

SPLITTINGS OF TRIANGLE ARTIN GROUPS

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ABSTRACT. We show that a triangle Artin group Art_{MNP} where $M \leq N \leq P$ splits as an amalgamated product or an HNN extension of finite rank free groups, provided that either $M > 2$, or $N > 3$. We also prove that all even three generator Artin groups are residually finite.

A triangle Artin group is given by the presentation

$$\text{Art}_{MNP} = \langle a, b, c \mid (a, b)_M = (b, a)_M, (b, c)_N = (c, b)_N, (c, a)_P = (a, c)_P \rangle,$$

where $(a, b)_M$ denote the alternating word $aba\dots$ of length M . Squier showed that the Euclidean triangle Artin group, i.e. Art_{236} , Art_{244} and Art_{333} , split as amalgamated products or an HNN extension of finite rank free groups along finite index subgroups [Squ87]. We generalize that result to other triangle Artin groups.

Theorem A. Suppose that $M \leq N \leq P$ where either $M > 2$, or $N > 3$. Then the Artin group Art_{MNP} splits as an amalgamated product or an HNN extension of finite rank free groups.

The assumption of the above theorem are satisfied for all triples of numbers except for $(2, 2, P)$ and $(2, 3, P)$. An Artin group is *spherical*, if the associated Coxeter group is finite. A three generator Artin group Art_{MNP} is spherical exactly when $\frac{1}{M} + \frac{1}{N} + \frac{1}{P} > 1$, i.e. $(M, N, P) = (2, 2, P)$ or $(2, 3, 3), (2, 3, 4), (2, 3, 5)$. All spherical Artin groups have infinite center, and none of them splits as a graph of finite rank free groups (see Proposition 2.10). The remaining cases are $(2, 3, P)$ where $P \geq 6$. The above theorem holds for triple $(2, 3, 6)$ by [Squ87]. It remains unknown for $(2, 3, P)$ with $P \geq 7$. The cases where $M > 2$ were considered in [Jan20, Thm B] and it was proven that they all split as amalgamated products of finite rank free groups.

Graphs of free groups form an important family of examples in geometric group theory. Graph of free groups with cyclic edge groups that contain no Baumslag-Solitar subgroups are virtually special [HW10], and contain quasiconvex surface subgroups [Wil18]. Graphs of free groups with arbitrary edge groups can exhibit various behaviors. For example, an amalgamated product $A *_C B$ of finite rank free groups where C is malnormal in A, B is hyperbolic [BF92], and virtually special [HW15]. On the other hand there are examples of amalgamated products of finite rank free groups that are not residually finite [Bha94], [Wis96], and even simple [BM97]. The last two arise as lattices in the automorphism group of a product of two trees.

By further analysis of the splitting, we are also able to show that some of the considered Artin groups are residually finite.

Theorem B. The Artin group Art_{2MN} where $M, N \geq 4$ and at least one of M, N is even, is residually finite.

An Artin group Art_{MNP} is *even* if all M, N, P are even. The above theorem combined with our result in [Jan20] (and the fact that $\text{Art}_{22P} = \mathbb{Z} \times \text{Art}_P$ is linear) gives us the following.

Corollary C. All even Artin groups on three generators are residually finite.

All linear groups are residually finite [Mal40], so residual finiteness can be viewed as testing for linearity. Spherical Artin groups are known to be linear ([Kra02], [Big01] for braid groups, and [CW02], [Dig03] for general spherical Artin groups). The right-angled Artin groups are also well known to be linear, but not much more is known about linearity of Artin groups. In last years, a successful approach in proving that groups are linear is by showing that they are virtually special. Artin groups whose defining graphs are forests are the fundamental groups of graph manifolds with boundary [Bru92], [HM99], and so they are virtually special [Liu13], [PW14]. Many Artin groups in certain classes (including 2-dimensional, or three generator) are not cocompactly cubulated even virtually, unless they are sufficiently similar to RAAGs [HJP16], [Hae20b]. In particular, the only (virtually) cocompactly cubulated three generator Artin group are $\text{Art}_{22M} = \mathbb{Z} \times \text{Art}_M$, $\text{Art}_{MN\infty}$ where M, N are both even, and $\text{Art}_{M\infty\infty} = \mathbb{Z} * \text{Art}_M$. Some triangle-free Artin groups act properly but not cocompactly on locally finite, finite dimensional CAT(0) cube complexes [Hae20a].

In [Jan20] we showed that Art_{MNP} are residually finite when $M, N, P \geq 3$, except for the cases where $(M, N, P) = (3, 3, 2p + 1)$ with $p \geq 2$. Few more families of Artin groups are known to be residually finite, e.g. even FC type Artin groups [BGMPP19], and certain triangle-free Artin groups [BGJP18].

Organization. In Section 1 we provide some background. In Section 2 we prove Theorem A as Proposition 2.5 and Corollary 2.9. We also show that the only irreducible spherical Artin groups splitting as graph of finite rank free groups are dihedral. In Section 3 we recall a criterion for residual finiteness of amalgamated products and HNN extensions of free groups from [Jan20] and prove Theorem B.

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1. BACKGROUND

1.1. **Graphs.** Let X be a finite graph with directed edges. We denote the vertex set of X by $V(X)$ and the edge set of X by $E(X)$. The vertices incident to an edge e are denoted by e^+ and e^- . A *map of graphs* $f : X_1 \rightarrow X_2$ sends vertices to vertices, and edges to concatenations of edges. A map f is a *combinatorial map* if single edges are mapped to single edges. A combinatorial map f is a *combinatorial immersion* if given two edges e_1, e_2 such that $e_1^- = e_2^-$ we have $f(e_1) = f(e_2)$ (as oriented edges) if and only if $e_1 = e_2$. Consider two edges e_1, e_2 with $e_1^- = e_2^-$. A *fold* is the natural combinatorial map $X \rightarrow X'$ where $V(X') = V(X)/e_1^+ \sim e_2^+$ and $E(X')/e_1 \sim e_2$. Stallings showed that every combinatorial map $X \rightarrow X'$ factors as $X \rightarrow \bar{X} \rightarrow X'$ where $X \rightarrow \bar{X}$ is a composition of finitely many folds, and $\bar{X} \rightarrow X'$ is a combinatorial immersion [Sta83]. We refer to $X \rightarrow \bar{X}$ as a *folding map*.

1.2. Maps between free groups. Let H, G be finite rank free groups. Let Y be a bouquet of $n = \text{rk } G$ circles. We can identify $\pi_1 Y \simeq F_n$ with G by orienting and labelling edges of Y with the generators of G . Every homomorphism $\phi : H \rightarrow G$ can be represented by a combinatorial immersion of graphs. Indeed, start with a map of graphs $X \rightarrow Y$ where X is a bouquet of $m = \text{rk } H$ circles. We think of each circle in X as subdivided with edges oriented and labelled by the generators of G , so that each circles is labelled by a word from a generating set of H . By Stallings, the map $X \rightarrow Y$ factors as $X \rightarrow \bar{X} \rightarrow Y$ where $X \rightarrow \bar{X}$ is a folding map, and $\bar{X} \rightarrow Y$ is a combinatorial immersion. Indeed, \bar{X} is obtained by identifying two edges with the same orientation and label that share an endpoint.

