RESEARCH STATEMENT

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My research is in geometric group theory. This is the study of the interplay of the algebraic properties of infinite finitely generated groups and the topology and geometry of spaces that the groups act on. In my work I use combinatorial and probabilistic techniques, tools from the algebraic topology, theory of computation, hyperbolic geometry, study of mapping class groups and the dynamics of the automorphisms of free groups. My research has connections with the study of 3- and 4-manifolds, which provide examples and tools. My work also has applications to these fields. I additionally study groups arising via algebraic constructions.

Here are the main themes of my research, which are discussed in more details in the next three sections of this document.

Artin groups. They generalize braid groups, and include free groups and free abelian groups. Artin groups are closely related to Coxeter groups and arise as fundamental groups of quotients of complexified hyperplane arrangement. They were introduced by Tits in the 60s. Spherical Artin groups became famous due to work of Bieskorn, Saito and Deligne in the 70s. Despite simple presentations, general Artin groups are poorly understood. Artin groups can be studied via their actions on CAT(0) spaces, following Davis, Charney and Salvetti. Combinatorial techniques, such as Garside methods, can also be utilized in understanding Artin groups. Right-angled Artin groups are a particularly nice family of Artin groups, and play an important role in the theory of special cube complexes.

Non-positive curvature. In the late 80s, in a foundational paper for geometric group theory, Gromov introduced two notions of "curvature" for metric spaces. The curvature of finitely generated groups can be studied via the action of the group on its *Cayley graphs*, which is a metric space that encodes the structure of the group. *Word-hyperbolic groups* capture many aspects of negative curvature of manifolds, and CAT(0) *spaces* are in a precise sense at least as nonpositively curved as the Euclidean plane. Both of these notions have far reaching algebraic consequences. The geometry of CAT(0) cube complexes and general CAT(0) spaces have strong connections with the algebraic properties of the groups that act on them. One of the most remarkable applications of CAT(0) cube complexes, to the theory of 3-manifolds was the resolution of Waldhausen's virtual Haken conjecture and Thurston's virtual fibering conjecture by Agol and Wise. I am interested in both constructing and obstructing actions on such spaces.

Algebraic fibering and coherence of groups. These properties can be thought of as group theoretic analogues of properties of 3-manifolds, decomposing as a fiber bundle over a circle, and the compact core property of 3-manifolds. Algebraic fibration can also be constructed using discrete methods, such as Bestvina-Brady Morse theory. The notion of coherence has been studied since the 70s, as it arose from the work of Scott, Shalen, and Stalings. A famous question of Baumslag asks whether all 1-relator groups are coherent.

1. Artin groups

Artin groups generalize braid groups. They admit presentations where every relation is of the form

$$\underbrace{aba\cdots}_{m} = \underbrace{bab\cdots}_{m}$$

where a, b are two distinct generators. *Spherical* Artin groups are those whose quotient Coxeter groups, obtained by imposing that all generators have order 2, are finite. Spherical Artin groups, like finite Coxeter groups, are classified by their Dynkin-Coxeter diagrams.

Actions on cube complexes. Together with Jingyin Huang and Piotr Przytycki we examined proper and cocompact actions of Artin groups on CAT(0) cube complexes and proved that many Artin groups do not admit such actions. This is an interesting result, because in general proving that a group cannot act (properly, cocompactly) on a CAT(0) cube complex can be hard, as there is no general procedure to do so. We gave a complete characterization of 2-dimensional or 3-generated Artin groups that admit such actions, in terms of the combinatorics of its defining graph.

Theorem 1 (Huang-Jankiewicz-Przytycki, 2016)**.** Let A be a three-generator Artin group. Then the following are equivalent.

(1) The group A acts properly and cocompactly on a CAT(0) cube complex.
(2) A finite index subgroup of A acts properly and cocompactly on a CAT(0) cube complex.
(3) The defining graph of A is one of:

This theorem give a partial answer to the question of Charney about actions of Artin groups on CAT(0) cube complexes, under the assumption that the action is cocompact. The following is a noteworthy corollary.

Corollary 2 (Huang-Jankiewicz-Przytycki, 2016)**.** The braid group B_4 on 4 strands does not have a finite index subgroup that acts properly and cocompactly on a CAT(0) cube complex.

The above corollary was surprising to the experts in the field, since B_4 is known to be CAT(0) by Brady-McCammond. We also obtained a similar result for all 2-dimensional Artin groups.

