A_{j+1}.
(4) If $T = \{s', t_1, \dots, t_j\}$ where $1 \leq j \leq n-2$, then A_T has type A_{j+1}.

We consider the Perron-Vannier representation $\rho : A \to \mathrm{Mod}(\Sigma)$ (see Section 2.6 and Figure 5). By Lemma 5.2 the generator z_T of RA is sent to a multitwist about the boundary of a subsurface Σ_T.

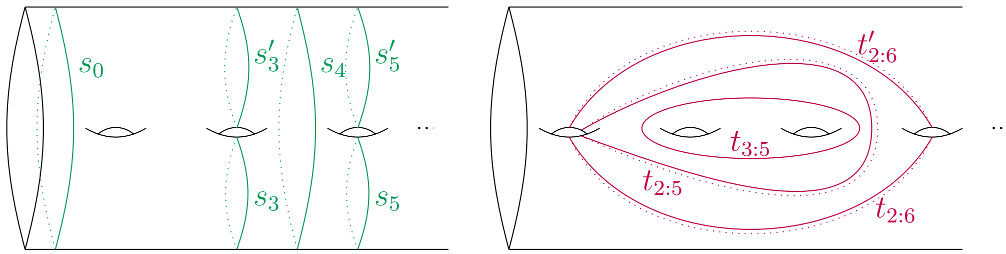

FIGURE 9. Curves $s_0, s_1, s_1', \ldots s_{2k}, s_{2k+1}, s_{2k+1}' \ldots$ and curves $t_{i:j}, t_{i:j}'$ in the surface Σ corresponding to the Artin group of type A_n or D_n. In the case of D_n, the boundary of Σ includes the gray curve.

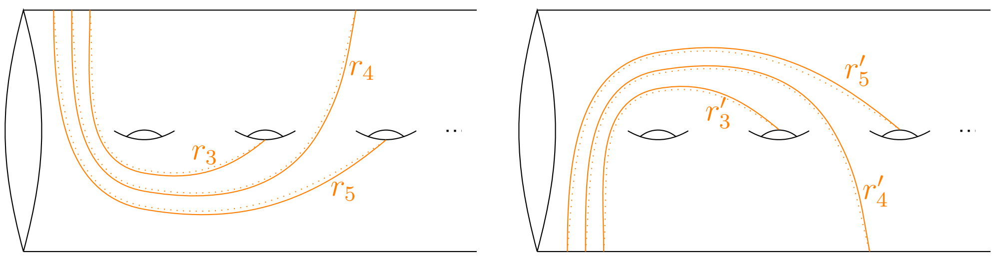

FIGURE 10. Curves $r_2, r_3, \ldots$ and $r_2', r_3', \ldots$.

We pick a connected component of, respectively, Σ_+ of $\Sigma - \{t_2, t_4, \ldots, t_{n-2}, k\}$ when n is even, and of $\Sigma - \{t_2, t_4, \ldots, t_{n-3}, k, \ell\}$ when n is odd. In both cases k denotes an arc with both endpoints in the outer boundary component of Σ that intersects t_1 and no other curves in S. When n is odd, then ℓ denotes an arc with both endpoints in the unique inner boundary component of Σ, that intersects t_{n-2} and no other curves in S. We denote the chosen component by Σ_+. We can also pick a connected component $\Sigma_\#$ of, respectively, $\Sigma - \{t_1, t_3, \ldots, t_{n-3}, k_1, k_2, k_3\}$ when n is even, and of $\Sigma - \{t_1, t_3, \ldots, t_{n-2}, k_1, k_2\}$ when n is odd. By k_1, k_2, k_3 we denote arc with endpoints in distinct connected components of $\partial\Sigma$ that intersect only respectively s, s', t_{n-2} (the last one only when n is even) among curves of S.

Let us now analyze what those multicurves are for the four families of irreducible special subgroups. Let s_0 denote the outer boundary curve of $\partial\Sigma$.

For each irreducible set $T = \{s, s', t_1, \ldots, t_j\}$, the multicurve $\partial\Sigma_T$ is of the form $\{s_0, s_{j+1}, s_{j+1}'\}$ when $|T|$ is even, and of the form $\{s_0, s_j\}$ when $|T|$ is odd. We also set $s_1 = s$ and $s_1' = s'$. See the left side of Figure 9. Without loss of generality, we can assume that all the curves s_j are contained in Σ_+.

Now consider an irreducible subset $T = \{t_i, \ldots, t_j\}$ where $1 \le i < j \le n-2$. If $|T|$ is even, let $t_{i:j}$ be the unique boundary curve of the subsurface Σ_T. If $|T|$ is odd, let $t_{i:j}, t_{i:j}'$ be the two boundary curves of the subsurface containing $t_i, \ldots, t_j$. See the right side of Figure 9. If i is even, we assume that $t_{i:j}$ lies in Σ_+, and if i is odd, we assume that $t_{i:j}$ is contained in $\Sigma_\#$.

