Uniform spanning forests and the bi-Laplacian Gaussian field

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Abstract

For $N = n(\log n)^{\frac{1}{4}}$, on the ball in $\mathbb{Z}^4$ with radius N we construct a ± 1 spin model coupled with the wired spanning forests. We show that as $n \to \infty$ the spin field has bi-Laplacian Gaussian field on $\mathbb{R}^4$ as its scaling limit. For $d \geq 5$, the model can be defined on $\mathbb{Z}^d$ directly without the N-cutoff. The same scaling limit result holds. Our proof is based on Wilson's algorithm and a fine analysis of intersections of simple random walks and loop erased walks. In particular, we improve results on the intersection of a random walk with a loop-erased walk in four dimensions. To our knowledge, this is the first natural discrete model (besides the discrete bi-Laplacian Gaussian field) that has been proved to converge to the bi-Laplacian Gaussian field.

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1 Introduction

This paper connects two natural objects that have four as the critical dimension: the bi-Laplacian Gaussian field and uniform spanning trees (UST) (and the associated loop-erased random walk (LERW)). We will construct a

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sequence of random fields on the integer lattice $\mathbb{Z}^d$ ($d \geq 4$) using LERW and UST and show that they converge in distribution to the bi-Laplacian field.

We will now describe the construction for each positive integer n, let $N = N_n = n(\log n)^{1/4}$. Let $A_N = \{x \in \mathbb{Z}^d : |x| \leq N\}$. We will construct a ± 1 valued random field on A_N as follows. Recall that a *wired spanning tree* on A_N is a tree on the graph $A_N \cup \{\partial A_N\}$ where we have viewed the boundary ∂A_N as "wired" to a single point. Such a tree produces a *spanning forest* on A_N by removing the edges connected to ∂A_N. We define the uniform spanning forest (USF) on A_N to be the forest obtained by choosing the wired spanning tree of $A_N \cup \{\partial A_N\}$ from the uniform distribution. (Note this is not the same thing as choosing a spanning forest uniformly among all spanning forests of A_N.) The field is then defined as follows. Let a_n be a sequence of positive numbers (we will be more precise later).

- Choose a USF on A_N. This partitions A_N into (connected) components.
- For each component of the forest, flip a coin and assign each vertex in the component value 1 or -1 based on the outcome. This gives a field of spins $\{Y_{x,n} : x \in A_N\}$. If we wish we can extend this to a field on $x \in \mathbb{Z}^d$ by setting $Y_{x,n} = 1$ for $x \notin A_N$.
- Let $\phi_n(x) = a_n\, Y_{nx,n}$ which is a field defined on $L_n := n^{-1}\,\mathbb{Z}^d$.

This random function is constructed in a manner similar to the Edward-Sokal coupling of the FK-Ising model [7]. The content of our main theorem which we now state is that for $d \geq 4$, we can choose a_n that ϕ_n converges to the bi-Laplacian Gaussian field on $\mathbb{Z}^d$. If $h \in C_0^\infty(\mathbb{R}^d)$, we write

$$\langle h, \phi_n \rangle = n^{-d/2} \sum_{x \in L_n} h(x)\, \phi_n(x).$$

Theorem 1.1.

- *If $d \geq 5$, there exists a such that if $a_n = a\, n^{(d-4)/2}$, then for every $h_1, \ldots, h_m \in C_0^\infty(\mathbb{R}^d)$, the random variables $\langle h_j, \phi_n \rangle$ converge in distribution to a centered joint Gaussian random variable with covariance*

$$\int \int h_j(z)\, h_k(w)\, |z - w|^{4-d}\, dz\, dw.$$

- *If $d = 4$, if $a_n = 1/\sqrt{3 \log n}$, then for every $h_1, \ldots, h_m \in C_0^\infty(\mathbb{R}^d)$ with*

$$\int h_j(z)\, dz = 0, \quad j = 1, \ldots, m,$$

  the random variables $\langle h_j, \phi_n \rangle$ converge in distribution to a centered Gaussian random variable with variance

$$-\int \int h_j(z)\, h_k(w)\, \log |z - w|\, dz\, dw.$$

Remark 1.2.

- For $d = 4$, we could choose the cutoff $N = n(\log n)^\alpha$ for any $\alpha > 0$. For $d > 4$, we could do the same construction with no cutoff ($N = 0$) and get the same result. For convenience, we choose a specific value of α.

- We will prove the result for $m = 1, h_1 = h$. The general result follows by applying the $k = 1$ result to any linear combination of $h_1, \ldots, h_m$.

- We will do the proof only for $d = 4$; the $d > 4$ case is similar but easier (see [22] for the $d \geq 5$ case done in detail). In the $d = 4$ proof, we will have a slightly different definition of a_n but we will show that $a_n \sim \sqrt{3 \log n}$.

Our main result uses strongly the fact that the scaling limit of the loop-erased random walk in four or more dimensions is Brownian motion, that is, has Gaussian limits [15, 14]. Although we do not use explicitly the results in those papers, they are used implicitly in our calculations where the probability that a random walk and a a loop-erased walk avoid each other is comparable to that of two simple random walks and can be given by the *expected number* of intersections times a non-random quantity. This expected number of intersections is the discrete biharmonic function that gives the covariance structure of the biLaplacian random field.

Gaussian fluctuations have been observed and studied for numerous physical systems. Many statistical physics models with long correlations converge are known to converge to the Gaussian free field. Typical examples come from domino tilings [9], random matrix theory [3, 19] and random growth models [6]. Our model can be viewed as an analogue of critical statistical mechanics

models in four dimension, in the sense that the correlation functions are scale invariant.

The discrete bi-Laplacian Gaussian field (in physics literature, this is known as the membrane model) is studied in [20, 10, 11], whose continuous counterpart is the bi-Laplacian Gaussian field. Our model can be viewed as another natural discrete object that converges to the bi-Laplacian Gaussian field. In one dimensional case, Hammond and Sheffield constructed a reinforced random walk with long range memory [8], which can be associated with a spanning forest attached to $\mathbb{Z}$. Our construction can also be viewed as a higher dimensional analogue of "forest random walks".

1.1 Uniform spanning trees

Here we review some facts about the uniform spanning forest (that is, wired spanning trees) on finite subsets of $\mathbb{Z}^d$ and loop erased random walks (LERW) on $\mathbb{Z}^d$. Most of the facts extend to general graphs as well. For more details, see [16, Chapter 9].

Given a finite subset $A \subset \mathbb{Z}^d$, the uniform wired spanning tree in A is a subgraph of the graph $A \cup \{\partial A\}$, choosing uniformly random among all spanning trees of $A \cup \{\partial A\}$. (A spanning tree $\mathcal{T}$ is a subgraph such that any two vertices in $\mathcal{T}$ are connected by a unique simple path in $\mathcal{T}$). We define the *uniform spanning forest (USF)* on A to be the uniform wired spanning tree restricted to the edges in A. One can also consider the uniform spanning forest on all of $\mathbb{Z}^d$ [18, 2], but we will not need this construction.

The uniform wired spanning tree, and hence the USF, on A can be generated by Wilson's algorithm [23] which we recall here:

- Order the elements of $A = \{x_1, \ldots, x_k\}$.
- Start a simple random walk at x_1 and stop it when in reaches ∂A giving a nearest neighbor path ω. Erase the loops chronologically to produce a self-avoiding path $\eta = LE(\omega)$. Add all the edges of η to the tree which now gives a tree $\mathcal{T}_1$ on a subset of $A \cup \{\partial A\}$ that includes ∂A.
- Choose the vertex of smallest index that has not been included and run a simple random walk until it reaches a vertex in $\mathcal{T}_1$. Erase the loops and add the new edges to the tree giving a tree $\mathcal{T}_2$.
- Continue until all vertices are included in the tree.

Wilson's theorem states that the distribution of the tree is independent of the order in which the vertices were chosen and is uniform chosen among all spanning trees. In particular we get the following. Suppose that at each $x \in A$, we have an independent simple random walk path ω^x from x stopped when it reaches ∂A.

- If $x, y \in A$, then the probability that x, y are in the same component of the USF is the same as

$$\mathbb{P}\{LE(\omega^x) \cap \omega^y = \emptyset\}. \tag{1}$$

Using this characterization, we can see the three regimes for the dimension d. Let us first consider the probabilities that neighboring points are in the same component. Let q_N be the probability that a nearest neighbor of the origin is in a different component as the origin.

$$q_\infty := \lim_{N\to\infty} q_N > 0, \qquad d \geq 5,$$

$$q_N \asymp (\log N)^{-1/3}, \qquad d = 4,$$

For $d < 4$, q_N decays like a power of N.

- If $d > 4$, and $|x| = n$, the probability that that 0 and x are in the same component is comparable to $|x|^{4-d}$. This is true even if $N = \infty$.
- If $d = 4$ and $|x| = n$, the probability that that 0 and x are in the same component is comparable to $1/\log n$. However, if we chose $N = \infty$, the probability would equal one.

The last fact can be used to show that the uniform spanning tree in all of $\mathbb{Z}^4$ is, in fact, a tree. For $d < 4$, the probability that that 0 and x are in the same component is asymptotic to 1 and our construction is not interesting. This is why we restrict to $d \geq 4$.

1.2 Intersection probabilities for loop-erased walk

The hardest part of the analysis of this model is estimation of the intersection probability (1) for $d = 4$. We will review what is known and then state the new results in this paper. We will be somewhat informal here; see Section 2 for precise statements.

Let ω denote a simple random walk path starting at the origin and let $\eta = LE(\omega)$ denote its loop erasure. We can write

$$\eta = \{0\} \cup \eta^1 \cup \eta^2 \cup \cdots ,$$

where η^j denotes the path from the first visit to $\{|z| \geq e^{j-1}\}$ stopped at the step immediately before the first visit to $\{|z| \geq e^j\}$. Let S be an independent simple random walk starting at the origin, write S for $S = [1, \infty)$, and let us define

$$Z_n = Z_n(\eta) = \mathbb{P}\left\{S \cap (\{0\} \cup \eta^1 \cup \eta^2 \cdots \cup \eta^n) = \emptyset \mid \eta\right\},$$
$$\overline{Z}_n = \overline{Z}_n(\eta) = \mathbb{P}\left\{S \cap (\eta^1 \cup \eta^2 \cdots \cup \eta^n) = \emptyset \mid \eta\right\},$$
$$\Delta_n = \Delta_n(\eta) = \mathbb{P}\{S \cap \eta^n \neq \emptyset \mid \eta\}.$$

With probability one, $Z_\infty = 0$, and hence $\mathbb{E}[Z_n] \to 0$ as $n \to \infty$. In [14] building on the work on slowly recurrent sets [13], it was shown that for integers $r, s \geq 0$,

$$p_{n,r,s} := \mathbb{E}\left[Z_n^r \, \overline{Z}_n^s\right] \asymp n^{-(r+s)/3},$$

where $\asymp$ means that the ratio of the two sides are bounded away from 0 and ∞ (with constants depending on r, s). The proof uses the "slowly recurrent" nature of the paths to derive a relation

$$p_{n,r,s} \sim p_{n-1,r,s} \left(1 - (r+s)\,\mathbb{E}[\Delta_n]\right). \tag{2}$$

We improve on this result by establishing an asymptotic result

$$\lim_{n\to\infty} n^{(r+s)/3}\, p_{n,r,s} = c_{r,s} \in (0, \infty).$$

The proof does not compute the constant $c_{r,s}$ except in one special case $r = 2, s = 1$. Let $\hat{p}_n = p_{n,2,1}$. We show that

$$\hat{p}_n \sim \frac{\pi^2}{24n}.$$

This particular value of r, s comes from a path decomposition in trying to estimate

$$\mathbb{P}\{S \cap \eta^n \neq \emptyset\} = \mathbb{E}\left[\Delta_n\right].$$

On the event $\{S \cap \eta^n \neq \emptyset\}$ there are many intersections of the paths. By focusing on a particular one using a generalization of a first passage decomposition, it turns out that

$$\mathbb{P}\{S \cap \eta^n \neq \emptyset\} \sim \hat{G}_n^2\, \hat{p}_n,$$

where $\hat{G}_n^2$ denotes the expected number of intersections of S with a simple random walk starting at the sphere of radius e^{n-1} stopped at the sphere of radius n. This expectation can be estimated well giving $\hat{G}_n^2 \sim 8/\pi^2$, and hence we get a relation

$$\mathbb{P}\{S \cap \eta^n \neq \emptyset\} \sim \frac{8}{\pi^2}\,\hat{p}_n.$$

Combining this with (2), we get a consistency relation,

$$\hat{p}_n \sim \hat{p}_{n-1}\left[1 - \frac{24}{\pi^2}\,\hat{p}_n\right],$$

which leads to $n\,\hat{p}_n \sim \pi^2/24$ and

$$\mathbb{P}\{S \cap \eta^n \neq \emptyset\} \sim \frac{1}{3n}.$$

From this we also get the asymptotic value of (1),

$$\mathbb{P}\{LE(\omega^x) \cap \omega^y = \emptyset\} \sim \frac{1}{3\log N}.$$

1.3 Estimates for Green's functions

Let $G_N(x,y)$ denote the usual random walk Green's function on A_N, and let

$$G^2(x,y) = \sum_{z\in\mathbb{Z}^d} G(x,z)\,G(z,y),$$

$$G_N^2(x,y) = \sum_{z\in A} G_N(x,z)\,G_N(z,y),$$

$$\tilde{G}_N^2(x,y) = \sum_{z\in A} G_N(x,z)\,G(z,y),$$

Note that

$$G_N^2(x,y) \le \tilde{G}_N^2(x,y) \le G^2(x,y).$$

Also, $G^2(x,y) < \infty$ if and only if $d > 4$.