Theorem 3 (Huang-Jankiewicz-Przytycki, 2016)**.** There is a complete classification of 2-dimensional Artin groups acting properly and cocompactly on CAT(0) cube complexes in terms of the combinatorics of their defining graphs.

Independently, the above results were also obtained by Haettel. Without the assumption on the cocompactness, Haettel recently proved that some triangle-free Artin group act properly on locally finite, finite dimensional CAT(0) cube complexes. The following question remains open, and I would like to answer it.

Question A. Do braid groups act properly on CAT(0) cube complexes?

I have so far been exploring the case of the Artin group A_{333} which is commensurable with the quotient of a braid group on four strands modulo its center. I am working on classifying all its codimension one subgroups.

Residual finiteness. In my recent work, I proved that many of these groups considered previously are residually finite. A group G is *residually finite* if for every $g \in G - \{1\}$ there exists a finite quotient $\phi : G \to \bar{G}$ such that $\phi(g) \neq 1$. All *linear* groups, i.e. groups with faithful representations into $GL_n(\mathbb{R})$ for some n, are residually finite by a theorem of Malcev.

Previously, there were very few families of Artin groups known to be residually finite. Spherical Artin groups were known to be linear by the work of Krammer, Bigelow, followed by Cohen-Wales and Digne. It can be deduced from the work of Squier that affine Artin groups on three generators are also residually finite. Some examples were provided by (Blasco-Garcia)-(Martinez-Perez)-Paris and (Blasco-Garcia)-Juhasz-Paris. In the latter paper, a question whether all 3-generator Artin groups are residually finite was posed. The question of residual finiteness of Artin groups was also raised by Hsu and Wise. I give a partial answer to these questions.

Theorem 4 (Jankiewicz, 2020)**.** The Artin group A_{MNP} is residually finite, provided that $M, N, P \geq 3$ and $(M, N, P) \neq (3, 3, 2k+1)$.

The above results holds for more general family of Artin groups, whose defining graph satisfies certain combinatorial criterion. My proof of residual finiteness of those Artin group, relies on the following splitting theorem.

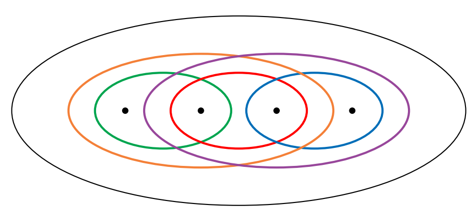
Theorem 5 (Jankiewicz, 2020)**.** All 3-generator large type (i.e. where all exponents are ≥ 3) Artin groups split as free products with amalgamation of finite rank free groups.

The above theorem holds for a larger family of large type Artin groups. The existence of a splitting does not guarantee residual finiteness. In fact, there exists examples of amalgamated product of finite rank free groups that are not residually finite, and can even be simple. Such examples were constructed by Bhattacharjee, Wise and Burger-Mozes (the latter two are lattices in a product of trees, which I briefly discuss in the next section). I would like to answer the following question.

Question B. Can you characterize residual finiteness of amalgamated products of finite rank free groups?

Generalized Tits conjecture on right-angled Artin subgroups of Artin groups. The Tits Conjecture, proved by Crisp and Paris, states that for every Artin group A with the standard set of generators S, the subgroup $\langle s^2 : s \in S\rangle$ is an obvious right-angled Artin group, i.e. where $[s^2, t^2] = 1$ if and only if $[s, t] = 1$.

With Schreve, we propose the following generalization of the Tits Conjecture. Given an irreducible, spherical subset $T \subset S$, let Δ_T^2 denote the corresponding center of the pure Artin subgroup PA_T. In the braid group, viewed as a mapping class group of a punctured disc, those new elements correspond to the Dehn twists about the curves enclosing more than two consecutive punctures.



Question C. Given $N \geq 1$, is the subgroup $\langle \Delta_T^{2N} : T \subset S\rangle$ of an Artin group A the obvious right-angled Artin group?

The above problem was conjectured to have a positive answer for $N = 1$ and every Artin group by Davis, Le and Schreve. However, we show that it fails for $N = 1$ for braid groups on at least eight strands. We give positive answers for certain families of Artin groups.

Artin group is *locally reducible* if every spherical special subgroup is a product of Artin groups of rank 1 or 2. This family of Artin groups was introduced by Charney in the study of the original Tits conjecture.

