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Chapter 1

Conformal mappings

1.1 Various preliminaries

We assume that the reader knows the definition of a one-dimensional Brownian motion. We also assume some familiarity with complex analysis although we will develop most of the facts that we need. Those very familiar with complex analysis can go quickly through some sections although it is recommended to read them to see how the results tie in with Brownian motion.

1.1.1 Notation

We write

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk and $\mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$ is the upper half plane.
- More generally, $\mathbb{D}_r = e^{-r}\,\mathbb{D}$ is the disk of radius e^{-r} about the origin and $\mathbb{D}_r(z) = z + \mathbb{D}_r$ is the disk of radius e^{-r} about z.
- $C_r = \partial\mathbb{D}_r, C_r(z) = \partial\mathbb{D}_r(z)$.
- $\mathbb{A}_r = \mathbb{D} \setminus \overline{\mathbb{D}_r} = \{e^{-r} < |z| < 1\}$ is the annulus with boundary $C_0 \cup C_r$.
- We write $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ for the Riemann sphere.

The potential theory of complex Brownian motion makes heavy use of the logarithm function. This affects our choice of notation. There are many times that we will want to write the logarithm of a radius, and by parametrizing our radii exponentially it will make our formulas nicer.

Definition A *(standard) complex Brownian motion* is a process of the form $B_t = B_t^1 + i\,B_t^2$ where B_t^1, B_t^2 are independent one-dimensional Brownian motions. Equivalently, B_t is a standard Brownian motion in $\mathbb{R}^2$ viewed as taking values in $\mathbb{C}$.

When we say complex Brownian motion, we will always mean standard complex Brownian motion. If the context is clear we will say just Brownian motion. If $D \subset \mathbb{C}$ is an open set, then we define

$$\tau_D = \inf\{t \geq 0 : B_t \notin D\},$$

$$\overline{\tau}_D = \inf\{t > 0 : B_t \notin D\}.$$

Note that $\tau_D = \overline{\tau}_D > 0$ if $B_0 \in D$ and $\tau_D = 0$ if $B_0 \in \mathbb{C} \setminus D$. If $B_0 \in \partial D$.

Definition We call a boundary point $z \in D$ *regular* if $\mathbb{P}^z\{\overline{\tau}_D = 0\} = 1$, that is, if Brownian motion starting at z immediately hits the boundary.

Here $\mathbb{P}^w$ means the probability assuming that $B_0 = w$. Note that isolated points of ∂D are not regular points. The next lemma shows that Brownian motion starting near a regular point exits the domain quickly with high probability.

Lemma 1.1.1. *Suppose D is an open subset of $\mathbb{C}$.*

- *Suppose z is a regular boundary point of ∂D. Then for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|w - z| < \delta$, then*
$$\mathbb{P}^w\{\operatorname{diam}(B[0, \overline{\tau}_D]) \geq \epsilon\} \leq \epsilon.$$
- *If z is an irregular boundary point of ∂D, then $\mathbb{P}^z\{\overline{\tau}_D > 0\} = 1$.*

Proof. Without loss of generality, assume that $z = 0$ and let $\xi_s = \inf\{t \geq 0 : |B_t| = e^{-s}\}$.

Suppose 0 is a regular point. Since $\mathbb{P}\{\overline{\tau}_D = 0\} = 1$, we know that $\mathbb{P}\{B(0, \xi_s] \subset D\} = 0$. Therefore, there exists u such that

$$\mathbb{P}\{B[\xi_u, \xi_s] \subset D\} \leq e^{-s}.$$

We can find $\delta > 0$ such that the distribution of $B(\xi_u)$ given that $|B_0| = \delta$ agrees with that assuming $B_0 = 0$ up to an error of e^{-s} (see comment below). Therefore, for $|w| < \delta$,

$$\mathbb{P}^w\{B[\xi_u, \xi_s] \subset D\} \leq 2e^{-s},$$

and hence

$$\mathbb{P}^w\{\operatorname{diam}[0, \overline{\tau}_D] \geq 2e^{-s}\} \leq 2e^{-s}.$$

Suppose 0 is an irregular point. Then there exists r with

$$\mathbb{P}\{B(0, \xi_r] \subset D\} = \rho > 0.$$

For ease we will assume that $r = 0$ (other r can be handled by scaling, see Exercise 1). For every $\epsilon > 0$ we can find $s \geq 0$ such that

$$\mathbb{P}\{B[\xi_s, \xi_0] \subset D\} \leq \rho + \epsilon.$$

As before, we can find u such that if $|z| \leq e^{-u}$,

$$\mathbb{P}^z\{B[\xi_s, \xi_0] \subset D\} \leq \rho + 2\epsilon.$$

Therefore,

$$\begin{aligned}\rho &= \mathbb{P}\{B(0,\xi_0] \subset D\} \\ &= \mathbb{P}\{B(0,\xi_u] \subset D\}\, \mathbb{P}\{B(0,\xi_0] \subset D \mid B(0,\xi_u] \subset D\} \\ &\le \mathbb{P}\{B(0,\xi_u] \subset D\}\, (\rho + 2\epsilon),\end{aligned}$$

which implies that

$$\mathbb{P}\{B(0,\xi_u] \subset D\} \ge \frac{\rho}{\rho + 2\epsilon}.$$

Hence,

$$\mathbb{P}\{\overline{\tau}_D > 0\} = \lim_{u\to\infty} \mathbb{P}\{B(0,\xi_u] \subset D\} = 1.$$

□

The astute reader will note that we "cheated" a little by not justifying why the hitting probability of C_u starting at w is almost the same as that starting at 0. We could give a proof of this here, but it follows easily from the exact form of the Poisson kernel in the disk which we do below, so we chose not to.

We will call an open set D *regular* if $\mathbb{P}^z\{\tau_D < \infty\} = 1$ for every $z \in D$.

Exercise 1. *Suppose B_t is a complex Brownian motion starting at the origin. Let $\theta \in \mathbb{R}, a > 0$ and*

$$Y_t = e^{i\theta}\, B_t, \quad Z_t = a^{-1}\, B_{a^2 t}.$$

Then Y_t, Z_t are (standard) complex Brownian motions.

Definition

- A *domain* is a connected open subset of $\mathbb{C}$.
- A domain is *simply connected* if $\hat{\mathbb{C}} \setminus D$ is connected.
- If D is an unbounded domain with ∂D compact, we will also consider $D \cup \{\infty\}$ as a domain in $\hat{\mathbb{C}}$.
- Let $\mathcal{D}$ denote the set of all domains $D \subsetneq \mathbb{C}$ such that every $z \in \partial D$ is regular.
- Let $\mathcal{D}^*$ denote the set of all domains $D \subsetneq \mathbb{C}$ such that there exists $z \in \partial D$ that is regular.

Clearly $\mathcal{D} \subset \mathcal{D}^*$. Giving the exact criterion to be in $\mathcal{D}$ is difficult. However, we will now derive a sufficient condition that will suffice for our purposes. We will use the following lemma that makes strong use of the planarity of $\mathbb{C}$.

