

# A Geometric Interpretation of Half-Plane Capacity

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Suppose  $A$  is a bounded, relatively closed subset of the upper half plane  $\mathbb{H}$ . We call  $A$  a compact  $\mathbb{H}$ -hull if  $A$  is bounded and  $\mathbb{H} \setminus A$  is simply connected. The *half-plane capacity* of  $A$ ,  $\text{hcap}(A)$ , is defined in a number of equivalent ways (see [1], especially Chapter 3). If  $g_A$  denotes the unique conformal transformation of  $\mathbb{H} \setminus A$  onto  $\mathbb{H}$  with  $g_A(z) = z + o(1)$  as  $z \rightarrow \infty$ , then  $g_A$  has the expansion

$$g_A(z) = z + \frac{\text{hcap}(A)}{z} + O(|z|^{-2}), \quad z \rightarrow \infty.$$

Equivalently, if  $B_t$  is a standard complex Brownian motion and  $\tau_A = \inf\{t \geq 0 : B_t \notin \mathbb{H} \setminus A\}$ ,

$$\text{hcap}(A) = \lim_{y \rightarrow \infty} y \mathbb{E}^{iy} [\text{Im}(B_{\tau_A})].$$

Let  $\text{Im}[A] = \sup\{\text{Im}(z) : z \in A\}$ . Then if  $y \geq \text{Im}[A]$ , we can also write

$$\text{hcap}(A) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathbb{E}^{x+iy} [\text{Im}(B_{\tau_A})] dx.$$

These last two definitions do not require  $\mathbb{H} \setminus A$  to be simply connected, and the latter definition does not require  $A$  to be bounded but only that  $\text{Im}[A] < \infty$ .

For  $\mathbb{H}$ -hulls (that is, for  $A$  for which  $\mathbb{H} \setminus A$  is simply connected), the half-plane capacity is comparable to a more geometric quantity that we define. This is not new (the second author learned it from Oded Schramm in oral communication), but we do not know of a proof in the literature. In this note, we prove the fact giving (nonoptimal) bounds on the constant. We start with the definition of the geometric quantity.

**Definition 1.** For an  $\mathbb{H}$ -hull  $A$ , let  $\text{hsiz}(A)$  be the 2-dimensional Lebesgue measure of the union of all balls centered at points in  $A$  that are tangent to the real line. In other words

$$\text{hsiz}(A) = \text{area} \left[ \bigcup_{x+iy \in A} \mathcal{B}(x+iy, y) \right],$$

where  $\mathcal{B}(z, \epsilon)$  denotes the disk of radius  $\epsilon$  about  $z$ .

In this paper, we prove the following.

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**Theorem 1.** For every  $\mathbb{H}$ -hull  $A$ ,

$$\frac{1}{66} \text{hsiz}(A) < \text{hcap}(A) \leq \frac{7}{2\pi} \text{hsiz}(A).$$

It suffices to prove this for weakly bounded  $\mathbb{H}$ -hulls, by which we mean  $\mathbb{H}$ -hulls  $A$  with  $\text{Im}(A) < \infty$  and such that for each  $\epsilon > 0$ , the set  $\{x + iy : y > \epsilon\}$  is bounded. Indeed, for  $\mathbb{H}$ -hulls that are not weakly bounded, it is easy to verify that  $\text{hsiz}(A) = \text{hcap}(A) = \infty$ .

We start with a simple inequality that is implied but not explicitly stated in [1]. Equality is achieved when  $A$  is a vertical line segment.

**Lemma 1.** If  $A$  is an  $\mathbb{H}$ -hull, then

$$\text{hcap}(A) \geq \frac{\text{Im}[A]^2}{2}. \quad (1)$$

*Proof.* Due to the continuity of  $\text{hcap}$  with respect to the Hausdorff metric on  $\mathbb{H}$ -hulls, it suffices to prove the result for  $\mathbb{H}$ -hulls that are path-connected. Further, by the monotonicity of  $\text{hcap}$  under containment,  $A$  can be assumed to be of the form  $\eta(0, T]$  where  $\eta$  is a simple curve with  $\eta(0+) \in \mathbb{R}$ , parameterized so that  $\text{hcap}[\eta(0, t)] = 2t$ . In particular,  $T = \text{hcap}(A)/2$ . If  $g_t = g_{\eta(0, t]}$ , then  $g_t$  satisfies the Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z, \quad (2)$$

where  $U : [0, T] \rightarrow \mathbb{R}$  is continuous. Suppose  $\text{Im}(z)^2 > 2 \text{hcap}(A)$  and let  $Y_t = \text{Im}[g_t(z)]$ . Then (2) gives

$$-\partial_t Y_t^2 \leq \frac{4Y_t}{|g_t(z) - U_t|^2} \leq 4,$$

which implies

$$Y_T^2 \geq Y_0^2 - 4T > 0.$$

This implies that  $z \notin A$ , and hence  $\text{Im}[A]^2 \leq 2 \text{hcap}(A)$ .  $\square$