Note that the rank of $\phi(H)$ is equal $\text{rk } \pi_1 \bar{X} = 1 - \chi(\bar{X})$ where χ denotes the Euler characteristic. In particular, a homomorphism ϕ is injective if and only if the folding map $X \rightarrow \bar{X}$ is a homotopy equivalence. In that case, \bar{X} is a precover of Y which can be completed to a cover of Y corresponding to the subgroup H of G via the Galois correspondence. In particular, every subgroup of G is uniquely represented by a combinatorial immersion $(X, x) \rightarrow (Y, y)$ where y is the unique vertex of Y , and X is a folded graph with basepoint x . We refer to [Sta83] for more details.

1.3. Intersections of subgroups of a free group. Let Y be a graph, and $\rho_i : (X_i, x_i) \rightarrow (Y, y)$ a combinatorial immersion for $i = 1, 2$. The *fiber product of X_1 and X_2 over Y* , denoted $X_1 \otimes_Y X_2$ is a graph with the vertex set

$$V(X_1 \otimes_Y X_2) = \{(v_1, v_2) \in V(Y_1) \times V(Y_2) : \rho_1(v_1) = \rho_2(v_2)\},$$

and the edge set

$$E(X_1 \otimes_Y X_2) = \{(e_1, e_2) \in E(Y_1) \times E(Y_2) : \rho_1(e_1) = \rho_2(e_2)\}.$$

The graph $Y_1 \otimes_X Y_2$ often has several connected components. There is a natural combinatorial immersion $Y_1 \otimes_X Y_2 \rightarrow X$, and it induces an embedding $\pi_1(Y_1 \otimes_X Y_2, (y_1, y_2)) \rightarrow \pi_1(X, x)$. We have the following.

Theorem 1.1 ([Sta83, Thm 5.5]). Let H_1, H_2 be two subgroups of $G = \pi_1 Y$, and for $i = 1, 2$ let $(X_i, x_i) \rightarrow (Y, y)$ be a combinatorial immersion of graphs inducing the inclusion $H_i \hookrightarrow G$. The intersection $H_1 \cap H_2$ is represented by a combinatorial immersion $(X_1 \otimes_Y X_2, (x_1, x_2)) \rightarrow (Y, y)$.

In particular, when Y is a bouquet of circles with $\pi_1 Y = G$, and $(X, x) \rightarrow (Y, y)$ is a combinatorial immersion inducing $H = \pi_1 X \hookrightarrow G$, then for every pair of (not necessarily distinct) vertices $x_1, x_2 \in X$, the group $\pi_1(X \otimes_Y X, (x_1, x_2))$ is an intersection $H^{g_1} \cap H^{g_2}$ for some $g_1, g_2 \in G$. In fact, every non-trivial intersection $H \cap H^g$ is equal $\pi_1(X \otimes_Y X, (x_1, x_2))$ where $x_1 = x$, and x_2 is some (possibly the same) vertex in X . When $x_2 = x$, the component of $X \otimes_Y X$ containing (x, x) is a copy of X , which we refer to as a *diagonal component*. The group $\pi_1(X \otimes_Y X, (x, x))$ is the intersection $H \cap H^g = H$, i.e. where $g \in H$. A connected component of $X \otimes_Y X$ that has no edges is called *trivial*.

1.4. Graph of groups and spaces. We recall the definitions of a graph of groups and a graph of spaces, following [SW79].

A *graph of spaces* consists of

- a graph Γ , called the *underlying graph*,
- a collection of CW-complexes X_v for each $v \in V(\Gamma)$, called *vertex spaces*,

- a collection of CW-complexes X_e for each $e \in E(\Gamma)$, called *edge spaces*, and
- a collection of continuous π_1 -injective maps $f_{(e,\pm)} : X_e \rightarrow X_{e^\pm}$ for each $e \in E(\Gamma)$.

The *total space* $X(\Gamma)$ is defined as

$$X(\Gamma) = \bigsqcup_{v \in V(\Gamma)} X_v \sqcup \bigsqcup_{e \in E(\Gamma)} X_e \times [-1, 1] / \sim$$

where $(x, \pm 1) \sim f_{(e,\pm)}(x)$ for $x \in X_e$.

Similarly, a *graph of groups* consists of

- the *underlying graph* Γ ,
- a collection of *vertex groups* G_v for each $v \in V(\Gamma)$,
- a collection of *edge groups* G_e for each $e \in E(\Gamma)$, and
- a collection of injective homomorphisms $\phi_{(e,\pm)} : G_e \rightarrow G_{e^\pm}$ for each $e \in E(\Gamma)$.

The *fundamental group of a graph of groups* is defined as the fundamental group of the graph of spaces $X(\Gamma)$, where $X_v = K(G_v, 1)$ for each $v \in V(\Gamma)$, $X_e = K(G_e, 1)$ for each edge $e \in E(\Gamma)$, and $f_{(e,\pm)}$ induces homomorphism $\phi_{(e,\pm)}$ on the fundamental groups. Note that the fundamental group $\pi_1\Gamma$ is a subgroup of G .

1.5. HNN extensions and doubles. We will denote the *HNN extension of A relative to $\beta : B \rightarrow A$* where $B \subseteq A$ by $A *_B, \beta$, i.e.

$$A *_B, \beta = \langle A, t \mid t^{-1}xt = \beta(x) \text{ for all } x \in B \rangle.$$

The generator t is called the *stable letter*. Note that $A *_B, \beta$ can be viewed as a graph of group $G(\Gamma)$ where Γ is a single vertex v with a single loop e , $G_v = A$, $G_e = B$, $\phi_{(e,-)}$ is the inclusion of B in A , and $\phi_{(e,+)} = \beta$.

A *double of A along C twisted by an automorphism $\beta : C \rightarrow C$* , denoted by $D(A, C, \beta)$, is an amalgamated product $A *_C A$, where C is mapped to the first factor via the standard inclusion, and to the second via the standard inclusion precomposed with β . As usual, $D(A, C, \beta)$ depends only on the outer automorphism class of β , and not a particular representative.

The double $D(A, C, \beta)$ can be viewed as a graph of groups $G(\Gamma)$ where Γ is a single edge e with distinct endpoints, $G_{e^\pm} = A$, $G_e = C$, and $\phi_{(e,-)}$ is the inclusion of C in A , and $\phi_{(e,+)}$ is the inclusion precomposed with β . Note that an amalgamated product $A *_C B$ where $[B : C] = 2$ has an index two subgroup $D(A, C, \beta)$ where $\beta : C \rightarrow C$ is conjugation by some (any) representative $g \in B$ of the non-trivial coset of B/C .

