

Mathematics 258
Final Exam
Due March 14, 2001

Solve the following problems, writing your answers clearly and explaining them as completely as possible. This should be turned in at Eckhart 307 by 3:00 on March 14.

1. If F is a field, show that

$$F[x, x^{-1}] = \left\{ \sum_{k=-N}^{k=N} a_k x^k \mid a_k \in F, N \geq 0 \right\}$$

is a Euclidean domain.

2. Suppose $F \subset K$ are fields, and K is n -dimensional as a vector space over F .
- (a) Show that, for any $\lambda \in K$, there is some irreducible polynomial $p(x) \in F[x] - \{0\}$ of degree at most n such that $p(\lambda) = 0$.
- (b) Show that the image of the homomorphism $\varphi : F[x] \rightarrow K$ given by $\varphi(q(x)) = q(\lambda)$ is a subfield of K .
3. How many conjugacy classes of elements of order 5 are there in the group $GL_4(\mathbb{F}_2)$? What are the rational canonical forms of the matrices in these conjugacy classes?
4. (a) Determine the number of isomorphism classes of $\mathbb{Z}[i]$ -modules with 26 elements.
(b) Determine the number of isomorphism classes of $\mathbb{Z}[i]$ -modules with 35 elements.
5. Let p be a prime number. Show that the polynomial $f(x) = 1 + x + x^2 + \cdots + x^{p-1}$ is irreducible in $\mathbb{Q}[x]$. (Hint: Put $y = x - 1$, and show that $f(y)$ is irreducible over $\mathbb{Z}[y]$.)
6. (a) Let F be a field, and $\alpha_1, \dots, \alpha_n$ be n distinct elements of F . Show that

$$\frac{F[x]}{\left(\prod_1^n (x - \alpha_i)\right)} \cong \bigoplus_1^n \frac{F[x]}{(x - \alpha_i)}$$

as $F[x]$ -modules. (Note: the Chinese Remainder Theorem says this holds as an equality of rings.)

- (b) Show that if T is a linear operator on a finite dimensional vector space V over F , and if the minimal polynomial of T splits into a product of distinct linear factors, then T is diagonalisable.
7. Find the number of solutions of the equation $x^7 + 1 = 0$ in the finite fields \mathbb{F}_{71} and \mathbb{F}_{89} .