Exercise 1 Use Itô’s formula and the product rule to find the SDEs satisfied by the following processes. Assume $B_t$ is a standard Brownian motion with $B_0 > 0$ and the equations need only be valid up to a (nontrivial) stopping time. For each of these find the appropriate $C^1$ process $C_t$ such that $M_t = C_t X_t$ is a local martingale and find the function $R_t$ such that

$$dM_t = R_t M_t dB_t.$$ 

$$X_t = B_t^q$$

$$X_t = (\sin B_t)^q$$

$$X_t = B_t^q (\sin B_t)^r.$$ 

Exercise 2 Suppose $D \subset \mathbb{R}^d$ is a domain and $f : \overline{D} \to \mathbb{R}$ is a continuous function, $C^2$ in $D$, satisfying

$$f(x) = 0, \quad x \in \partial D.$$ 

$$\frac{1}{2} \Delta f(x) = -1, \quad x \in D.$$ 

Let $B_t$ be a standard $d$-dimensional Brownian motion and let $\tau = \inf \{t \geq 0 : B_t \not\in D\}$.

1. Show that

$$M_t = f(B_{t \wedge \tau}) + (t \wedge \tau),$$

is a local martingale.

2. Prove that if $x \in D$,

$$E^x[\tau] = f(x).$$

Exercise 3 Suppose $B_t$ is a standard Brownian motion in $\mathbb{R}^d$ and $X_t = |B_t|$. Show that

$$\langle X \rangle_t = t.$$ 

(Note: we computed $dX_t$ in class as an example.)