Finally consider the irreducible subset $T = \{s, t_1, \dots t_j\}$. When j is odd, we denote the unique boundary component of Σ_T by r_{j+1}. When j is even, the boundary of the subsurface $\partial\Sigma_T$ has two connected components, one is s'_{j+1}, and the other is denoted by r_{j+1}. See Figure 10. We define curves $r'_2, r'_3, \dots$ analogously, so that $\partial\Sigma_T = \{r'_{j+1}\}$ or $\{s_{j+1}, r'_{j+1}\}$ for $T = \{s', t_1, \dots t_j\}$.

Let $\mathcal{C} = \{s_j, s'_j\} \cup \{t_j\} \cup \{t_{i:j}, t'_{i,j}\} \cup \{r_j, r'_j\}$ be the collection of all the curves that arise as boundary curves of subsurfaces Σ_T for irreducible subsets $T \subset S$. By Theorem 5.3 the subgroup generated by sufficiently large powers of Dehn twists around curves in $\mathcal{C}$ is a RAAG, which we denote by $\mathrm{RA}_{\mathcal{C}}$. Note that the words w_σ corresponding to centers of irreducible spherical Artin subgroups, regarded as elements of $\mathrm{RA}_{\mathcal{C}}$ do not satisfy property PP, see Figure 11.

We now show that generalized property PP implies the conjecture for Artin groups of type D_n. Let $\mathcal{C}_1 = \mathcal{C} - \{r'_k\}_{k\geq 2}$, $\mathcal{C}_2 = \mathcal{C} - \{r_k\}_{k\geq 2}$, and $\mathcal{C}_0 = \mathcal{C}_1 \cap \mathcal{C}_2 = \mathcal{C} - \{r_k, r'_k\}_{k\geq 2}$.

Let L be the flag complex on the vertex set $\mathcal{C}$ defining $\mathrm{RA}_{\mathcal{C}}$, and let L_0, L_1, L_2 be the full subcomplexes of L on the sets $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$ respectively. Since the curves r_i and r'_j intersect for every $i, j \geq 2$, the complex L decomposes as $L_1 \cup_{L_0} L_2$. Therefore we have a splitting of the $\mathrm{RA}_{\mathcal{C}}$ as $A_{L_1} *_{A_{L_0}} A_{L_2}$ where A_{L_i} is a RAAG with nerve L_i. Also, the RAAG RA associated with A splits as $\mathrm{RA} = \mathrm{RA}_{L_1} *_{\mathrm{RA}_{L_0}} \mathrm{RA}_{L_2}$ where

- RA_{L_1} omits all the generators corresponding to centers of the irreducible special subgroups from family (4), i.e. A_T with $T = \{s', t_1, \dots, t_k\}$ and $k \geq 1$,
- RA_{L_2} omits generators corresponding to centers of the irreducible special subgroups from family (3), i.e. A_T with $T = \{s, t_1, \dots, t_k\}$ and $k \geq 1$, and
- RA_{L_0} omits all the generators corresponding to centers of the irreducible special subgroups from families (3) and (4).

We claim that each L_i satisfies Property PP avoiding L_0. We only verify it for L_1, the proof for L_2 is analogous. For each irreducible special subgroup A_T (which corresponds to a simplex in L spanned by the boundary curves of its corresponding subsurface Σ_T) from families (1), (2) and (3), we need to make a choice of a boundary curve of Σ_T. For a subgroup from family (3) of type A_k we choose the curve r_k. Note that these subgroups are exactly the ones corresponding to simplices σ with nontrivial w_σ such that $\sigma \notin L_0$. We thus see that $p(\sigma) \notin L_0$ for such simplices σ. Also, w_τ is trivial for any other simplex τ containing $p(\sigma)$, because Σ_T is the unique subsurface corresponding to an irreducible special subgroup whose boundary contains r_k.

By Lemma 5.7, for each irreducible special subgroup A_T of type D_k (i.e. from family (1)), there is a unique boundary curve of Σ_T which intersects $\partial\Sigma_U$ for U in family (3). Indeed, it is curve s_{k-1}, see Figure 9. Finally, for irreducible subset in family (2), i.e. $T = \{t_i, \dots, t_j\}$, we pick the curve $t_{i:j}$, i.e. a curve in $\partial\Sigma_T$ that intersects r_k for every $i \leq k \leq j$, which is unique unless i, j are both odd (in which case either choice works).

