TOWARDS THE BOUNDARY RIGIDITY OF LATTICES IN PRODUCTS OF TREES

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ABSTRACT. We consider lattices in products of trees, whose action is free and vertex-transitive. We show that for every CAT(0) cube complex X with a geometric action of such a group, the visual boundary of X is a join of two copies of the Cantor set.

1. Introduction

A visual boundary is a compactification of a metric space, defined as a set of equivalence classes of asymptotic rays and endowed with an appropriate topology. For hyperbolic spaces, any quasi-isometry $X \to Y$ extends to a homeomorphism $\partial X \to \partial Y$. Consequently, the boundary of a hyperbolic group (meaning the boundary of any its Cayley graph) is a group invariant. The situation is different for CAT(0) groups. Bowers-Ruane showed that quasi-isometry between CAT(0) spaces does not need to extend to a homoemorphism between their boundaries [BR96]. Croke-Kleiner provided examples of a CAT(0) group G and two CAT(0) spaces X, Y with a geometric action of G such that ∂X and ∂Y are non-homeomorphic [CK00]. Wilson further showed that in fact G acts geometrically on uncountable many spaces with boundaries of distinct topological type [Wil05]. The group G in the Croke-Kleiner construction is the right-angled Artin group with the defining graph a path on four vertices.

A CAT(0) group G is *rigid*, if the boundaries of any two CAT(0) spaces with geometric action of G are homeomorphic. As noted above, not all CAT(0) groups are rigid. Ruane proved that product of hyperbolic groups are rigid [Rua99]. Hosaka extended that to show that product of rigid groups are rigid [Hos03]. Hruska proved that groups acting geometrically on CAT(0) spaces with isolated flats are rigid [Hru05].

In this note, we study a family of groups acting properly and cocompactly by isometries on specific CAT(0) spaces - product of two infinite, locally finite, regular trees. We refer to such groups as *(cocompact) lattices in a product of trees*. We are interested in the following.

Question. Are lattices in a product of trees rigid?

The simplest example of a lattice in a product of trees is a direct product $F_n \times F_m$ of two finite rank free groups. However, there exist lattices in product of trees are *irreducible*, i.e. they do not split as direct products, even after passing to a finite index subgroups. Lattices in products of trees were first studies by Mozes [Moz92], Burger-Mozes [BM97], [BM00] and by [Wis96]. Burger-Mozes constructed examples of simple lattices in product of trees, providing the first examples of simple CAT(0) groups, as well as the first examples of simple amalgamated product of free groups.

We do not give an answer to this question in full generality. We prove the following.

Theorem 1. Let G be a lattice in a product of trees acting freely and vertex-transitively. Suppose G acts geometrically on a CAT(0) cube complex X. Then ∂X is the join $\mathcal{C} * \mathcal{C}$ of two copies of the Cantor set.

Moreover, if the action of G on X is essential, or if X is geodesically complete, then X splits as a product of CAT(0) cube complexes $X_1 \times X_2$, where $\partial X_i = \mathcal{C}$ for each $i = 1, 2$.

2. Lattices in a product of trees

Let T be an infinite, regular tree of degree ≥ 3. We view T as a metric space, with the path metric, where each edge has length 1. The automorphism group $\text{Aut}(T)$ is a group of all isometries $T \to T$, i.e. permutations of the vertex sets that preserve the adjacency. The group $\text{Aut}(T)$ endowed with a compact-open topology, is a locally compact group. We now consider two infinite regular trees T_n and T_m of degree $n, m \geq 3$ respectively. The product $T_n \times T_m$ has a natural structure of a square complex. It is easy to verify that each vertex link is a complete bipartite graph $K(n, m)$ and, in particular, it is a flag simplicial complex. Thus $T_n \times T_m$, with the path metric induced by the Euclidean metric on each square, is a CAT(0) square complex.

