

**Math 312, Autumn 2007**  
**Problem Set 3**

**Rudin:** Chapter 3: 4,5,9,11,14,16,18,23,25

(Note: Problem 5 is really a problem about random variables, and you might want to “translate” the problem into probability notation.)

**Probability Notes:** Exercise 4.1.

**Exercise 1** Let  $X$  be a random variable with  $\mathbf{E}[|X|] < \infty$ . Let  $X_n = X 1\{|X_n| \leq n\}$ . Show that if  $\alpha > 1$ ,

$$\lim_{n \rightarrow \infty} \frac{\mathbf{E}[|X_n|^\alpha]}{n^{\alpha-1}} = 0.$$

**Exercise 2** Suppose  $X_1, X_2, \dots$  are independent random variables each with mean zero. Let

$$S_n = X_1 + \dots + X_n.$$

Prove that for every  $\lambda > 0$ ,

$$\mathbf{P} \left\{ \omega : \max_{1 \leq j \leq n} S_j(\omega) \geq \lambda \right\} \leq \frac{\mathbf{E}[S_n^2]}{\lambda^2}.$$

(Hint: Let  $T = \min\{j : S_j \geq \lambda\}$ , let  $A_j$  be the event  $\{T = j\}$ . Show that for every  $j \leq n$ ,

$$\mathbf{E}[S_n^2 1_{A_j}] \geq \mathbf{E}[S_j^2 1_{A_j}] \geq \lambda^2 \mathbf{P}(A_j). \quad )$$

**Exercise 3** Suppose  $X_1, X_2, \dots$  are independent, identically distributed random variables with mean zero and let  $\hat{X}_n = X_n 1\{|X_n| \leq n\}$ ,

$$S_n = \hat{X}_1 + \dots + \hat{X}_n.$$

The goal of this exercise is to show

$$\sum_{n=1}^{\infty} 2^{-2n} \text{Var}[S_{2^n}] < \infty,$$

by proving the following steps.

1. For each  $n$ ,

$$\text{Var}[S_{2^n}] \leq 2^n \mathbf{E}[X_1^2 1\{|X_1| \leq 2^n\}].$$

2.

$$\sum_{n=1}^{\infty} 2^{-2n} \text{Var}[S_{2^n}] \leq \mathbf{E} \left[ X_1^2 \sum_{n=1}^{\infty} 2^{-n} 1\{|X_1| \leq 2^n\} \right].$$

3.

$$\sum_{n=1}^{\infty} 2^{-n} 1\{|X_1| \leq 2^n\} \leq \frac{1}{|X_1|}.$$