We now claim that these choices satisfy the Property PP relative to L_0. For each simplex in $L_1 - L_0$, we have chosen curves not contained in L_0. Also, for each edge in $L_1 - L_0$, we have chosen a simple closed curve that is not a boundary curve of

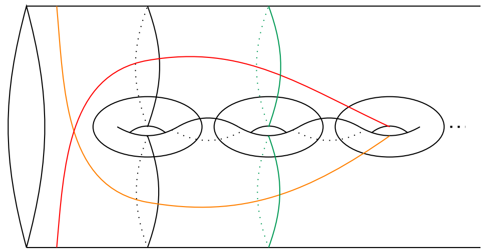

Figure 11. These curves show the failure of property PP for the images of centers of D_n. The red and orange curves are boundary components of subsurfaces Σ_{A_5} and $\Sigma_{A'_5}$. The green curves are the boundary components of the subsurface Σ_{D_4}. Since the other boundary component misses γ and γ' respectively, we are forced to choose these. However, this implies that we cannot correctly choose a boundary component of $\partial\Sigma_{D_4}$.

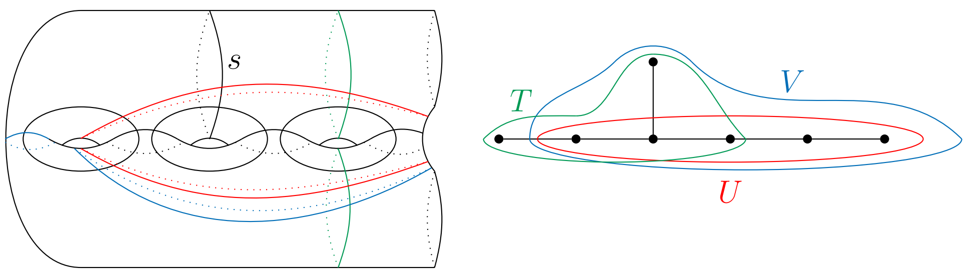

Figure 12. High powers of Dehn twists around the multicurves above do not generate the obvious RAAG subgroup. The figure on the left shows the corresponding centers of irreducible Artin subgroups. To see this, start with the Dehn twist around s, then conjugate by the red multitwist, then conjugate by the green multitwist. This element commutes with the blue multitwist.

any subsurface corresponding to a different irreducible Artin subgroup. Therefore, condition (2) is satisfied. By Theorem 4.10, the conclusion follows.

□

5.4. **Types E_6, E_7 and E_8.** We now show that any homomorphism $A_{E_7} \to \mathrm{Mod}(\Sigma)$ which sends generators to Dehn twists has nontrivial kernel which intersects the image of Φ_m. Wajnryb has previously shown that there is no injective homomorphism from the Artin groups of type E_6, E_7, E_8 to any mapping class group which maps generators to Dehn twists [40]. It is still open whether these Artin groups admit other faithful representations into mapping class groups.

The Dehn twists in Figure 12 show an element in the kernel of the Perron-Vannier representation $\rho : A \to \mathrm{Mod}(\Sigma)$ where A is of type E_7. A specific element in the

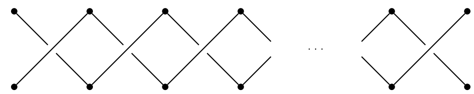

FIGURE 13. The graph $\Gamma(m)$ on $2(m-1)$ vertices.

kernel of ρ is the commutator

$$[\Delta_T^{2m}\Delta_U^{2m}s^{2m}\Delta_U^{-2m}\Delta_T^{-2m}, \Delta_V^{2m}]$$

where $T = \{t_1, t_2, t_3, t_4, s\}$, $U = \{t_2, t_3, t_4, t_5, t_6\}$ and $V = \{t_2, t_3, t_4, t_5, t_6, s\}$. The nerve of the subgroup of RA on the generators corresponding to $T, \{s\}, V, U$ is a path on 4 vertices. It is easy to verify that the corresponding commutator in RA is nontrivial. The commutator above is also nontrivial in A. This can be verified by computing the Deligne's normal form for its positive representatives. We have performed the computation in GAP 3 with the package CHEVIE [39] [24]. Note that the above does not imply that the Generalized Tits Conjecture does not hold for Artin group of type E_7 (or E_8). It only shows that our strategy, using the Perron-Vannier representation, does not work in that case.

Remark 5.11. The multicurves produced by the Perron-Vannier representation of the Artin group of type E_6 do not satisfy generalized property PP, but we cannot find a word in RA as above in the kernel of the representation.

6. Generalized Tits Conjecture for other spherical Artin groups

In this section, we show the Generalized Tits Conjecture holds for all spherical Artin groups which are not small type. The main tool is a folding trick due to Crisp (see [11] or [12, Section 6]), which embeds any of these Artin groups into small type spherical Artin groups.