The next lemma is a variant of the vital covering lemma. If  $c > 0$  and  $z = x + iy \in \mathbb{H}$ , let

$$\mathcal{I}(z, c) = (x - cy, x + cy),$$

$$\mathcal{R}(z, c) = \mathcal{I}(z, c) \times (0, y] = \{x' + iy' : |x' - x| < cy, 0 < y' \leq y\}.$$

**Lemma 2.** Suppose  $A$  is a weakly bounded  $\mathbb{H}$ -hull and  $c > 0$ . Then there exists a finite or countably infinite sequence of points  $\{z_1 = x_1 + iy_1, z_2 = x_2 + iy_2, \dots\} \subset A$  such that:

- $y_1 \geq y_2 \geq y_3 \geq \dots$ ;
- the intervals  $\mathcal{I}(x_1, c), \mathcal{I}(x_2, c), \dots$  are disjoint;
- 

$$A \subset \bigcup_{j=1}^{\infty} \mathcal{R}(z_j, 2c). \quad (3)$$

*Proof.* We define the points recursively. Let  $A_0 = A$  and given  $\{z_1, \dots, z_j\}$ , let

$$A_j = A \setminus \left[ \bigcup_{k=1}^j \mathcal{R}(z_k, 2c) \right].$$

If  $A_j = \emptyset$  we stop, and if  $A_j \neq \emptyset$ , we choose  $z_{j+1} = x_{j+1} + iy_{j+1} \in A$  with  $y_{j+1} = \text{Im}[A_j]$ . Note that if  $k \leq j$ , then  $|x_{j+1} - x_k| \geq 2cy_k \geq c(y_k + y_{j+1})$  and hence  $\mathcal{I}(z_{j+1}, c) \cap \mathcal{I}(z_k, c) = \emptyset$ . Using the weak boundedness of  $A$ , we can see that  $y_j \rightarrow 0$  and hence (3) holds.  $\square$

**Lemma 3.** For every  $c > 0$ , let

$$\rho_c := \frac{2\sqrt{2}}{\pi} \arctan\left(e^{-\theta}\right), \quad \theta = \theta_c = \frac{\pi}{4c}.$$

Then, for any  $c > 0$ , if  $A$  is a weakly bounded  $\mathbb{H}$ -hull and  $x_0 + iy_0 \in A$  with  $y_0 = \text{Im}(A)$ , then

$$\text{hcap}(A) \geq \rho_c^2 y_0^2 + \text{hcap}[A \setminus \mathcal{R}(z, 2c)].$$

*Proof.* By scaling and invariance under real translation, we may assume that  $\text{Im}[A] = y_0 = 1$  and  $x_0 = 0$ . Let  $S = S_c$  be defined to be the set of all points  $z$  of the form  $x + iy$  where  $x + iy \in A \setminus \mathcal{R}(i, 2c)$  and  $0 < u \leq 1$ .

Clearly,  $S \cap A = A \setminus \mathcal{R}(i, 2c)$ .

Using the capacity inequality [1, (3.10)]

$$\text{hcap}(A_1 \cup A_2) - \text{hcap}(A_2) \leq \text{hcap}(A_1) - \text{hcap}(A_1 \cap A_2), \quad (4)$$

we see that

$$\text{hcap}(S \cup A) - \text{hcap}(S) \leq \text{hcap}(A) - \text{hcap}(S \cap A).$$

Hence, it suffices to show that

$$\text{hcap}(S \cup A) - \text{hcap}(S) \geq \rho_c^2.$$

Let  $f$  be the conformal map of  $\mathbb{H} \setminus S$  onto  $\mathbb{H}$  such that  $z - f(z) = o(1)$  as  $z \rightarrow \infty$ . Let  $S^* := S \cup A$ . By properties of halfplane capacity [1, (3.8)] and (1),

$$\text{hcap}(S^*) - \text{hcap}(S) = \text{hcap}[f(S^* \setminus S)] \geq \frac{\text{Im}[f(i)]^2}{2}.$$

Hence, it suffices to prove that

$$\text{Im}[f(i)] \geq \sqrt{2} \rho = \frac{4}{\pi} \arctan\left(e^{-\theta}\right). \quad (5)$$

By construction,  $S \cap \mathcal{R}(z, 2c) = \emptyset$ . Let  $V = (-2c, 2c) \times (0, \infty) = \{x + iy : |x| < 2c, y > 0\}$  and let  $\tau_V$  be the first time that a Brownian motion leaves the domain. Then [1, (3.5)],

$$\text{Im}[f(i)] = 1 - \mathbb{E}^i[\text{Im}(B_{\tau_S})] \geq \mathbb{P}\{B_{\tau_S} \in [-2c, 2c]\} \geq \mathbb{P}\{B_{\tau_V} \in [-2c, 2c]\}.$$