Every lattice in a product of trees acting freely and vertex-transitively on $T_1 \times T_2$ has a special presentation of the form $\langle A \cup B \mid R \rangle$ where R is a set of relators such that

- R is a collection of relators of the form $a_1 b_1 a_2 b_2$ where $a_1, a_2 \in A^{\pm 1}, b_1, b_2 \in B^{\pm 1}$,
- For every $a_1 \in A^{\pm 1}, b_1 \in B^{\pm 1}$, there exist unique $a_2 \in A^{\pm 1}, b_2 \in B^{\pm 1}$ such that a cyclic permutation of $a_1 b_1 a_2 b_2$ belongs to $R^{\pm 1}$.

Such group presentation are referred to as *BMW presentations* by Caprace in [Cap19, Sec 4.1]. A proof that lattices acting freely and vertex-transitively on a product of trees admit BMW presentations can be found in [Rat04]. The presentation complex of a BMW presentation is a square complex with a unique vertex, whose link is a complete bipartite graph $K(n,m)$. Such square complexes are examples of *complete square complexes*. See [Wis07] for more details on complete square complexes.

3. Ends of a space and visual boundary

We recall the definitions and relevant facts about the space of ends of a topological space, and the visual boundary of a metric space. For more details, see [BH99].

Let Y be a topological space. A *ray* in Y is a (proper) map $r : [0,\infty] \to X$. A *ray at* x_0 where y_0 is a basepoint of Y, is a ray with $r(0) = y_0$. An *end* e of Y is an equivalence class of rays in Y where $r_1 \simeq r_2$ if and only if for every compact set $K \subset Y$ there exists $N \geq 0$ such that $r_1([N,\infty])$ and $r_2([N,\infty])$ are contained in the same connected component of $Y - K$. We denote the equivalence class of the ray r by $e(r)$. The set of ends of Y is denoted by $Ends(Y)$ and it can be endowed with topology, which we now describe, following [BH99, I.8.27]. Let r_n be a sequence of rays. The convergence $e(r_n) \to e(r)$ is defined as follows: for every compact set $K \subset Y$ there exists $n_0 > 0$ such that for every $n > n_0$ there exists N_n such that $r_n([N_n,\infty])$ and $r(N_n,\infty)$ lie in the same connected component of $Y - K$. A subset $B \subseteq Ends(Y)$ is closed, if for every sequence of rays r_n such that $e(r_n) \in B$ and $e(r_n) \to e(r)$, we have $e(r) \in B$.

Assume that Y is a metric space with metric d. Two rays r, r' are *asymptotic*, if there exists a constant $C > 0$ such that $d(r(t), r'(t)) < C$ for all $t \in [0,\infty]$. The *visual boundary* ∂Y of Y is the set of equivalence classes of rays at y_0, where two rays are equivalent if they are asymptotic. The *cone topology* on ∂Y is given by the neighborhood basis $\{U(r,N,C) : [r] \in \partial Y,\, t, C > 0\}$ where

$$U(r,t,C) = \{[r'] \in \partial Y : r'(0) = y_0, d(r(t), r'(t)) < C\}.$$

See [BH99, Chap II.8] for more details.

If X is a proper CAT(0) space, then there is a natural well-defined map $\partial Y \to Ends(Y)$ sending a ray r to $e(r)$. This map does not depend on the choice of a ray in the equivalence class of asymptotic rays. The map is a continuous surjection [BH99, Rem II.8.10].

4. Main proof

Let G be a lattice in a product of trees acting freely and vertex-transitively. Assume that G acts geometrically on a geodesically complete CAT(0) cube complex X.

Lemma 2. Suppose the action of G on a CAT(0) cube complex X is essential. Then X is a product of two CAT(0) cube complexes $X_1 \times X_2$, with $\dim \partial X_1 = \dim \partial X_2 = 0$.

