Research Statement

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1 Introduction

In 1961, Vaught introduced the concepts of atomic, saturated, and homogeneous models [1961]. The ideas that arose from these models (which will be defined below) played a huge role in the development of modern model theory. In the 1970s and early 1980s, researchers including Goncharov, Harrington, Millar, Morley, and Peretyat'kin became interested in the computable content of these models as a way to more deeply understand their structure. In the past decade, Csima, Hirschfeldt, Knight, and Soare extended earlier results on the computability of atomic models, and Hirschfeldt, Shore, and Slaman examined atomic models from the proof theoretic perspective of reverse mathematics. I am engaged in a program of research on the complexity of homogeneous models from the perspectives of both computability theory and reverse mathematics. This work generalizes some of the earlier results in this tradition of research and deepens our understanding of homogeneous models.

I originally came to the subject from the vantage point of computable model theory. One aim of computable model theory is to study relative complexity of structures (such as groups or fields) by using different computational reducibilities. A set A is *Turing reducible* to B, denoted $A \leq_T B$, if there is an algorithm to compute elements of A using information about B. To obtain a more complete understanding of homogeneous structures, I then studied these structures from the complementary perspective of reverse mathematics. Reverse mathematics deems two mathematical statements to have the same proof theoretic strength if each statement can be used to prove the other over some weak base system of axioms, typically the *Recursive Comprehension Axioms* (RCA_0). The axiom system RCA_0 consists of a weak form of induction and set existence axioms that are just strong enough to prove the existence of computable sets.

We assume throughout that all objects considered are countable. Given a language $\mathcal{L}$, we fix a complete theory T. A *type* of T is a maximal set of formulas on a fixed finite set of free variables that is consistent with T. We let $S(T)$ denote the set of all types of T. We say a type $p(\bar{x})$ is *principal* if there is a formula $\varphi \in p$ such that $T \vdash \varphi \to \theta$ for all $\theta \in p$. Let $S^P(T)$ denote the set of principal types in T. A theory is called *atomic* if every formula φ is an element of some principal type of T. We call a complete theory T in a language $\mathcal{L}$ *decidable* if the set of sentences in T is computable, and we say a model $\mathcal{A}$ is (D-)*decidable* if its elementary diagram, the set of statements φ about elements in $\mathcal{A}$ such that $\mathcal{A} \models \varphi$, is ($D$-) computable (for a given set $D \subseteq \omega$).

Researchers, including Goncharov and Nurtazin [1973], Millar [1978]; [1980], and Peretyat'kin [1978], showed that decidable copies of atomic and homogeneous models do not always exist. A model $\mathcal{P}$ of a theory T is called *atomic* if $\mathbb{T}(\mathcal{P}) = S^P(T)$, where $\mathbb{T}(\mathcal{P})$ denotes the set of types satisfied by some tuple in $\mathcal{P}$. Every atomic theory T has an atomic model. A model $\mathcal{A}$ is called *homogeneous* if for any n-tuples $\bar{a}$ and $\bar{b}$ that satisfy the same type $p(\bar{x})$, there exists an automorphism of $\mathcal{A}$ sending $\bar{a}$ to $\bar{b}$. A *saturated* model is another related structure. All atomic and saturated models are homogeneous, and two homogeneous models $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if and only if $\mathbb{T}(\mathcal{A}) = \mathbb{T}(\mathcal{B})$. Homogeneous models intuitively have a uniform internal structure, and examples of homogeneous models are found throughout mathematics. For example, the set of natural numbers is a homogeneous (in fact atomic) model of the theory of successor with a least element. Studying the computational complexity of homogeneous models provides greater insight into the underlying combinatorial processes that govern this important class of models.