The map  $\Phi(z) = \sin(\theta z)$  maps  $V$  onto  $\mathbb{H}$  sending  $[-2c, 2c]$  to  $[-1, 1]$  and  $\Phi(i) = i \sinh \theta$ . Using conformal invariance of Brownian motion and the Poisson kernel in  $\mathbb{H}$ , we see that

$$\mathbb{P}\{B_{\tau_V} \in [-2c, 2c]\} = \frac{2}{\pi} \arctan\left(\frac{1}{\sinh \theta}\right) = \frac{4}{\pi} \arctan\left(e^{-\theta}\right).$$

The second equality uses the double angle formula for the tangent. □

**Lemma 4.** Suppose  $c > 0$  and  $x_1 + iy_1, x_2 + iy_2, \dots$  are as in Lemma 2. Then

$$\text{hsiz}(A) \leq [\pi + 8c] \sum_{j=1}^{\infty} y_j^2. \quad (6)$$

If  $c \geq 1$ , then

$$\pi \sum_{j=1}^{\infty} y_j^2 \leq \text{hsiz}(A). \quad (7)$$

*Proof.* A simple geometry exercise shows that

$$\text{area} \left[ \bigcup_{x+iy \in \mathcal{R}(z_j, 2c)} \mathcal{B}(x+iy, y) \right] = [\pi + 8c] y_j^2.$$

Since

$$A \subset \bigcup_{j=1}^{\infty} \mathcal{R}(z_j, 2c),$$

the upper bound in (6) follows. Since  $c \geq 1$ , and the intervals  $\mathcal{I}(z_j, c)$  are disjoint, so are the disks  $\mathcal{B}(z_j, y_j)$ . Hence,

$$\text{area} \left[ \bigcup_{x+iy \in A} \mathcal{B}(x+iy, y) \right] \geq \text{area} \left[ \bigcup_{j=1}^{\infty} \mathcal{B}(z_j, y_j) \right] = \pi \sum_{j=1}^{\infty} y_j^2.$$

□

*Proof of Theorem 1.* Let  $V_j = A \cap \mathcal{R}(z_j, c)$ . Lemma 3 tells us that

$$\text{hcap} \left[ \bigcup_{k=j}^{\infty} V_k \right] \geq \rho_c^2 y_j^2 + \text{hcap} \left[ \bigcup_{k=j+1}^{\infty} V_k \right],$$

and hence

$$\text{hcap}(A) \geq \rho_c^2 \sum_{j=1}^{\infty} y_j^2.$$

Combining this with the upper bound in (6) with any  $c > 0$  gives

$$\frac{\text{hcap}(A)}{\text{hsiz}(A)} \geq \frac{\rho_c^2}{\pi + 8c}.$$

Choosing  $c = \frac{8}{5}$  gives us

$$\frac{\text{hcap}(A)}{\text{hsiz}(A)} > \frac{1}{66}.$$

For the upper bound, choose a covering as in Lemma 2 with  $c = 1$ . Subadditivity and scaling give

$$\text{hcap}(A) \leq \sum_{j=1}^{\infty} \text{hcap}[\mathcal{R}(z_j, 2y_j)] = \text{hcap}[\mathcal{R}(i, 2)] \sum_{j=1}^{\infty} y_j^2.$$

Combining this with the lower bound in (6) gives

$$\frac{\text{hcap}(A)}{\text{hsiz}(A)} \leq \frac{\text{hcap}[\mathcal{R}(i, 2)]}{\pi}.$$

Note that  $\mathcal{R}(i, 2)$  is the union of two real translates of  $\mathcal{R}(i, 1)$ ,  $\text{hcap}[\mathcal{R}(i, 2)] \leq 2 \text{hcap}[\mathcal{R}(i, 1)]$  whose intersection is the interval  $(0, i]$ . Using (4), we see that

$$\text{hcap}(\mathcal{R}(i, 2)) \leq 2 \text{hcap}(\mathcal{R}(i, 1)) - \text{hcap}((0, i]) = 2 \text{hcap}(\mathcal{R}(i, 1)) - \frac{1}{2}.$$

But  $\mathcal{R}(i, 1)$  is strictly contained in  $A' := \{z \in \mathbb{H} : |z| \leq \sqrt{2}\}$ , and hence

$$\text{hcap}[\mathcal{R}(i, 1)] < \text{hcap}(A') = 2.$$

The last equality can be seen by considering  $h(z) = z + 2z^{-1}$  which maps  $\mathbb{H} \setminus A'$  onto  $\mathbb{H}$ . Therefore,

$$\text{hcap}[\mathcal{R}(i, 2)] < \frac{7}{2},$$

and hence

$$\frac{\text{hcap}(A)}{\text{hsiz}(A)} \leq \frac{7}{2\pi}.$$

□

## References

- [1] G. Lawler, *Conformally Invariant Processes in the Plane*, American Mathematical Society, 2005.