Math 274 (Spring 2002): Homework 8; due Thursday, May 23

1. (Hodge star operation) Given a \( p \)-form \( \omega \) in \( \mathbb{R}^n \), define an \((n - p)\)-form \( \ast \omega \) by setting
\[
\ast(dx_{i_1} \wedge \ldots \wedge dx_{i_p}) = \text{sgn}(\tau)(dx_{j_1} \wedge \ldots \wedge dx_{j_{n-p}})
\]
and extending it linearly, where \( i_1 < \ldots < i_p, j_i < \ldots < j_{n-p} \) and \( \tau \) is the premutation \((i_1, \ldots, i_p, j_1, \ldots, j_{n-p})\) of \((1, 2, \ldots, n)\). Show that:

a) If \( \omega = a_{12}dx_1 \wedge dx_2 + a_{13}dx_1 \wedge dx_3 + a_{23}dx_2 \wedge dx_3 \) is a 2-form in \( \mathbb{R}^3 \), then
\[
\ast \omega = a_{12}dx_3 - a_{13}dx_2 + a_{23}dx_1;
\]

b) If \( \omega = a_1dx_1 + a_2dx_2 \) is a 1-form in \( \mathbb{R}^2 \), then
\[
\ast \omega = a_1dx_2 - a_2dx_1;
\]

c) \( \ast \ast \omega = (-1)^{p(n-p)} \omega \).

2. (The divergence) A differentiable vector field \( v \) in \( \mathbb{R}^n \) may be considered as a differentiable map \( v: \mathbb{R}^n \to \mathbb{R}^n \). Define a function \( \text{div}v : \mathbb{R}^n \to \mathbb{R}^n \), called the divergence of \( v \), as follows:
\[
(\text{div}v)(x) = \text{trace}(dv)_x, \quad x \in \mathbb{R}^n.
\]
Show that:

a) If \( v = \sum_i a_ie_i \), where \( \{e_i\} \) is the canonical basis of \( \mathbb{R}^n \), then
\[
\text{div}v = \sum_i \frac{\partial a_i}{\partial x_i};
\]

b) If \( \omega \) denotes the differential 1-form obtained from \( v \) by the canonical isomorphism induced by the inner product \( \langle \cdot, \cdot \rangle \) and \( \nu = dx_1 \wedge \ldots \wedge dx_n \) is the volume element of \( \mathbb{R}^n \), the divergence can be obtained as follows:
\[
v \mapsto \omega \mapsto \ast \omega \mapsto d(\ast \omega) = (\text{div}v)\nu.
\]
3. (The gradient) Given a differentiable function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), define a vector field \( \text{grad} f \in \mathbb{R}^n \), called the gradient of \( f \), by
\[
\langle \text{grad} f(x), u \rangle = df_x(u), \quad x \in \mathbb{R}^n, u \in T_x \mathbb{R}^n.
\]
Notice that \( \text{grad} f \) is the vector field corresponding to the 1-form \( df \) in the canonical isomorphism. Show that:

a) In the canonical basis \( \{e_i\} \) of \( \mathbb{R}^n \)
\[
\text{grad} f = \sum_i \frac{\partial f}{\partial x_i} e_i;
\]

b) If \( x \in \mathbb{R}^n \) is such that \( \text{grad} f(x) = 0 \), then \( \text{grad} f(x) \) is perpendicular to the “level set” \( \{ y \in \mathbb{R}^n \mid f(y) = f(x) \} \);

c) The linear map \( df_x : T_x \mathbb{R}^n \rightarrow \mathbb{R} \) restricted to the unit sphere centered at \( x \) reaches its maximum at \( v = \text{grad} f / |\text{grad} f| \).

4. (The Laplacian) Given a differentiable function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), define the Laplacian \( \Delta f : \mathbb{R}^n \rightarrow \mathbb{R} \) by
\[
\Delta f = \text{div}(\text{grad} f).
\]
Show that:

a) \( \Delta f = \sum \frac{\partial^2 f}{\partial x_i^2} \);

b) \( \Delta(fg) = f \Delta g + g \Delta f + 2 \langle \text{grad} f, \text{grad} g \rangle \);

c) \( d \ast (df) = (\Delta f) \nu \), where \( \nu \) is the volume element of \( \mathbb{R}^n \).

5. (The curl) Let \( v \) be a differentiable vector field in \( \mathbb{R}^n \). The curl (or rotational) \( \text{curl} v \) is the \((n-2)\)-form defined by
\[
v \mapsto \omega \mapsto d\omega \mapsto *(d\omega) = \text{curl} v.
\]
a) Prove that \( \text{curl} (\text{grad} f) = 0 \).
b) In the particular case when \( n = 3 \), the 1-form \( \text{curl} v \) corresponds to a vector field which is also denoted by \( \text{curl} v \). Show that, for \( n = 3 \),

\[
\text{curl} \left( \sum_{i=1}^{3} a_i e_i \right) = \left( \frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3} \right) e_1 + \left( \frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1} \right) e_2 + \left( \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \right) e_3
\]

and \( \text{div} (\text{curl} v) = 0 \).