6.1. **Folding homomorphisms.** In this section it is more convenient to work with Coxeter diagrams rather than nerves. So, suppose that A_Γ is an Artin group with a connected Coxeter diagram Γ that has no ∞-labels. Crisp and Paris define a *folding homomorphism* $\Psi : A_\Gamma \to A_{\widetilde{\Gamma}}$ where $A_{\widetilde{\Gamma}}$ is small-type. Here is the construction: let $N = \text{lcm}\{m_{st} - 1 | s \neq t \in S\}$. For each vertex $s \in S$, let $I(s)$ be a set with N elements. For $m \geq 3$, let $\Gamma(m)$ be the Coxeter diagram for $A_{m-1} \times A_{m-1}$, i.e. $\Gamma(m)$ is a disjoint union of two copies of the Coxeter graph of type A_{m-1}. See Figure 13.

Let $\widetilde{\Gamma}$ be a Coxeter diagram so that:

- The vertex set of $\widetilde{\Gamma}$ is the disjoint union of the sets $I(s)$.
- If there is no edge in Γ between s and t, there is no edge between vertices in $I(s)$ and the vertices of $I(t)$. In particular, there are no edges between any two vertices in $I(s)$.
- If $m_{st} \geq 3$, then the subgraph of $\widetilde{\Gamma}$ spanned by $I(s) \cup I(t)$ is isomorphic to $\frac{N}{m_{st}-1}$ copies of $\Gamma(m_{st})$.

It is easy to see that such a $\widetilde{\Gamma}$ can always be constructed, though it will not be unique. There is a map of graphs $\widetilde{\Gamma} \to \Gamma$ sending every vertex in $I(s)$ to s. Crisp and Paris show that the map $\Psi : A_\Gamma \to A_{\widetilde{\Gamma}}$ which sends a generator s to the product $\prod_{s_i \in I(s)} s_i$ extends to a homomorphism [12, Prop 13]. Crisp showed that this homomorphism is injective when restricted to the Artin monoid [11], and it follows that it is injective when A_Γ is spherical. It is still open whether Ψ is injective in general.

We now verify that Ψ has some additional properties. For each $T \subset S$, let $\widetilde{T}$ denote the preimage of T under $\widetilde{\Gamma} \to \Gamma$, i.e. $\widetilde{T}$ contains the vertices in $I(s)$ for each $s \in T$.

Lemma 6.1. *Let A_Γ be an Artin group with connected Coxeter diagram Γ and no ∞-labels. Let $\widetilde{\Gamma}$ as above, and let $\Psi : A_\Gamma \to A_{\widetilde{\Gamma}}$ be the above homomorphism. Then Ψ satisfies the following:*

- *For every spherical Artin subgroup $A_T \subset A_\Gamma$, $A_{\widetilde{T}}$ is spherical.*
- *If A_T is an irreducible spherical subgroup of A_Γ , then*

$$\Psi(\Delta_T^2) = \prod_{T_i \subset \widetilde{T}} \Delta_{T_i}^2$$

where the $\{T_i\}$ are the irreducible components of $\widetilde{T}$.

Proof. If A_T is small type, then the Coxeter diagram of $A_{\widetilde{T}}$ is a disjoint union of finitely many copies of Coxeter diagram for A_T, and hence $A_{\widetilde{T}}$ is a direct product of copies of A_T. In this case, both statements follow easily. Suppose σ is not small type. If A_T is irreducible, the Coxeter subdiagram Γ_T for A_T is a line, with exactly one edge e labeled with number greater than 3. Let $\widetilde{\Gamma}^i_T$ be a connected component of the Coxeter diagram for $\widetilde{\Gamma}_T$. Then $\widetilde{\Gamma}_{\widetilde{T}^i}$ is obtained from Γ_T by replacing e with a copy of $\Gamma(m)$ and then attaching copies of $T - e$. In each case we check directly that this produces a spherical Artin group.

For the second statement, by the definition of Ψ, a Coxeter element for W_Γ maps to a Coxeter element for $W_{\widetilde{\Gamma}}$. The Coxeter number is preserved in the above identifications (see Lemma 6.2 below), in that if $\widetilde{\Gamma}_{\widetilde{T}^i}$ is connected component of $\widetilde{\Gamma}_T$, then the Coxeter number of A_T is the same as the number for $A_{\widetilde{T}^i}$). This immediately implies the second statement by Lemma 2.1. □

We now record the following specific cases as a lemma (these were previously tabulated in [11]).