Lemma 1.1.2. *There exists $\beta > 0, c < \infty$ such that if B_t is a complex Brownian motion starting at $z \in \mathbb{D}$ and $\tau = \tau_{\mathbb{D}} = \inf\{t : |B_t| = 1\}$, then the probability that the origin lies in the unbounded component of $\mathbb{C} \setminus B[0,\tau]$ is no more than $c\,|z|^{\beta}$.*

It follows that if D is a domain, $0 \notin D$, and the connected component of $\mathbb{C} \setminus D$ containing 0 also contains at least one point on $\partial\mathbb{D}$, then

$$\mathbb{P}^z \{B[0, \tau_D] \not\subset \mathbb{D}\} \leq c\,|z|^\beta.$$

Proof. Let $p(z)$ be the probability that 0 lies in the unbounded component given $B_0 = z$, and note that $p(z) = p(|z|)$. Let q be the probability that a Brownian motion starting on C_1, the circle of radius e^{-1} about the origin, disconnects C_1 from C_0 before time τ. By constructing a particular event, it is easy to see that $q > 0$ and is independent of the angle of the starting point. Using the strong Markov property and the scaling property of Brownian motion we can see that for $r \geq 1$, $p(e^{-r}) \leq (1-q)\,p(e^{-r-1})$. Hence for integer $n \geq 0$, $p(e^{-n}) \leq e^{n \log(1-q)}$, and more generally $p(e^{-u}) \leq p(e^{-\lfloor u \rfloor}) \leq e^{-\log(1-q)}\, e^{u \log(1-q)}$. □

The optimal value of β is called the *disconnection exponent* and is known to be 1/4. This is significantly harder to show and we will not need it.

Corollary 1.1.3. *Suppose $D \subsetneq \mathbb{C}$ is a domain such that all the connected components of $\mathbb{C} \setminus D$ are larger than one point. Then $D \in \mathcal{D}$. In particular, all simply connected $D \subsetneq \mathbb{C}$ are in $\mathcal{D}$.*

The proof of Lemma 1.1.2 establishes the following stronger fact.

Definition If $z \in \mathbb{C}, K \subset \mathbb{C}$, we define

$$\mathrm{rad}_K(z) = \sup\{|w - z| : w \in K\}, \quad \mathrm{rad}_K = \mathrm{rad}_K(0),$$

$$\mathrm{diam}(K) = \sup\{|w - z| : w, z \in K\}.$$

Note that if $z \in K$, then $\mathrm{rad}_K(z) \leq \mathrm{diam}(K) \leq 2\,\mathrm{rad}_K(z)$.

Proposition 1.1.4. *Suppose D is a domain and w, w' are in the same component of $\mathbb{C} \setminus D$. Then, for all z,*

$$\mathbb{P}^z \left\{\mathrm{diam}\,(B[0, \tau_D]) \geq 2\,|w' - w|\right\} \leq c \left(\frac{|z - w|}{|w' - w|}\right)^\beta,$$

where c, β are as in Lemma 1.1.2.

Proposition 1.1.5. *Suppose D is a domain and z is an irregular boundary point of ∂D. Let h be a strictly positive harmonic function on D. Then there exists a sequence $z_n \in D$ with $z_n \to z$ with $\liminf_{n\to\infty} h(z_n) > 0$.*

Proof. Since z is irregular, we can find a compact $V \subset D$ and $\delta > 0$. such that $\mathbb{P}^z\{\tau_{\mathbb{C}\setminus V} < \overline{\tau}_D\} \geq \delta > 0$. This implies that there exists $z_n \to z$ with $\mathbb{P}^{z_n}\{\tau_{\mathbb{C}\setminus V} < \tau_D\} \geq \delta\}$ and hence $h(z_n) \geq \delta\, \min\{h(w) : w \in V\}$.

□

Proposition 1.1.6. *If D is a domain and $z \in D$, then with probability one, $B(\tau_D)$ is a regular point of ∂D.*

Proof. Let V be a subset of ∂D and let E be the event $\{B(\tau_D) \in V\}$. Let M_t be the martingale $M_t = \mathbb{P}[E \mid \mathcal{F}_{t\wedge\tau}]$. For $t < \tau_D$, $M_t = h(B_t)$ where h is the positive harmonic function $h(w) = \mathbb{P}^w\{B_{\tau_D} \in V\}$. Assume that V is such that $0 < h < 1$ on D. Note that $M_\tau = 1_E$ and the martingale convergence theorem implies that $M_{\tau-} = M_\tau$ with probability one. In particular, with probability one, if $\zeta := B(\tau_D) \notin V$, then there exists $\gamma : [0,1) \to D$ with $\gamma(1-) = \zeta$ such that

$$\lim_{t\uparrow 1} h(\gamma(t)) = 0.$$

Since h is a bounded function, one can see using Lemma 1.1.2 that this last condition implies: if $z_n \to \zeta$, then $h(z_n) \to 0$. This is impossible if ζ is an irregular point by the previous proposition. □

1.2 Brownian motion and harmonic functions

1.2.1 Harmonic functions

We will consider only functions on $\mathbb{R}^2$, or equivalently, $\mathbb{C}$ although much of we state here applies (with appropriate modifications) to functions on $\mathbb{R}^d$. Recall that the Laplacian of a C^2 function is defined by

$$\Delta f(z) = \partial_{xx} f(z) + \partial_{yy} f(z).$$

Definition If D is a domain, then a measurable function $\phi : D \to \mathbb{R}$, is *harmonic (in D)* if it is locally bounded and satisfies the mean-value property. In other words, if $z \in D$ and $\operatorname{dist}(z, \partial D) > \epsilon$, then

$$\phi(z) = MV(\phi, z, \epsilon) := \frac{1}{2\pi} \int_0^{2\pi} \phi(z + \epsilon e^{i\theta})\, d\theta. \tag{1.1}$$

The next lemma shows that harmonic functions are smooth.

Lemma 1.2.1. *If ϕ is a harmonic function on a domain, then ϕ is a C^∞ function.*

Proof. It suffices to show that ϕ is C^∞ in a neighborhood of every point and without loss of generality we will assume that $0 \in D$ and we will differentiate in a neighborhood of 0. By shrinking D if necessary, we may assume that

$$\int_D |\phi(z)|\, dA(z) < \infty.$$

Let $\epsilon < \operatorname{dist}(0, \partial D)/2$, and let ψ denote a radially symmetric, nonnegative, C^∞ function on $\mathbb{C}$ that vanishes on $\{|w| \geq \epsilon\}$ and satisfies

$$\int_{\mathbb{C}} \psi(w)\, dA(w) = 1.$$

Since ϕ satisfies the mean value property, we can use polar coordinates to see that that for $|z| < \epsilon$.

$$\phi(z) = \int_{\mathbb{C}} \phi(w)\, \psi(w - z)\, dA(w).$$

Since ϕ is L^1 and ψ is C^∞ with compact support, the right-hand side is infinitely differentiable. □

Proposition 1.2.2. *A function ϕ is harmonic on D if and only if it is C^2 and satisfies $\Delta\phi(z) = 0$ for every $z \in D$. Moreover, if $D' \subset D$ is a subdomain bounded by a finite disjoint union of C^1 curves in D, then*

$$\int_{\partial D'} \partial_n \phi(z)\, |dz| = 0, \tag{1.2}$$

where n denotes the unit inward normal.

Proof. We first claim that if ϕ is C^2 then at each z,

$$\frac{1}{4}\,\Delta\phi(z) = \lim_{\epsilon \downarrow 0} \frac{MV(\phi, z, \epsilon) - \phi(z)}{\epsilon^2}. \tag{1.3}$$

Without loss of generality we may assume that $z = 0$ and $\phi(z) = 0$ in which case we can write

$$\phi(z) = b_x\, x + b_y y + \frac{1}{2}\,[b_{xx}\, x^2 + b_{yy}\, y^2 + 2\, b_{xy}\, xy] + o(\epsilon^2).$$

Here $b_x, b_y, b_{xx}, b_{yy}, b_{xy}$ are the first and second partial derivatives evaluated at 0. Here we use complex notation $z = x + iy$. Since

$$MV(x^2, 0, \epsilon) + MV(y^2, 0, \epsilon) = MV(x^2 + y^2, 0, \epsilon) = \epsilon^2,$$

we see that $MV(x^2, 0, \epsilon) = MV(y^2; 0, \epsilon) = \epsilon^2/2$, and

$$MV(\phi, 0, \epsilon) = \frac{\Delta\phi(0)}{4}.$$

This gives (1.3) and it follows immediately that $\Delta\phi \equiv 0$ for harmonic functions.