In both situations where there is unique edge in the underlying graph Γ , we will skip the label e , and we will denote $\phi_{(e,\pm)}$ simply by ϕ_\pm . Similarly, in a graph of spaces with a unique edge e , we will write f_\pm instead of $f_{(e,\pm)}$.

1.6. Triangle groups. A *triangle (Coxeter) group* is given by the presentation

$$W_{MNP} = \langle a, b, c \mid a^2, b^2, c^2, (ab)^M, (bc)^N, (ca)^P \rangle.$$

The group W_{MNP} acts as a reflection group on

- the sphere, if $\frac{1}{M} + \frac{1}{N} + \frac{1}{P} > 1$,
- the Euclidean plane, if $\frac{1}{M} + \frac{1}{N} + \frac{1}{P} = 1$,
- the hyperbolic plane, if $\frac{1}{M} + \frac{1}{N} + \frac{1}{P} < 1$.

Hyperbolic triangle groups are commensurable with the fundamental groups of negatively curved surfaces, and therefore they are locally quasiconvex, virtually special, and Gromov-hyperbolic. A *von Dyck triangle group* is an index 2 subgroup of W_{MNP} with the presentation

$$\langle x, y \mid x^M, y^N, (x^{-1}y)^P \rangle$$

obtained by setting $x = ba$ and $y = bc$.

2. SPLITTINGS

The goal of this section is to prove Theorem A. In [Jan20] we proved the following.

Theorem 2.1 ([Jan20, Thm B]). The Artin group Art_{MNP} with $M, N, P \geq 3$ splits as an amalgamated product or an HNN extension of finite rank free groups.

The remaining cases in Theorem A are Art_{2MN} where $M, N \geq 4$. The case where M, N are both even is Proposition 2.5, and the other case is Corollary 2.9. We start with considering a non-standard presentation of Art_{2MN} . In the next two subsections we construct the splittings. Then we show that the only spherical Artin groups that split as graphs of free groups are dihedral Artin groups. Spherical Artin group on 3 generators Art_{MNP} correspond to one of the following triples $(2, 2, P)$ for any $P \geq 2$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$. For completeness, in the last subsection we include the proof that the three generator Artin groups with at least one ∞ label admit splittings as HNN extensions or amalgamated products of finite rank free groups.

The Artin group Art_{236} splits as $F_3 *_{F_7} F_4$ by Squier [Squ87]. The only remaining three generator Artin groups are Art_{23M} where $M \geq 7$, and the following remains unanswered.

Question 2.2. Does the Artin group Art_{23M} where $M \geq 7$ split as a graph of finite rank free groups?

We conjecture that the answer is positive. More generally, we ask the following.

Question 2.3. Do all 2-dimensional Artin group split as a graph of finite rank free groups?

2.1. Presentations of Art_{2MN} . Here is the standard presentation of Artin group Art_{2MN} :

$$\langle a, b, c \mid (a, b)_M = (b, a)_M, (b, c)_N = (c, b)_N, ac = ca \rangle.$$

Let $x = ab$ and $y = cb$ and consider a new presentation of Art_{2MN} with generators b, x, y . The relation $(a, b)_M = (b, a)_M$ is replaced by $bx^mb^{-1} = x^m$ when $M = 2m$, and by $bx^mb = x^{m+1}$ when $M = 2m+1$. We denote this relation by $r_M(b, x)$. Note that $yx^{-1} = ca^{-1}$, so relation $ac = ca$ can be replaced by $yx^{-1} = bx^{-1}yb^{-1}$. See Figure 1(a).

This gives us the following presentation

$$(*) \quad \text{Art}_{2MN} = \langle b, x, y \mid r_M(b, x), r_N(b, y), bx^{-1}yb^{-1} = yx^{-1} \rangle.$$

Let X_{2MN} be the presentation complex associated to the presentation (*). Let X_A be the bouquet of two loops labelled by x and y . The complex X_{2MN} can be viewed as a union of the graph X_A and for each relation in (*): a cylinder (for relations $bx^{-1}yb^{-1} = yx^{-1}$ and each $r_M(b, x)$ with M even) or a Mobius strip (for each relation $r_M(b, x)$ with M odd) with boundary cycles are glued to X_A . We can metrize them so that the height of each cylinder/Mobius strip is equal 2.

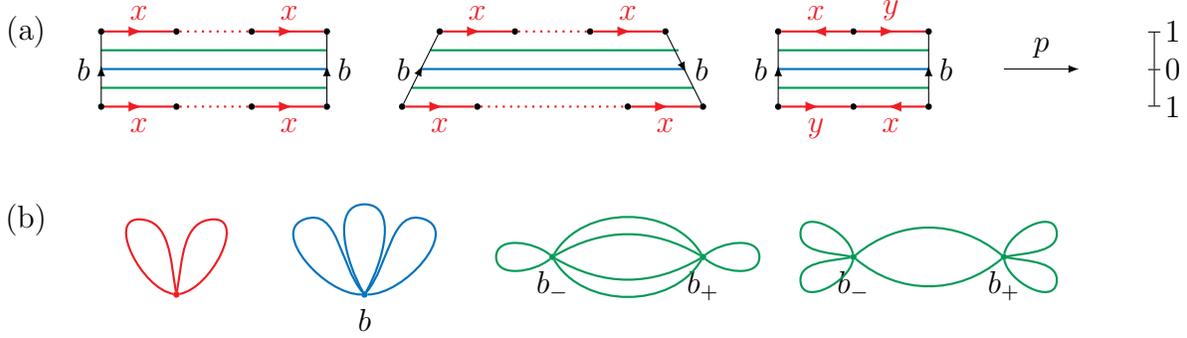


FIGURE 1. (a) The relation $R_M(b, x)$ where M is even (left) and odd (middle), and the relation $bx^{-1}yb^{-1} = yx^{-1}$ (right), with the projection p of the cells onto the interval $[0, 1]$. The horizontal graphs X_A, X_B, X_C are the preimages $p^{-1}(1), p^{-1}(0), p^{-1}(\frac{1}{2})$ respectively. (b) Graphs X_A, X_B and two versions of X_C depending on whether M, N are both odd (left green), or one of M, N is even (right green).

We now define a map $p : X_{2MN} \rightarrow [0, 1]$ by describing the restriction of p to each cylinder/Mobius strip of X_{2MN} . Each point of the cylinder/Mobius strip is mapped to its distance from the center circle of that cylinder/Mobius strip. In particular, the center circle of each cylinder or Mobius strip is mapped to 0, and the boundary circles of the cylinder or Mobius strip are mapped to 1. See Figure 1(a).