Lemma 6.2. *Let A_Γ be a spherical Artin group as above, and let $A_{\widetilde{\Gamma}_c}$ be an Artin group in the image of Ψ where $\widetilde{\Gamma}_c$ is a connected component of $\widetilde{\Gamma}$. Let h be the Coxeter number of the Coxeter system of A_Γ. Then the Coxeter number of the Coxeter system of $A_{\widetilde{\Gamma}_c}$ is h. Moreover:*

1. *If $A_\Gamma = B_n$, then $A_{\widetilde{\Gamma}_c}$ has type D_{n+1} or A_{2n-1}, and $h = 2n$.*
2. *If $A_\Gamma = I_2(p)$, then $A_{\widetilde{\Gamma}_c}$ has type A_{p-1} and $h = p$.*

(3) *If* $A_\Gamma = H_3$*, then* $A_{\widetilde{\Gamma}_c}$ *has type* D_6 *and* $h = 10$*.*
(4) *If* $A_\Gamma = H_4$*, then* $A_{\widetilde{\Gamma}_c}$ *has type* E_8 *and* $h = 30$*.*
(5) *If* $A_\Gamma = F_4$*, then* $A_{\widetilde{\Gamma}_c}$ *has type* E_6 *and* $h = 12$*.*

Let A_Γ be an Artin group with no ∞-labels and connected Γ. The RAAG with presentation 1.1 associated with A_Γ is denoted by RA_Γ, and the RAAG associated with $A_{\widetilde{\Gamma}}$ is denoted by $\mathrm{RA}_{\widetilde{\Gamma}}$. We denote the homomorphism $\mathrm{RA}_{\widetilde{\Gamma}} \to A_{\widetilde{\Gamma}}$ from the statement of the Generalized Tits Conjecture by $\widetilde{\Phi}_N$.

We consider the composition $\Psi \circ \Phi_N : RA_\Gamma \to A_{\widetilde{\Gamma}}$. Our aim is to verify the Generalized Tits Conjecture for A_Γ, i.e. to show that Φ_N is injective for some N. Let $F : \mathrm{RA}_\Gamma \to \mathrm{RA}_{\widetilde{\Gamma}}$. If T is an irreducible, special subset, then $\widetilde{T}$ will generally not be irreducible. For a generator z_T of RA_Γ its image $F(z_T)$ is the product $\prod_{\widetilde{T}_i \subseteq \widetilde{T}} z_{\widetilde{T}_i}$ of generators in $\mathrm{RA}_{\widetilde{\Gamma}}$ corresponding to irreducible subsets $\widetilde{T}_i \subseteq \widetilde{T}$. For each N, we have the following equality $\widetilde{\Phi}_N \circ F = \Psi \circ \Phi_N$. Crisp and Paris show the following lemma for the RAAG generated by $\{s^2 : s \in S\}$. The same proof works for our RAAGs. For the benefit of the reader, we provide the proof.

Lemma 6.3. *The homomorphism* F *is injective. Therefore, if* $\widetilde{\Phi}_N$ *is injective, then* Φ_N *is injective.*

Proof. Note that if $T, U \subseteq S$ are distinct irreducible subsets, then $z_{\widetilde{T}^i} \neq z_{\widetilde{U}^j}$ for all i and j. Furthermore, if z_T and z_U do not commute in RA_Γ, then for every $z_{\widetilde{T}^i}$ there exists a $z_{\widetilde{U}^i}$ that does not commute with $s_{\widetilde{T}^i}$ in $\mathrm{RA}_{\widetilde{\Gamma}}$. To see this, note that if two spherical subsets T and U do not commute, there is $t \in T - U$ and $u \in U - T$ with $m_{tu} \neq 2$. Any component $\widetilde{T}^i$ of $\widetilde{T}$ contains a vertex $\widetilde{t}$ of $I(t)$. There exists a vertex $\widetilde{u} \in I(u)$ with $m_{\widetilde{t}\widetilde{u}} \neq 2$. Therefore, the component of $\widetilde{U}$ containing $\widetilde{u}$ does not commute with $\widetilde{T}_i$. By the normal form for RAAGs this implies F is injective, as it takes a reduced word in RA_Γ to a reduced word in $\mathrm{RA}_{\widetilde{\Gamma}}$. □

6.2. **Generalized Tits Conjecture for** $B_n, F_4, H_3, H_4, I_2(p)$**.** We now finish the remaining spherical cases. By Lemma 6.2 Artin groups of types B_n, H_3, and $I_2(p)$ embed via the folding homomorphisms in Artin groups of types A_m and D_m, for which we already know that the conjecture holds by Proposition 5.9 and Theorem 5.10. By Lemma 6.1 and Lemma 6.3 we get the following.