Proof. Every element of G bounds a half-plane in $T_1 \times T_2$, so G contains no rank-1 elements. By rank rigidity for CAT(0) cube complexes [CS11, Cor B], X is a product of two CAT(0) cube complexes $X_1 \times X_2$. Thus $\partial X = \partial X_1 \star \partial X_2$. Since the dimension of the boundary is a quasi-isometry invariant of CAT(0) groups [GO07], we must have $\dim \partial X = 1$, and it follows that $\dim \partial X_1 = \dim \partial X_2 = 0$. □

Lemma 3. Each of $\partial X_1, \partial X_2$ has at least two points, and at least one of them has at least three points.

Proof. Since G contains $\mathbb{Z}^2$ subgroups, by the Flat Torus theorem there is a flat F embedded in X as a convex subspace. Thus, $S^1 \simeq \partial F$ embeds in ∂X. In particular, both $|\partial X_1|, |\partial X_2| \geq 2$. If both $|\partial X_1|, |\partial X_2| = 2$, then $\partial X \simeq S^1$. Following the proof of [Rua06, Thm 3.5] we see that X contains a G-invariant copy of $\mathbb{E}^2 \times K$ and in particular, G is virtually abelian, which is a contradiction. □

Lemma 4. The boundary ∂X_i is homeomorphic to the space of ends $Ends(X_i)$.

Proof. Let us first show that each X_i is hyperbolic. The complex X_i is a visibility space by [BH99, Lem II.9.22]. Since X_i is a proper CAT(0) space, X_i is uniformly visible by [BH99, Prop II.9.32]. By [BH99, Prop III.1.4] X_i is hyperbolic.

Since X_1 is a proper hyperbolic space, the natural map $\partial X_1 \to Ends(X_1)$ is continuous and the fibers of that map are the connected components of ∂X_1 [BH99, Exer III.H.3.9]. By Lemma 2 ∂X_1 is 0-dimensional, so the connected components are single points. Thus the map $\partial X_1 \to Ends(X_1)$ is a continuous bijection. Every continuous bijection from a compact space to a Hausdorff space is a homeomorphism. □

Lemma 5. For every compact set $K \subset X_i$ and every open neighborhood of an end of X_i, there exists $g \in G$ such that $gK \subset U$.

Proof. We show that the claim holds for X_1. The argument for X_2 is identical. Let $K \subset X_1$ and $K' \subset X_2$ be compact sets. Then $K \times K'$ is a compact set in X. By Lemma 4 ∂X_1 is homeomorphic to $Ends(X_1)$. Let $\xi \in \partial X_1$, and let U be an open neighborhood of ξ. Then $U \times X_2$ is an open neighborhood of ξ, viewed as a point in $\partial X = \partial X_1 \star \partial X_2$. Since the action of G on X is geometric, there exists $g \in G$ such that $g(K \times K') \subset U \times X_2$ (see e.g. [Bes96]). It follows that in the action of G on X_1, we have $gK \subset U$. □

The following is not stated explicitly, but it is proved in [Hop44]. It is also proved in a similar form in [Ber05]. We include the proof for completeness.

Proposition 6 ([Hop44])**.** Let Y be a connected, locally connected (Hausdorff, locally compact, with a countable basis) space with at least three ends. Let G be a group acting on Y cocompactly so that the following holds: for every compact set $K \subset Y$, and every open neighborhood U of an end of Y, there exists $g \in G$ such that $gK \subset U$. Then the space of ends $Ends(Y)$ is perfect.