If ϕ is C^2 and satisfies $\Delta\phi \equiv 0$, then the divergence theorem shows that

$$\int_{\partial D'} \partial_n \phi(z)\, |dz| = -\int_{D'} \Delta\phi(w)\, dA(w) = 0.$$

Applying this to the circles centered at z shows that

$$\frac{dMV(\phi, z, \epsilon)}{d\epsilon} = \frac{1}{2\pi\epsilon} \int_{|z|=\epsilon} \partial_n \phi(z)\, |dz| = \frac{c}{\epsilon},$$

for some $c \in \mathbb{R}$. Since $MV(\phi, z, 0+) = \phi(z)$, we get that $c = 0$ and $\phi(z, \epsilon) = \phi(z)$ for all ϵ.

□

1.2.2 Optional sampling theorem

There is a close relationship between harmonic function and martingales. Before proceeding we will prove a lemma that is one version of the "optional sampling" or "optional stopping" theorem for martingales. The assumptions we make are significantly stronger than is needed for the result, but it will suffice for our purposes.

Proposition 1.2.3. *Suppose $M_t, 0 \le t \le T$ is a uniformly bounded continuous martingale, and τ is a stopping time, each with respect to the filtration $\{\mathcal{F}_t\}$. Then $M_{t\wedge\tau}, 0 \le t \le T$ is a continuous martingale with respect to the filtration $\{\mathcal{F}_{\tau\wedge t}\}$.*

We recall that if τ is a stopping time, then $\mathcal{F}_\tau$ is the σ-algebra of all events E such that for all t, $E \cap \{\tau \le t\} \in \mathcal{F}_t$. One thinks of this as all the events that depend on the process only up to the stopping time τ.

Proof. Let $Y_t = M_{t\wedge\tau}$. It is immediate that Y is a continuous process. Let us first assume that τ takes on only a discrete number of values $0 = s_0 < s_1 < \cdots < s_k < \infty$. If $s < t$, then $M_{t\wedge\tau}$ can be written as

$$M_{t\wedge\tau} = \sum_{s_j<t} M_{s_j}\, 1\{\tau = s_j\} + M_t\, 1\{\tau > s_j\}.$$

Using the definition, it is not hard to show that this is a martingale. To illustrate this, we consider the case $s_j = s < t$, in which case

$$\mathbb{E}\left[M_{t\wedge\tau} \mid \mathcal{F}_s\right] = \sum_{k\le j} M_{s_k}\, 1\{\tau = s_k\} + \mathbb{E}\left[M_t\, 1\{\tau > s_j\} \mid \mathcal{F}_{s_j}\right].$$

Since the event $\{\tau > s_j\}$ is the complement of the event $1\{\tau \le s_j\}$ which is $\mathcal{F}_{s_j}$-measurable,

$$\mathbb{E}\left[M_t\, 1\{\tau > s_j\} \mid \mathcal{F}_{s_j}\right] = 1\{\tau > s_j\}\, \mathbb{E}\left[M_t \mid \mathcal{F}_{s_j}\right] = 1\{\tau > s_j\}\, M_{s_j},$$

and hence

$$\begin{aligned}\mathbb{E}\left[M_{t\wedge\tau} \mid \mathcal{F}_{s_j}\right] &= \sum_{k\le j} M_{s_j}\, 1\{\tau = s_j\} + M_{s_j}\, 1\{\tau > s_j\} \\ &= \sum_{k<j-1} M_{s_k}\, 1\{\tau = s_k\} + M_{s_j}\, 1\{\tau \ge s_{j+1}\} = Y_{s_j}.\end{aligned}$$

For more general τ, we approximate τ by discrete stopping times,

$$\tau^j = \tau_{(j+1)/n}, \quad \frac{j}{n} \le \tau < \frac{j+1}{n}.$$

The random variables $M_{t\wedge\tau^j} \to M_t$ with probability one. Since they are bounded, they also converge in L^1.

□

- Let B_t be a standard one-dimensional Brownian motion starting at the origin and suppose that $a, b > 0$. Let $\tau = \inf\{t : B_t = b \text{ or } B_t = -a\}$. Then $B_{t\wedge\tau}$ is a martingale. Therefore, for each t,

  $$0 = \mathbb{E}[B_0] = \mathbb{E}\left[B_{t\wedge\tau}\right].$$

  With probability one $B_{t\wedge\tau} \to B_\tau$. Since $B_{t\wedge\tau}$ is uniformly bounded, we can use the dominated convergence theorem to see that

  $$\mathbb{E}[B_\tau] = \lim_{t\to\infty} \mathbb{E}\left[B_{t\wedge\tau}\right] = 0.$$

  But,

  $$\mathbb{E}[B_\tau] = b\,\mathbb{P}\{B_\tau = b\} - a\,\left[1 - \mathbb{P}\{B_\tau = b\}\right].$$

Solving, we get

$$\mathbb{P}\{B_\tau = b\} = \frac{a}{a+b}.$$

This relation is often referred to as the *gambler's ruin* estimate for one-dimensional Brownian motion. From this we can easily see that one-dimensional Brownian motion is *recurrent*, that is, it keeps returning to the origin.

- One must be careful in using this proposition. If $\mathbb{P}\{\tau < \infty\} = 1$, then with probability one

$$\lim_{t\to\infty} M_{t\wedge\tau} = M_\tau.$$

However, it is not always the case that this limit is in L^1. Indeed, it is possible for

$$\mathbb{E}[M_\tau] \neq \lim_{t\to\infty} \mathbb{E}[M_{t\wedge\tau}].$$

As an example, let $M_t = B_t$ be a standard one-dimensional Brownian motion starting at the origin and let $\tau = \inf\{t : B_t = 1\}$. Recurrence of one-dimensional Brownian motion implies that $\mathbb{P}\{\tau < \infty\} = 1$. However, $\mathbb{E}[B_\tau] \neq \mathbb{E}[B_0]$.

1.2.3 Itô's formula calculation

Suppose D is a domain, $h : D \to \mathbb{R}$ is a harmonic function and $B_t = B_t^1 + iB_t^2$ is a complex Brownian motion starting at $z \in D$. Let $\tau = \tau_D = \inf\{t : B_t \notin D\}$. Then, for $t < \tau$, Itô's formula implies that

$$dh(B_t) = h_x(B_t)\, dB_t^1 + h_y(B_t)\, dB_t^2.$$

Suppose $K \subset D$ is a compact set with $z \in \text{int}(K)$, and let $\tau' = \inf\{t : B_t \in \partial K\}$. Then $h(B_{t\wedge\tau'})$ is a bounded martingale. It follows that

$$\mathbb{E}^z\left[h(B_{t\wedge\tau'})\right] = \mathbb{E}^z[h(B_0)] = h(z).$$

The left-hand side is the same as $MV(z; f, \partial K)$.

Proposition 1.2.4 (Dirichlet problem)**.** *Suppose $D \in \mathcal{D}$, and h is a bounded continuous function on ∂D. Then there exists a unique bounded continuous function $h : \overline{D} \to \mathbb{R}$ that extends h and is harmonic in D. In fact, for every $z \in D$,*

$$h(z) = \mathbb{E}^z\left[h(\tau_D)\right]. \tag{1.4}$$

Proof. If h is defined by (1.4), then h is locally integrable and satisfies the mean value property. Hence, h is harmonic. Conversely if h is harmonic in D and continuous on ∂D, then $M_t = h_{t\wedge\tau_D}$ is a continuous martingale, and (1.4) satisfies the mean value property. We need to show that h defined as in (1.4) is continuous on ∂D, and this uses the fact that every point in ∂D is a regular point. □

The assumption that h is bounded is necessary for uniqueness. For example if $D = (0, \infty)$ and $h(0) = 0$, there are an infinite number of harmonic extensions to D given by $h(x) = cx$.