Corollary 6.4. *The Generalized Tits Conjecture holds for spherical Artin group of type* B_n*,* H_3*, and* $I_2(p)$*.*

In the case of F_4 and H_4, we do not know the conjecture for E_6 and E_8 respectively. However, in each of those cases, we can still show that the generalized Tits conjecture holds by considering

$$\mathrm{RA}_\Gamma \to A_\Gamma \hookrightarrow A_{\widetilde{\Gamma}} \to \prod_{\Gamma^c \in \Gamma} \mathrm{Mod}(\Sigma_{\Gamma^c})$$

where A_Γ is of type F_4 (respectively H_4), RA_Γ its associated RAAG, and each component Γ^c of $\widetilde{\Gamma}$ is of type E_6 (respectively E_8) with its Perron-Vannier representation

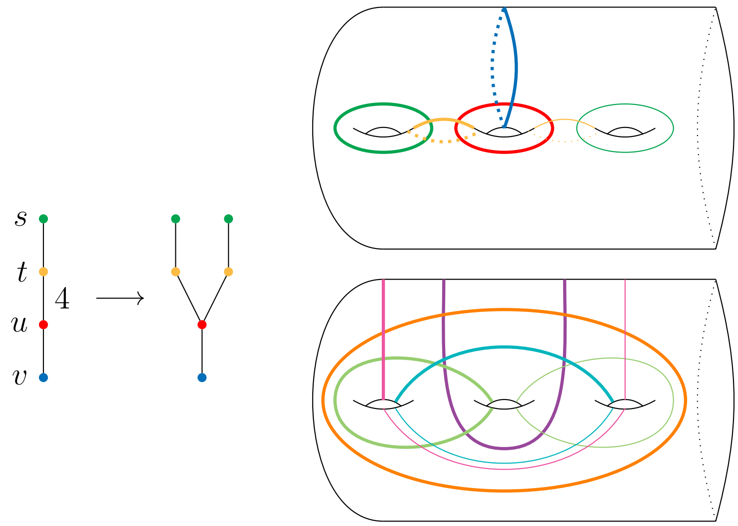

FIGURE 14. The restriction of the folding homomorphism for A_Γ of type F_4 to one connected component. The generators of A_Γ are mapped to the Dehn twists and multitwists in the left surface. The right surface has all the multicurves that arise as subsurfaces corresponding to irreducible special subgroups of A_Γ. Thick curves represent an example of a choice of curves satisfying Property PP.

in a mapping class group (see Section 2.6). In both cases, we will only concentrate on one component of $\widetilde{\Gamma}$. An identical argument will work for all components, and so we get a faithful representation of RA_Γ into a direct product of RAAG's.

Theorem 6.5. *The Generalized Tits Conjecture holds for the spherical Artin group of type F_4.*

Proof. Let $S = \{s, t, u, v\}$ be the standard generators of A_Γ of type F_4 where $m_{st} = m_{uv} = 3$, $m_{tu} = 4$. Consider the homomorphism $\Psi_i : A_\Gamma \to A_{\widetilde{\Gamma}_i}$, where $A_{\widetilde{\Gamma}}$ has type E_8, and where Ψ_i is the composition of the folding homomorphism Ψ with the projection $A_{\widetilde{\Gamma}} \to A_{\widetilde{\Gamma}_i}$ where $\widetilde{\Gamma}_i$ is a connected component of $\widetilde{\Gamma}$. See Figure 15. We also consider the Perron-Vannier representation $\rho : A_{\widetilde{\Gamma}_i} \to \mathrm{Mod}(\Sigma)$, as discussed in Section 2.6. The images of the elements Δ_T^4 for irreducible subsets $T \subseteq S$ are powers of Dehn twists around curves in Σ, pictured in Figure 14. Let $\mathcal{C}$ denote the collection of all these curves. Again, by Theorem 5.3 sufficiently large powers k of Dehn twists around single curves in $\mathcal{C}$ generate a RAAG. By Lemma 5.4 to show that A satisfies the Generalized Tits Conjecture, we need to make a choice $p(T)$ of a curve in $\partial\Sigma_T$ for each of the irreducible subsets T of S. For $T = \{u\}$ or $\{v\}$, the choice is unique. It remains to make choices for $\{s\}, \{t\}, \{s, t\}, \{t, u\}, \{s, t, u\}, \{t, u, v\}$. There is a unique curve in $\Sigma_{t,u}$ that intersects the curve $p(\{v\})$, we choose that curve for $p(\{t, u\})$. Note that that curve intersects both boundary components of $\Sigma_{\{s\}}$, both boundary components of $\Sigma_{\{s,t\}}$ and the unique boundary component of $\Sigma_{\{u,v\}}$. For