The Green's function $G(x,y)$ is well understood for $d > 2$. We will use the following asymptotic estimate [16, Theorem 4.3.1]

$$G(x) = C_d\,|x|^{2-d} + O(|x|^{-d}), \qquad \text{where } C_d = \frac{d\,\Gamma(d/2)}{(d-2)\,\pi^{d/2}}. \tag{3}$$

(Here and throughout we use the convention that if we say that a function on $\mathbb{Z}^d$ is $O(|x|)^{-r})$ with $r > 0$, we still imply that it is finite at every point. In other words, for lattice functions, $O(|x|^{-r})$ really means $O(1 \wedge |x|^{-r})$. We do not make this assumption for functions on $\mathbb{R}^d$ which could blow up at the origin.) It follows immediately, and will be more convenience for us, that

$$G(x) = C_d\, I_d(x)\left[1 + O(|x|^{-2})\right], \tag{4}$$

where we write

$$I_d(x) = \int_V \frac{d^d\zeta}{|\zeta - x|^{d-2}},$$

and $V = V_d = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : |x_j| \le 1/2\}$ denotes the d-dimensional cube of side length one centered at the origin. One aesthetic advantage is that $I_d(0)$ is not equal to infinity.

Proposition 1.3. *If $d \ge 5$, there exists c_d such that*

$$G^2(x,y) = c_d\,|x-y|^{4-d} + O\left(|x-y|^{2-d}\right).$$

Moreover, there exists $c < \infty$ such that if $|x|, |y| \le N/2$, then

$$G^2(x,y) - G_N^2(x,y) \le c\,N^{4-d}.$$

We note that it follows immediately that

$$G^2(x,y) = c_d\, J_d(x-y)\,\left[1 + O\left(|x-y|^{-2}\right)\right],$$

where

$$J_d(x) = \int_V \frac{d^d\zeta}{|\zeta - x|^{d-4}}.$$

The second assertion shows that it does not matter in the limit whether we use $G^2(x,y)$ or $G_N^2(x,y)$ as our covariance matrix.

Proof. We may assume $y = 0$. We use (3) to write

$$\begin{aligned} G^2(0,x) &= C_d^2 \sum_{z\in\mathbb{Z}^d} I_d(z)\, I_d(z-x)\,\left[1 + O(|z|^{-2}) + O(|z-x|^{-2})\right] \\ &= C_d^2 \int\int \frac{d\zeta_1\, d\zeta_2}{|\zeta_1|^{d-2}\,|\zeta_2 - x|^{d-2}} + O(|x|^{2-d}) \\ &= C_d^2\,|x|^{4-d} \int\int \frac{d\zeta_1\, d\zeta_2}{|\zeta_1|^{d-2}\,|\zeta_2 - u|^{d-2}} + O(|x|^{2-d}), \end{aligned}$$

where u denotes a unit vector in $\mathbb{R}^d$. This gives the first expression with

$$c_d = C_d^2 \int \int \frac{d\zeta_1 \, d\zeta_2}{|\zeta_1|^{d-2} \, |\zeta_2 - u|^{d-2}}.$$

For the second assertion, Let S^x, S^y be independent simple random walks starting at x, y stopped at times T^x, T^y, the first time that they reach ∂A_N. Then,

$$G^2(x,y) - G_N^2(x,y) \le \mathbb{E}\left[G^2(S^x(T^x), y)\right] + \mathbb{E}\left[G^2(x, S^y(T^y))\right],$$

and we can apply the first result to the right-hand side.

□

Lemma 1.4. *If $d = 4$, there exists $c_0 \in \mathbb{R}$ such that if $|x|, |y|, |x-y| \le N/2$, then*

$$G_N^2(x,y) = \frac{8}{\pi^2} \log\left[\frac{N}{|x-y|}\right] + c_0 + O\left(\frac{|x|+|y|+1}{N} + \frac{1}{|x-y|}\right).$$

Proof. Using the martingale $M_t = |S_t|^2 - t$, we see that

$$\sum_{w \in A_N} G_N(x,w) = N^2 - |x|^2 + O(N). \tag{5}$$

Using (4), we see that

$$\sum_{w \in A_N} G(0,w) = O(N) + \frac{2}{\pi^2} \int_{|s| \le N} \frac{ds}{|s|^2} = 2N^2 + O(N).$$

Let $\delta = N^{-1}[1 + |x| + |y|]$ and note that $|x-y| \le \delta \, N$. Since

$$\begin{aligned} \sum_{|w| < N(1-\delta)} G_n(x-y,w) \, G_n(0,w) &\le \sum_{w \in D_n} G_n(x,w) \, G_n(y,w) \\ &\le \sum_{|w| \le N(1+\delta)} G_n(x-y,w) \, G_n(0,w), \end{aligned}$$

it suffices to prove the result when $y = 0$ in which case $\delta = (1 + |x|)/N$. Recall that

$$\min_{N \le |z| \le N+1} G(x,z) \le G(x,w) - G_n(x,w) \le \max_{N \le |z| \le N+1} G(x,z).$$

In particular,

$$G_n(0,w) = G(0,w) - \frac{2}{\pi^2 N^2} + O(N^{-3}),$$
$$G_n(x,w) = G(x,w) - \frac{2}{\pi^2 N^2} + O(\delta N^{-2}).$$

Using (5), we see that

$$\begin{aligned}\sum_w G_n(x,w)\, G_n(0,w) &= \sum_{w\in D_n} \left[G(x,w) - \frac{2}{\pi^2\, N^2} + O\left(\delta N^{-2}\right)\right] G_n(0,w) \\ &= O(\delta) - \frac{2}{\pi^2} + \sum_{w\in D_n} G(x,w)\, G_n(0,w).\end{aligned}$$

Similarly, we write,

$$\sum_{w\in D_n} G(x,w)\, G_n(0,w) = \sum_{w\in D_n} G(x,w) \left[G(0,w) - \frac{2}{\pi^2\ N^2} + O\left(N^{-3}\right)\right],$$

and use (5) to see that

$$\sum_{w\in D_n} G(x,w)\, G_n(0,w) = -\frac{2}{\pi^2} + O(\delta^2) + \sum_{w\in D_n} G(x,w)\, G(0,w).$$

Hence it suffices to show that there exists c' such that

$$\sum_{w\in D_n} G(x,w)\, G(0,w) = \log(N/|x|) + \hat{c} + O(|x|^{-1}) + O(\delta).$$

Define $\epsilon(x,w)$ by

$$\epsilon(x,w) = G(x,w)\, G(0,w) - \left(\frac{2}{\pi^2}\right)^2 \int_{S_w} \frac{ds}{|x-s|^2\, |s|^2}.$$

The estimate for the Green's function implies that

$$|\epsilon(x,w)| \le \begin{cases} c\, |w|^{-3}\, |x|^{-2}, & |w| \le |x|/2 \\ c\, |w|^{-2}\, |x-w|^{-3}, & |x-w| \le |x|/2 \\ c\, |w|^{-5}, & \text{other } w, \end{cases}$$

and hence,

$$\begin{aligned}\sum_{w\in D_n} G(x,w)\,G(0,w) &= O(|x|^{-1}) + \sum_{w\in D_n}\left(\frac{2}{\pi^2}\right)^2 \int_{S_w} \frac{ds}{|x-s|^2\,|s|^2} \\ &= O(|x|^{-1}) + \left(\frac{2}{\pi^2}\right)^2 \int_{|s|\le N} \frac{ds}{|x-s|^2\,|s|^2}.\end{aligned}$$

(The second equality had an error term of $O(N^{-1})$ but this is smaller than the $O(|x|^{-1})$ term already there.) It is straightforward to check that there exists c' such that

$$\int_{|s|\le N} \frac{d^4 s}{|x-s|^2\,|s|^2} = 2\pi^2 \log\left(\frac{N}{|x|}\right) + c' + O\left(\frac{|x|}{N}\right).$$

□

1.4 Proof of Theorem

Here we give the proof of the theorem leaving the proof of the most important lemma for Section 2. The proofs for the $d=4$ and $d>4$ are similar with a few added difficulties for $d=4$. We will only consider the $d=4$ case here. We fix $h\in C_0^\infty$ with $\int h = 0$ and allow implicit constants to depend on h. We will write just Y_x, for $Y_{x,n}$. Let K be such that $h(x)=0$ for $|x|\le K$.

Let us write $L_n = n^{-1}\,\mathbb{Z}^4 \cap \{|x|\le K\}$ and

$$\langle h,\phi_n\rangle = n^{-4}\,a_n \sum_{x\in L_n} h(x)\,Y_{nx} = n^{-4}\,a_n \sum_{nx\in A_{nK}} h(x)\,Y_{nx},$$

where a_n is still to be specified. We will now give the value. Let ω be a LERW from 0 to ∂A_N in A_N and let $\eta = LE(\omega)$ be its loop-erasure. Let ω_1 be another simple random walk starting at the origin stopped when it reaches ∂A_N and let ω_1' be ω_1 with the initial step removed. Let

$$u(\eta) = \mathbb{P}\{\omega_1\cap\eta = \{0\} \mid \eta\},$$

$$\tilde{u}(\eta) = \mathbb{P}\{\omega_1'\cap\eta = \emptyset \mid \eta\}.$$

We define

$$p_n = \mathbb{E}\left[u(\eta)\,\tilde{u}(\eta)^2\right].$$

In [] it was shown that $p_n \asymp 1/\log n$ if $d=4$. One of the main goals of Section 2 is to give the following improvement.

Lemma 1.5. *If $d = 4$ and p_n is defined as above, then*

$$p_n \sim \frac{\pi^2}{24 \log n}.$$

We can write the right-hand side as

$$\frac{1}{(8/\pi^2)} \frac{1}{3 \log n},$$

where $8/\pi^2$ is the nonuniversal constant in Lemma 1.4 and $1/3$ is a universal constant for loop-erased walks. If $d \geq 5$, then $p_n \sim c\, n^{4-d}$. We do not prove this here.

For $d = 4$, we define

$$a_n = \sqrt{\frac{8}{\pi^2 p_n}} \sim \frac{1}{\sqrt{3 \log n}}.$$

Let $\delta_n = \exp\{-(\log \log n)^2\}$; this is a function that decays faster than any power of $\log n$. Let $L^k_{n,*}$ be the set of $\bar{x} = (x_1, \ldots, x_k) \in L^k_n$ such that $|x_j| \leq K$ for all j and $|x_i - x_j| \geq \delta_n$ for each $i \neq j$. Note that $\#(L^k_n) \asymp n^{4k}$ and $\#(L^k_n \setminus L^k_{n,*}) \asymp k^2\, n^{4k}\, \delta_n$. In particular,

$$n^{-4k}\, a_n^{2k} \sum_{\bar{x} \in L^{2k}_{n,*}} h(x_1)\, h(x_2) \cdots h(x_{2k}) = o_{2k}(\sqrt{\delta_n}).$$

Let $q_N(x, y)$ be the probability that x, y are in the same component of the USF on A_N, and note that

$$\mathbb{E}\left[Y_{x,n}\, Y_{y,n}\right] = q_N(x, y),$$

$$\mathbb{E}\left[\langle h, \phi_n \rangle^2\right] = n^{-4} \sum_{x \in L_n} h(x)\, h(y)\, a_n^2\, q_N(nx, ny).$$

An upper bound for $q_N(x, y)$ can be given in terms of the probability that the paths of two independent random walks starting at x, y, stopped when then leave A_N, intersect. This gives

$$q_N(x, y) \leq \frac{c \log[N/|x - y|]}{\log N}.$$

In particular,

$$q_N(x,y) \le c\,\frac{(\log\log n)^2}{\log n}, \qquad |x-y| \ge n\,\delta_n. \tag{6}$$

To estimate $\mathbb{E}\left[\langle h,\phi_n\rangle^2\right]$ we will use the following lemma, the proof of which will be delayed.