As we will see below, if a Brownian motion starts at $z \in \mathbb{D}$, then,

$$\mathbb{P}^z\{B_{\tau_{\mathbb{D}}} \in V\} = \frac{1}{2\pi} \int_{|\zeta|=1} \frac{1-|z|^2}{|\zeta - z|^2}\, 1_V(\zeta)\, d|\zeta|,$$

and hence

$$\mathbb{E}^z\left[F(B_{\tau_{\mathbb{D}}})\right] = \frac{1}{2\pi} \int_{|\zeta|=1} \frac{1-|z|^2}{|\zeta - z|^2}\, F(\zeta)\, d|\zeta|. \tag{1.5}$$

One can verify the last equality using the previous proposition and checking that the right-hand side is harmonic in $\mathbb{D}$ and obtains the correct boundary value. Using this we can derive the fundamental facts about harmonic functions.

Proposition 1.2.5 (Derivative estimates)**.** *For every positive integer k, there exists $c_k < \infty$ such that if D is a domain, $h : D \to \mathbb{R}$ is harmonic, and $z \in D$ with $\operatorname{dist}(z, D) \geq \epsilon$, then for $0 \leq j \leq k$,*

$$|\partial_x^j \partial_y^{k-j} h(z)| \leq c_k\, \epsilon^{-k} \sup\{|h(w)| : |w - z| < \epsilon\}.$$

Proof. By considering $\tilde{h}(w) = h(z + \epsilon w)$, we can see that it suffices to prove the result for $D = \mathbb{D}, z = 0$. In this case we can differentiate under the integral in (1.5). □

Proposition 1.2.6 (Harnack principle)**.** *If D is a domain, then for every compact $K \subset D$ there exists $c = c(K, D) < \infty$ such that if $h : D \to (0, \infty)$ is harmonic, then*

$$h(z) \leq c\, h(w), \quad z, w \in K.$$

Proof. Since D is connected, by choosing K larger if necessary, we may assume that K is connected. If $D \supset \mathbb{D}$ and $K = \{|w| \leq 1/2\}$, then the result follows from the explicit form of h in (1.5). More generally, we can cover K by a finite collection of open balls $\mathbb{D}_r(\zeta_j), j = 1, \ldots, N$ with $\mathbb{D}_{r-\log 2}(\zeta) \subset D$. We write $w \sim z$, if one of these balls contains both w, z. We say that w and z are connected if there exists a sequence $z = z_0, z_1, \ldots, z_k = w$ with $z_{j-1} \sim z_j$ for each j. We claim that all points in K are connected. Indeed, let U_1 be the union of all the disks $\mathbb{D}_r(\zeta_j), j = 1, \ldots, N$ for which z is connected to ζ_j and let U_2 be the union of the other disks. If $U_2 \neq \emptyset$, then U_1, U_2 disconnect K. Hence, for $z, w \in K$ we can find a sequence $z = z_0, z_1, \ldots, z_k = w$ with $z_{j-1} \sim z_j$ for all j. By "erasing loops" if necessary, we can guarantee that $k \leq N$, and hence, and $f(z) \leq c_0^N\, f(w)$. □

It will be useful to have the following convention.

Convention. Suppose D is a domain and $h : D \to \mathbb{R}$ is a harmonic function. We say $h : \overline{D} \to \mathbb{R}$ is an extension of the harmonic function to the boundary, if $\overline{h}$ is continuous at all regular points of ∂D.

While we have stated the derivative estimates and Harnack principle for harmonic functions in $\mathbb{R}^2$, the analogous results hold for harmonic functions in $\mathbb{R}^d$.

Proposition 1.2.7 (Schwarz reflection, harmonic functions)**.** *Suppose h is a harmonic function defined on $\mathbb{H}\cap\mathbb{D}$ with boundary value 0 on $(-1,1)$. If h is extended to $\mathbb{D}$ by $h(x-iy)=-h(x+iy)$ then h is harmonic on $\mathbb{D}$.*

Proof. We show that h has the mean-value property. In other words, we need to show that if $z\in\mathbb{D}$, $r<\operatorname{dist}(z,\partial\mathbb{D})$, $\mathcal{B}=\{w:|z-w|<r\}$, and $\tau=\inf\{t:B_t\in\partial\mathcal{B}\}$, then

$$h(z)=\mathbb{E}^z[h(B_\tau)].$$

If $z\in\mathbb{R}$ this is immediate by symmetry and the case $\operatorname{Im}(z)<0$ is identical to $\operatorname{Im}(z)>0$ so we will assume $\operatorname{Im}(z)>0$. We also assume that $r>\operatorname{Im}(z)$, for otherwise this follows immediately from the harmonicity of h on $\mathbb{H}\cap\mathbb{D}$. Let $V_-=\partial\mathcal{B}\cap\{\operatorname{Im}(\zeta)<0\}$, $V_+=\{w:\overline{w}\in V_-\}$ and $V=V_-\cup V_+$. Note that $V_+\subset\mathcal{B}$. Let $\partial^*=\partial[\mathcal{B}\cap\mathbb{H}]$. We define a sequence of stopping times. Let $\rho_0=0$ and

$$\sigma_j=\tau\wedge\inf\{t\geq\rho_{j-1}:B_t\in\partial^*\},$$

$$\rho_j=\tau\wedge\inf\{t\geq\sigma_j:B_t\in V\},$$

Harmonicity in $\mathbb{H}\cap\mathbb{D}$ shows that for each j, $\mathbb{E}^z[h(B_{\sigma_{j+1}})]=\mathbb{E}^z[h(B_{\rho_j})]$ and symmetry shows that $\mathbb{E}^z[h(B_{\rho_j})]=\mathbb{E}^z[h(B_{\sigma_j})]$. Therefore,

$$\mathbb{E}[h(B_\tau)]=\lim_{j\to\infty}\mathbb{E}^z[h(B_{\sigma_j})]=h(z).$$

□

1.2.4 Harmonic functions and holomorphic functions

We assume the following facts from an undergraduate course in complex variables.

Definition A function $f:u+iv$ on a domain $D\subset\mathbb{C}$ is called *holomorphic* or *analytic* on D if any of the following equivalent facts hold.

- The derivative
$$f'(z)=\lim_{w\to z}\frac{f(w)-f(z)}{w-z}$$
exists.
- (u,v) is a C^2 function satisfying the Cauchy-Riemann equations
$$\partial_x u=\partial_y v,\quad \partial_y u=-\partial_x v.$$
- For each $z\in D$, we can expand f in a power series
$$f(w)=\sum_{j=0}^{\infty}a_j\,(w-z)^j,$$
where the radius of convergence is at least $\operatorname{dist}(z,\partial D)$.

From the Cauchy-Riemann equations, one can see that u, v are harmonic functions. The next proposition, which is a standard result from a first course in complex variables, gives a partial converse to this fact.