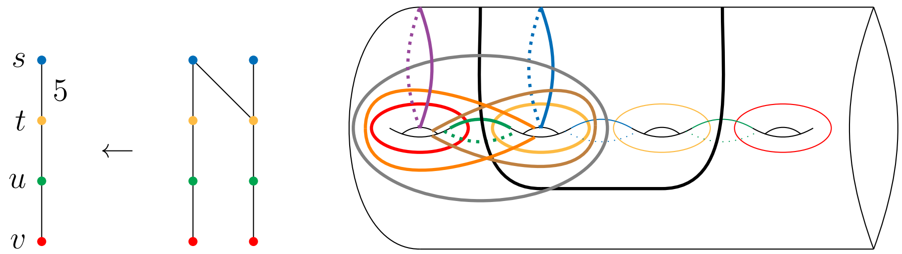

Figure 15. The restriction of folding homomorphism for A_Γ of type E_8 to one connected component. The thick curves are the ones we choose for the standard generators in verifying Property PP. The gray and purple curves are the curves we choose for $\{t,u,v\}$ and $\{s,t,u\}$ respectively. The black, brown, and orange curves are the unique curves for the irreducible rank 2 subsets.

each choice of a curve in $\partial\Sigma_{\{s\}}$, there is a unique intersecting curve in $\partial\Sigma_{\{t\}}$, and there is a unique curve in $\partial\Sigma_{\{s,t\}}$ that is the boundary of a subsurface containing those two curves. We make such a consistent choice of $p(\{s\}), p(\{t\}), p(\{s,t\})$. Then there is a unique choice of $p(\{t,u,v\})$ that intersects $p(\{s\})$ and both components of $\Sigma_{\{s,t,u\}}$. Finally, any choice of $p(\{s,t,u\})$ works. □

Theorem 6.6. *The Generalized Tits Conjecture holds for the spherical Artin group of type H_4.*

Proof. Let $S = \{s,t,u,v\}$ be the standard generators of A_Γ of type H_4 where $m_{st} = 5$ and $m_{tu} = m_{uv} = 3$. Consider the restriction of a folding homomorphism to a single component of Coxeter diagram. We get a homomorphism $\Phi : A_{H_4} \to A_{E_8}$ as pictured in the left of Figure 15. We also consider the Perron-Vannier representation $A_{E_6} \to \mathrm{Mod}(\Sigma)$, as discussed in Section 2.6. The images of the squares of the original generators, and of the elements Δ_T^2 for other irreducible subsets $T \subseteq S$, are Dehn twists around curves in two copies of Σ in Figure 14. Let $\mathcal{C}$ denote the collection of all these curves. As in previously considered cases, by Theorem 5.3 sufficiently large powers k of Dehn twists around single curves in $\mathcal{C}$ generate a RAAG. We need to verify that Property PP is satisfied for $\{\Phi(\Delta_T^{2k})\}$. Specifically, we need to make choices for the following irreducible subsets of S: $\{s\},\{t\},\{s,t\},\{t,u\},\{s,t,u\},\{t,u,v\}$. There is a unique curve in $\Sigma_{t,u}$ that intersects the curve corresponding to v, we choose that curve for $p(\{t,u\})$. Note that that curve intersects both boundary components of $\Sigma_{\{s\}}$, both boundary components of $\Sigma_{\{s,t\}}$ and the unique boundary component of $\Sigma_{\{u,v\}}$. For each choice of a curve in $\partial\Sigma_{\{s\}}$, there is a unique intersecting curve in $\partial\Sigma_{\{t\}}$, and there is a unique curve in $\partial\Sigma_{\{s,t\}}$ that is the composition of those two curves. We make such a consistent choice of $p(\{s\}), p(\{t\}), p(\{s,t\})$. That also forces a choice of $p(\{t,u,v\})$. Finally, any choice of $p(\{s,t,u\})$ works. □

7. Applications

If the Generalized Tits Conjecture holds in full generality, one immediate application is that the subgroups of Artin groups are as complicated as subgroups of right-angled Artin groups. The latter are currently more well understood. In this section, we give a few applications of this idea.

7.1. **Incoherence.** Recall that a group G is *coherent* if every finitely generated subgroup of G is finitely presented. Droms had showed that the right-angled Artin group RA_L was coherent if and only if L was a chordal graph [21]. Gordon showed that if the Artin group of type H_3 was incoherent, there was a similar classification of coherent Artin groups. Wise answered this in the affirmative in 2013.

Theorem 7.1 ([41])**.** *The Artin group of type H_3 is incoherent.*

Since A of type H_3 satisfies the generalized Tits conjecture, we can give an alternative proof of Wise's theorem. In this case, the nerve $L_\oslash$ of the RAAG subgroup RA in A is the cone on a pentagon. The RAAG based on a pentagon is well known to be incoherent, for example the Bestvina-Brady subgroup of the RAAG (the kernel of the map to $\mathbb{Z}$ which sends every generator to 1) is finitely generated and not finitely presented.