Lemma 1.6. *There exists a sequence r_n with $r_n \le O(\log\log n)$, a sequence $\epsilon_n \downarrow 0$, such that if $x,y \in L_n$ with $|x-y| \ge 1/\sqrt{n}$,*

$$\left|a_n^2\, q(nx,ny) - r_n + \log|x-y|\right| \le \epsilon_n.$$

It follows from the lemma, (6), and the trivial inequality $q_N \le 1$, that

$$\begin{aligned}
\mathbb{E}\left[\langle h,\phi_n\rangle^2\right] &= o(1) + n^{-d} \sum_{x,y\in L_n} \left[r_n - \log|x-y|\right]\, h(x)\,h(y) \\
&= o(1) - n^{-d} \sum_{x,y\in L_n} \log|x-y|\, h(x)\,h(y) \\
&= o(1) - \int h(x)\,h(y)\,\log|x-y|\,dx\,dy,
\end{aligned}$$

which shows that the second moment has the correct limit. The second equality uses $\int h = 0$ to conclude that

$$\frac{r_n}{n^d} \sum_{x,y\in L_n} h(x)\,h(y) = o(1).$$

We now consider the higher moments. It is immediate from the construction that the odd moments of $\langle h,\phi_n\rangle$ are identically zero, so it suffices to consider the even moments $\mathbb{E}[\langle h,\phi_n\rangle^{2k}]$. We fix $k \ge 2$ and allow implicit constants to depend on k as well. Let $L_* = L_{n,*}^k$ be the set of $\bar{x} = (x_1,\dots,x_k) \in L_n^k$ such that $|x_j| \le K$ for all j and $|x_i - x_j| \ge \delta_n$ for each $i \ne j$. We write $h(\bar{x}) = h(x_1)\,\dots\,h(x_{2n})$. Then we see that

$$\begin{aligned}
\mathbb{E}\left[\langle h,\phi_n\rangle^{2k}\rangle\right] &= n^{-2kd}\, a_n^{2k} \sum_{\bar{x}\in L_N^{2k}} h(\bar{x})\,\mathbb{E}\left[Y_{nx_1}\cdots Y_{nx_{2k}}\right] \\
&= O(\sqrt{\delta_n}) + n^{-2kd}\, a_n^{2k} \sum_{\bar{x}\in L_n^k\setminus L_*} h(\bar{x})\,\mathbb{E}\left[Y_{nx_1}\cdots Y_{nx_{2k}}\right]
\end{aligned}$$

Lemma 1.7. *For each k, there exists $c < \infty$ such that the following holds. Suppose $\bar{x} \in L_{n,*}^{2k}$ and let $\omega^1, \ldots, \omega^{2k}$ be independent simple random walks started at $nx_1, \ldots, nx_{2k}$ stopped when they reach ∂A_N. Let N denote the number of integers $j \in \{2, 3, \ldots, 2k\}$ such that*

$$\omega^j \cap (\omega^1 \cup \cdots \cup \omega^{j-1}) \neq \emptyset\}.$$

Then,

$$\mathbb{P}\{N \geq k+1\} \leq c \left[\frac{(\log\log n)^3}{\log n}\right]^{k+1}.$$

We write $y_j = nx_j$ write Y_j for Y_{y_j}. To calculate $\mathbb{E}[Y_1 \cdots Y_{2k}]$ we first sample our USF which gives a random partition $\mathcal{P}$ of $\{y_1, \ldots, y_{2k}\}$ Note that $\mathbb{E}[Y_1 \cdots Y_{2k} \mid \mathcal{P}]$ equals 1 if it is an "even" partition in the sense that each set has an even number of elements. Otherwise, $\mathbb{E}[Y_1 \cdots Y_{2k} \mid \mathcal{P}] = 0$. Any even partition, other than a partition into k sets of cardinality 2, will have $N \geq k+1$. Hence

$$\mathbb{E}[Y_1 \cdots Y_{2k}] = O\left(\left[\frac{(\log\log n)^3}{\log n}\right]^{k+1}\right) + \sum \mathbb{P}\left(\mathcal{P}_{\bar{y}}\right),$$

where the sum is over the $(2k-1)!!$ perfect matchings of $\{1, 2, \ldots, 2k\}$ and $\mathbb{P}\left(\mathcal{P}_{\bar{y}}\right)$ denotes the probability of getting this matching for the USF for the vertices $y_1, \ldots, y_{2k}$.

Let us consider one of these perfect matchings that for convenience we will assume is $y_1 \leftrightarrow y_2, y_3 \leftrightarrow y_4, \ldots, y_{2k-1} \leftrightarrow y_{2k}$. We claim that

$$\mathbb{P}(y_1 \leftrightarrow y_2, y_3 \leftrightarrow y_4, \ldots, y_{2k-1} \leftrightarrow y_{2k}) =$$
$$O\left(\left[\frac{(\log\log n)^3}{\log n}\right]^{k+1}\right) + \mathbb{P}(y_1 \leftrightarrow y_2)\,\mathbb{P}(y_3 \leftrightarrow y_4) \cdots \mathbb{P}(y_{2k-1} \leftrightarrow y_{2k}).$$

Indeed, this is just inclusion-exclusion using our estimate on $\mathbb{P}\{N \geq k+1\}$.

If we write $\epsilon_n = \epsilon_{n,k} = (\log\log n)^{3(k+1)}/\log n$, we now see from symmetry that

$$\begin{aligned}
&\mathbb{E}\left[\langle h, \phi_n\rangle^{2k}\rangle\right] \\
&= O(\epsilon_n) + n^{-2kd} a_n\,(2k-1)!! \sum_{\bar{x} \in L_*} \mathbb{P}\{nx_1 \leftrightarrow nx_2, \ldots, nx_{2k-1} \leftrightarrow nx_{2k}\} \\
&= O(\epsilon_n) + (2k-1)!! \left[\mathbb{E}\left(\langle h, \phi_n\rangle^2\right)\right]^k.
\end{aligned}$$

1.5 Proof of Lemma 1.7

Here we fix k and let $y_1, \dots, y_{2k}$ be points with $|y_j| \le Kn$ and$|y_i - y_j| \ge n\,\delta_n$ where we recall $\log \delta_n = (\log\log n)^2$. Let $\omega^1, \dots, \omega^{2k}$ be independent simple random walk starting at y_j stopped when they get to ∂A_N. We let $E_{i,j}$ denote the event that $\omega^i \cap \omega^j \ne \emptyset$, and let $R_{i,j} = \mathbb{P}(E_{i,j} \mid \omega^j)$.

Lemma 1.8. *There exists $c < \infty$ such that for all i, j, and all n sufficiently large,*

$$\mathbb{P}\left\{R_{i,j} \ge c\,\frac{(\log\log n)^3}{\log n}\right\} \le \frac{1}{(\log n)^{4k}}.$$

Proof. We know that there exists $c < \infty$ such that if $|y - z| \ge n\,\delta_n^2$, then the probability that simple random walks starting at y, z stopped when the reach ∂A_N intersect is at most $c(\log\log n)^2/\log n$. Hence there exists c_1 such that

$$\mathbb{P}\left\{R_{i,j} \le \frac{c_1\,(\log\log n)^2}{\log n}\right\} \ge \frac{1}{2}. \tag{7}$$

Start a random walk at z and run it until one of three things happens:

- It reaches ∂A_N
- It gets within distance $n\,\delta_n^2$ of y
- The path is such that the probability that a simple random walk starting at y intersects the path before reaching ∂A_N is greater than $c_1(\log\log n)^2/\log n$.

If the third option occurs, then we restart the walk at the current site and do this operation again. Eventually one of the first two options will occur. Suppose it takes r trials of this process until one of the first two events occur. Then either $R_{i,j} \le r\,c_1\,(\log\log n)^2/\log n$ or the original path starting at z gets within distance δ_n^2 of y. The latter event occurs with probability $O(\delta_n) = o((\log n)^{-4k})$. Also, using (7), we can see the probability that it took at least r steps is bounded by $(1/2)^r$. By choosing $r = c_2 \log\log n$, we can make this probability less than $1/(\log n)^{4k}$. □

Proof of Lemma 1.7. Let R be the maximum of $R_{i,j}$ over all $i \ne j$ in $\{1, \dots, 2k\}$. Then, at least for n sufficiently large,

$$\mathbb{P}\left\{R \ge c\,\frac{(\log\log n)^3}{\log n}\right\} \le \frac{1}{(\log n)^{3k}}.$$

Let

$$E_{\cdot,j} = \bigcup_{i=1}^{j-1} E_{i,j}.$$

On the event $R < c(\log\log n)^3/\log n$, we have

$$\mathbb{P}\left\{E_{\cdot,j} \mid \omega^1, \ldots, \omega^{j-1}\right\} \leq \frac{c(j-1)\,(\log\log n)^3}{\log n}.$$

If N denotes the number of j for which $E_{\cdot,j}$ occurs, we see that

$$\mathbb{P}\{N \geq k+1\} \leq c\left[\frac{(\log\log n)^3}{\log n}\right]^{k+1}.$$

□

2 Loop-erased walk in $\mathbb{Z}^4$

The purpose of the remainder of this paper is to improve results about the escape probability for loop-erased walk in four dimensions.

We start by setting up our notation. Let S be a simple random walk in $\mathbb{Z}^4$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ starting at the origin. Let $G(x,y)$ be the Green's function for simple random walk which we recall satisfies

$$G(x) = \frac{2}{\pi^2\,|x|^2} + O(|x|^{-4}), \quad |x| \to \infty.$$

Using this we see that there exists a constant λ such that as $r \to \infty$,

$$\sum_{|x|\leq r} G(x)^2 = \frac{8}{\pi^2}\,\log r + \lambda + O(r^{-1}). \tag{8}$$

It will be easiest to work on geometric scales, and we let C_n, A_n be the discrete balls and annuli defined by

$$C_n = \{z \in \mathbb{Z}^4 : |z| < e^n\}, \quad A_n = C_n \setminus C_{n-1} = \{z \in C_n : |z| \geq e^{n-1}\}.$$

Let

$$\sigma_n = \min\{j \geq 0 : S_j \notin C_n\},$$

and let $\mathcal{F}_n$ denote the σ-algebra generated by $\{S_j : j \le \sigma_n\}$. If $V \subset \mathbb{Z}^4$, we write

$$H(x,V) = \mathbb{P}^x\{S[0,\infty) \cap V \neq \emptyset\}, \quad H(V) = H(0,V), \quad \mathrm{Es}(V) = 1 - H(V),$$

$$\overline{H}(x,V) = \mathbb{P}^x\{S[1,\infty) \cap V \neq \emptyset\}, \quad \overline{\mathrm{Es}}(V) = 1 - \overline{H}(0,V).$$

Note that $\overline{\mathrm{Es}}(V) = \mathrm{Es}(V)$ if $0 \notin V$ and otherwise a standard last-exit decomposition shows that

$$\mathrm{Es}(V^0) = G_{V^0}(0,0)\,\overline{\mathrm{Es}}(V),$$

where $V^0 = V \setminus \{0\}$.

We have to be a little careful about the definition of the loop-erasures of the random walk and loop-erasures of subpaths of the walk. We will use the following notations.

- $\hat{S}[0,\infty)$ denotes the (forward) loop-erasure of $S[0,\infty)$ and

$$\Gamma = \hat{S}[1,\infty) = \hat{S}[0,\infty) \setminus \{0\}.$$

- ω_n denotes the finite random walk path $S[\sigma_{n-1},\sigma_n]$
- $\eta^n = LE(\omega_n)$ denotes the loop-erasure of $S[\sigma_{n-1},\sigma_n]$.
- $\Gamma_n = LE(S[0,\sigma_n]) \setminus \{0\}$, that is, Γ_n is the loop-erasure of $S[0,\sigma_n]$ with the origin removed.

Note that $S[1,\infty)$ is the concatenation of the paths $\omega_1, \omega_2, \ldots$. However, it is not true that Γ is the concatenation of $\eta^1, \eta^2, \ldots$, and that is one of the technical issues that must be addressed in the proof.