Proof. Suppose that there exists an end $e \in Ends(Y)$ that is isolated, i.e. there exists a neighborhood U of e that does not contain any other ends. First we show that without loss of generality, we can assume that $Y - U$ is connected. Indeed, if $Y - U$ is not connected, we construct a neighborhood $U' \subset U$ of e such that $Y - U'$ is connected. Since $\overline{Y}$ is compact, so is $\overline{Y} - U$. Let $V_1, \ldots, V_n$ be a finite collection of open sets covering $\overline{Y} - U$ such that each V_i is connected and does not contain e, and its closure $\overline{V}_i$ in $\overline{Y}$ is compact. The union $\bigcup_{i=1^n} V_i$ has finitely many components, and since $Ends Y$ is nowhere dense in $\overline{Y}$, each V_i contains points of Y. Each two points in Y can be joined by an interval in Y. In particular, there exists a closed connected set Q which is the union of $\bigcup_{i=1^n} V_i$ and a finite number of intervals in Y. Note that Q does not contain e. The set $U' = \overline{Y} - Q$ is an open neighborhood of e and since $Y - U \subseteq \bigcup V_i \subseteq Q$, we have $U' \subset U$.

By assumption, there are at least three distinct ends e_1, e_2, e_3 in Y. Let K be a compact set in Y such that each of the ends e_1, e_2, e_3 lies in a different connected component Y_1, Y_2, Y_3 of $Y - K$. By assumption, there exists an element $g \in G$ such that $gK \subset U$. We claim that for each $i = 1, 2, 3$, either $gY_i \subset U$, or $g(Y - Y_i) \subset U$. Indeed, otherwise there exists point $p \in gY_i - U$ and $q \in g(Y - Y_i) - U$. Since $Y - U$ is connected, there exists a path γ in $Y - U$ joining p and q. Note that p, q lie in distinct connected component of $Y - gK$, so γ has to pass through gK. This is a contradiction, since $gK \subset U$.

Since U contains only one end, we have $g(Y - Y_i) \subset U$ for at least two i's among $1, 2, 3$, say 1 and 2. It follows that $Y - U \subset gY_i$ for $i = 1, 2$.

The subsets Y_1 and Y_2 are disjoint, and so are gY_1 and gY_2. This is a contradiction and therefore the space of $Ends(Y)$ has no isolated points. □

Lemma 7. If $|\partial X_i| \geq 3$, then ∂X_1 is the Cantor set $\mathcal{C}$.

Proof. Since X_1 is a proper CAT(0) space, the boundary ∂X_1 is compact and metrizable [BH99]. By Lemma 2 $\dim \partial X_1 = 0$, so ∂X_1 is totally disconnected. The action of G on X_1 is cocompact, since so is the action of G on X. By Lemma 5, for every compact set $K \subset X_1$ and every open neighborhood $U \subset X_1$ of an end of X_1, there exists $g \in G$ such that $gK \subset U$. By Proposition 6 $Ends(X_1) = \partial X_1$ is a perfect space. By the characterization of the Cantor set, as a non-empty, perfect, totally disconnected, compact metrizable space, we conclude that ∂X_1 is the Cantor set. □

Proof of Theorem 1. Since G is finitely generated, and the action of G on X is proper, by [CS11, Prop 3.12] the G-essential core Y of X embeds as a convex G-invariant subcomplex of X. Since the action of G on X is cocompact, Y has to be N-dense in X for some $N \geq 0$. By Lemma 2, Y splits as a product $X_1 \times X_2$ where $\dim \partial X_1 = \dim \partial X_2 = 0$. By Lemma 3, without loss of generality, $|\partial X_1| \geq 3$ and $|\partial X_2| \geq 2$. By Lemma 4, X_1 has at least three ends. By Lemma 7, ∂X_1 is the Cantor set $\mathcal{C}$.

Now, we show that $|\partial X_1| \geq 3$. Otherwise, if $|\partial X_2| = 2$, then ∂X is a suspension of the Cantor set. By [Rua06, Thm 4.4] G is virtually a direct product of a free group with $\mathbb{Z}$, which is a contradiction. Thus $|\partial X_2| \geq 3$. By Lemma 7, ∂X_2 is the Cantor set. Finally, since Y is N-dense in X, we have $\partial X = \partial Y = \mathcal{C} * \mathcal{C}$. □

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