Proposition 1.2.8. *Suppose D is a simply connected domain and u is harmonic function on D. Then there is a harmonic function v on D, which is unique up to an additive constant, such that $f(z) := u(z) + iv(z)$ is holomorphic.*

Sketch of proof. We use the Cauchy-Riemann equations to find v. Let us fix $z_0 \in D$ and arbitrarily choose $v(z_0) = 0$. If $\gamma : [0, 1] \to D$ is a smooth curve with $\gamma(0) = z_0, \gamma(1) = z$, then we define

$$v(z) = \int_\gamma \partial_n u \cdot d\gamma := \int_0^1 \left[\partial_y u(\gamma(t))\, \partial_x \gamma(t) - \partial_x u(\gamma(t))\, \partial_y \gamma(t)\right] dt.$$

In order to show this is well defined, we need to show that we get the same value for $v(z)$ regardless of the curve γ. Equivalently, we need to show that if $\gamma(1) = z_0$, then

$$\int_\gamma \partial_n u \cdot d\gamma = 0.$$

If D is simply connected, then the region(s) bounded by γ are entirely in D, and this identity follows from Green's theorem and the fact that u is harmonic. By construction, u, v satisfy the Cauchy-Riemann equation and hence $f = u + iv$ is holomorphic. To show uniqueness, suppose that $\tilde{f} = u + i\tilde{v}$ is also holomorphic. Then $f - \tilde{f}$ is holomorphic and only takes on imaginary values. Hence $f - \tilde{f}$ is constant.

□

The function v is often called the *complex conjugate.* We have used simple connectedness to conclude that there exists an complex conjugate v defined on all of D. Such an extension does not necessarily exist if the domain is not simply connected. For example, if $u(z) = \log|z|$ on $D = \{z : 0 < |z| < 1\}$, then u is harmonic, but there is no holomorphic extension to all of D. However, regardless of the topology of D, we can always find conjugates v defined in a neighborhood of z_0. When trying to determine if two complex domains are conformally equivalent, it is often the case that one can determine the real or imaginary part (or, perhaps, the radial part which is the real part of the exponential, or something similar). This determines the function (up to a constant) locally and then the question becomes whether or not one can extend it to the entire domain D.

Proposition 1.2.9 (Schwarz reflection, holomorphic functions)**.** *Suppose $f = u + iv$ is a holomorphic function on $\mathbb{D}_+$ with $\lim_{y \downarrow 0} v(x + iy) = 0$ for all $-1 < x < 1$. Then f can be extended to a holomorphic function on $\mathbb{D}$ satisfying $f(\bar{z}) = \overline{f(z)}$.*

Proof. By Proposition 1.2.7, if we extend v to $\mathbb{D}$ by $v(x) = 0$ and $v(\bar{z}) = -v(z)$, then v is harmonic in $\mathbb{D}$. By Proposition 1.2.8, there exists u^* (unique up to an additive constant) such that $u^* + iv$ is holomorphic. By uniqueness, we can choose the constant so that $u^* \equiv u$ on $\mathbb{D}_+$. Since $\hat{f}(z) = u(\bar{z}) - iv(\bar{z})$ is holomorhphic in $-\mathbb{D}_+$, we see that $u^* \equiv u + c_0$ in $-\mathbb{D}_+$ for some $c_0 \in \mathbb{R}$. Continuity at the real axis shows that $c_0 = 0$.

□

We have stated Schwarz reflection for functions in $\mathbb{D}_+$, but there is an immediate corollary for functions in $\epsilon\mathbb{D}_+$.

The term conjugate is overused in complex variables! I have given lectures where I have used conjugate three different ways in the same lecture — as the conjugate of a number, as the complex conjugate function above, and also in the algebraic sense of a conjugate function $\tilde{f} = g^{-1} \circ f \circ g$.

1.2.5 Conformal invariance

If f is a holomorphic function with $f(0) = 0, f'(0) \neq 0$, then locally near zero f looks like a dilation by $|f'(0)|$ and a rotation by $\arg f'(0)$. Brownian motion is invariant under rotation and is also invariant under scaling if one changes the parametrization appropriately. This is the basic reason why the following theorem holds.

Theorem 1.2.10. *Suppose $D \subset \mathbb{C}$ is a domain and B_t is a complex Brownian motion starting at $z \in D$. Suppose $f : D \to \mathbb{C}$ is a nonconstant holomorphic function. Let*

$$\xi = \int_0^{\tau_D} |f'(B_s)|^2 \, ds \in (0, \infty],$$

and for $t < \xi$, define $\sigma(t) < \tau_D$ by

$$\int_0^{\sigma(t)} |f'(B_s)|^2 \, ds = t.$$

Then $Y_t = f(B_{\sigma(t)}), 0 \leq t < \xi$ is a complex Brownian motion.

We will need a lemma that states in some sense that all stochastic integrals are time changes of standard Brownian motions. Indeed, a stronger fact is true that we will not prove — all continuous martingales are time changes of standard Brownian motions.

Lemma 1.2.11. *Suppose B_t is a standard one-dimensional Brownian motion with filtration $\{\mathcal{F}_t\}$ and suppose that A_t is a continuous, adapted process such that there exist $0 < c_1 < c_2 < \infty$ with $c_1 \leq |A_t| \leq c_2$. Let*

$$X_t = \int_0^t A_s \, dB_s,$$

and let

$$\sigma(r) = \inf\{t : \langle X \rangle_t = r\},$$

that is

$$\int_0^{\sigma(r)} A_s^2 \, ds = r.$$

Suppose that for all t, $\mathbb{P}\{\sigma(t) < \infty\} = 1$. Then $W_r := X_{\sigma(r)}$ is a standard Brownian motion with respect to $\tilde{\mathcal{F}}_r = \mathcal{F}_{\sigma(r)}$.

Sketch of Proof. To prove this, one shows that conditioned on $\tilde{\mathcal{F}}_s$ the distribution of $W_{r+s} - W_s$ is that of a Brownian motion with variance r^2. We will do this in the case $s = 0$; the general case is similar. If $\lambda \in \mathbb{R}$, let

$$K_t = \exp\{i\lambda X_t\}.$$

(If one does not want to use Itô's formula with with complex valued processes one can write this as $\cos(\lambda X_t) + i \sin(\lambda X_t)$.) Itô's formula shows that

$$dK_t = K_t \left[i\,\lambda\, A_t\, dB_t - \frac{\lambda^2}{2}\, A_t^2\, dt \right] = K_t \left[i\,\lambda\, A_t\, dB_t - \frac{\lambda^2}{2}\, d\langle X \rangle_t \right].$$

If $M_t = \exp\{\lambda^2\, \langle X \rangle_t / 2\}\, K_t$, then M_t is a local martingale satisfying,

$$dM_t = i\,\lambda M_t\, dB_t.$$

Note that $M_{t\wedge\sigma(r)}$ is a bounded martingale, and hence the optional sampling theorem implies that

$$\mathbb{E}\left[M_{\sigma(r)}\right] = \mathbb{E}[M_0] = 1.$$

But, $M_{\sigma(r)} = e^{\lambda^2 r/2}\, \exp\{i\lambda X_{\sigma(r)}\}$, and hence

$$\mathbb{E}\left[e^{i\lambda W_r}\right] = e^{-\lambda^2 r/2}.$$

Since the characteristic function determines the distribution, we see that $W_r \sim N(0, r)$. □

Proof of Theorem 1.2.10. We will give a sketch of the proof relying on some facts from stochastic calculus.