Let Y_n, Z_n be the $\mathcal{F}_n$-measurable random variables

$$Y_n = H(\eta^n), \qquad Z_n = \mathrm{Es}[\Gamma_n], \quad G_n = G_{\mathbb{Z}^4 \setminus \Gamma_n}(0,0).$$

As noted above,

$$\overline{\mathrm{Es}}(\Gamma_n \cup \{0\}) = G_n^{-1}\, Z_n.$$

It is easy to see that $1 \le G_n \le 8$, and using transience, we can see that with probability one

$$\lim_{n\to\infty} G_n = G_\infty := G_{\Gamma\setminus\{0\}}(0,0).$$

Theorem 2.1. *For every $0 \le r, s < \infty$, there exists $c_{r,s} < \infty$, such that*

$$\mathbb{E}\left[Z_n^r\, G_n^{-s}\right] \sim \frac{c_{r,s}}{n^{r/3}}.$$

Moreover, $c_{3,2} = \pi^2/24$.

Our methods do not compute the constant $c_{r,s}$ except in the case case $r = 3, s = 2$. The proof of this theorem requires several steps which we will outline now. For the remainder of this paper we fix $0 < r \le r_0$ and allow constants to depend on r_0 but not otherwise on r. We write

$$p_n = \mathbb{E}\left[Z_n^r\right], \quad \hat{p}_n = \mathbb{E}\left[Z_n^3\, G_n^{-2}\right].$$

Here are the steps in order with a reference to the section that the argument appears. Let

$$\phi_n = \prod_{j=1}^{n} \exp\left\{-\mathbb{E}[H(\eta^j)]\right\}.$$

- **Section 2.1.** Show that $\mathbb{E}\left[H(\eta^n)\right] = O(n^{-1})$, and hence,

$$\phi_n = \phi_{n-1}\, \exp\{-\mathbb{E}[H(\eta^n)]\} = \phi_{n-1}\left[1 + O\left(\frac{1}{n}\right)\right].$$

- **Section 2.2.** Show that

$$p_n = p_{n-1}\left[1 + O\left(\frac{\log^4 n}{n}\right)\right]. \tag{9}$$

- **Section 2.4.** Find a function $\tilde{\phi}_n$ such that there exist $\tilde{c}_r$ with

$$p_{n^4} = \tilde{c}_r\, \tilde{\phi}_n^r\left[1 + O\left(\frac{\log^2 n}{n}\right)\right].$$

- **Section 2.5.** Find $u, \hat{c} > 0$ such that

$$\tilde{\phi}_n = \hat{c}\, \phi_{n^4}\left[1 + O(n^{-u})\right].$$

- Combining the previous estimates we see that there exist $c' = c'_r, u$ such that

$$p_n = c'\, \phi_n\left[1 + O\left(\frac{\log^4 n}{n^{1/4}}\right)\right].$$

- **Section 2.6.** Show that there exists $c'_{r,s}, u$ such that

$$\mathbb{E}\left[Z_n^r\, G_n^{-s}\right] = c'_{r,s}\, \phi_n \left[1 + O\left(\frac{\log^4 n}{n^{1/4}}\right)\right].$$

- **Section 2.7.** Use a path decomposition to show that

$$\mathbb{E}[H(\eta^n)] = \frac{8}{\pi^2}\, \hat{p}_n \left[1 + O(n^{-u})\right],$$

  and combine the last two estimates to conclude that

$$\hat{p}_n \sim \frac{\pi^2}{24n}.$$

The error estimates are probably not optimal, but they suffice for proving our theorem. We say that a sequence $\{\epsilon_n\}$ of positive numbers is *fast decaying* if it decays faster than every power of n, that is

$$\lim_{n\to\infty} n^k\, \epsilon_n = 0,$$

for every $k > 0$. We will write $\{\epsilon_n\}$ for fast decaying sequences. As is the convention for constants, the exact value of $\{\epsilon_n\}$ may change from line to line. We will use implicitly the fact that if $\{\epsilon_n\}$ is fast decaying then so is $\{\epsilon'_n\}$ where

$$\epsilon'_n = \sum_{m\geq n} \epsilon_m.$$

2.1 Preliminaries

In this subsection we prove some necessary lemmas about simple random walk in $\mathbb{Z}^4$. The reader could skip this section now and refer back as necessary. We will use the following facts about intersections of random walks in $\mathbb{Z}^4$, see [12, 13].

Proposition 2.2. *There exist $0 < c_1 < c_2 < \infty$ such that the following is true. Suppose S is a simple random walk starting at the origin in $\mathbb{Z}^4$ and $a > 2$. Then*

$$\begin{aligned}
\frac{c_1}{\sqrt{\log n}} &\leq \mathbb{P}\left\{S[0,n] \cap S[n+1,\infty] = \emptyset\right\} \\
&\leq \mathbb{P}\left\{S[0,n] \cap S[n+1,2n] = \emptyset\right\} \\
&\leq \frac{c_2}{\sqrt{\log n}},
\end{aligned}$$

$$\mathbb{P}\left\{S[0,n]\cap S[n\,(1+a^{-1}),\infty)\neq\emptyset\right\}\leq c_2\,\frac{\log a}{\log n}.$$

Moreover, if S^1 is an independent simple random walk starting at $z\in\mathbb{Z}^4$,

$$\mathbb{P}\left\{S[0,n]\cap S^1[0,\infty)\neq\emptyset\right\}\leq c_2\,\frac{\log\alpha}{\log n}, \tag{10}$$

where

$$\alpha=\max\left\{2,\frac{\sqrt{n}}{|z|}\right\}.$$

An important corollary of the proposition is that

$$\sup_n n\,\mathbb{E}\left[H(\eta^n)\right]\leq\sup_n n\,\mathbb{E}\left[H(\omega_n)\right]<\infty, \tag{11}$$

and hence

$$\exp\left\{-\mathbb{E}\left[H(\eta^n)\right]\right\}=1-\mathbb{E}\left[H(\eta^n)\right]+O(n^{-2}),$$

and if $m<n$,

$$\phi_n=\phi_m\left[1+O(m^{-1})\right]\prod_{j=m+1}^{n}\left[1-\mathbb{E}\left[H(\eta^j)\right]\right].$$

Corollary 2.3. *There exists $c<\infty$ such that if $n>0,\alpha\geq 2$ and we let $m=m_{n,\alpha}=(1+\alpha^{-1})\,n$,*

$$Y=Y_{n,\alpha}=\max_{j\geq m}H(S_j,S[0,n]),$$

$$Y'=Y'_{n,\alpha}=\max_{0\leq j\leq n}H(S_j,S[m,2n]),$$

then for every $0<u<1$,

$$\mathbb{P}\left\{Y\geq\frac{\log\alpha}{(\log n)^u}\right\}\leq\frac{c}{(\log n)^{1-u}},\quad \mathbb{P}\left\{Y'\geq\frac{\log\alpha}{(\log n)^u}\right\}\leq\frac{c}{(\log n)^{1-u}}.$$

Proof. Let

$$\tau=\min\left\{j\geq m: H(S_j,S[0,n])\geq\frac{\log\alpha}{(\log n)^u}\right\}.$$

The strong Markov property implies that

$$\mathbb{P}\left\{S[0,n] \cap S[m,\infty) \neq \emptyset \mid \tau < \infty\right\} \geq \frac{\log \alpha}{(\log n)^u},$$

and hence Proposition 2.2 implies that

$$\mathbb{P}\{\tau < \infty\} \leq \frac{c_2}{(\log n)^{1-u}}.$$

This gives the first inequality and the second can be done similarly looking at the reversed path.

□

Lemma 2.4. *There exists $c < \infty$ such that for all n:*

- *if $m < n$,*
$$\mathbb{P}\{S[\sigma_n,\infty) \cap C_{n-m} \neq \emptyset \mid \mathcal{F}_n\} \leq c\, e^{-2m}; \tag{12}$$
- *if ϕ is a positive (discrete) harmonic function on C_n and $x \in C_{n-1}$,*
$$|\log[\phi(x)/\phi(0)]| \leq c\,|x|\, e^{-n}. \tag{13}$$

Proof. These are standard estimates, see, e.g., [16, Theorem 6.3.8, Proposition 6.4.2].

□

Lemma 2.5. *Let*

$$\sigma_n^- = \sigma_n - \lfloor n^{-1/4}\, e^{2n} \rfloor, \quad \sigma_n^+ = \sigma_n + \lfloor n^{-1/4}\, e^{2n} \rfloor,$$

$$\mathcal{S}_n^- = S[0,\sigma_n^-], \quad \mathcal{S}_n^+ = S[\sigma_n^+,\infty),$$

$$R = R_n = \max_{x \in \mathcal{S}_n^-} H(x,\mathcal{S}_n^+) + \max_{y \in \mathcal{S}_n^+} H(y,\mathcal{S}_n^-),$$

Then, for all n sufficiently large,

$$\mathbb{P}\{R_n \geq n^{-1/3}\} \leq n^{-1/3}. \tag{14}$$

Our proof will actually give a stronger estimate, but (14) is all that we need and makes for a somewhat cleaner statement.

Proof. Using the fact that $\mathbb{P}\{\sigma_n \leq (k+1)\, e^{2n} \mid \sigma_n \geq k\, e^{2n}\}$ is bounded uniformly away from zero, we can see that there exists c_0 with

$$\mathbb{P}\{\sigma_n \geq c_0\, e^{2n} \log n\} \leq O(n^{-1}).$$

Let $N = N_n = \lceil c_0\, e^{2n} \log n \rceil, k = k_n = \lfloor n^{-1/4}\, e^{2n}/4 \rfloor$ and let $E_j = E_{j,n}$ be the event that either

$$\max_{i \leq jk} H(S_i, S[j(k+1), N]) \geq \frac{\log n}{n^{1/4}},$$

or

$$\max_{(j+1)k \leq i \leq N} H(S_i, S[0, jk]) \geq \frac{\log n}{n^{1/4}},$$

Using the previous corollary, we see that $\mathbb{P}(E_i) \leq O(n^{-3/4})$ and hence

$$\mathbb{P}\left[\bigcup_{jk \leq N} E_j\right] \leq O\left(\frac{\log n}{n^{1/2}}\right).$$

□

Lemma 2.6. *Let*

$$M_n = \max |S_j - S_k|\,,$$

where the maximum is over all $0 \leq j \leq k \leq n\, e^{2n}$ *with* $|j-k| \leq n^{-1/4}\, e^{2n}$. *Then,*

$$\mathbb{P}\left\{M_n \geq n^{-1/5}\, e^n\right\}$$

is fast decaying.

Proof. This is a standard large deviation estimate for random walk, see, e.g., [16, Corollary A.2.6]. □

Lemma 2.7. *Let* U_n *be the event that there exists* $k \geq \sigma_n$ *with*

$$LE(S[0,k]) \cap C_{n-\log^2 n} \neq \Gamma \cap C_{n-\log^2 n}.$$

Then $\mathbb{P}(U_n)$ *is fast decaying.*

Proof. By the loop-erasing process, we can see that the event U_n is contained in the event that either

$$S[\sigma_{n-\frac{1}{2}\log^2 n}, \infty) \cap C_{n-\log^2 n} \neq \emptyset$$

or

$$S[\sigma_n, \infty) \cap C_{n-\frac{1}{2}\log^2 n} \neq \emptyset.$$

The probability that either of these happens is fast decaying by (12). □

Proposition 2.8. *If $\Lambda(m,n) = S[0,\infty) \cap A(m,n)$, then the sequences*

$$\mathbb{P}\left\{H[\Lambda(n-1,n)] \geq \frac{\log^2 n}{n}\right\}, \quad \mathbb{P}\{\left\{H(\omega_n) \geq \frac{\log^4 n}{n}\right\},$$

are fast decaying.

Proof. Using Proposition 2.2 we can see that there exists $c < \infty$ such that for all $|z| \geq e^{n-1}$,

$$\mathbb{P}^z\left\{H(S[0,\infty)) \geq \frac{c}{n}\right\} \leq \frac{1}{2}.$$

Using this and the strong Markov property, we can by induction that for every positive integer k,

$$\mathbb{P}\left\{H[\Lambda(n-1,n)] \geq \frac{ck}{n}\right\} \leq 2^{-k}.$$

Setting $k = \lfloor c^{-1}\log^2 n\rfloor$, we see that the first sequence is fast decaying.

For the second, we use (12) to see that

$$\mathbb{P}\left\{\omega_n \not\subset A(n-\log^2 n, n)\right\}$$

is fast decaying, and if $\omega_n \subset A(n-\log^2 n, \log n)$,

$$H(\omega_n) \leq \sum_{n-\log^2 n \leq j \leq n} H\left[\Lambda(j-1,j)\right].$$

□

2.2 Sets in $\mathcal{X}_n$

Simple random walk paths in $\mathbb{Z}^4$ are "slowly recurrent" sets in the terminology of [13]. In this section we will consider a subcollections $\mathcal{X}_n$ of the collection of slowly recurrent sets and give uniform bounds for escape probabilities for such sets. We will then apply these uniform bounds to random walk paths and their loop-erasures.