We can identify X_A with the preimage $p^{-1}(1)$. We define a graph X_B as the union of all the center circles, i.e. the preimage $p^{-1}(0)$. We also define a subgraph X_C of X_{2MN} as the preimages $p^{-1}(\frac{1}{2})$. The graphs X_A, X_B and X_C are illustrated in Figure 1(b). The graph X_C has two vertices, which are its intersection with the edge b . We denote them by b_-, b_+ , so that b_- , the midpoint of the edge b , b_+ are ordered consistently with the orientation of the edge b . When M, N are both even, then the graph X_C is not connected. Indeed, each of its connected components is a copy of X_B . We denote the connected component containing the vertex b_- by X_B^- , and the component containing the vertex b_+ by X_B^+ . Otherwise, if at least one M, N is odd, then X_C is a connected double cover of X_B . In the next two sections, we describe the graph of spaces decomposition of X_{2MN} associated to the map p , and the induced graph of groups decomposition of Art_{2MN} . We consider separately the case where M, N are both even, and the case where at least one of them is odd.

2.2. Both even. In the case where both M, N are even ≥ 4 , Presentation $(*)$ of Art_{2MN} is the standard presentation of an HNN-extension.

Proposition 2.4. Let $M = 2m$ and $N = 2n$ be both even and ≥ 4 . Then Art_{2MN} splits as an HNN-extension $A *_B \beta$ where $A = \langle x, y \rangle \simeq F_2$ and $B = \langle x^m, y^n, x^{-1}y \rangle \simeq F_3$ and $\beta : B \rightarrow A$ is given by $\beta(x^m) = x^m$, $\beta(y^n) = y^n$ and $\beta(x^{-1}y) = yx^{-1}$.

Proof. Presentation $(*)$ is the standard presentation of the HNN-extension $A *_B \beta$ with the stable letter b . \square

Alternatively, the splitting of Art_{2MN} as above, can be deduced from a graph of spaces decomposition of X_{2MN} .

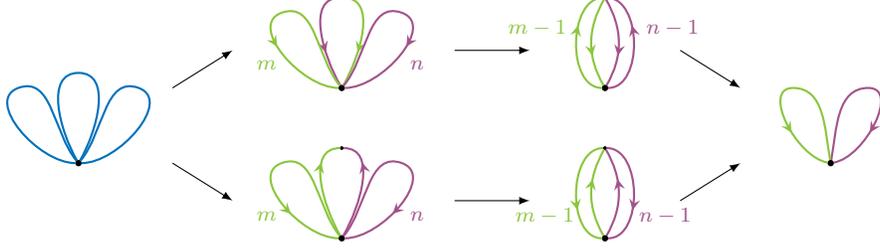


FIGURE 2. The map $f_- : X_B \rightarrow \bar{X}_B^- \rightarrow X_A$ (top), and $f_+ : X_B \rightarrow \bar{X}_B^+ \rightarrow X_A$ (bottom).

Proposition 2.5. Let $M = 2m$ and $N = 2n$ be both even and ≥ 4 . Then X_{2MN} is a graph of spaces $X(\Gamma)$ where Γ is a single vertex with a single loop. The vertex space is the graph X_A , the edge space is the graph X_B and the two maps $X_B \rightarrow X_A$ are given in the Figure 2.

Proof. Indeed, the map p factors as $X_{2MN} \xrightarrow{\tilde{p}} S^1 = [-1, 1]/(-1 \sim 1) \rightarrow [0, 1]$ where the second map is the absolute value, and where \tilde{p} sends the loop b isometrically onto S^1 and is extended linearly. By construction, the preimage $\tilde{p}^{-1}(t)$ is homeomorphic X_B when $t \in (-1, 1)$, and to X_A when $t = 1$. In particular, X_{2MN} can be expressed as a graph of spaces, induced by \tilde{p} , where the cellular structure of S^1 consists of a single vertex $v = 1$ and a single edge e . Indeed,

$$X_{2MN} = X_A \cup X_B \times [-1, 1]/(x, -1) \sim f_-(x), (x, 1) \sim f_+(x)$$

where $f_-, f_+ : X_B \rightarrow X_A$ are the two maps obtained by “pushing” the graph X_B in Figure 1(a) “upwards” and “downwards” respectively. See Figure 2 for f_-, f_+ expressed as a composition of Stallings fold and a combinatorial immersion. \square

Remark 2.6. The subgroup B and $\beta(B)$ are conjugate. See Figure 2. Indeed, the graphs \bar{X}_B^- and \bar{X}_B^+ are identical (but have different basepoints).

Example 2.7 (Group Art_{244}). In the case where $M = N = 4$, Proposition 2.5 provides the splitting of $\text{Art}_{244} = A *_{B, \beta}$ where $A = \langle x, y \rangle$ and $B = \langle x^2, y^2, x^{-1}y \rangle$, and $\beta : B \rightarrow B$ is given by $\beta(x^2) = x^2, \beta(y^2) = y^2, \beta(x^{-1}y) = yx^{-1}$. In particular, B has index 2 in A . This splitting was first proven by Squier [Squ87].

2.3. At least one odd. We now assume that at least one M, N is odd. We have the following description of the complex X_{2MN} .

Proposition 2.8. The complex X_{2MN} is a graph of spaces with the underlying graph is an interval the vertex spaces are graphs X_A and X_B , and the edge space is X_C . The attaching map $X_C \rightarrow X_B$ is a double cover, and the attaching map $X_C \rightarrow X_A$ factors as $X_C \rightarrow \bar{X}_C \rightarrow X_A$, illustrated in Figure 3, where the first map is a homotopy equivalence and the second map is a combinatorial immersion.

Proof. Indeed, X_{2MN} can be obtained as a union of X_A, X_B and $X_C \times [0, 1]$ where $X_C \times \{1\}$ is glued to X_A and $X_C \times \{0\}$ is glued to X_B . Note that the preimage $p^{-1}([0, \frac{1}{2}])$ is a union of “half” cylinders and Mobius strip, and its boundary is the graph X_C . The projection onto the center circle of the boundary of each cylinder or Mobius strip is a (connected or not) double cover of the center circle. It follows that $X_C \rightarrow X_B$ is a double cover. The map $X_C \rightarrow X_A$ is induced by “pushing” X_C “downwards” and “upwards” onto X_A , and it can

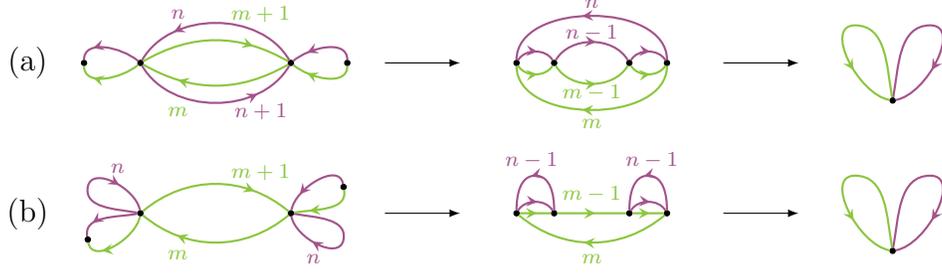


FIGURE 3. The maps $X_C \rightarrow \bar{X}_C \rightarrow X_A$ when (a): $M = 2m + 1, N = 2n + 1$ are both odd; and (b): $M = 2m + 1, N = 2n$.

be described by the labelling of the right graphs in Figure 3. The factorization $X_C \rightarrow \bar{X}_C$ is obtained by performing Stallings folds. Note that the middle graphs in Figure 3 are fully folded, provided that $m - 1, n - 1 > 0$, which is equivalent to the condition that $M, N \geq 4$. It follows that the map $\bar{X}_C \rightarrow X_A$ is a combinatorial immersion. Since the rank of $\pi_1 X_C$ and $\pi_1 \bar{X}_C$ are both equal 5, the folding map is a homotopy equivalence. \square

Corollary 2.9. Suppose at most one of M, N is even and $M, N \geq 4$. Then Art_{2MN} splits as a free product with amalgamation $A *_C B$ where $A = F_2$ and $B = F_3$, and $C = F_5$.

