

Math 312, Autumn 2007
Final Problem Set

Rudin, Chapter 5, 17; Chapter 6, 11 (you may assume Problem 10 as being given); Chapter 9: 2 (but please change the word characteristic to indicator in the problem!), 6, 8, 19

Exercise 1 A Hamel basis for a complex vector space V is a collection of vectors $\{v_\alpha\}$ such that each $v \in V$ can be written in a unique way as

$$v = \sum_{\alpha} c_{\alpha} v_{\alpha},$$

where $c_{\alpha} \in \mathbb{C}$ and all but a finite number of the c_{α} are zero. The existence of Hamel bases can be shown using the axiom of choice. Regardless of whether or not you believe in the axiom of choice, show that if H is a Hilbert space then any Hamel basis is either finite or uncountable.

Exercise 2 If f is a measurable function on \mathbb{R} , let

$$I_f = \{1 \leq p < \infty; \|f\|_p < \infty\},$$

where we are using Lebesgue measure.

1. Which subsets of $[1, \infty)$ equal I_f for some f ?
2. Which subsets of $[1, \infty)$ equal I_f for some f that vanishes on $\{|x| \geq 1\}$?

Exercise 3 We have considered primarily bounded linear functionals on a Hilbert space. Let H be a Hilbert space with a countably infinite basis. Prove or disprove the following: every linear map $\Phi : H \rightarrow \mathbb{C}$ is bounded. (You may assume axiom of choice.)

Exercise 4 Let X_1, X_2, \dots be independent random variables with mean zero and such that $\mathbf{P}\{|X_j| > 1\} = 0$.

1. Suppose that

$$\sum \text{Var}[X_n] < \infty.$$

Show that with probability one that the limit

$$\sum_{j=1}^n X_j \tag{1}$$

converges.

2. Show that if

$$\sum \text{Var}[X_n] = \infty,$$

then with probability one the limit in (1) does not exist.

Exercise 5 Let X_1, X_2, \dots be independent random variables each normal mean zero, variance one. Let

$$S_n = X_1 + \dots + X_n, \quad S_n^* = \max |S_j|; j = 1, 2, \dots, n.$$

and let

$$Y = \limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{n \log \log n}}, \quad Y^* = \limsup_{n \rightarrow \infty} \frac{S_n^*}{\sqrt{n \log \log n}}.$$

1. Show that there exist $c_0, c_1 \in [0, \infty]$ such that with probability one $Y = c_0, Y^* = c_1$. (Use the Kolmogorov Zero-One Law).

2. Explain why for every $\lambda > 0$ and every integer n ,

$$\mathbf{P}\{S_n^* \geq \lambda\} \leq 2 \mathbf{P}\{|S_n| \geq \lambda\} \leq 4 \mathbf{P}\{S_n \geq \lambda\}.$$

3. Find a $c_2 < \infty$ such that

$$\sum_{n=3}^{\infty} \mathbf{P}\{S_{2^n}^* \geq c_2 \sqrt{2^n \log \log 2^n}\} < \infty.$$

4. Show that $c_1 < \infty$.

5. Find a $c_3 > 0$ such that

$$\sum_{n=3}^{\infty} \mathbf{P}\{S_{2^n} - S_{2^{n-1}} \geq c_3 \sqrt{2^n \log \log 2^n}\} = \infty.$$

6. Show that with probability one

$$\limsup_{n \rightarrow \infty} \frac{S_{2^n} - S_{2^{n-1}}}{\sqrt{2^n \log \log 2^n}} \geq c_3.$$

7. Show that $c_0 > 0$.

(If you get stuck on parts of the problem, feel free to do other parts assuming previous parts.)