

## HOMEWORK 8

Due on Wed, May 27 in class.

**Exercise 1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be monotone. Show that  $f$  is integrable.

**Exercise 2.** Let  $E = \{\frac{1}{n} | n \in \mathbb{N}\}$  and  $f : [0, 1] \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} 1 & x \in E \\ 0 & \text{otherwise} \end{cases} .$$

Show that  $f$  is integrable and that  $\int_0^1 f(x)dx = 0$ .

**Exercise 3.** Let  $f : [0, 1] \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q}, \text{ with } p, q \in \mathbb{N} \text{ co-prime} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

- (1) Show that the set of points where  $f$  is not continuous is  $\mathbb{Q}$ .
- (2) Show that for any  $\epsilon > 0$  there are only finitely many points,  $t_1, \dots, t_n$ , satisfying  $f(t_j) < \epsilon$ .
- (3) Given  $\epsilon > 0$  construct a partition such that  $U(f, P) < \epsilon$ .
- (4) Conclude that  $f$  is integrable and that  $\int_0^1 f(x)dx = 0$ .

**Exercise 4.** Let  $P_n = \{\frac{j}{n} | j = 0, \dots, n\}$  be partitions of  $[0, 1]$ .

- (1) Show that for any bounded function  $f : [0, 1] \rightarrow \mathbb{R}$ ,  $f$  is integrable if and only if

$$\lim_{n \rightarrow \infty} U(f, P_n) = \lim_{n \rightarrow \infty} L(f, P_n) = I_0(f),$$

In which case  $I_0(f) = \int_0^1 f(x)dx$ .

- (2) Compute  $U(f, P_n)$  and  $L(f, P_n)$  and  $\int_0^1 f(x)dx$  for  $f(x) = x$  and for  $f(x) = e^x$ .

**Exercise 5.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function.

- (1) Show that if  $f$  is continuous at  $x_0$  and  $f(x_0) \neq 0$  then  $\int_a^b |f(x)|dx > 0$ .
- (2) Assume that  $f$  is continuous on  $[a, b]$ . Show that  $\int_a^b |f(x)|dx = 0$  iff  $f(x) = 0$  for all  $x \in [a, b]$ .
- (3) Is this true without the continuity assumption?

**Exercise 6.** Let  $f$  be integrable on  $[a, b]$  and  $E \subset [a, b]$  a finite set. Show that if  $g : [a, b] \rightarrow \mathbb{R}$  is a function satisfying  $f(x) = g(x)$  for all  $x \notin E$  then  $g$  is integrable and  $\int_a^b f(x)dx = \int_a^b g(x)dx$ .

**Exercise 7.** Suppose that  $g_n : [0, 1] \rightarrow \mathbb{R}$  are positive integrable functions such that  $\lim_{n \rightarrow \infty} g_n(x) = 0$  pointwise for any  $x \in [0, 1]$ .

- (1) Is it true that  $\lim_{n \rightarrow \infty} \int_0^1 g_n(x)dx = 0$ ? Prove or give a counterexample.
- (2) Assume that  $\lim_{n \rightarrow \infty} \int_0^1 g_n(x)dx = 0$  and show that  $\lim_{n \rightarrow \infty} \int_0^1 f(x)g_n(x)dx = 0$  for any integrable  $f$ .

**Exercise 8.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be an integrable function and let  $\{x_k\}_{k=0}^{\infty}$  be a sequence in  $[a, b]$  with  $\lim_{k \rightarrow \infty} x_k = b$ . Show that

$$\lim_{n \rightarrow \infty} \int_a^{x_n} f(x) dx = \int_a^b f(x) dx.$$