Proof. This directly follows from Proposition 2.8. We get that $\text{Art}_{2MN} = \pi_1 X_{2MN} = A *_C B$ where $A = \pi_1 X_A$, $B = \pi_1 X_B$ and $C = \pi_1 X_C$. From Figure 1, we see that $\text{rk } A = 2$, $\text{rk } B = 3$ and $\text{rk } C = 5$. \square

2.4. Splittings of spherical Artin groups. All spherical Artin groups have non-trivial center and their cohomological dimension is equal to the number of standard generators [Del72], [BS72]. We now give a characterization of graph of finite rank free groups with non-trivial center. This allows us to deduce that the only spherical Artin groups that split as graphs of finite rank free groups are the dihedral Artin groups (i.e. on two generators).

Proposition 2.10. Let $G = G(\Gamma)$ be a finite graph of free groups. Then the following conditions are equivalent.

- (1) the center of G is non-trivial,
- (2) $G(\Gamma)$ satisfies one of the following:
 - (a) all the vertex groups and edge groups are \mathbb{Z} and for every $h \in G_v$ and $g \in \pi_1 \Gamma$ such that $g^{-1}hg \in G_v$ we have $g^{-1}hg = h$,
 - (b) $\pi_1 \Gamma \simeq \mathbb{Z} = \langle g \rangle$, all vertex and edge groups are isomorphic, and there exists $n \in \mathbb{N}$ for every $v \in V(\Gamma)$ and $h \in G_v$ we have $g^{-n}hg^n = h$.

In particular, every finite graph of free groups with non-trivial center is virtually $F \times \mathbb{Z}$ where F is a free group.

Proof. We first show that $G(\Gamma)$ satisfying (2) has non-trivial center. Suppose that all the vertex groups and edge groups of G are \mathbb{Z} . That in particular implies that all the inclusions $G_e \rightarrow G_v$ are inclusions of finite index subgroup. Since there are finitely many vertex and edge groups, their intersection H is non-empty and has finite index in each vertex and edge group. If for every $h \in G_v$ and $g \in \pi_1 \Gamma$ such that $g^{-1}hg \in G_v$ we have $g^{-1}hg = h$, then H is

central in G . Note that the quotient G/H is a finite graph of finite groups, so it is virtually free. It follows that G is virtually $F \times \mathbb{Z}$ for some free group F .

Now suppose that $\pi_1\Gamma \simeq \mathbb{Z}$, all vertex and edge groups are isomorphic, and for every $g \in \pi_1\Gamma$ and every $h \in G_v$ we have $g^{-1}hg = h$. Then $G \simeq G_v \times \pi_1\Gamma = F \times \mathbb{Z}$, and it has non-trivial center.

We now prove the other direction. Consider a graph of groups $G(\Gamma)$ with the fundamental group G whose center is non-trivial. We have $Z(G) \cap G_v \subset Z(G_v)$ for every vertex v , so either G_v is infinite cyclic, or it intersects the center trivially. The same is true for edge groups. If any G_v intersects the center non-trivially, then all the vertex group and edge groups must intersect the center nontrivially, and in particular they are all infinite cyclic. Then for $h \in Z(G) \cap G_v$, and $g \in \pi_1\Gamma$, we must have $g^{-1}hg = h$, so $G(\Gamma)$ satisfies Condition (a).

Now suppose that every vertex and edge group intersects the center trivially. The center must be contained in the center of $\pi_1\Gamma$. Since $\pi_1\Gamma$ is a free group, it must be \mathbb{Z} , and it must be an element g^n such that $g^{-n}hg^n = h$ for every $h \in G_v$ and every $v \in V(\Gamma)$. The condition (b) is satisfied. □

Corollary 2.11. The only irreducible spherical Artin group that split as non-trivial graphs of free groups, are the dihedral Artin groups.

Proof. Let Art_M be a dihedral Artin group with the presentation $\langle a, b \mid (a, b)_M = (b, a)_M \rangle$. Let $M = 2m$ be even, and let $x = ab$. Then $\text{Art}_M \simeq \langle a, x \mid ax^ma^{-1} = x^m \rangle$. In particular, $\text{Art}_M = \langle x \rangle *_{\langle x^m \rangle} \mathbb{Z} *_{\mathbb{Z}}$. Now if $M = 2m + 1$, let $x = ab$ and $y = (a, b)_M$. Then $\text{Art}_M \simeq \langle x, y \mid x^M = y^2 \rangle$. In particular, $\text{Art}_M \simeq \langle x \rangle *_{\langle x^M \rangle} \langle y \rangle = \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$. By Proposition 2.10, spherical Artin groups on more than two generators do not split as amalgamated products of HNN extension of nontrivial finite rank free groups, since their virtual cohomological dimension is greater than 2, which is the virtual cohomological dimension of $F \times \mathbb{Z}$. □

2.5. Splittings of three generator Artin groups with ∞ labels. To complete the picture, the remaining three generator Artin groups, i.e. those with at least one ∞ label, also split as graphs of finite rank free groups. The Artin group $\text{Art}_{\infty\infty\infty}$ is the free group on three generators. The group $\text{Art}_{M\infty\infty} = \text{Art}_M * \mathbb{Z}$ can be described as $\langle x, c \rangle *_{x^M=y^2} \langle y \rangle = F_2 *_{\mathbb{Z}} \mathbb{Z}$ where $x = ab$ and $y = (a, b)_M$. Finally, for the Artin group $\text{Art}_{MN\infty}$ we can use the presentation (*) skipping the relation $bx^{-1}yb^{-1} = yx^{-1}$, i.e.

$$\text{Art}_{MN\infty} = \langle x, y, b \mid r_M(b, x), r_N(b, y) \rangle$$

We get that $\text{Art}_{MN\infty}$ splits as

- $A *_{B, \beta}$ where $A = \langle x, y \rangle \simeq F_2$, $B = \langle x^m, y^n \rangle \simeq F_2$ $\beta = \text{id}_C$, when $M = 2m$ and $N = 2n$,
- $A *_C B$ where $A = \langle x, y \rangle \simeq F_2$, $B \simeq F_2$ and $C \simeq F_3$, when at least one of M, N is odd. The splitting is obtained in the same way as in the case of Art_{2MN} .