2 Deciding Copies of Homogeneous Models

We say a model $\mathcal{A}$ has a $\mathbf{0}$-*basis* if there exists a uniformly computable listing of $\mathbb{T}(\mathcal{A})$. An atomic model $\mathcal{P}$ of a complete atomic decidable theory T has a decidable copy precisely when $\mathcal{P}$ has a $\mathbf{0}$-*basis* [Goncharov-Nurtazin 1973], [Harrington 1974]. In [Morley 1976], Morley asked whether having a $\mathbf{0}$-basis for a homogeneous model $\mathcal{A}$ guarantees that $\mathcal{A}$ has a decidable copy. In [Goncharov 1978], [Millar 1980], and [Peretyat'kin 1978], the authors gave very complex constructions of homogeneous models with $\mathbf{0}$-bases that do not have decidable copies. I greatly extended this result.

Let X' denote the halting problem relative to any set X (the halting problem is a key idea in computability theory). A set D is called *low* if $D' = \emptyset'$. Low sets can be thought of as having low information content similar to computable sets. A set D is called low$_2$ if $D'' \leq_{\mathrm{T}} \emptyset''$. Sets that are low$_2$ still have relatively low information content. I proved the following.

Theorem 2.1. *Let set $D \leq_{\mathrm{T}} \emptyset'$ be low$_2$. Then there exists a homogeneous model with a* **0***-basis but with no D-decidable copy.*

In addition to building their examples, Goncharov [1978] and Peretyat'kin [1978] showed that a homogeneous model of a complete decidable theory with a **0**-basis X has a decidable copy if and only if a certain computable type amalgamation condition holds for X. This result can be done relative to any set D.

Let $D \leq_{\mathrm{T}} \emptyset'$ be low$_2$. We construct a **0**-basis X for a homogeneous $\mathcal{A}$ such that no D-computable function computes the additional information required to build a D-decidable copy. We construct X by meeting certain homogeneity conditions and diagonalizing against every D-computable function. The homogeneity and diagonalization requirements, however, are directly at odds with one another, and this conflict must be handled very delicately.

A natural question to ask is whether the result in Theorem 2.1 is the best possible. We say a set D is **0***-basis homogeneous bounding* if any homogeneous model with a **0**-basis has a D-decidable isomorphic copy. By Theorem 2.1, any low$_2$ set $D \leq_{\mathrm{T}} \emptyset'$ is not **0**-basis homogeneous bounding. I showed that this is the best possible result for $D \leq_{\mathrm{T}} \emptyset'$ in [Lange ta].

Theorem 2.2. *Let $D \leq_{\mathrm{T}} \emptyset'$. If D is nonlow$_2$ then D is* **0***-basis homogeneous bounding.*

The aim in this proof (and in all later positive theorems) is to D-compute the needed type amalgamation information for any **0**-basis of a homogeneous model $\mathcal{A}$ for any set D specified in the theorem. Combining Theorems 2.1 and 2.2, we obtain a characterization of the Δ^0_2 **0**-basis homogeneous bounding sets that (not obviously) corresponds to the analogous result on atomic models found in [CHKS 2004].

Theorem 2.3. *Let $D \leq_{\mathrm{T}} \emptyset'$. The set D is* **0***-basis homogeneous bounding if and only if D is nonlow$_2$.*

Additionally, we can consider the sets that can decide an isomorphic copy to a single fixed homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis. Csima, Hirschfeldt, Knight, and Soare first studied this problem in the atomic case, and I then examined the more general homogeneous case. Among other results, I prove the following in [Lange ta].

Theorem 2.4. *Let T be a complete decidable theory, and let $\mathcal{A}$ be a homogeneous model of T with a $\mathbf{0}$-basis. There exists a model $\mathcal{B} \cong \mathcal{A}$ and a low set D such that $\mathcal{B}$ is D-decidable.*

A stronger version of this result gives Csima's analogous result for atomic models as a corollary.

Corollary 2.5 (Csima 2004)**.** *Let T be a complete atomic decidable theory. Then there exists a low set D and an atomic model $\mathcal{A}$ of T such that $\mathcal{A}$ is D-decidable.*

Let $S(T)$ denote the set of all types of T. In the case that all the types in $S(T)$ are computable, we obtain the following strong result.