If $V \subset \mathbb{Z}^4$ we write

$$V_n = V \cap A(n-1,n), \quad h_n = H(V_n).$$

Using the Harnack inequality, we can see that there exist $0 < c_1 < c_2 < \infty$ such that

$$c_1\, h_n \le H(z, V_n) \le c_2\, h_n, \qquad z \in C_{n-2} \cup A(n+1, n+2).$$

Let $\mathcal{X}_n$ denote the collection of subsets of V of $\mathbb{Z}^4$ such that for all integers $m \ge \sqrt{n}$,

$$H(V_m) \le \frac{\log^2 m}{m}.$$

Proposition 2.9. $\mathbb{P}\{S[0,\infty) \notin \mathcal{X}_n\}$ *is a fast decaying sequence.*

Proof. This is an immediate corollary of Proposition 2.8. □

Let E_n denote the event

$$E_n = E_{n,V} = \{S[1, \sigma_n] \cap V = \emptyset\}.$$

Proposition 2.10. *There exists $c < \infty$ such that if $V \in \mathcal{X}_n$, $m \ge n/10$, and $\mathbb{P}(E_{m+1} \mid E_m) \ge 1/2$, then*

$$\mathbb{P}(E_{m+2}^c \mid E_{m+1}) \le \frac{c \log^2 n}{n}.$$

Proof. For $z \in \partial C_{m+1}$, let

$$u(z) = \mathbb{P}\{S(\sigma_{m+1}) = z \mid E_m\}.$$

Then,

$$\mathbb{P}(E_{m+2}^c \mid E_{m+1}) = \sum_{z \in C_{m+1}} u(z)\, \mathbb{P}^z\{S[0, \sigma_{m+2}] \cap V \ne \emptyset\}.$$

Note that

$$\begin{aligned} u(z) &= \frac{\mathbb{P}\{S(\sigma_{m+1}) = z, E_{m+1}\}}{\mathbb{P}(E_{m+1})} \\ &\le 2\, \frac{\mathbb{P}\{S(\sigma_{m+1}) = z, E_{m+1}\}}{\mathbb{P}(E_m)} \le 2\, \mathbb{P}\{S(\sigma_{m+1}) = z \mid E_m\}. \end{aligned}$$

Using the Harnack principle, we see that

$$\mathbb{P}\{S(\sigma_{m+1}) = z \mid E_m\} \le \sup_{w \in \partial C_m} \mathbb{P}^w\{S(\sigma_{m+1}) = z\} \le c\, \mathbb{P}\{S(\sigma_{m+1}) = z\}.$$

Therefore,

$$\begin{aligned}\mathbb{P}(E_{m+2}^c \mid E_{m+1}) &\le c\,\mathbb{P}\{S[\sigma_{m+1},\sigma_{m+2}]\cap V\neq\emptyset\}\\ &\le c\sum_{k=1}^{m+2}\mathbb{P}\{S[\sigma_{m+1},\sigma_{m+2}]\cap V_k\neq\emptyset\}.\end{aligned}$$

Note that

$$\begin{aligned}\mathbb{P}\{S[\sigma_{m+1},\sigma_{m+2}]\cap(V_m\cup V_{m+1}\cup V_{m+2})\neq\emptyset\} &\le H(V_m\cup V_{m+1}\cup V_{m+2})\\ &\le c\,\frac{\log^2 n}{n}.\end{aligned}$$

Using (12), we see that for λ large enough,

$$\mathbb{P}\{S[\sigma_{m+1},\sigma_{m+2}]\cap C_{m-\lambda\log m}\neq\emptyset\}\le cn^{-2}.$$

For $m-\lambda\log m\le k\le m-1$, we estimate

$$\mathbb{P}\{S[\sigma_{m+1},\sigma_{m+2}]\cap V_k\neq\emptyset\}\le\mathbb{P}[E']\,\mathbb{P}\{S[\sigma_{m+1},\sigma_{m+2}]\cap V_k\neq\emptyset\mid E'\},$$

where $E'=\{S[\sigma_{m+1},\sigma_{m+2}]\cap C_{k+1}\neq\emptyset\}$. The first probability on the right-hand side is $\exp\{-O(k-m)\}$ and this second is $O(\log^2 n/n)$. Summing over k we get the result.

□

Proposition 2.11. *There exists $c<\infty$ and a fast decaying sequence $\{\epsilon_n\}$ such that if $V\in\mathcal{X}_n$, and $\mathbb{P}(E_n)\ge\epsilon_n$, then*

$$\mathbb{P}(E_{j+1}^c\mid E_j)\le\frac{c\,\log^2 n}{n},\qquad\frac{3n}{4}\le j\le n.$$

Proof. It suffices to consider n sufficiently large. If $\mathbb{P}(E_{m+1}\mid E_m)\le 1/2$ for all $n/4\le m\le n/2$, then $\mathbb{P}(E_n)\le(1/2)^{n/4}$ which is fast decaying. If $\mathbb{P}(E_{m+1}\mid E_m)\ge 1/2$ for some $n/4\le m\le n/2$, then (for n sufficiently large) we can use the previous proposition and induction to conclude that $\mathbb{P}(E_{k+1}\mid E_k)\ge 1/2$ for $m\le k\le n$.

□

Corollary 2.12. *There exists c such that for all n,*

$$|\log(p_n/p_{n+1})|\le\frac{c\,\log^4 n}{n}.$$

In particular, there exists c such that if $n^4 \le m \le (n+1)^4$,

$$|\log(p_m/p_{n^4})| \le \frac{c \log^4 n}{n}.$$

Proof. Except for an event with fast decaying probability, we have

$$\Gamma_n \triangle \Gamma_{n+1} \subset \Lambda(n - \log^2 n, n+1).$$

and

$$H\left(\Lambda(n - \log^2 n, n+1)\right) \le \frac{c \log^4 n}{n}.$$

This implies that there exists a fast decaying sequence ϵ_n such that, except for an event of probability at most ϵ_n, either $Z_n \le \epsilon_n$ or

$$|\log Z_n - \log Z_{n+1}| \le \frac{c \log^4 n}{n}.$$

□

2.3 Loop-free times

One of the technical nuisances in the analysis of the loop-erased random walk is that if $j < k$, it is not necessarily the case that

$$LE(S[j,k]) = LE(S[0,\infty)) \cap S[j,k].$$

However, this is the case for special times called *loop-free* times. We say that j is a *(global) loop-free time* if

$$S[0,j] \cap S[j+1,\infty) = \emptyset.$$

Proposition 2.2 shows that the probability that j is loop-free is comparable to $1/(\log j)^{1/2}$. From the definition of chronological loop erasing we can see the following:

- If $j < k$ and j, k are loop-free times, then for all $m \le j < k \le n$,

$$LE\left(S[m,n]\right) \cap S[j,k] = LE\left(S[0,\infty)\right) \cap S[j,k] = LE\left(S[j,k]\right). \quad (15)$$

It will be important for us to give upper bounds on the probability that there is *no* loop-free time in a certain interval of time. If $m \le j < k \le n$, let $I(j,k;m,n)$ denote the event that for all $j \le i \le k-1$,

$$S[m,i] \cap S[i+1,n] \neq \emptyset.$$

Proposition 2.2 gives a lower bound on $\mathbb{P}\left[I(n,2n;0,3n)\right]$,

$$\mathbb{P}\left[I(n,2n;0,3n)\right] \ge \mathbb{P}\{S[0,n] \cap S[2n,3n] \neq \emptyset\} \asymp \frac{1}{\log n}.$$

The next lemma shows that $\mathbb{P}\left[I(n,2n;0,3n)\right] \asymp 1/\log n$ by giving the upper bound.

Lemma 2.13. *There exists $c < \infty$ such that*

$$\mathbb{P}\left[I(n,2n;0,\infty)\right] \le \frac{c}{\log n}.$$

Proof. Let $E = E_n$ denote the complement of $I(n,2n;0,\infty)$. We need to show that $\mathbb{P}(E) \ge 1 - O(1/\log n)$.

Let $k_n = \lfloor n/(\log n)^{3/4} \rfloor$ and let $A_i = A_{i,n}$ be the event that

$$A_i = \left\{S[n+(2i-1)k_n, n+2ik_n] \cap S[n+2ik_n+1, n+(2i+1)k_n] = \emptyset\right\}.$$

and consider the events $A_1, A_2, \ldots, A_r$ where $r = \lfloor (\log n)^{3/4}/4 \rfloor$. These are r independent events each with probability greater than $c\,(\log n)^{-1/2}$. Hence the probability that none of them occurs is $\exp\{-O((\log n)^{-1/4})\} = o((\log n)^{-3})$, that is,

$$\mathbb{P}(A_1 \cup \cdots \cup A_r) \ge 1 - o\left(\frac{1}{\log^3 n}\right).$$

Let $B_i = B_{i,n}$ be the event

$$B_i = \left\{S[0, n+(2i-1)k_n] \cap S[n+2ik_n, \infty) = \emptyset\right\},$$

and note that

$$E \supset (A_1 \cup \cdots \cup A_r) \cap (B_1 \cap \cdots \cap B_r).$$

Since $\mathbb{P}(B_i^c) \le c \log\log n/\log n$, we see that

$$\mathbb{P}(B_1 \cap \cdots \cap B_r) \ge 1 - \frac{cr \log\log n}{\log n} \ge 1 - O\left(\frac{\log\log n}{(\log n)^{1/4}}\right),$$

and hence,

$$\mathbb{P}(E) \geq \mathbb{P}\left[(A_1 \cup \cdots \cup A_r) \cap (B_1 \cap \cdots \cap B_r)\right] \geq 1 - O\left(\frac{\log\log n}{(\log n)^{1/4}}\right). \quad (16)$$

This is a good estimate, but we need to improve on it.

Let $C_j, j = 1., \ldots, 5$, denote the independent events (depending on n)

$$I\left(n\left[1 + \frac{3(j-1)+1}{15}\right], n\left[1 + \frac{3(j-1)+2}{15}\right]; n + \frac{(j-1)n}{5}, n + \frac{jn}{5}\right).$$

By (16) we see that $\mathbb{P}[C_j] \leq o\left(1/(\log n)^{1/5}\right)$, and hence

$$\mathbb{P}(C_1 \cap \cdots \cap C_5) \leq o\left(\frac{1}{\log n}\right).$$

Let $D = D_n$ denote the event that at least one of the following ten things happens:

$$S\left[0, n\left(1 + \frac{j-1}{5}\right]\right) \cap S\left[n\left(1 + \frac{3(j-1)+1}{15}\right), \infty\right) \neq \emptyset, \quad j = 1, \ldots, 5.$$

$$S\left[0, n\left(1 + \frac{3(j-1)+2}{15}\right)\right] \cap S\left[n\left(1 + \frac{j}{5}\right), \infty\right) \neq \emptyset, \quad j = 1, \ldots, 5.$$

Each of these events has probability comparable to $1/\log n$ and hence $\mathbb{P}(D) \asymp 1/\log n$. Also,

$$I(n, 2n; 0, \infty) \subset (C_1 \cap \cdots \cap C_n) \cup D.$$

Therefore, $\mathbb{P}\left[I(n, 2n; 0, \infty)\right] \leq c/\log n$.

□

Corollary 2.14.

1. *There exists $c < \infty$ such that if $0 \leq j \leq j + k \leq n$, then*

$$\mathbb{P}\left[I(j, j+k; 0, n)\right] \leq \frac{c\,\log(n/k)}{\log n}.$$

2. *There exists $c < \infty$ such that if $0 < \delta < 1$ and*

$$I_{\delta,n} = \bigcup_{j=0}^{n-1} I(j, j + \delta n; 0, n),$$

then

$$\mathbb{P}\left[I_{\delta,n}\right] \leq \frac{c\,\log(1/\delta)}{\delta\,\log n}.$$

Proof.