3. RESIDUAL FINITENESS

In the section we prove Theorem B. We do so separately in the case where M, N are both even (Theorem 3.6), and where exactly one of M, N is odd (Theorem 3.7). We start with recalling a criterion for residual finiteness of amalgamated products and HNN extensions of

finite rank free groups, proved in [Jan20], which relies on Wise’s result on residual finiteness of *algebraically clean* graphs of free groups [Wis02]. In the second subsection we compute the intersections of conjugates of the amalgamating subgroup in the factor groups. Finally, we give proofs of the main theorems.

We start with a motivating example.

Example 3.1 (Group Art_{244}). By Example 2.7 the group Art_{244} fits in the following short exact sequence of groups

$$1 \rightarrow C \rightarrow \text{Art}_{244} \rightarrow \mathbb{Z}/2\mathbb{Z} * \mathbb{Z} \rightarrow 1.$$

In particular, Art_{244} is a (finite rank free group)-by-(virtually free group), and so it is virtually (finite rank free group)-by-(free group). The residual finiteness of Art_{244} follows from the fact that every split extension of a finitely generated residually finite group by residually finite group is residually finite [Mal56].

To summarize, the only three generator Artin groups that are not known to be residually finite are $\text{Art}_{33(2m+1)}$ for $m \geq 2$, $\text{Art}_{2(2m+1)(2n+1)}$ for $m + n \geq 4$ and $\text{Art}_{23(2m)}$ for $m \geq 4$. Indeed, if at least one label is ∞ then the defining graph is a tree, and hence virtually special [Bru92], [HM99], [Liu13], [PW14]. Artin groups Art_{22M} for any $M \geq 2$, and Art_{23M} where $M \in \{3, 4, 5\}$ are spherical, and so linear [CW02], [Dig03]. The cases $(3, 3, 3)$, $(2, 4, 4)$ and $(2, 3, 6)$ follows from [Squ87]. The cases where $M, N, P \geq 3$, except the case of $(3, 3, 2m + 1)$, were covered by [Jan20]. The rest of the cases are proven here.

3.1. Criteria for residual finiteness. Recall that a subgroup C is *malnormal* in a group A , if $C \cap g^{-1}Cg = \{1\}$ for every $g \in A - C$. Similarly, C is *almost malnormal* in A , if $|C \cap g^{-1}Cg| < \infty$ for every $g \in A - C$. More generally, a collection of subgroup $\{C_i\}_{i \in I}$ is an *almost malnormal* collection, if $|C_i \cap gC_jg^{-1}| < \infty$ whenever $g \notin C_i$ or $i \neq j$.

We now assume that the inclusion of free groups $C \rightarrow A$ is induced by a map $f : X_C \rightarrow X_A$ of graphs, and the automorphism $\beta : C \rightarrow C$ is induced by some graph automorphism $X_C \rightarrow X_C$. The following theorem was proven in [Jan20].

Theorem 3.2 ([Jan20, Thm 2.9]). Let $\hat{f} : \hat{X}_C \rightarrow \hat{X}_A$ be a map of 2-complexes that restricted to the 1-skeletons is equal f , and let $\pi : A \rightarrow \hat{A}$ be the natural quotient induced by the inclusion $X_A \hookrightarrow \hat{X}_A$ of the 1-skeleton. Suppose that the following conditions hold.

- (1) \hat{A} is a locally quasiconvex, virtually special hyperbolic group.
- (2) $\pi(C) = \pi_1 \hat{X}_C$ and that the lift of \hat{f} to the universal covers is an embedding.
- (3) $\pi(C)$ is almost malnormal in \hat{A} .
- (4) β projects to an isomorphism $\pi(C) \rightarrow \pi(C)$.

Then $D(A, C, \beta)$ is residually finite.

The theorem above is a combination of Thm 2.9 and Lem 2.6 in [Jan20]. Condition (2) in the statement of [Jan20, Thm 2.9] is that $\pi(C)$ is *malnormal* in \hat{A} . However, the proof is identical when we replace it with *almost malnormal*. Indeed, the Bestvina-Feighn combination theorem [BF92], as well as the Hsu-Wise combination theorem [HW15] only require almost malnormality.

We now state a version for HNN extension. Similarly as above, combining Thm 2.12 and Lem 2.6 from [Jan20], we obtain the following.



FIGURE 4. A connected component of $\bar{X}_C \otimes_{X_A} \bar{X}_C$ when (a): $M = 2m+1, N = 2n$, and (b): $M = 2m+1, N = 2n+1$ (right).

Theorem 3.3 ([Jan20, Thm 2.12]). Let $\hat{f}_-, \hat{f}_+ : \hat{X}_B \rightarrow \hat{X}_A$ be two maps of 2-complexes that restricted to the 1-skeletons are equal to f_-, f_+ respectively, and let $\pi : A \rightarrow \hat{A}$ be the natural quotient induced by the inclusion $X_A \hookrightarrow \hat{X}_A$. Suppose that the following conditions hold.

- (1) \hat{A} is a locally quasiconvex, virtually special hyperbolic group
- (2) $\pi(B) = \hat{f}_{-*}(\pi_1 \hat{X}_B)$, and $\pi(\beta(B)) = \hat{f}_{+*}(\pi_1 \hat{X}_B)$ and the lifts of \hat{f}_-, \hat{f}_+ to the universal covers are both embeddings.
- (3) The collection $\{\pi(B), \pi(\beta(B))\}$ is almost malnormal in \hat{A} ,
- (4) $\beta : B \rightarrow \beta(B)$ projects to an isomorphism $\pi(B) \rightarrow \pi(\beta(B))$.

Then $A *_{B, \beta}$ is residually finite.

3.2. Intersection of the conjugates of the amalgamating subgroup. Let $G = \text{Art}_{2MN}$ where $M, N \geq 4$ and at least one of them is odd. Let A, B, C be free groups of ranks 2, 3, 5 respectively, provided by Corollary 2.9. In this section we describe intersections $C \cap g^{-1}Cg$ where $g \in A$.

Proposition 3.4. Let $M = 2m+1$ be odd, $N = 2n$ even, and let A, B, C be as in Corollary 2.9. Let $g \in A - C$. Then the intersection $C \cap g^{-1}Cg$ is one of: $\langle x^{2m+1}, y^n, x^{-1}y \rangle$, $\langle x^{2m+1}, y^n, yx^{-1} \rangle$, $\langle x^{2m+1}, y^n \rangle$, $\langle x^{2m+1} \rangle$, $\langle y^n \rangle$.