7.2. **Hyperbolic surface subgroups.** In [26], Gordon, Long, and Reid studied which Coxeter groups and Artin groups contained hyperbolic surface subgroups. They showed that all finite type Artin groups except types $A_1, I_2(m)$, and H_3 contained these subgroups. The first two classes do not contain such subgroups (More generally, any Artin group where the nerve L is a tree does not contain a hyperbolic surface subgroup), and type H_3 was left as an open question. It follows from the Generalized Tits Conjecture that the answer to their question is yes.

Theorem 7.2. *Every Artin group such that there exists $s, t, u \in S$ with $m_{st}, m_{tu}, m_{us} < \infty$ and at most one of them equals 2, contains a surface subgroup.*

Proof. The subgroup $A_{\{s,t,u\}}$ satisfies the Generalized Tits Conjecture and the nerve of $\mathrm{RA}_{\{s,t,u\}}$ is a pentagon, hexagon or a cone on a pentagon. The RAAG based on a pentagon or a hexagon is commensurable to a right-angled Coxeter group which has a pentagon or a hexagon as a full subcomplex [15]. It follows that the Coxeter group contains a hyperbolic surface subgroup, hence so does the RAAG based on a pentagon, and hence so does the Artin group. □

Corollary 7.3. *The Artin group of type H_3 contains a hyperbolic surface subgroup.*

In [26], they also considered Artin groups of Euclidean type, and asked specifically if the Artin group of type $\tilde{B}_2$ or type $\tilde{G}_2$ contain a hyperbolic surface subgroup. The nerves of these Artin groups are triangles with labels $(2, 3, 6)$ and $(2, 4, 4)$ respectively. In unpublished work, Sang-hyun Kim can show the existence of such a subgroup in the $(2, 4, 4)$ case, but we believe the $(2, 3, 6)$-case was still open. In either case, these groups are locally reducible. Therefore, the answer to this question is also yes, and follows as above from the Generalized Tits Conjecture.

Corollary 7.4. *The Artin groups of types $\tilde{B}_2$ and $\tilde{G}_2$ contain hyperbolic surface subgroups.*

At this point, we do not have a complete characterization of which Artin groups contain hyperbolic surface subgroups. This is still open in the right-angled case. On the other hand, these should be easier to construct outside of the right-angled case. For example, if the nerve L is a 4-cycle, then the RAAG is $F_2 \times F_2$, which does not contain such a subgroup. On the other hand, if any edge has a label > 2 then the Artin group is locally reducible and the associated RAAG subgroup has nerve a pentagon.

7.3. **Subgroups of type F_n and not F_{n+1}.** Spherical Artin groups are generally too high-dimensional for coherence to be an interesting question. A more interesting question is when a spherical Artin group contains a subgroup which is type F_n but not type F_{n+1} where $n+2$ is the cohomological dimension (recall a group is type F_n is it admits a classifying space with finite n-skeleton). For example, the type H_3 Artin group has cohomological dimension 3, so coherence is an interesting question. It follows again from the generalized Tits Conjecture that Artin groups of type H_4 or F_4 have subgroups which are F_2 but not F_3.

In [42], Zaremsky showed that the pure braid groups PB_n contained subgroups N so that N was type F_{m-3} but not F_{m-2} for $3 \leq m \leq n$. The existence of these subgroups again follows from the fact that the generalized Tits conjecture holds for the braid groups (on the other hand Zaremsky's examples are normal and have $PB_n/N \cong \mathbb{Z}$; our subgroups have neither of these properties). In this case, the nerve of the RAAG subgroup is the cone on a flag triangulation of S^{n-1}. The Bestvina-Brady subgroup of this RAAG is type F_{n-3} but not type F_{n-2}. To get subgroups that are F_{m-3} but not F_{m-2} for $3 \leq m < n$, one can instead map RA to $\mathbb{Z}$ by sending some generators to 0.

Since the generalized Tits conjecture holds for type D_n, we have an analogous theorem with the same proof as above.

Theorem 7.5. *The Artin group of type D_n contains subgroups N so that N is type F_{m-3} but not F_{m-2} for $3 \leq m \leq n$.*

Of course, the Artin group of type D_n contains the braid group A_{n-1}, so the only improvement on what Zaremsky's theorem provides is $n = m$.

8. Questions

We end the paper with some open questions which

Question 8.1. *Does the generalized Tits conjecture hold for spherical Artin groups of type E_n?*

If one can show the conjecture holds for all spherical Artin groups, a natural next step is the Artin groups of FC type.

Question 8.2. *Does the conjecture hold for all Artin groups of FC type?*

It would be very interesting to know some geometric properties of these subgroups:

Question 8.3. *Each standard free abelian subgroup quasi-isometrically embeds into the Artin group A. When the generalized Tits conjecture is true, when are the entire* RAAG *subgroups quasi-isometrically embedded? Does a quasi-isometry between two Artin groups coarsely preserve these* RAAG *subgroups?*

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