1. We will assume that $k \geq n^{1/2}$, since otherwise $\log(n/k)/\log n \geq 1/2$. Note that

$$I(j, j+k; 0, n) \subset I(j, j+k; j-k, j+2k) \cup \{S[0, j-k] \cap S[k+2k, n] \neq \emptyset\},$$

and

$$\mathbb{P}\left[I(j, j+k; j-k, j+2k)\right] \leq \frac{c}{\log k} = O\left(\frac{1}{\log n}\right),$$

$$\mathbb{P}\{S[0, j-k] \cap S[k+2k, n] \neq \emptyset\} \leq \frac{c \log(n/k)}{\log n}.$$

2. We can cover $I_{\delta,n}$ by the union of $O(1/\delta)$ events each of probability $O(\log(1/\delta)/\log n)$.

□

2.4 Along a subsequence

It will be easier to prove the main result first along the subsequence $\{n^4\}$. We let $b_n = n^4$, $\tau_n = \sigma_{n^4}, q_n = p_{n^4}$, $\mathcal{G}_n = \mathcal{F}_{n^4}$. Let $\hat{\eta}_n$ denote the (forward) loop-erasure of $S[\sigma_{(n-1)^4+(n-1)}, \sigma_{n^4-n}]$. and

$$\tilde{\Gamma}_n = \hat{\eta}_n \cap A(b_{n-1} + 4(n-1), b_n - 4n),$$

$$\Gamma_n^* = \Gamma \cap A(b_{n-1} + 4(n-1), b_n - 4n).$$

We state the main result that we will prove in this subsection.

Proposition 2.15. *There exists c_0 such that as $n \to \infty$,*

$$q_n = \left[c_0 + O\left(\frac{\log^4 n}{n^2}\right)\right] \exp\left\{-r \sum_{j=1}^{n} \tilde{h}_j\right\},$$

where

$$\tilde{h}_j = \mathbb{E}\left[H(\tilde{\Gamma}_j)\right].$$

Lemma 2.16. *There exists $c < \infty$ such that*

$$\mathbb{P}\left\{\tilde{\Gamma}_n \neq \Gamma_n^*\right\} \leq \frac{c}{n^4}.$$

$$\mathbb{P}\left\{\exists m \geq n \text{ with } \tilde{\Gamma}_m \neq \Gamma_m^*\right\} \leq \frac{c}{n^3}.$$

Proof. The first follows from Corollary 2.14, and the second is obtained by summing over $m \geq n$.

□

Let S' be another simple random walk defined another probability space $(\Omega', \mathbb{P}')$ with corresponding stopping times τ_n'. Let

$$Q_n = \mathrm{Es}\left[\tilde{\Gamma}_1 \cup \cdots \cup \tilde{\Gamma}_n\right], \quad Y_n = H(\tilde{\Gamma}_n).$$

Using only the fact that

$$\tilde{\Gamma}_n \subset \left\{\exp\{b_{n-1} + 4(n-1)\} \leq |z| \leq \exp\{b_n - 4n\}\right\},$$

we can see that

$$Q_{n+1} = Q_n\,[1 - Y_n]\,[1 + O(\epsilon_n)]. \tag{17}$$

for some fast decaying sequence $\{\epsilon_n\}$. On the event that $S[0,\infty) \in \mathcal{X}_n$,

$$\mathrm{Es}[\Gamma_{n+1}] = \mathrm{Es}[\Gamma_n]\left[1 - Y_n + O\left(\frac{\log^4 n}{n^3}\right)\right]$$

$$H(\eta^n) = Y_n + O\left(\frac{\log^4 n}{n^3}\right).$$

and for $m \geq n$,

$$Z_{m^4} = Z_{n^4}\,\exp\left\{-\sum_{j=n+1}^{m} Y_j\right\}\left[1 + O\left(\frac{\log^4 n}{n^2}\right)\right]. \tag{18}$$

Lemma 2.17. *There exists a fast decaying sequence $\{\epsilon_n\}$ such that we can define $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \ldots$ and $\tilde{\Gamma}_1', \tilde{\Gamma}_2', \ldots$ on the same probability space so that $\tilde{\Gamma}_j$ has the same distibution as $\tilde{\Gamma}_j'$ for each j; $\tilde{\Gamma}_1', \tilde{\Gamma}_2', \ldots$ are independent; and*

$$\mathbb{P}\{\tilde{\Gamma}_n \neq \tilde{\Gamma}_n'\} \leq \epsilon_n.$$

In particular, if $Y'_n = H(\tilde{\Gamma}'_n)$,

$$\begin{aligned}\mathbb{E}\left[\prod_{j=n}^{m}[1-Y_j] \mid \mathcal{G}_{n-1}\right] &= [1+O(\epsilon_n)]\prod_{j=n}^{m}\mathbb{E}[1-Y_j] \\ &= \left[1+O\left(\frac{\log^4 n}{n^4}\right)\right]\exp\left\{-\sum_{j=n}^{m}\mathbb{E}(Y_j)\right\}.\end{aligned}$$

Also, using (18), except on an event of fast decaying probability,

$$\mathbb{E}\left[Z^r_{m^4} \mid \mathcal{G}_n\right] = Z^r_{n^4}\,\exp\left\{-\sum_{j=n}^{m}\mathbb{E}(Y_j)\right\}\left[1+O\left(\frac{\log^4 n}{n^2}\right)\right].$$

Taking expectations, we see that

$$q_m = q_n\,\exp\left\{-\sum_{j=n}^{m}\mathbb{E}(Y_j)\right\}\left[1+O\left(\frac{\log^4 n}{n^2}\right)\right],$$

which implies the proposition.

2.5 Comparing hitting probabilities

The goal of this section is to prove the following. Recall that $b_n = n^4, h_n \asymp n^{-1}, \tilde{h}_n \asymp n^{-1}$.

Proposition 2.18. *There exists* $c < \infty, u > 0$ *such that*

$$\left|\tilde{h}_n - \sum_{j=b_{n-1}+1}^{b_n} h_j\right| \le \frac{c}{n^{1+u}}.$$

Proof. For a given n we will define a set

$$U = U_{b_{n-1}+1} \cup \cdots \cup U_{b_n},$$

such that the following four conditions hold:

$$U \subset \tilde{\Gamma}_n, \tag{19}$$

$$U_j \subset \eta^j, \quad j = b_{n-1}+1, \ldots b_n, \tag{20}$$

$$\mathbb{E}\left[H(\tilde{\Gamma}_n \setminus U)\right] + \sum_{j=b_{n-1}+1}^{b_n} \mathbb{E}\left[H(\eta^n \setminus U_j)\right] \leq O(n^{-(1+u)}), \tag{21}$$

$$\max_{b_{n-1}<j\leq b_n} \max_{x\in U_j} H(x, U \setminus U_j) \leq n^{-u}. \tag{22}$$

We claim that finding such a set proves the result. Indeed,

$$H(U) \leq H(\tilde{\Gamma}_n) \leq H(U) + H(\tilde{\Gamma}_n \setminus U),$$

$$\sum_{j=b_{n-1}+1}^{b_n} H(U_j) \leq \sum_{j=b_{n-1}+1}^{b_n} H(\eta^j) \leq \sum_{j=b_{n-1}+1}^{b_n} \left[H(U_j) + H(\eta^j \setminus U_j)\right].$$

Also it follows from (22) and the strong Markov property that

$$H(U) = \left[1 - O(n^{-u})\right] \sum_{j=b_{n-1}+1}^{b_n} H(U_j) = -O(n^{-1-u}) + \sum_{j=b_{n-1}+1}^{b_n} H(U_j).$$

Taking expectations and using (19)–(21) we get

$$\tilde{h}_n = \mathbb{E}\left[H(\tilde{\Gamma}_n)\right] = O(n^{-1-u}) + \sum_{j=b_{n-1}+1}^{b_n} \mathbb{E}\left[H(\eta^j)\right] = O(n^{-1-u}) + \sum_{j=b_{n-1}+1}^{b_n} h_j.$$

Hence, we need only find the sets U_j.

Recall that $\sigma_j^{\pm} = \sigma_j \pm \lfloor j^{-1/4} e^{2j} \rfloor$ and let $\tilde{\omega}_j = S[\sigma_{j-1}^+, \sigma_j^-]$. Using Lemma 2.6, we can see that

$$\mathbb{P}\left\{\operatorname{diam}\left(S[\sigma_{j-1}^+, \sigma_j^-]\right) \geq j^{-1/5}\, e^j\right\}$$

is fast decaying. We will set U_j equal to $\eta^j \cap \tilde{\omega}_j$ unless one of the following six events occurs in which case we set $U_j = \emptyset$. We assume $(n-1)^4 < j \leq n^4$.

1. if $j \leq (n-1)^4 + 8n$ or $j \geq n^4 - 8n$.
2. If $H(\omega_j) \geq n^{-4} \log^2 n$.
3. If $\omega_j \cap C_{j-8\log n} \neq \emptyset$.
4. If $H(\omega_j \setminus \tilde{\omega}_j) \geq n^{-4-u}$.
5. If it is not true that there exist loop-free points in both $[\sigma_{j-1}, \sigma_{j-1}^+]$ and $[\sigma_j^-, \sigma_j]$.

6. If

$$\sup_{x \in \tilde{\omega}_j} H(x, S[0,\infty) \setminus \omega_j) \geq n^{-u}.$$

The definition of U_j immediately implies (20). Combining conditions 1 and 3, we see that (for n sufficiently large which we assume throughout) that $U_j \subset A(b_{n-1} + 6n, b_n - 6n)$. Moreover, if there exists loop-free points in $[\sigma_{j-1}, \sigma_{j-1}^+]$ and $[\sigma_j^-, \sigma_j]$, then $\tilde{\eta}_n \cap \tilde{\omega}_j = \eta_j \cap \tilde{\omega}_j$. Therefore, (19) holds. Also, condition 6 immediately gives (22).

In order to establish (21) we first note that

$$(\tilde{\Gamma}_n \cup \eta^{b_{n-1}+1} \cup \cdots \cup \eta^{b_n}) \setminus U \subset \bigcup_{j=b_{n-1}+1}^{b_n} V_j,$$

where

$$V_j = \begin{cases} \omega_j & \text{if } U_j = \emptyset \\ \omega_j \setminus \tilde{\omega}_j & \text{if } U_j = \eta^j \cap \tilde{\omega}_j \end{cases}.$$

Hence, it suffices to show that

$$\sum_{j=b_{n-1}+1}^{b_n} \mathbb{E}\left[H(\omega_j); U_j = \emptyset\right] + \sum_{j=b_{n-1}+1}^{b_n} \mathbb{E}\left[H(\omega_j \setminus \tilde{\omega}_j)\right] \leq c\, n^{-1-u}.$$

We write the event $\{U_j = \emptyset\}$ as a union of six disjoint events

$$\{U_j = \emptyset\} = E_j^1 \cup \cdots \cup E_j^6,$$

where E_j^i is the event that the ith condition above holds but none of the previous ones hold. It suffices to show that for $b_{n-1} < j \leq b_n$,

$$\mathbb{E}\left[H(\omega_j \setminus \tilde{\omega}_j)\right] + \sum_{i=1}^{6} \mathbb{E}\left[H(\omega_j)\, 1_{E_j^i}\right] \leq n^{-4-u}.$$

- In order to estimate $\mathbb{E}\left[H(\omega_j \setminus \tilde{\omega}_j)\right]$, we use standard estimates on random walk to show that for $\delta > 0$, except perhaps on an event of probability $o(n^{-2})$,

  $$\mathrm{Cap}[\omega_j \setminus \tilde{\omega}_j] \leq O(n^{-\frac{1}{8}+\delta})\, n^{-1/2}\, e^n, \qquad \mathrm{diam}[\omega_j \setminus \tilde{\omega}_j] \leq n^{-\frac{1}{2}+\delta}\, e^n$$

  and by choosing $\delta = 1/64$, we have off the bad event,

  $$H(\omega_j \setminus \tilde{\omega}_j) \leq O(n^{-1-\frac{1}{16}}). \tag{23}$$

We now consider the events $E_j^1, \ldots, E_j^6$.

1. Since for each j, $\mathbb{E}[H(\omega_j)] \asymp n^{-4}$

$$\sum_{j \le (n-1)^4 + 8n \text{ or } j \ge n^4 - 8n} \mathbb{E}[H(\omega_j)] = O(n^{-3}).$$

2. We have already seen that

$$\mathbb{P}\left\{ H(\omega_j) \ge \frac{\log^2 n}{n^4} \right\} \le O(\epsilon_n).$$

   We note that given this, for $j > 2$,

$$\mathbb{E}\left[H(\omega_j)\, 1_{E_n^j} \right] \le \frac{\log^2 n}{n^4}\, \mathbb{P}(E_n^j).$$

Hence it suffices for $j = 3, \ldots, 6$ to prove that $\mathbb{P}(E_n^j) \le n^{-u}$ for some u.

3. The standard estimate gives

$$\mathbb{P}\{\omega_j \cap C_{j - \log j} \ne \emptyset\} \le O(j^{-2}).$$

4. This we did in (23).
5. This follows from Corollary 2.14 for any $u < 1/4$.
6. This follows from Lemma 2.5.

□

2.6 $s \ne 0$

The next proposition is easy but it is important for us.