Theorem 2.6. *Let T be a complete decidable theory with all types in $S(T)$ computable. Let $\mathcal{A}$ be a homogeneous model with a $\mathbf{0}$-basis, and let D be any noncomputable set. Then there exists a D-decidable $\mathcal{B} \cong \mathcal{A}$.*

The above computability theoretic results turn out to be the same as the analogous results in the atomic model case (Theorems 2.3, 2.4, and 2.6 correspond with the results on atomic models found in [CHKS 2004], [Csima 2004], and [Hirschfeldt 2006], respectively). We will compare the atomic and homogeneous cases further in §3.

There are many questions I want to explore related to the above results. Although in general a homogeneous model requires a $\mathbf{0}$-basis and certain additional information to guarantee a decidable copy, it would be interesting to see if these conditions could be weakened for specific algebraic or model theoretic classes. Furthermore, many other problems share the model theoretic and combinatorial processes used to prove the above theorems. An Ehrenfeucht theory is a complete theory with only finitely many nonisomorphic countable models. A well-known question of Morley asks whether every countable model of a decidable Ehrenfeucht theory with all types in $S(T)$ computable has a decidable copy. I would like to study several related open problems concerning Ehrenfeucht theories and almost homogeneous models, which would provide insight into Morley's question. These problems might also lead to new ideas on another important open question that asks whether any complete decidable theory with only countably many countable models must have a decidable atomic or even homogeneous model.

3 Reverse Math of Homogeneous Models

As mentioned above, the computability theoretic results on homogeneous models turn out to be the same as the analogous results in the atomic model case. A natural question is whether there is some intrinsic structural connection between atomic and homogeneous models. For this reason, I became interested in the reverse mathematical strength of statements about homogeneous models and, in particular, the relative strength of the following two statements. Hirschfeldt, Shore, and Slaman studied the second statement in [HSS ta] and showed that it is a fairly weak principle that sits in an interesting place in the reverse mathematics universe.

HMT: "If X is a set of types satisfying certain homogeneity conditions, there exists a homogeneous model realizing exactly the types in X."

(The homogeneity conditions are the conditions that are noneffectively necessary and sufficient to guarantee that a set of types equals $\mathbb{T}(\mathcal{A})$ for some homogeneous model $\mathcal{A}$.)

AMT: "Every atomic theory has an atomic model."

I have shown AMT implies HMT in the computable setting, and I am currently working out the details that AMT implies HMT over RCA_0. Specifically, I have shown the following.

Theorem 3.1. *Let T be a complete decidable theory. Let $\mathcal{A}$ be a homogeneous model of T with a $\mathbf{0}$-basis X. There is a uniformly computable procedure to build a complete atomic decidable theory $\hat{T}$ from T and X such that any atomic model $\mathcal{P}$ of $\hat{T}$ can decide a copy of $\mathcal{A}$.*

Another main axiom system in reverse mathematics is WKL_0. This system consists of RCA_0 and the statement (known as Weak König's Lemma) that every infinite subtree of $2^{<\mathbb{N}}$ has an infinite path. Csima, Harizanov, Hirschfeldt, and Soare in [CHKS 2007] construct a complete decidable theory T such that any homogeneous model of T can compute infinite paths through infinite computable trees. Among other results, I proved this argument can be done in RCA_0. This result shows that building homogeneous models in some sense requires finding infinite paths through trees.

Theorem 3.2. *The statement "Every theory has a homogeneous model" is equivalent to WKL_0 over RCA_0.*

My first future goal is to prove that HMT implies AMT over RCA_0, which I conjecture is true. I would also like to understand how these statements relate to other well-known combinatorial principles. For example, Cohesive ADS (CADS) is the principle that every infinite linear order has a subset S of order type ω, ω^*, or $\omega + \omega^*$. In [HSS ta], the authors show that a related principle to CADS implies AMT over RCA_0. It remains unknown, however, whether CADS implies AMT over RCA_0. Furthermore, I would like to mine other classical model theoretic concepts (for example submodel completeness) for their reverse mathematical content.

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