Let $U \subset D$ be a subdomain with $\overline{U}$ compact containing none of the zeros of f', and let $\tau = \tau_U < \tau_D$. Let us write $B_t = B_t^1 + iB_t^2$ and let $X_t = u(B_t^1, B_t^2), Y_t = v(B_t^1, B_t^2)$. Using the fact that u, v are harmonic functions, Itô's formula and the Cauchy-Riemann equations give

$$dX_t = u_x(B_t)\, dB_t^1 + u_y(B_t)\, dB_t^2,$$

$$\begin{aligned} dY_t &= v_x(B_t)\, dB_t^1 + v_y(B_t)\, dB_t^2 \\ &= -u_y(B_t)\, dB_t^1 + u_x(B_t)\, dB_t^2. \end{aligned}$$

Note that $\partial_t \sigma(t) = |f'(B_t)|^{-2} = |\nabla u(B_t)|^{-2}$. If we let $\hat{X}_t = X_{\sigma(t)}, \hat{Y}_t = Y_\sigma(t)$, then

$$d\hat{X}_t = \frac{u_x(\hat{B}_t)}{|\nabla u(\hat{B}_t)|}\, dW_t^1 + \frac{u_y(\hat{B}_t)}{|\nabla u(\hat{B}_t)|}\, dW_t^2,$$

$$d\hat{Y}_t = \frac{-u_y(\hat{B}_t)}{\nabla u(\hat{B}_t)}\, dW_t^1 + \frac{u_x(\hat{B}_t)}{|\nabla u(\hat{B}_t)|}\, dW_t^2,$$

where W_t^1, W_t^2 are independent, standard Brownian motions. This means that $(\hat{X}_t, \hat{Y}_t)$ are independent standard Brownian motions, that is,

$$\hat{B}_t = \hat{X}_t + i\,\hat{Y}_t,$$

is a standard complex Brownian motion at least for $t \le \tau$. Since this holds for every U, and with probability one the Brownian motion avoids the singular points of U, we can conclude that it holds for $t < \tau_D$.

□

The statement of Theorem 1.2.10 is a little nicer if f is a conformal transformation. We say that $f : D \to D'$ is a *conformal transformation* if f is holomorphic, one-to-one, and onto.

Theorem 1.2.12. *Suppose D is a domain in $\mathbb{C}$ and $f : D \to f(D)$ is a conformal transformation. Suppose B_t is a complex Brownian motion starting at $z \in D$. Let*

$$\xi = \int_0^{\tau_D} |f'(B_s)|^2 \, ds \in (0, \infty],$$

and for $s < \xi$, define $\sigma(s) < \tau_D$ by

$$\int_0^{\sigma(s)} |f'(B_u)|^2 \, du = s.$$

Then $Y_s := f(B_{\sigma(s)}), 0 \le s < \xi$ is a complex Brownian motion, and $\xi = \tau_{f(D)} = \inf\{t : Y_t \notin f(D)\}$.

Proof. Note that if $\xi < \infty$, we can extend $Y_s, 0 \le s \le \xi$ by continuity. If $\xi < \infty$, we claim that $Y_\xi \in \partial f(D)$. Indeed, if $Y_\xi = w \in f(D)$, then $B_{\tau_D -} = f^{-1}(w) \in D$.

□

Example: Recurrence

Here we show that two-dimensional Brownian motion is *neighborhood recurrent*. To be more precise, with probability one, for all $z \in \mathbb{C}, \epsilon > 0, T < \infty$, there exists $t > T$ with $|B_t - z| < \epsilon$. It suffices to prove this for z with rational coordinates, and the proof is essentially the same for all of them, so let us consider $z = 0$. Let B_t be a complex Brownian motion and let $f(z) = e^z, Y_t = f(B_t) = e^{B_t} = e^{B_t^1} \, e^{iB_t^2}$. Then Y_t is a time change of a Brownian motion. Since $|Y_t| = e^{B_t^1}$, we see from the recurrence of the one-dimensional Brownian motion B^1 that

$$\liminf_{t \to \infty} |Y_t| = 0.$$

We can also see from this that the Brownian motion is not *pointwise recurrent*. Indeed with probability one, a Brownian motion never visits the origin after time zero. This is obvious for Y_t since 0 is not in the range of the exponential function.

An important corollary of the neighborhood recurrence of Brownian motion is the following: if D is a domain with at least one regular boundary point, then for all $z \in D$,

$$\mathbb{P}^z\{\tau_D < \infty\} = 1.$$

Indeed, if w is a regular point, then there exists a δ such that if the Brownian motion is within distance δ of w, then with probability $1/2$ it leaves the domain before it goes distance one from z. Since we keep returning to the δ neighborhood of z we get infinitely many chances to escape D near w and we will eventually succeed. This fact is used implicitly in the following definition.

Definition If $D \in \mathcal{D}^*$ and $z \in D$, then the *harmonic measure* $\mathrm{hm}_D(z, \cdot)$ is defined to be the distribution of $B(\tau_D)$ assuming $B_0 = z$. In other words, the probability that a Brownian motion starting at z exits D at V is $\mathrm{hm}_D(z, V)$. More generally, if f is a function defined on ∂D, we let

$$\mathrm{hm}_D(z, f) = \mathbb{E}^z \left[f(B_\tau) \right] = \int_{\partial D} f(w) \, d\mathrm{hm}_D(z, w).$$

If $f : D \to f(D)$ is a conformal transformation, that extends to a homemomorphism of $\overline{D}$, then

$$\mathrm{hm}_{f(D)}(f(z), f(V)) = \mathrm{hm}_D(z, V). \tag{1.6}$$

There is a similar formula that holds if f does not extend to a homemorphism, but to explain it requires a discussion of prime ends which we will do later. We For example, if $D = \mathbb{D} \cap \mathbb{H}$, then $f(z) = z^2$ is a conformal transformation of D onto $f(D) = \mathbb{D} \setminus [0, 1)$. The boundary point $1/4 \in \partial f(D)$ corresponds to two equivalences classes, one for curves approaching $1/4$ from above and the other for curves approaching $1/4$ from below. These correspond to curves in D that leave D at $1/2$ and $-1/2$ respectively. Examples like this where there are "two-sided" points are easy to handle, even if we have to be careful about our notation. Once we have defined prime ends, the generalization will be straightforward.

The definition of harmonic measure does not require any smoothness of the boundary. However, if the boundary is nice, then one can write harmonic measure as an integral of a kernel called the *Poisson kernel*. Rotational invariance of Brownian motion shows that harmonic measure on $\mathbb{D}$ centered at zero is the uniform distribution. We define $H_D(z, w)$ to be the Poisson kernel normalized so that

$$H_{\mathbb{D}}(0, e^{i\theta}) = \frac{1}{2}.$$

In other words, if V is sufficiently smooth,

$$\mathrm{hm}_D(z, V) = \frac{1}{\pi} \int_V H_D(z, w)\, |dw|. \tag{1.7}$$

Here $|dw|$ represents integration with respect to arc length, that is, a traditional line integral from vector calculus rather than a complex integral along a curve. The term "sufficiently smooth" is a little vague. We will only use the Poisson kernel at places where ∂D is locally an analytic curve. If $f : D \to f(D)$ is a conformal transformation, D is locally analytic at w,. and $f(D)$ is locally analytic at $f(w)$, then the Poisson kernel satisfies the "conformal covariance" relation

$$H_D(z, w) = |f'(w)|\, H_{f(D)}(f(z), f(w)).$$

Suppose $f : \mathbb{D} \to D$ is a conformal transformation. Then boundary ∂D can be very rough. For example, there can be points $w \in \partial D$ such that there is no continuous path $\eta : [0, 1) \to D$ with $\eta(1-) = w$. However, the harmonic measure of such points has to be zero. This is immediate from the definition of harmonic measure in terms of Brownian motion.

The Poisson kernel is naturally defined up to a constant. We made a nonstandard choice in (1.7) which is convenient for later work with Schramm-Loewner evolution. Under this normalization, if $x > 0$,

$$H_{\mathbb{H}}(0, x) = x^{-2}.$$

Example: the annulus

Recall that $\mathbb{A}_r$ is the annulus $\mathbb{A}_r = \mathbb{D} \setminus \overline{\mathbb{D}_r} = \{z : e^{-r} < |z| < 1\}$.