Proposition 2.19. *For every r, s there exists $c_{r,s}$ such that*

$$\mathbb{E}\left[Z_n^r\, G_n^{-s} \right] = c_{r,s}\, \phi_n^r \left[1 + \left(\frac{\log^4 n}{n^{1/4}} \right) \right]$$

Proof. Using Lemma 2.7, we can see that there exists a fast decaying sequence $\{\epsilon_n\}$ such that if $m \geq n$,

$$\mathbb{P}\{|G_n - G_m| \geq \epsilon_n\} \leq \epsilon_n.$$

Since $1 \leq G_n \leq 8$, this implies that

$$\left[\phi_n^{-1}\, Z_n^r\, G_n^{-s}\right] - \left[\phi_n^{-1}\, Z_n^r\, G_\infty^{-s}\right]$$

is fast decaying and

$$\left|\mathbb{E}\left(\hat{\phi}_n^{-1}\, Z_n^r\, G_\infty^{-s}\right) - \mathbb{E}\left(\hat{\phi}_m^{-1} Z_m^r\, G_\infty^{-s}\right)\right| \leq c\, \left|\mathbb{E}\left(\phi_n^{-1}\, Z_n^r\right) - \mathbb{E}\left(\phi_m^{-1} Z_m^r\right)\right|.$$

□

2.7 Determining the constant

The goal of this section is to prove

$$\hat{p}_n \sim \frac{\pi^2}{24n}. \tag{24}$$

We will give the outline of the proof using the lemmas that we will prove afterwards. By combining the results of the previous sections, we see that

$$\mathbb{E}\left[Z_n^r\, G_n^{-s}\right] = c'_{r,s}\, \left[1 + O(n^{-u})\right]\, \exp\left\{-r \sum_{j=1}^{n} h_j\right\},$$

where $h_j = \mathbb{E}\left[H(\eta^j)\right]$, and $c'_{r,s}$ are unknown constants. In particular,

$$\hat{p}_n = c'_{3,2}\, \left[1 + O(n^{-u})\right]\, \exp\left\{-3 \sum_{j=1}^{n} h_j\right\}. \tag{25}$$

We show in Section 2.7.2 that

$$h_n = \frac{8}{\pi^2}\, \hat{p}_n + O(n^{-1-u}).$$

It follows that the limit

$$\lim_{n\to\infty} \left[\log \hat{p}_n + \frac{24}{\pi^2} \sum_{j=1}^{n} \hat{p}_j\right],$$

exists and is finite. Using this and $\hat{p}_{n+1} = \hat{p}_n \left[1 + O(\log^4 n/n)\right]$ (see Corollary 2.12) we can conclude (see Lemma 2.20) the result. We note that it follows from this and (25) that there exists c such that

$$\phi_n^3 = \exp\left\{-3\sum_{j=1}^{n} h_j\right\} \sim \frac{c}{n}.$$

Let W denote a simple random walk independent of S. Then, $h_n = \mathbb{P}(E)$, where where

$$E = E_n = \{W[0,\infty) \cap \eta^n \neq \emptyset\}.$$

2.7.1 A lemma about a sequence

Lemma 2.20. *Suppose $p_1, p_2, \ldots$ is a sequence of positive numbers with $p_{n+1}/p_n \to 1$ and such that*

$$\lim_{n\to\infty}\left[\log p_n + \beta\sum_{j=1}^{n} p_j\right]$$

exists and is finite. Then

$$\lim_{n\to\infty} n\, p_n = 1/\beta.$$

Proof. It suffices to prove the result for $\beta = 1$, for otherwise we can consider $\tilde{p}_n = \beta\, p_n$. Let

$$a_n = \log p_n + \sum_{j=1}^{n} p_j.$$

We will use the fact that $\{a_n\}$ is a Cauchy sequence.

We first claim that for every $\delta > 0$, there exists $N_\delta > 0$ such that if $n \geq N_\delta$ and $p_n = (1+2\epsilon)/n$ with $\delta \leq \epsilon$, then there does not exists $r > n$ with

$$p_k \geq \frac{1}{k}, \quad k = n, \ldots, r-1.$$

$$p_r \geq \frac{1+3\epsilon}{r}.$$

Indeed, suppose these inequalities hold for some n, r. Then,

$$\log(p_{n+r}/p_n) \geq \log\frac{1+3\epsilon}{1+2\epsilon} + \log(r/n),$$

$$\sum_{j=n+1}^{r} p_j \geq \log(r/n) - O(n^{-1}).$$

and hence for n sufficiently large,

$$a_r - a_n \geq \frac{1}{2} \log \frac{1+3\epsilon}{1+2\epsilon} \geq \frac{1}{2} \log \frac{1+3\delta}{1+2\delta}.$$

Since a_n is a Cauchy sequence, this cannot be true for large n.

We next claim that for every $\delta > 0$, there exists $N_\delta > 0$ such that if $n \geq N_\delta$ and $p_n = (1+2\epsilon)/n$ with $\delta \leq \epsilon$, then there exists r such that

$$\frac{1+\epsilon}{k} \leq p_k < \frac{1+3\epsilon}{k}, \quad k = n, \ldots, r-1. \tag{26}$$

$$p_r < \frac{1+\epsilon}{r}.$$

To see this, we consider the first r such that $p_{n+r} \leq \frac{1+\epsilon}{n+r}$. By the previous claim, if such an r exists, then (26) holds for n large enough. If no such r exists, then by the argument above for all $r > n$,

$$a_r - a_n \geq \log \frac{1+\epsilon}{1+2\epsilon} + \frac{\epsilon}{2} \log(r/n) - (1+\epsilon)\, O(n^{-1}).$$

Since the right-hand side goes to infinity as $r \to \infty$, this contradicts the fact that a_n is a Cauchy sequence.

By iterating the last assertion, we can see that for every $\delta > 0$, there exists $N_\delta > 0$ such that if $n \geq N_\delta$ and $p_n = (1+2\epsilon)/n$ with $\delta \leq \epsilon$, then there exists $r > n$ such that

$$p_r < \frac{1+2\delta}{r},$$

$$p_k \leq \frac{1+3\epsilon}{k}, \quad k = n, \ldots, r-1.$$

Let s be the first index greater than r (if it exists) such that either

$$p_k \leq \frac{1}{k} \text{ or } p_k \geq \frac{1+2\delta}{k}.$$

Using $p_{n+1}/p_n \to 1$, we can see, perhaps by choosing a larger N_δ if necessary, that

$$\frac{1-\delta}{k} \leq p_s \leq \frac{1+4\delta}{k}.$$

If $p_s \geq (1+2\delta)/k$, then we can iterate this argument with $\epsilon \leq 2\delta$ to see that

$$\limsup_{n\to\infty} n\, p_n \leq 1 + 6\delta.$$

The lim inf can be done similarly.

□

2.7.2 Exact relation

We will prove the following. Let S, W be simple random walks with corresponding stopping times σ_n and let $G_n = G_{C_n}$. We will assume that $S_0 = w, W_0 = 0$. Let $\eta = LE\,(S[0,\ \sigma_n])$. Let

$$\hat{G}_n^2(w) = \sum_{z\in\mathbb{Z}^4} G(0,z)\, G_n(w,z) = \sum_{z\in C_n} G(0,z)\, G_n(w,z).$$

Note that we are stopping the random walk S at time σ_n but we are allowing the random walk W to run for infinite time. If $w \in \partial C_{n-1}$, then

$$\hat{G}_n^2(w) = \frac{8}{\pi^2} + O(e^{-n}). \tag{27}$$

This is a special case of the next lemma.

Lemma 2.21. *If $w \in C_n$, then*

$$\hat{G}_n^2(w) = \frac{8}{\pi^2}\ [n - \log|w|]\ + O(e^{-n}) + O(|w|^{-2}\ \log|w|).$$

Proof. Let $f(x) = \frac{8}{\pi^2}\ \log|x|$ and note that

$$\Delta f(x) = \frac{2}{\pi^2\,|x|^2} + O(|x|^{-4}) = G(x) + O(|x|^{-4}).$$

where Δ denotes the discrete Laplacian. Also, we know that for any function f,

$$f(w) = \mathbb{E}^w\,[f(S_{\sigma_n})] - \sum_{z\in C_n} G_n(w,z)\,\Delta f(z).$$

Since $e^n \leq |S_{\sigma_n}| \leq e^n + 1$,

$$\mathbb{E}^w\,[f(S_{\sigma_n})] = \frac{8n}{\pi^2} + O(e^{-n}).$$

Therefore,

$$\sum_{z\in C_n} G_n(w,z)\, G(z) = \frac{8}{\pi^2}\, [n - \log|w|] + O(e^{-n}) + \epsilon,$$

where

$$|\epsilon| \le \sum_{z\in C_n} G_n(w,z)\, O(|z|^{-4}) \le \sum_{z\in C_n} O(|w-z|^{-2})\, O(|z|^{-4}).$$

We split the sum into three pieces.

$$\begin{aligned}\sum_{|z|\le|w|/2} O(|w-z|^{-2})\, O(|z|^{-4}) &\le c\,|w|^{-2} \sum_{|z|\le|w|/2} O(|z|^{-4}) \\ &\le c\,|w|^{-2} \log|w|,\end{aligned}$$

$$\begin{aligned}\sum_{|z-w|\le|w|/2} O(|w-z|^{-2})\, O(|z|^{-4}) &\le c\,|w|^{-4} \sum_{|x|\le|w|/2} O(|x|^{-2}) \\ &\le c\,|w|^{-2},\end{aligned}$$

If we let C_n' the the set of $z \in C_n$ with $|z| > |w|/2$ and $|z-w| > |w|/2$, then

$$\sum_{z\in C_n'} O(|w-z|^{-2})\, O(|z|^{-4}) \le \sum_{|z|>|w|/2} O(|z|^{-6}) \le c\,|w|^{-2}.$$

□

Proposition 2.22. *There exists $\alpha < \infty$ such that if $n^{-1} \le e^{-n}\,|w| \le 1 - n^{-1}$, then*

$$\left|\log \mathbb{P}\{W[0,\infty] \cap \eta \ne \emptyset\} - \log[\hat{G}_n^2(w)\, \hat{p}_n]\right| \le c\, \frac{\log^\alpha n}{n}.$$

In particular,

$$\mathbb{E}\,[H(\eta^n)] = \frac{8\,\hat{p}_n}{\pi^2}\, \left[1 + O\left(\frac{\log^\alpha n}{n}\right)\right].$$

The second assertion follows immediately from the first and (27). We can write the conclusion of the proposition as

$$\mathbb{P}\{W[0,\infty] \cap \eta \ne \emptyset\} = \hat{G}_n^2(w)\, \hat{p}_n\, [1 + \theta_n]\,,$$

where θ_n throughout this subsection denotes an error term that decays at least as fast as $\log^\alpha n/n$ for some α (with the implicit uniformity of the estimate over all $n^{-1} \le e^{-n}\,|w| \le 1 - n^{-1}$).

We start by giving the sketch of the proof which is fairly straightforward. On the event $\{W[0,\infty] \cap \eta \neq \emptyset\}$ there are typically many points in $W[0,\infty] \cap \eta$. We focus on a particular one. This is analogous to the situation when one is studying the probability that a random walk visits a set. In the latter case, one usually focuses on the first or last visit. In our case with two paths, the notions of "first" and "last" are a little ambiguous so we have to take some care. We will consider the first point on η that is visited by W and then focus on the last visit by W to this first point on η.

To be precise, we write

$$\eta = [\eta_0, \ldots, \eta_m].$$

$$i = \min\{t : \eta_t \in W[0,\infty)\},$$
$$\rho = \max\{t \ge 0 : S_t = \eta_i\},$$
$$\lambda = \max\{t : W_t = \eta_i\}.$$

Then the event $\{\rho = j; \lambda = k; S_\rho = W_\lambda = z\}$ is the event that:

$$\begin{aligned}
\mathbf{I} &: \quad S_j = z, \quad W_k = z, \\
\mathbf{II} &: \quad LE\,(S[0,j]) \cap (S[j+1,\sigma_n] \cup W[0,k] \cup W[k+1,\infty)) = \{z\}, \\
\mathbf{III} &: \quad z \notin S[j+1,\sigma_n] \cup W[k+1,\infty).
\end{aligned}$$

Using the slowly recurrent nature of the random walk paths, we expect that as long as z is not too close to 0,w, or ∂C_n, then **I** is almost independent of (**II** and **III**) and

$$\mathbb{P}\,\{\mathbf{II} \text{ and } \mathbf{III}\} \sim \hat{p}_n.$$

This then gives

$$\mathbb{P}\{\rho = j; \lambda = k; S_\rho = W_\lambda = z\} \sim \mathbb{P}\{S_j = W_k = z\}\,\hat{p}_n,$$

and summing over j, k, z gives

$$\mathbb{P}\{W[0,\infty] \cap \eta \neq \emptyset\} \sim \hat{p}_n\,\hat{G}_n^2(w).$$

The rest of this section is devoted to making this precise.