Theorem 6 (Jankiewicz-Schreve, 2020)**.** Let A be a locally reducible. Then the subgroup of A generated by sufficiently large powers of all the Garside elements of irreducible parabolic subgroups, is a right-angle Artin group.

We also examines spherical Artin groups and showed the following.

Theorem 7 (Jankiewicz-Schreve, 2020)**.** Let A be a spherical Artin group of any type other than E_6, E_7, E_8. Then for sufficiently large N the subgroup $\langle \Delta_T^{2N} : T \subset S \rangle$ of A is a right-angled Artin group for N sufficiently large.

Our proof uses representations of Artin groups in mapping class groups, due to Perron and Vannier, and utilizes Koberda's theorem on right-angled Artin subgroups of mapping class groups. In the case of exotic types E_n, the Perron-Vannier representation is not injective on $\langle \Delta_T^{2N} : T \subset S \rangle$. It remains open whether Question C has positive answer for those three cases.

2. CAT(0) cube complexes

A CAT(0) cube complex is a CW complex obtained from a collection of Euclidean cubes by identifying their sides via isometries that satisfies a combinatorial condition guaranteeing that the complex admits a CAT(0) metric in which the cubes are Euclidean unit cubes. CAT(0) cube complexes were first introduced by Gromov as easily constructible examples of CAT(0) spaces. The combinatorial nature of CAT(0) cube complexes provided by their *hyperplanes* makes them natural generalizations of trees and makes them very accessible to work with.

There are strong relations between actions on CAT(0) cube complexes and algebraic properties of the group. The most fundamental such correspondence comes from the *Sageev's construction* relating essential actions on CAT(0) cube complexes with codimension-1 subgroups. Other examples are the Tits alternative or the rank rigidity conjecture of Ballman-Buyalo, which are known to hold for groups acting properly and cocompactly on CAT(0) cube complexes, but remain unknown for general CAT(0) groups.

Cubical dimension and uniform exponential growth. The *cubical dimension* of a group G, denoted by cub-dim G, is the minimum n such that G acts properly on an n-dimensional CAT(0) cube complex. Similarly, the *CAT(0) dimension* CAT(0)dim G is the minimum n such that G acts properly by semi-simple isometries on an n-dimensional CAT(0) space. We have

$$\mathrm{cd}\, G \leq \mathrm{CAT(0)\text{-}dim}\, G \leq \mathrm{cub\text{-}dim}\, G$$

where $\mathrm{cd}\, G$ is the cohomological dimension of G. Examples of group with (bounded) gaps between the cohomological and CAT(0) dimensions were provided by Bridson, and Brady-Crisp.

I constructed the first examples of groups with arbitrarily large finite gaps between their cubical dimension and their cohomological and CAT(0) dimensions.

Theorem 8 (Jankiewicz, 2019)**.** For every n there exist hyperbolic groups with $\mathrm{cd}\, G = \mathrm{CAT(0)\text{-}dim}\, G = 2$, and $\mathrm{cub\text{-}dim}\, G > n$.

My construction uses small cancellation theory, and analysis of free semi-subgroups in isometry groups of CAT(0) cube complexes. The case of $n = 2$ in the above theorem, can be deduced from the work of Kar and Sageev who study uniform exponential growth of groups acting freely on CAT(0) square complexes. My work generalizes some of their methods to higher dimensions.

In a subsequent work, joint with Daniel Wise, we use my result to construct the following example.

Theorem 9 (Jankiewicz-Wise, 2019)**.** There exists a finitely generated group of cohomological dimension 2 that acts freely (and thus metrically properly) on a locally-finite CAT(0) cube complex, but does not act properly on any finite dimensional CAT(0) cube complex.

The *exponential growth rate of G with respect to a finite generating set S* is defined as

$$w(G,S) := \lim_{n\to\infty} |B(n,S)|^{1/n}.$$

A group G has *exponential growth* if $w(G,S) > 1$ for some (equivalently every) finite generating set S, and it has *uniform exponential growth* if there exists $k > 1$ such that $w(G,S) > k$ for every finite generating set S. Kar-Sageev proved that a group acting freely on a CAT(0) square complex has uniform exponential growth, unless it is virtually abelian.

In a recent joint work with Radhika Gupta and Thomas Ng, we extend the result of Kar-Sageev to certain other groups acting on CAT(0) cube complexes. We show that the same is true for groups with torsion:

Theorem 10 (Gupta-Jankiewicz-Ng, 2020)**.** Every group acting properly on a CAT(0) square complex has uniform exponential growth, unless it is virtually abelian.