Proposition 1.2.13. *If $z \in \mathbb{A}_r$ and $\tau = \tau_{\mathbb{A}_r}$, then*

$$\mathbb{P}^z\{|B_\tau| = e^{-r}\} = \frac{-\log|z|}{r}. \tag{1.8}$$

Proof. Let us give two similar proofs. First, note that $\phi(z) := -\log|z|$ is a bounded harmonic function in $\mathbb{A}_r$ that is continuous on $\overline{\mathbb{A}_r}$. This can be checked by differentiation or by noting that it is the real part of the (locally) analytic function $-\log z$. Therefore, by Proposition 1.2.4,

$$\phi(z) = \phi(B_0) = \mathbb{E}^z\left[\phi(B_\tau)\right] = r\,\mathbb{P}^z\left\{|\phi(B_\tau)| = e^{-r}\right\}.$$

Alternatively we can let $Y_t = \exp\{B_t\}$. Then the probability is the same as the probability that the one-dimensional Brownian motion B^1_t starting at $\log|z|$ reaches level $-r$ before reaching level 0 which by the gambler's ruin estimate is $-\log|z|/r$.

□

Proposition 1.2.14. *Suppose ϕ is a harmonic function on the annulus $\mathbb{A}_r = \{e^{-r} < |z| < 1\}$. Let*

$$M_s = MV(\phi, 0, e^{-s}) = \frac{1}{2\pi}\int_0^{2\pi} \phi(e^{-s+i\theta})\,d\theta,$$

be the average value of ϕ on the circle of radius e^{-s}. Then there exist a, b such that

$$M_s = as + b, \quad 0 < s < r.$$

Moreover, for each s,

$$\int_{C_s} \partial_n\phi(w)\,|dw| = 2\pi a,$$

where n is the inward unit normal.

Proof. Suppose that $0 < p < s < q < r$. Let B_t be a Brownian motion starting uniformly on C_s, the circle of radius e^{-s}, and let T be the first time t that $B_t \in C_p \cup C_q$. Then

$$\mathbb{P}\{B_T \in C_p\} = \frac{q-s}{q-p}.$$

By rotational symmetry, given that $B_T \in C_p$ (or given $B_T \in C_q$), the distribution of the angular part is uniform. Therefore,

$$M_s = \frac{q-s}{q-p}\,M_p + \frac{s-p}{q-p}\,M_q = \frac{M_q - M_p}{q-p}\,s + \frac{qM_p - pM_q}{q-p}.$$

This establishes the result for $p < s < q$ and by letting $p \to 0, q \to r$, we get the first assertion. For the final assertion note that

$$2\pi a = 2\pi\partial_s M_s = \partial_s \int_0^{2\pi} \phi(e^{-s+i\theta})\,d\theta = \partial_s \int_{C_s} \phi(w)\,e^s\,|dw| = \int_{C_s} \partial_n\phi(w)\,|dw|.$$

□

1.3 Green's function

The Green's function $G_D(z,w)$ is the normalized probability that a Brownian motion starting at z visits w before leaving D. As stated this does not make sense since the probability that the Brownian motion visits w is zero. However, we can make sense of it as a limit,

$$G_D(z,w) = \lim_{\epsilon \downarrow 0} \log[1/\epsilon]\, \mathbb{P}^z\{\text{dist}(w, B[0,\tau_D]) \le \epsilon\}.$$

We will show this limit exists in this section and derive some properties. We will first consider the case $w = 0$ and write just $G_D(z)$ for $G_D(z,0)$. Throughout this section, we let

$$\sigma_s = \inf\{t : |B_t| = e^{-s}\}.$$

We will make no topological assumptions about the domain D. We only require that the boundary contain a regular point, that is, that $\mathbb{P}^z\{\tau_D < \infty\} = 1$.

Definition Let $\mathcal{U}_r$ (resp., $\mathcal{U}_r^{\rm s}$) denote the set of domains (resp., simply connected domains) containing the origin with at least one regular boundary point and $\text{dist}(0, \partial D) \ge r$. If $r = 1$ we write just $\mathcal{U}, \mathcal{U}^{\rm s}$. There are natural bijections $\mathcal{U} \leftrightarrow \mathcal{U}_r$ and $\mathcal{U}^{\rm s} \leftrightarrow \mathcal{U}_r^{\rm s}$ given by $D \leftrightarrow rD$.

Proposition 1.3.1. *Suppose $D \in \mathcal{U}_r$ and $z \in D \setminus \{0\}$. Then the limit*

$$G_D(z) = \lim_{s \to \infty} s\, \mathbb{P}^z\{\sigma_s < \tau_D\},$$

exists and lies in $(0, \infty)$. Moreover, $G_{rD}(rz) = G_D(z)$.

We note that (1.8) establishes the result for $D = \mathbb{D}$ for which

$$G_D(z) = -\log|z|.$$

Proof. We write $\tau = \tau_D$. It suffices to prove the result for $D \in \mathcal{U}$, after which we can use conformal invariance of Brownian motion to see that

$$G_{e^u D}(e^u z) = \lim_{s\to\infty} s\, \mathbb{P}^{e^u z}\{\sigma_s < \tau_{e^u D}\} = \lim_{s\to\infty} s\, \mathbb{P}^z\{\sigma_{s-u} < \tau\} = \lim_{s\to\infty} (s-u)\, \mathbb{P}^z\{\sigma_{s-u} < \tau\} = G_D(z).$$

Since $\mathbb{D} \subset D$ and ∂D contains a regular point, the Harnack principle (Proposition 1.2.6) implies that

$$\inf_{|\zeta|=1} \mathbb{P}^\zeta\{\tau < \sigma_1\} =: \rho = \rho_D > 0. \tag{1.9}$$

For $s > 1$, let

$$q_s = \sup_{|\zeta|=1} \mathbb{P}^\zeta\{\sigma_s < \tau\}.$$

By the strong Markov property, this is the same as the supremum over $|\zeta| \ge 1$. We claim that

$$q_s \le \frac{1}{s\rho}, \tag{1.10}$$

To see this, note that if $|\zeta| = 1$ and $s > 1$, then

$$\mathbb{P}^\zeta\{\sigma_s < \tau\} \le \mathbb{P}^\zeta\{\sigma_1 < \tau\} \sup_{|\zeta'|=1/e} \mathbb{P}^{\zeta'}\{\sigma_s < \tau\} \le (1-\rho) \sup_{|\zeta'|=1/e} \mathbb{P}^{\zeta'}\{\sigma_s < \tau\}.$$

If $|\zeta'| = 1/e$, then using (1.8), we get

$$\begin{aligned}\mathbb{P}^{\zeta'}\{\sigma_s < \tau\} &= \mathbb{P}^{\zeta'}\{\sigma_s < \sigma_0\} + \mathbb{P}^{\zeta'}\{\sigma_s > \sigma_0\}\,\mathbb{P}^{\zeta'}\{\sigma_s < \tau \mid \sigma_s > \sigma_0\} \\ &\le \frac{1}{s} + \frac{s-1}{s}\,q_s.\end{aligned}$$

By taking the supremum over $|\zeta| = 1$, we see that

$$q_s \le (1-\rho)\,\left[\frac{1}{s} + \frac{s-1}{s}\,q_s\right] \le \frac{1}{s} + (1-\rho)\,q_s.$$

which gives (1.10).