Proof. Let X_A, \bar{X}_C be as in Figure 3(b). By Theorem 1.1 the conjugates $C \cap g^{-1}Cg$ can be represented by the connected components of the fiber product $\bar{X}_C \otimes_{X_A} \bar{X}_C$. Let x_1, x_2, x_3, x_4 be the four vertices of valence 4 in \bar{X}_C such that x_1, x_2 belong to the same y -cycle and x_3, x_4 belong to the same y -cycle, and so that the ordering of the indices is consistent with the order of the vertices on the x -cycle. Then the connected component of $\bar{X}_C \otimes_{X_A} \bar{X}_C$ containing one of (x_i, x_j) is

- the graph in Figure 4(a), if $|i - j| = 2$, in which case $C \cap g^{-1}Cg$ is $\langle x^{2m+1}, y^n, x^{-1}y \rangle$ or $\langle x^{2m+1}, y^n, yx^{-1} \rangle$, or
- a bouquet of two circles, labelled by x^{2m+1} and y^n otherwise, in which case $C \cap g^{-1}Cg$ is $\langle x^{2m+1}, y^n \rangle$.

Every other non-diagonal connected component of $\bar{X}_C \otimes_{X_A} \bar{X}_C$ is either trivial or a single circle, which is labelled by either x^{2m+1} or y^n , in which case $C \cap g^{-1}Cg$ is $\langle x^{2m+1} \rangle$ or $\langle y^n \rangle$ respectively. \square

We finish with the following observation regarding the case where M, N are both odd.

Remark 3.5. Let $M = 2m+1, N = 2n+1$ be both odd, and let A, B, C be as in Corollary 2.9. Let $g \in A - C$. Then the intersection $C \cap g^{-1}Cg$ is one of: $\langle x^{2m+1}, y^n, x^{-1}y \rangle$,

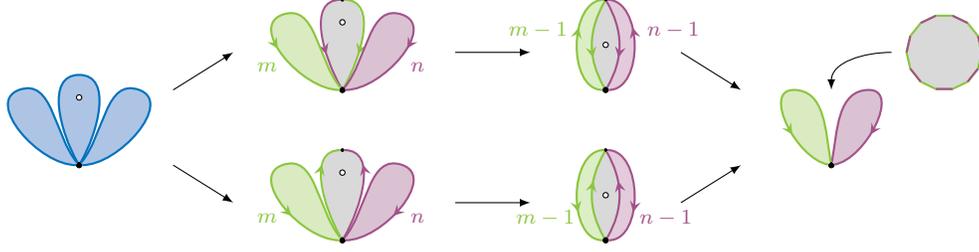


FIGURE 5. The map $\hat{f}_- : \hat{X}_B \rightarrow \hat{X}_B^- \rightarrow \hat{X}_A$ (top), and $\hat{f}_+ : \hat{X}_B \rightarrow \hat{X}_B^+ \rightarrow \hat{X}_A$ (bottom). White nodes are contained in the 2-cells whose boundary is mapped p -to-1.

$\langle x^{2m+1}, y^n \rangle, \langle x^{2m+1} \rangle, \langle y^n \rangle$. Let X_A, \bar{X}_C be as in Figure 3. Let x_1, x_2, x_3, x_4 be the four vertices of valence 4 in \bar{X}_C ordered consistently with the orientation of the x -cycle such that x_1 and x_4 are at distance m in the x -cycle. Then the connected component of $\bar{X}_C \otimes_{X_A} \bar{X}_C$ containing vertices $(x_1, x_3), (x_2, x_4), (x_4, x_1)$ (or vertices $(x_3, x_1), (x_4, x_2), (x_1, x_4)$) looks like the right graph in Figure 4.

3.3. Proof of residual finiteness. We now use Theorem 3.3 to prove that Art_{2MN} where M, N are even and equal at least 4 is residually finite.

Theorem 3.6. Let $M = 2m$ and $N = 2n$ be both even and ≥ 4 . Then Art_{2MN} is residually finite.

Proof. The case where $M = N = 4$ is proven in Example 3.1, so we assume that $M \geq 6$. By Proposition 2.5, Art_{2MN} splits as an HNN-extension $A *_B \beta$ where $A = \langle x, y \rangle$ and $B = \langle x^m, y^n, x^{-1}y \rangle$, and $\beta : B \rightarrow A$ is given by $x^m \mapsto x^m, y^n \mapsto y^n$ and $x^{-1}y \mapsto yx^{-1}$. We deduce residual finiteness of Art_{2MN} from Theorem 3.3. We now check that all its assumptions are satisfied.

Let $\hat{A} = \langle x, y \mid x^m, y^n, (x^{-1}y)^p \rangle$ where $p \geq 7$, and let \hat{X}_A be the presentation complex of \hat{A} . Let $\pi : A \rightarrow \hat{A}$ be the natural quotient. Since $m \geq 3$, the group \hat{A} is a hyperbolic von Dyck triangle group, and in particular Condition (1) of Theorem 3.3 is satisfied.

The image $\pi(B)$ is a finite cyclic group \mathbb{Z}/p of order p generated by $x^{-1}y$, and the image $\pi(\beta(B))$ is a copy of \mathbb{Z}/p generated by yx^{-1} . Since $\pi(B), \pi(\beta(B))$ are finite groups, they form an almost malnormal collection in \hat{A} , so Condition (3) in Theorem 3.3 is satisfied. Let \hat{X}_B be obtained from X_B by attaching a 2-cell to each of the left and the right loop of X_B (left and right in Figure 2) via a 1-to-1 map (corresponding to the relations x^m, y^n), and one 2-cell to the middle loop via a p -to-1 map (corresponding to the relation $(x^{-1}y)^p$). It is immediate that Conditions (2) and (4) of Theorem 3.3 holds. See Figure 5. The proof is complete. □

Similarly, we use Theorem 3.2 to prove that Art_{2MN} where one of M, N is odd, is residually finite.

Theorem 3.7. Let $M = 2m + 1$ and $N = 2n$ be both ≥ 4 . Then Art_{2MN} is residually finite.

Proof. We will use Theorem 3.2. Let $\hat{A} = \langle x, y \mid x^{2m+1}, y^n, (x^{-1}y)^p \rangle$ where $p \geq 6$ and let \hat{X}_A be its presentation 2-complex.

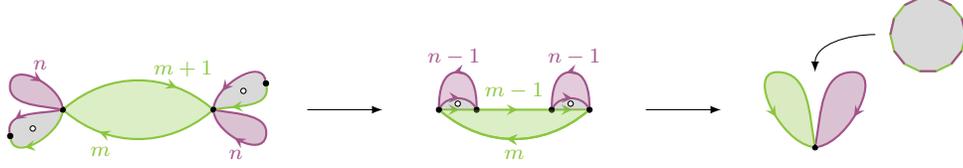


FIGURE 6. The maps $\hat{X}_C \rightarrow \hat{X}_C \rightarrow \hat{X}_A$. White nodes are in the 2-cells whose boundary is mapped p -to-1.