6. Let \( g : \mathbb{R}^3 \rightarrow \mathbb{R} \), \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \) be differentiable functions, and let \( X \subseteq \mathbb{R}^3 \) be a compact differentiable 3-manifold with boundary \( \partial X \). Prove that:

a) (1st Green’s identity)

\[
\int_X \langle \text{grad} f, \text{grad} g \rangle \nu + \int_X f \Delta^2 g \nu = \int_{\partial X} f \langle \text{grad} g, N \rangle \sigma,
\]

where \( \nu \) and \( \sigma \) are the volume element of \( X \) and the area element of \( \partial X \), respectively, and \( N \) is the unit normal of \( \partial X \);

Hint: Set \( v = f \text{grad} g \) in the Divergence Theorem.

b) (2nd Green’s identity)

\[
\int_X (f \Delta^2 g - g \Delta^2 f) \nu = \int_{\partial X} (f \langle \text{grad} g, N \rangle - g \langle \text{grad} f, N \rangle) \sigma.
\]

7. (Introduction to potential theory in \( \mathbb{R}^3 \))

A differentiable function \( g : \mathbb{R}^3 \rightarrow \mathbb{R} \) is said to be harmonic in a subset \( B \subseteq \mathbb{R}^3 \) if \( \Delta^2 g = 0 \) for all \( x \in B \). Let \( X \subseteq \mathbb{R}^3 \) be a bounded region with regular boundary \( \partial X \). Prove that:

a) If \( g_1 \) and \( g_2 \) are harmonic in \( X \) and \( g_1 = g_2 \) in \( \partial X \), then \( g_1 = g_2 \) in \( X \);

Hint: Use Green’s first identity with \( f = g = g_1 - g_2 \).

b) If \( g \) is harmonic in \( X \) and

\[
\frac{\partial g}{\partial N} := \langle \text{grad} g, N \rangle = 0
\]

in \( \partial X \), where \( N \) is the unit normal vector to \( \partial X \), then \( g = \text{const} \) in \( X \);

Hint: Use Green’s first identity with \( f = g \).
c) If \( g_1 \) and \( g_2 \) are harmonic in \( X \) and
\[
\frac{\partial g_1}{\partial N} = \frac{\partial g_2}{\partial N}
\]
in \( \partial X \), then \( g_1 = g_2 + \text{const} \) in \( X \);
d) If \( g \) is harmonic in \( X \), then
\[
\int_{\partial X} \frac{\partial g}{\partial N} \sigma = 0;
\]
e) The function \( \frac{1}{(x^2+y^2+z^2)^{1/2}} \) is harmonic in \( \mathbb{R}^3 - \{0\} \);
f) (Mean Value Theorem) Let \( f \) be harmonic in the region
\[
B_r = \{ x \in \mathbb{R}^3 | |x - x_0|^2 \leq r^2 \}
\]
whose boundary is the sphere \( S_r \) centered at \( x_0 \). Then
\[
f(x_0) = \frac{1}{4\pi r^2} \int_{S_r} f \sigma;
\]

Hint: Use Green’s second identity in the region \( D = B_r - B_\rho, \rho < r \), with \( f = f \) and \( g = 1/r \). Since \( g \) and \( f \) are harmonic,
\[
\int_{S_\rho} \left( f \frac{\partial}{\partial N} \left( \frac{1}{r} \right) - \frac{1}{r} \frac{\partial f}{\partial N} \right) \sigma = \int_{S_r} \left( f \frac{\partial}{\partial N} \left( \frac{1}{r} \right) - \frac{1}{r} \frac{\partial f}{\partial N} \right) \sigma.
\]
Since \( \frac{\partial}{\partial N} (1/r) = \frac{\partial}{\partial r} (1/r) = -1/r^2 \), we obtain from d) that
\[
\frac{1}{4\pi \rho^2} \int_{S_\rho} f \sigma = \frac{1}{4\pi r^2} \int_{S_r} f \sigma.
\]
Now let \( \rho \to 0 \) to obtain the desired conclusion.
g) (The Maximum Principle) Let \( f \) be a harmonic function in a closed bounded region \( X \subseteq \mathbb{R}^3 \) (i.e. \( X \) is the union of a bounded connected open set with its boundary which is not necessarily regular). Then \( f \) reaches the maximum and minimum in the boundary \( \partial X \) of \( X \).

Hint: Assume that \( f(x) \) is a maximum, \( x \in X - \partial X \), and consider a ball \( B \subseteq X - \partial X \) centered at \( x \) such that \( f(x) \geq f(y) \), for all \( y \in B \). Show that this contradicts f).
8. Let $X$ be a compact differentiable $n$-manifold with boundary. Show that $X$ is orientable if and only if there exists a differential $n$-form $\omega$ on $X$ which is everywhere non-zero.

*Hint:* For the "only if" part use a partition of unity to construct a non-zero $n$-form globally defined on $X$.

9. Let $X$ be a compact orientable differentiable manifold without boundary. Show that $X$ is not contractible to a point.

*Hint:* Use Problem 8, Poincaré’s Lemma and Stokes Theorem.

10. Let $A, B$ and $C$ be differentiable functions in $\mathbb{R}^3$ and consider the differential system

\[
\begin{align*}
\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} &= A \\
\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} &= B \\
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= C
\end{align*}
\]

where $P, Q$ and $R$ are unknown functions in $\mathbb{R}^3$.

Show that a necessary and sufficient condition for a solution to the above system to exist is that

\[
\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = 0.
\]

*Hint:* Consider in $\mathbb{R}^3$ the differential form

\[
\omega = A \, dy \wedge dz + B \, dz \wedge dx + C \, dx \wedge dy.
\]

Then $d\omega = \left( \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \right) dx \wedge dy \wedge dz$. By Poincaré’s Lemma, $d\omega = 0$ if and only if there exists a form $\alpha = P \, dx + Q \, dy + R \, dz$ with $d\alpha = \omega$; this last condition is precisely the above system.