Math 274 (Spring 2002): Homework 8; due Thursday, May 23

1. (*Hodge star operation*) Given a p-form ω in $\mathbb{R}^n$, define an $(n-p)$-form $*\omega$ by setting

$$*(dx_{i_1} \wedge \ldots \wedge dx_{i_p}) = \text{sgn}(\tau)(dx_{j_1} \wedge \ldots \wedge dx_{j_{n-p}})$$

and extending it linearly, where $i_1 < \ldots < i_p$, $j_i < \ldots < j_{n-p}$ and τ is the premutation $(i_1, \ldots, i_p, j_1, \ldots, j_{n-p})$ of $(1, 2, \ldots, n)$. Show that:

a) If $\omega = a_{12}\, dx_1 \wedge dx_2 + a_{13}\, dx_1 \wedge dx_3 + a_{23}\, dx_2 \wedge dx_3$ is a 2-form in $\mathbb{R}^3$, then

$$*\omega = a_{12}dx_3 - a_{13}dx_2 + a_{23}dx_1;$$

b) If $\omega = a_1 dx_1 + a_2 dx_2$ is a 1-form in $\mathbb{R}^2$, then

$$*\omega = a_1 dx_2 - a_2 dx_1;$$

c) $* * \omega = (-1)^{p(n-p)}\omega$.

2. (*The divergence*) A differentiable vector field v in $\mathbb{R}^n$ may be considered as a differentiable map $v : \mathbb{R}^n \to \mathbb{R}^n$. Define a function $\text{div}\, v : \mathbb{R}^n \to \mathbb{R}^n$, called the *divergence* of v, as follows:

$$(\text{div}\, v)(x) = \text{trace}(dv)_x, \quad x \in \mathbb{R}^n.$$

Show that:

a) If $v = \sum_i a_i e_i$, where $\{e_i\}$ is the canonical basis of $\mathbb{R}^n$, then

$$\text{div}\, v = \sum_i \frac{\partial a_i}{\partial x_i};$$

b) If ω denotes the differential 1-form obtained from v by the canonical isomorphism induced by the inner product $\langle \cdot, \cdot \rangle$ and $\nu = dx_1 \wedge \ldots \wedge dx_n$ is the volume element of $\mathbb{R}^n$, the divergence can be obtained as follows:

$$v \mapsto \omega \to *\omega \to d(*\omega) = (\text{div}\, v)\nu.$$

3. (*The gradient*) Given a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, define a vector field $\operatorname{grad} f \in \mathbb{R}^n$, called the *gradient* of f, by

$$\langle \operatorname{grad} f(x), u \rangle = df_x(u), \quad x \in \mathbb{R}^n, u \in T_x\mathbb{R}^n.$$

Notice that $\operatorname{grad} f$ is the vector field corresponding to the 1-form df in the canonical isomorphism. Show that:

a) In the canonical basis $\{e_i\}$ of $\mathbb{R}^n$

$$\operatorname{grad} f = \sum_i \frac{\partial f}{\partial x_i} e_i;$$

b) If $x \in \mathbb{R}^n$ is such that $\operatorname{grad} f(x) = 0$, then $\operatorname{grad} f(x)$ is perpendicular to the "level set" $\{y \in \mathbb{R}^n \mid f(y) = f(x)\}$;

c) The linear map $df_x : T_x\mathbb{R}^n \to \mathbb{R}$ restricted to the unit sphere centered at x reaches its maximum at $v = \operatorname{grad} f / |\operatorname{grad} f|$.

4. (*The Laplacian*) Given a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, define the *Laplacian* $\Delta f : \mathbb{R}^n \to \mathbb{R}$ by

$$\Delta f = \operatorname{div}(\operatorname{grad} f).$$

Show that:

a) $\Delta f = \sum \frac{\partial^2 f}{\partial {x_i}^2}$;

b) $\Delta(fg) = f\Delta g + g\Delta f + 2\langle \operatorname{grad} f, \operatorname{grad} g \rangle$;

c) $d * (df) = (\Delta f)\nu$, where ν is the volume element of $\mathbb{R}^n$.

5. (*The curl*) Let v be a differentiable vector field in $\mathbb{R}^n$. The *curl* (or *rotational*) $\operatorname{curl} v$ is the $(n-2)$-form defined by

$$v \mapsto \omega \to d\omega \to *(d\omega) = \operatorname{curl} v.$$

a) Prove that $\operatorname{curl}(\operatorname{grad} f) = 0$.

b) In the particular case when $n = 3$, the 1-form $\operatorname{curl} v$ corresponds to a vector field which is also denoted by $\operatorname{curl} v$. Show that, for $n = 3$,

$$\operatorname{curl}\,(\sum_{i=1}^{3} a_i e_i) = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right) e_1 + \left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right) e_2 + \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right) e_3$$

and $\operatorname{div}(\operatorname{curl} v) = 0$.