Since $n \geq 2$ and $2m + 1 \geq 5$, the group \hat{A} is a hyperbolic von Dyck triangle group, and in particular, \hat{A} satisfies Condition (1) of Theorem 3.2. Let $\pi : A \rightarrow \hat{A}$ be the natural quotient. Let \hat{X}_C be a 2-complex obtained from \bar{X}_C by attaching five 2-cells: one along the unique cycle labelled x^{2m+1} , one along each of the two cycles labelled by y^n , and one along each of two cycles xy^{-1} via a p -to-one map. See Figure 6. In Lemma 3.8 (below) we verify that Condition (2) of Theorem 3.2 is satisfied.

By Proposition 3.4, the intersection of distinct conjugates of $\pi(C)$ in \hat{A} is either \mathbb{Z}/p or trivial. In particular, $\pi(C)$ is almost malnormal in \hat{A} , so Condition (3) of Theorem 3.2 is satisfied. Finally, we note that the 2-cells of \hat{X}_C can be pulled back via $X_C \rightarrow \bar{X}_C$, and are preserved under the (unique) nontrivial deck transformation of $X_C \rightarrow X_B$. See Figure 6. Condition (4) of Theorem 3.2 follows. This completes the proof. \square

Lemma 3.8. The image $\pi(C)$ is isomorphic to $\mathbb{Z}/p * \mathbb{Z}/p$. In particular, $\pi(C) = \pi_1 \hat{X}_C$. Moreover, \hat{f} lifts to an embedding in the universal covers.

Proof. For simplicity we set $z = xy^{-1}$. The image $\pi(C)$ is generated by z and $z' = x^m z x^{-m}$. The universal cover of the complex \hat{X}_A can be identified with the hyperbolic plane. Consider the tiling of \mathbb{H}^2 by a triangle with angles $\frac{\pi}{2m+1}, \frac{\pi}{n}, \frac{\pi}{p}$. Each vertex of the tiling is a fixed point of a conjugate of one of x, y, z , and the action of \hat{A} preserve the type of a vertex (i.e. whether it is fixed by a conjugate of x, y or z). We abuse the notation and identify each vertex v with the conjugate x^g, y^g or z^g which generates the stabilizer of v (where g is some element of \hat{A}). The tiling is the dual of the universal cover of the complex \hat{X}_A , in the way that the vertices of types x, y, z correspond to the 2-cells with boundary words $x^{2m+1}, y^n, (xy^{-1})^p$ respectively.

Consider the Bass-Serre tree T of the free product $\mathbb{Z}/p * \mathbb{Z}/p$, i.e. a regular tree of valence p , where each vertex is stabilized by a conjugate of one of two \mathbb{Z}/p factors. In order to prove that $\pi(C)$ splits as a free product and that \hat{f} lifts to embedding of the universal covers, we show that there is $\mathbb{Z}/p * \mathbb{Z}/p$ -equivariant embedding of T in \mathbb{H}^2 where the action of $\mathbb{Z}/p * \mathbb{Z}/p$ on \mathbb{H}^2 is the action of the group $\langle z, z' \rangle$.

First consider the union of the orbits of z , and of z' under the action of $\pi(C)$. Note that it is a collection of vertices of type z . We join z^g and $(z')^g$ by a path consisting of two edges of the tiling meeting at a vertex of type x . See Figure 7.

In order to see that T is embedded, we verify that any bi-infinite path γ always turning rightmost in the image of T never crosses itself. In Figure 7 a part of that path is presented as the path with vertices $x_0, z_1, x_2, z_3, x_4, z_5, x_6$. We claim that another path γ' (whose part is labelled by $q_0, s_0, q_1, s_1, \dots, q_5, s_5, q_6$ in Figure 7) stays in a final distance from γ , and

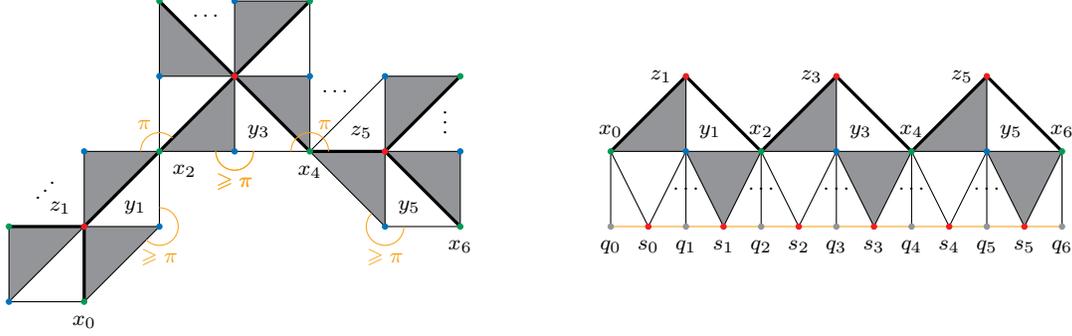


FIGURE 7. The vertices z_1, z_5 are $\pi(C)$ -conjugates of s , and the vertex z_3 is a $\pi(C)$ -conjugate of z . Thick edges are in the image of the tree T . Odd gray vertices (q_1, q_3, q_5) are either green or red, depending on parity of n . Even gray vertices (q_0, q_2, q_4, q_6) are either blue or red, depending on parity of m . Note that the orange segments are not necessarily edges in the tiling.

separates γ from a subspace of \mathbb{H}^2 which is a union of halfspaces. Let us explain how the vertices q_i, s_i are defined. The vertex s_i with odd i is the unique vertex other than z_i that forms a triangle with y_i and x_{i+1} . The vertex s_i with even i is the unique vertex other than z_{i+1} that form a triangle with x_i and y_{i+1} . In particular, vertices s_i are always of type z . The vertices q_i with odd i are images of a z_i under the π -rotation at y_i , i.e. the vertex y_i is a midpoint of the segment $[z_i, q_i]$. The vertices q_i with even i are chosen so that the angle $\angle s_{i-1}x_iq_i$ and $\angle s_ix_iq_i$ are equal. The path γ' is obtained by joining each pair s_{i-1}, q_i and q_i, s_i be a geodesic segment (which are not necessarily edges of the tiling).

The vertices s_{i-1}, q_i, s_i are not necessarily distinct. If $n = 2$, then $s_{i-1} = q_i = s_i$ for each odd i . Also, if $2m + 1 = 5$, then $s_{i-1} = q_i = s_i$ for $i = 4k + 2$. In that extreme case γ' is a geodesic line, as long as $p \geq 6$. In more general case, the “upper” angle between the segments of γ' at each vertex q_i or s_i (i.e. the angle of the sector containing y_i or x_i depending on the parity of i) is always at most π . This proves our claim. \square

Our approach in the last two theorems fails in the case where $M = 2m + 1, N = 2n + 1$. Indeed, the fiber product $\bar{X}_C \otimes_{X_A} \bar{X}_C$ is too “large”. The fiber product was computed in Remark 3.5. After attaching 2-cells along x^{2m+1}, y^{2n+1} and $(x^{-1}y)^p$, the resulting 2-complex has fundamental group $\mathbb{Z} * \mathbb{Z}/p$.

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