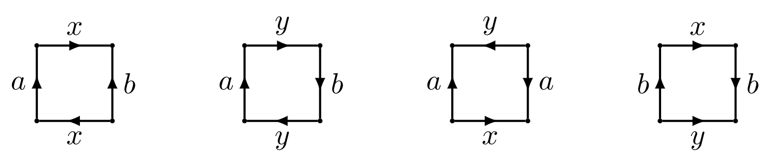
Building on my work on cubical dimension, Gupta, Ng and I also generalize the result to arbitrary dimension, under the assumption that X has *isolated flats*. CAT(0) spaces with isolated flats can be non-hyperbolic, but they have many features of hyperbolic spaces.

Theorem 11 (Gupta-Jankiewicz-Ng, 2020)**.** Let G acts freely on a CAT(0) cube complex of dimension d with isolated flats that admits a geometric group action. Then G has uniform exponential growth depending on d only, unless G is virtually abelian.

We have also shown uniform exponential growth of certain groups acting improperly on CAT(0) square complexes with vertex stabilizers that are either virtually abelian, or have uniform exponential growth. It includes subgroups of a triangle free Artin group or of the Higman's group.

Lattices in product of trees. Let T, T' be regular, locally finite trees of degree ≥ 3. The product $T \times T'$ has a natural structure of a CAT(0) square complex. A *(uniform) lattice* in $\mathrm{Aut}(T \times T')$ is a discrete group acting properly and cocompactly on $T \times T'$. The simplest example of such group is $F_2 \times F_2$ which acts on a product of two copies of a regular valence 4 tree.

A lattice that does not virtually split as a direct product of free group is called *irreducible* and can be characterized as having non-discrete projections to each of the factors $\mathrm{Aut}(T)$. Famous examples of irreducible lattices in product of trees are due to Burger-Mozes, who constructed example that are virtually simple. Wise independently constructed non-residually finite examples. An example of an irreducible lattice is a group generated by a, b, x, y with the relations below.



A *visual boundary* $\partial_\infty X$ of a proper CAT(0) space X is a well-defined compactification of X. Unlike for hyperbolic groups, a visual boundary of a CAT(0) group does not need to be unique. Croke-Kleiner constructed an example of a CAT(0) group acting geometrically on two CAT(0) spaces X_1, X_2 whose visual boundaries $\partial X, \partial X'$ are non-homeomorphic. We say a CAT(0) group G is *rigid* if any two boundaries of CAT(0) spaces that G acts on are homeomorphic.

Annette Karrer, Kim Ruane, Bakul Sathaye and I are interested in addressing the following question.

Question D. Are lattices in a product of trees rigid?

The visual boundary $\partial_\infty(T \times T')$ of the product of two regular trees is a join $C \star C$ of two copies of the Cantor set. Hence the above question asks if the boundary of every CAT(0) space that given lattice in $\mathrm{Aut}(T \times T')$ acts on geometrically, is homeomorphic to $C \star C$. We have proved the following.

Theorem 12 (Jankiewicz-Karrer-Ruane-Sathaye, 2020)**.** Let $G < \mathrm{Aut}(T \times T')$ act freely and vertex-transitively on $T \times T'$. Let G act properly and cocompactly on a CAT(0) cube complex X. Then X splits as a product of CAT(0) cube complexes $X_1 \times X_2$ with $\partial X_i = C$. In particular, $\partial X \simeq C \star C$.

We are working towards generalizing the above result to show that the boundary of arbitrary CAT(0) space with a geometric group action of G is a join of two copies of the Cantor set.

Genevieve Walsh and I are investigating the question of incoherence of lattices in products of trees. A group is *coherent* if every finitely generated subgroup is finitely presented, and otherwise it is *incoherent*. Example of coherent groups include free groups, surface groups, fundamental groups of 3-manifolds, free-by-cyclic groups. A classical example of a group that is incoherent is $F_2 \times F_2$, which is an example of a reducible lattice in product of trees. Genevieve Walsh and I are investigating the following question.

Question E. Are all lattices in a product of trees incoherent?

A negative answer would imply that coherence is not a quasi-isometry invariant. A positive answer would be an indication that coherence might be a "geometric" property, i.e. that two group that act geometrically on the same space either both coherent, or both incoherent. There are many collections of coherent groups share geometry (such as acting on a tree, hyperbolic plane, hyperbolic 3-space), and most proofs of coherence are geometric in nature. On the other hand most of the known proofs of incoherence rely on existence of an incoherent subgroup such as $F_2 \times F_2$ or the kernel of an algebraic fibering in a group of dimension 2. Thus, it would be very interesting to find geometric reasons for incoherence.