6. Let $g : \mathbb{R}^3 \to \mathbb{R}$, $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable functions, and let $X \subseteq \mathbb{R}^3$ be a compact differentiable 3-manifold with boundary ∂X. Prove that:

a) (*1st Green's identity*)

$$\int_X \langle \operatorname{grad} f, \operatorname{grad} g\rangle\, \nu + \int_X f \Delta^2 g\, \nu = \int_{\partial X} f \langle \operatorname{grad} g, N\rangle\, \sigma,$$

where ν and σ are the volume element of X and the area element of ∂X, respectively, and N is the unit normal of ∂X;

Hint: Set $v = f \operatorname{grad} g$ in the Divergence Theorem.

b) (*2nd Green's identity*)

$$\int_X (f\Delta^2 g - g\Delta^2 f)\, \nu = \int_{\partial X} (f\langle \operatorname{grad} g, N\rangle - g\langle \operatorname{grad} f, N\rangle)\, \sigma.$$

7. (*Introduction to potential theory in* $\mathbb{R}^3$)
A differentiable function $g : \mathbb{R}^3 \to \mathbb{R}$ is said to be *harmonic* in a subset $B \subseteq \mathbb{R}^3$ if $\Delta^2 g = 0$ for all $x \in B$. Let $X \subseteq \mathbb{R}^3$ be a bounded region with regular boundary ∂X. Prove that:

a) If g_1 and g_2 are harmonic in X and $g_1 = g_2$ in ∂X, then $g_1 = g_2$ in X;

Hint: Use Green's first identity with $f = g = g_1 - g_2$.

b) If g is harmonic in X and

$$\frac{\partial g}{\partial N} := \langle \operatorname{grad} g, N\rangle = 0$$

in ∂X, where N is the unit normal vector to ∂X, then $g = \text{const}$ in X;

Hint: Use Green's first identity with $f = g$.

c) If g_1 and g_2 are harmonic in X and

$$\frac{\partial g_1}{\partial N} = \frac{\partial g_2}{\partial N}$$

in ∂X, then $g_1 = g_2 + \text{const}$ in X;

d) If g is harmonic in X, then

$$\int_{\partial X} \frac{\partial g}{\partial N}\, \sigma = 0;$$

e) The function $\frac{1}{(x^2+y^2+z^2)^{1/2}}$ is harmonic in $\mathbb{R}^3 - \{0\}$;

f) (*Mean Value Theorem*) Let f be harmonic in the region

$$B_r = \{x \in \mathbb{R}^3 \,|\, |x - x_0|^2 \leq r^2\}$$

whose boundary is the sphere S_r centered at x_0. Then

$$f(x_0) = \frac{1}{4\pi r^2} \int_{S_r} f\, \sigma;$$

Hint: Use Green's second identity in the region $D = B_r - B_\rho$, $\rho < r$, with $f = f$ and $g = 1/r$. Since g and f are harmonic,

$$\int_{S_\rho} \left(f \frac{\partial}{\partial N}(\frac{1}{r}) - \frac{1}{r}\frac{\partial f}{\partial N} \right) \sigma = \int_{S_r} \left(f \frac{\partial}{\partial N}(\frac{1}{r}) - \frac{1}{r}\frac{\partial f}{\partial N} \right) \sigma.$$

Since $\frac{\partial}{\partial N}(1/r) = \frac{\partial}{\partial r}(1/r) = -1/r^2$, we obtain from d) that

$$\frac{1}{4\pi\rho^2} \int_{S_\rho} f\, \sigma = \frac{1}{4\pi r^2} \int_{S_r} f\, \sigma.$$

Now let $\rho \to 0$ to obtain the desired conclusion.

g) (*The Maximum Principle*) Let f be a harmonic function in a closed bounded region $X \subseteq \mathbb{R}^3$ (i.e. X is the union of a bounded connected open set with its boundary which is not necessarily regular). Then f reaches the maximum and minimum in the boundary ∂X of X.

Hint: Assume that $f(x)$ is a maximum, $x \in X - \partial X$, and consider a ball $B \subseteq X - \partial X$ centered at x such that $f(x) \geq f(y)$, for all $y \in B$. Show that this contradicts f).

8. Let X be a compact differentiable n-manifold with boundary. Show that X is orientable if and only if there exists a differential n-form ω on X which is everywhere non-zero.

Hint: For the "only if" part use a partition of unity to construct a non-zero n-form globally defined on X.

9. Let X be a compact orientable differentiable manifold without boundary. Show that X is not contractible to a point.

Hint: Use Problem 8, Poincaré's Lemma and Stokes Theorem.

10. Let A, B and C be differentiable functions in $\mathbb{R}^3$ and consider the differential system

$$\begin{aligned} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} &= A \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} &= B \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= C \end{aligned}$$

where P, Q and R are unknown functions in $\mathbb{R}^3$.

Show that a necessary and sufficent condition for a solution to the above system to exist is that

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = 0.$$

Hint: Consider in $\mathbb{R}^3$ the differential form

$$\omega = A\, dy \wedge dz + B\, dz \wedge dx + C\, dx \wedge dy.$$

Then $d\omega = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dx \wedge dy \wedge dz$. By Poincaré's Lemma, $d\omega = 0$ if and only if there exists a form $\alpha = Pdx + Qdy + Rdz$ with $d\alpha = \omega$; this last condition is preciesely the above system.