Let V be the event

$$V = \{w \notin S[1, \sigma_n], 0 \notin W[1, \infty)\}.$$

A simple argument which we omit shows that there is a fast decaying sequence $\{\epsilon_n\}$ such that

$$|\mathbb{P}(V) - G(0,0)^{-2}| \le \epsilon_n,$$

$$|\mathbb{P}\{W[0,\infty] \cap \eta \neq \emptyset \mid V\} - \mathbb{P}\{W[0,\infty] \cap \eta \neq \emptyset\}| \le \epsilon_n.$$

Hence it will suffice to show that

$$\mathbb{P}\left[V \cap \{W[0,\infty] \cap \eta \neq \emptyset\}\right] = \frac{\hat{G}_n^2(w)}{G(0,0)^2}\, \hat{p}_n \, \left[1 + \theta_n\right]. \tag{28}$$

Let $E(j,k,z), E_z$ be the events

$$E(j,k,z) = V \cap \{\rho = j; \lambda = k; S_\rho = W_\lambda = z\}, \quad E_z = \bigcup_{j,k=0}^{\infty} E(j,k,z).$$

Then,

$$\mathbb{P}\left[V \cap \{W[0,\infty] \cap \eta \neq \emptyset\}\right] = \sum_{z \in C_n} \mathbb{P}(E_z).$$

Let

$$C_n' = C_{n,w}' = \{z \in C_n : |z| \ge n^{-4} e^n, \ |z - w| \ge n^{-4}\, e^n, \ |z| \le (1 - n^{-4}) e^n\}.$$

We can use the easy estimate

$$\mathbb{P}(E_z) \le G_n(w,z)\, G(0,z),$$

to see that

$$\sum_{z \in C_n \setminus C_n'} \mathbb{P}(E_n) \le O(n^{-6}),$$

so it suffices to estimate $\mathbb{P}(E_z)$ for $z \in C_n'$.

We will translate so that z is the origin and will reverse the paths $W[0,\lambda]$, $S[0,\rho]$. Using the fact that reverse loop-erasing has the same distribution as loop-erasing, we see that $\mathbb{P}[E(j,k,z)]$ can be given as the probability of the following event where $\omega^1, \ldots, \omega^4$ are independent simple random walks

starting at the origin and l^i denotes the smallest index l such that $|\omega^i_l - y| \geq e^n$.

$$\left(\omega^3[1,l^3] \cup \omega^4[1,\infty)\right) \cap LE(\omega^1[0,j]) = \emptyset,$$
$$\omega^2[0,k] \cap LE(\omega^1[0,j]) = \{0\},$$
$$j < l^1, \quad \omega^1(j) = x, \quad x \notin \omega^1[0,j-1],$$
$$\omega^2(k) = y, \quad y \notin \omega^2[0,k-1],$$

where $x = w - z, y = -z$. Note that

$$n^{-1}\, e^n \leq |y|, |x-y|, |x| \leq e^n\, [1 - n^{-1}],$$

We now rewrite this. We fix x, y and let $C^y_n = y + C_n$. Let W^1, W^2, W^3, W^4 be independent random walks starting at the origin and let $T^i = T^i_n = \min\{j : W^i_j \notin C^y_n\}$,

$$\tau^1 = \min\{j : W^1_j = x\}, \quad \tau^2 = \min\{k : W^2_k = y\}.$$

Let $\hat{\Gamma} = \hat{\Gamma}_n = LE\left(W^1[0,\tau^1]\right)$. Note that

$$\mathbb{P}\{\tau^1 < T^1\} = \frac{G_n(0,x)}{G_n(x,x)} = \frac{G_n(0,x)}{G(0,0)} + o(e^{-n}),$$

$$\mathbb{P}\{\tau^2 < \infty\} = \frac{G(0,y)}{G(y,y)}.$$

Let $p'_n(x,y)$ be the probability of the event

$$\hat{\Gamma} \cap \left(W^2[1,\tau^2] \cap W^3[1,T^3]\right) = \emptyset, \quad \hat{\Gamma} \cap W^4[0,T^4] = \{0\},$$

where W^1, W^2, W^3, W^4 are independent starting at the origin; W^3, W^4 are usual random walks; W^1 has the distribution of random walk conditioned that $\tau^1 < T^2$ stopped at τ^1; and W^2 has the distribution of random walk conditioned that $\tau^2 < \infty$ stopped at τ^2. Then in order to prove (28), it suffices to prove that

$$p'_n(x,y) = \hat{p}_n\, [1 + \theta_n]\,.$$

We write Q for the conditional distribution. Then for any path $\omega = [0, \omega^1, \ldots, \omega^k]$ with $0, \omega^1, \ldots, \omega^{k-1} \in C^y_n \setminus \{x\}$,

$$Q\{[W^1_0, \ldots, W^1_k] = \omega\} = \mathbb{P}\{[S_0, \ldots, S_k] = \omega\}\, \frac{\phi_x(\omega_k)}{\phi_x(0)}, \tag{29}$$

where

$$\phi_x(\zeta) = \phi_{x,n}(\zeta) = \mathbb{P}^\zeta\{\tau^1 < T_n^1\}.$$

Similarly if $\phi_y = \mathbb{P}^\zeta\{\tau^2 < \infty\}$ and $\omega = [0, \omega^1, \ldots, \omega^k]$ is a path with $y \notin \{0, \omega^1, \ldots, \omega^{k-1}\}$,

$$Q\{[W_0^1, \ldots, W_k^1] = \omega\} = \mathbb{P}\{[S_0, \ldots, S_k] = \omega\} \, \frac{\phi_y(\omega_k)}{\phi_y(0)}.$$

Using (13), we can see that if $\zeta \in \{x, y\}$, and if $|z| \le e^n \, e^{-\log^2 n}$,

$$\phi_\zeta(z) = \phi_\zeta(0) \, [1 + O(\epsilon_n)] \, .$$

where ϵ_n is fast decaying.

Let σ_k be defined for W^1 as before,

$$\sigma_k = \min\{j : |W_j^1| \ge e^k\}.$$

Using this we see that we can couple W^1, S on the same probability space such that, except perhaps on an event of probability $O(\epsilon_n)$,

$$\hat{\Gamma} \cap C_{n-\log^3 n} = LE\,(S[0,\infty)) \cap C_{n-\log^3 n}.$$

Using this we see that

$$q_n \le p_{n-\log^3 n} + O(\epsilon_n) \le p_n \, [1 + \theta_n] \, .$$

Hence,

$$\hat{\Gamma} \cap C_{n-\log^3 n} \subset \hat{\Gamma} \subset \Theta_n \cup (\hat{\Gamma} \cap C_{n-\log^3 n}),$$

where

$$\Theta_n = W^1[0, \tau^1] \cap A(n - \log^3 n, 2n).$$

We consider two events. Let $\hat{W} = W^2[1, \tau^2] \cup W^3[1, T^3] \cup W^4[0, T^4]$ and let E_0, E_1, E_2 be the events

$$E_0 = \{0 \notin W^2[1, \tau^2] \cup W^3[1, T^3]\},$$

$$E_1 = E_0 \cap \left\{\hat{W} \cap \hat{\Gamma} \cap C_{n-\log^3 n} = \{0\}\right\},$$

$$E_2 = E_1 \cap \left\{\hat{W} \cap \Theta_n = \emptyset\right\},$$

Then we show that

$$Q(E_1) = p_n\ [1+\theta_n]\,,$$
$$Q(E_1 \setminus E_2) \le n^{-1}\,\theta_n,$$

in order to show that

$$Q(E_2) = p_n\ [1+\theta_n]\,.$$

Find a fast decaying sequence $\bar{\epsilon}_n$ as in Proposition 2.8 such that

$$\mathbb{P}\left\{H\left(S[0,\infty)\cap A(m-1,m)\right) \le \frac{\log^2 m}{m}\right\} \le \bar{\epsilon}_m, \quad \forall m \ge \sqrt{n}.$$

Let $\delta_n = \bar{\epsilon}_n^{1/10}$ which is also fast decaying and let $\rho = \rho_n = \min\{j : |W_j^1 - x| \le e^n\,\delta_n\}$. Using the Markov property we see that

$$\mathbb{P}\left\{W^1[\rho,\tau^x] \not\subset \{|z-x| \le e^n\,\sqrt{\delta_n}\}\right\} = O(\delta_n),$$

and since $|x| \ge e^{-n}\,n^{-1}$, we know that

$$H\left[\{|z-x| \le e^n\,\sqrt{\delta_n}\}\right] \le n^2\,\delta_n.$$

Hence, we see that there is a fast decaying sequence δ_n such that such that

$$\mathbb{P}\left\{H\left(S[0,\infty)\cap A(m-1,m)\right) \le \frac{\log^2 m}{m}\right\} \le \delta_m, \quad \forall m \ge \sqrt{n}.$$

In particular, using (29), we see that

$$Q\left\{H\left(W[0,\tau^x]\cap A(n-\log^4 n, n+1)\right) \le \frac{\log^7 n}{n}\right\}$$

is fast decaying. Then using Proposition 2.11 we see that there exists a fast decaying sequence ϵ_n such that, except for an event of Q probability at most ϵ_n, either $\mathrm{Es}[\hat{\Gamma}] \le \epsilon_n$ or

$$\mathrm{Es}[\hat{\Gamma}] = \mathrm{Es}[\hat{\Gamma}\cap C_{n-\log^3 n}]\,[1+\theta_n].$$

This suffices for W^3 and W^4 but we need a similar result for W^2, that is, if

$$\mathrm{Es}^y[\hat{\Gamma}] = Q\left\{W^2[1,\tau^y] = \emptyset\cap\hat{\Gamma} \mid \hat{\Gamma}\right\},$$

then we need that except for an event of Q probability at most ϵ_n, either $\mathrm{Es}[\hat{\Gamma}] \le \epsilon_n$ or

$$\mathrm{Es}^y[\hat{\Gamma}] = \mathrm{Es}[\hat{\Gamma} \cap C_{n-\log^3 n}]\,[1+\theta_n].$$

Since the measure Q is almost the same as P for the paths near zero, we can use Proposition 2.11 to reduce this to showing that

$$Q\{W^2[0,\tau^y] \cap (\hat{\Gamma} \setminus C_{n-\log^3 n}) \neq \emptyset \mid \hat{\Gamma}\} \le \theta_n.$$

This can be done with similar argument using the fact that the probability that random walks starting at y and x ever intersection is less than θ_n.

Corollary 2.23. *If $n^{-1} \le e^{-n}\,|w| \le 1-n^{-1}$, then*

$$\mathbb{P}\{W[0,\infty) \cap \eta \neq \emptyset\} \sim \frac{\pi^2\,\hat{G}_n^2(w)}{24\,n}.$$

More precisely,

$$\lim_{n\to\infty} \max_{n^{-1} \le e^{-n}\,|w| \le 1-n^{-1}} \Big| 24\,n\,\mathbb{P}\{W[0,\infty) \cap \eta \neq \emptyset\} - \pi^2\,\hat{G}_n(w) \Big| = 0.$$

By a very similar proof one can show the following. Let

$$G_n^2(z) = \sum_{z\in\mathbb{Z}^4} G_n(0,z)\,G_n(w,z) = \sum_{z\in C_n} G_n(0,z)\,G_n(w,z),$$

and

$$\sigma_n^W = \min\{j : W_j \notin C_n\}.$$

We note that

$$\tilde{G}_n^2(w) - G_n^2(w) = \sum_{z\in C_n} [G(0,z) - G_n(0,z)]\,G_n(w,z) = O(1).$$

The following can be proved in the same way.

Proposition 2.24. *There exists $\alpha < \infty$ such that if $n^{-1} \le e^{-n}\,|w| \le 1-n^{-1}$, then*

$$\big|\log \mathbb{P}\{W[0,\sigma_n^W] \cap \eta \neq \emptyset\} - \log[G_n^2(w)\,\hat{p}_n]\big| \le c\,\frac{\log^\alpha n}{n}.$$

We note that if $|w| = n^{-r}\,e^n$, then

$$G_n^2 \sim \hat{G}_n^2(w) \sim \frac{\log(1/r)}{n}.$$

This gives Lemma 1.6 in the case that x or y equals zero. The general case can be done similarly.

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