We have so far focused on understanding some small examples, and proved incoherence of some of them by finding subgroups that algebraically fiber. We are working on finding geometric or algorithmic method of constructing subgroups of lattices in products of trees.

3. Algebraic fibering

As mentioned in the previous section, a group G is *coherent*, if every finitely generated subgroup of G is finitely presented. One of my project concerning incoherence is described in the previous section. We say G *algebraically fibers*, if it admits an epimorphism onto $\mathbb{Z}$ with finitely generated kernel. Stallings' fibration theorem states that a compact irreducible 3-manifold M fibers as a surface bundle over a circle if and only if $\pi_1 M$ algebraically fibers. Algebraic fibering of any group G of cohomological dimension 2 yields incoherence of G, unless G is free-by-cyclic, by a theorem of Bieri.

With Sergey Norin and Daniel Wise we give a sufficient condition for a right-angled Coxeter group W to virtually algebraically fiber, in terms of the combinatorics of the defining graph Γ of W. We prove the following.

Theorem 13 (Jankiewicz-Norin-Wise, 2019)**.** If Γ has a legal system, then W_Γ virtually algebraically fibers.

A *legal system* in the above statement is a coloring of vertices of Γ, along with a partition of the vertex set of Γ which satisfy certain combinatorial condition. The proof relies on construction of a Morse function in the sense of Bestvina–Brady for the CAT(0) cube complex associated with W.

Among the most noteworthy examples of Γ with legal systems, are the 24-cell and the 600-cell. The Coxeter groups that they define, are 4-dimensional hyperbolic reflection groups with fundamental domain respectively right-angled ideal hyperbolic 24-cell, and right-angled hyperbolic 120-cell. This yields to the first examples of higher dimensional hyperbolic manifolds with algebraically fibered fundamental groups. No hyperbolic 4-manifolds can fiber over a circle, because of the Gauss-Bonnet theorem, so algebraic fibering is a natural candidate for generalizing virtual fibering of 3-manifolds to the higher dimensions.

Example 14 (Jankiewicz-Norin-Wise, 2019)**.** We give explicit examples of a compact hyperbolic 4-manifold and a non-compact finite volume hyperbolic 4-manifold, whose fundamental groups algebraically fiber.

More recently, Kielak showed that groups satisfying RFRS condition (introduced by Agol and used in his famous proof of virtual fibering of closed hyperbolic 3-manifold), virtual algebraic fibering is equivalent to vanishing of the first L^2-Betti number.

Our work inspired a number of groups of other mathematicians. Fiz-Pontiveros, Glebov and Karpas proved that the satisfying our criterion is generic for random graphs, and consequently random right-angled Coxeter groups almost surely admit virtual algebraic fibering. Most interestingly, recently, Battista and Martelli, extended and smoothened our Morse function in the case of the 24-cell. That led to the first known example of a 4-manifold M with *perfect circle-valued* Morse function, i.e. a smooth function $f : M \to S^1$ with the minimal possible number of critical points, which is equal $|\chi(M)|$. Our construction was also revisited by Ma and Zheng.

In another direction, I am interested in using our methods to answer the following question.

Question F. Are there hyperbolic Coxeter groups of arbitrarily high virtual cohomological dimension that virtually algebraically fiber?

A nice feature of right-angled Coxeter groups is that many properties of the group can be read from the defining graph. In particular, Moussong proved that a right-angled Coxeter group is word hyperbolic if and only if every 4-cycle in its defining graph has a diagonal. Along with the dimension bounds of right-angled Coxeter groups coming from non vanishing homology of the defining graph, by Davis, the above question can be reduced to the question about the existence of a simplicial complex with a legal system, that is a pseudo-manifolds and where every 4-cycle has a diagonal.

In another joint work with Daniel Wise we apply Bestvina–Brady Morse theory to the Coxeter groups $G_{(r,m)}$ of rank r and a constant exponent m (i.e. $(ab)^m = 1$ for all pairs of generators a, b). We prove the following.

Theorem 15 (Jankiewicz-Wise, 2016)**.** For each $m \geq 3$ for all sufficiently large r the group $G_{(r,m)}$ virtually algebraically fibers, and in particular is incoherent.