We now fix z and let $f(s) = \mathbb{P}^z\{\sigma_s < \tau\}$. Then, if $|z| > e^{-s}$,

$$\begin{aligned}f(s+1) = \mathbb{P}^z\{\sigma_{s+1} < \tau\} &= \mathbb{P}^z\{\sigma_s < \tau\}\,\mathbb{P}^z\{\sigma_{s+1} < \tau \mid \sigma_s < \tau\} \\ &= f(s)\,\mathbb{P}^z\{\sigma_{s+1} < \tau \mid \sigma_s < \tau\}.\end{aligned}$$

If $|\zeta| = e^{-s}$, then (1.8) and (1.10) imply that

$$\begin{aligned}\mathbb{P}^{\zeta}\{\sigma_{s+1} < \tau\} &= \mathbb{P}^{\zeta}\{\sigma_{s+1} < \sigma_0\} + \mathbb{P}^{\zeta}\{\sigma_0 < \sigma_{s+1}\}\,\mathbb{P}^{\zeta}\{\sigma_{s+1} < \tau \mid \sigma_0 < \sigma_{s+1}\} \\ &\le \frac{s}{s+1} + \frac{1}{s+1}\,\frac{1}{\rho\,(s+1)} = \frac{s}{s+1} + O(s^{-2}),\end{aligned}$$

where here and throughout the remainder of the proof, the $O(\cdot)$ terms can depend on D. Therefore,

$$f(s+1) = f(s)\,\left[1 - \frac{1}{s+1} + O(s^{-2})\right],$$

$$\log f(s+1) = \log f(s) - \frac{1}{s+1} + O(s^{-2}).$$

This equation implies the existence of a constant which we call $G_D(z)$ such that for integer s,

$$\mathbb{P}^z\{\sigma_s < \tau\} = f(s) = \frac{G_D(z)}{s}\,\left[1 + O(s^{-1})\right]. \tag{1.11}$$

If $0 \le u \le 1$, the same argument shows that

$$f(s+u) = f(s)\,\frac{s}{s+u}\,\left[1 + O(s^{-1})\right] = \frac{G_D(z)}{s+u}\,\left[1 + O(s^{-1})\right],$$

and hence (1.11) holds for all s.

□

We extend $G_D(z)$ to be a function on $\mathbb{C}\setminus\{0\}$ by setting $G_D(z) = 0,\ z \notin D$. If D is open but not connected and $\tilde{D}$ is the connected component of D containing the origin, we define $G_D(z) = G_{\tilde{D}}(z)$. If $\mathbb{C} \setminus D$ is compact, then we can extend G_D to infinity,

$$G_D(\infty) = \lim_{z\to\infty} G_D(z).$$

We state the next proposition for $D \in \mathcal{U}$ but it extends immediately to a result about $D \in \mathcal{U}_r$ by scaling.

Proposition 1.3.2. *There exists $c < \infty$ such that if $D \in \mathcal{U}$, the following holds.*

1. *G_D is a positive harmonic function on $D \setminus \{0\}$ that vanishes on ∂D.*
2. *There exists $c_D < \infty$ such that for all z,*
$$G_D(z) \le \log_+(1/|z|) + c_D.$$
3. *G_D is continuous at every regular point of ∂D.*
4. *If h_D is defined by*
$$h_D(z) = G_D(z) + \log|z|, \quad z \in \mathbb{D} \setminus \{0\},$$
$$h_D(0) = \frac{1}{2\pi}\int_0^{2\pi} G_D(e^{r+i\theta})\, d\theta,$$
then h_D is a harmonic function on D.
5. *If $|z| < e^{-1}$,*
$$|h_D(z) - h_D(0)| \le c\, h_D(0)\, |z|. \tag{1.12}$$
6. *If $z \in D$,*
$$h_D(z) = \mathbb{E}^z\left[\log|B_\tau|\right] - \lim_{r\to\infty} r\, \mathbb{E}^z[h_D(B_{\sigma_{-r}}); \sigma_{-r} < \tau],$$
where $\tau = \tau_D = \inf\{t > 0 : B_t \notin D\}$. In particular, if D is bounded, then
$$h_D(z) = \mathbb{E}^z\left[\log|B_\tau|\right].$$
7. *If $D' \subset D$, then $G_{D'}(z) \le G_D(z)$.*
8. *If $D_1 \subset D_2 \subset \cdots$ and $D = \bigcup_{n=1}^\infty D_n$, then for $z \in D$,*
$$G_D(z) = \lim_{n\to\infty} G_{D_n}(z). \tag{1.13}$$
9. *If $f : D \to f(D)$ is a conformal transformation with $f(0) = 0$, then $G_{f(D)}(f(z)) = G_D(z)$.*

Proof.

1. We have shown that $G_D(z) > 0$ if $z \in D$ and it is defined to be zero on ∂D. Suppose $U \subset D$ is a disk with $\overline{U} \subset D \setminus \{0\}$. Then by the definition of G_D we can see that
$$G_D(z) = \mathbb{E}^z\left[G_D(B_{\tau_U})\right] = \mathrm{MV}(z, G_D, U)$$
from which we see that G_D is harmonic on $D \setminus \{0\}$.
2. Note that (1.10) implies that $G_D(z) \le 1/\rho$ for $|z| = 1$. If $|z| = e^{-r}$ with $r < s$, then
$$\mathbb{P}^z\{\sigma_s < \tau_D\} \le \mathbb{P}^z\{\sigma_s < \sigma_0\} + \mathbb{P}^z\{\sigma_0 < \sigma_s < \tau_D\} \le \frac{r}{s} + \frac{s-r}{s^2\rho}.$$
Letting $s \to \infty$, we see that $G_D(z) \le r + (1/\rho)$.

3. Let z be a regular boundary point and let $\xi = \inf\{t : |B_t - z| = |z|/2\}$. Then for $|w-z| < |z|/2$,

$$G_D(w) \le \alpha\, \mathbb{P}^w\{\xi < \tau_D\}$$

where $\alpha = \sup\{G_D(\zeta) : |z-\zeta| = |z|/2\} < \infty$. For every $\epsilon > 0$, we can find $\delta > 0$ such that $|w-z| < \delta$ implies that $\mathbb{P}^w\{\xi < \tau_D\} < \epsilon$ and hence $G_D(w) \le \epsilon\,\alpha$.

4. For $z \in D \setminus \{0\}$, let

$$h_D(z) = G_D(z) + \log|z|.$$

Suppose $|z| < 1$. Then,

$$\begin{aligned}\mathbb{P}^z\{\sigma_s < \tau\} &= \mathbb{P}^z\{\sigma_s < \sigma_0\} + \mathbb{P}^z\{\sigma_0 < \sigma_s < \tau\} \\ &= \frac{-\log|z|}{s} + \int_{C_0} \mathbb{P}^w\{\sigma_s < \tau\}\, \mathrm{hm}_{\mathbb{A}_s}(z, dw).\end{aligned}$$

If we multiply both sides by s and take the limit as $s \to \infty$, we get

$$G_D(z) = -\log|z| + \int_{C_0} G_D(w)\, \mathrm{hm}_{\mathbb{D}}(z, dw) = -\log|z| + \frac{1}{\pi}\int_0^{2\pi} G_D(e^{i\theta})\, H_{\mathbb{D}}(z, e^{i\theta})\, d\theta.$$

In other words,

$$h_D(z) = \frac{1}{\pi}\int_0^{2\pi} G_D(e^{i\theta})\, H_{\mathbb{D}}(z, e^{i\theta})\, d\theta, \quad 0 < |z| < 1,$$

which can be extended to 0 by setting $z = 0$ on the right-hand side.

5. Using the exact form of the Poisson kernel, we can see that

$$H_{\mathbb{D}}(z, e^{i\theta}) = \frac{1}{2} + O(|z|),$$

and hence

$$|G_D(z) + \log|z| - h_D(0)| \le c\,|z|\, h_D(0).$$

6. Let

$$\theta = \theta_z = \liminf_{r\to\infty} r\, \mathbb{P}^z\left\{\tau_D > \sigma_{-r}\right\}.$$

We claim that

$$\theta = \lim_{r\to\infty} r\, \mathbb{P}^z\left\{\tau_D > \sigma_{-r}\right\}.$$

To see this we first note that since ∂D is regular, if

$$q(r) = \sup_{|\zeta|=1} \mathbb{P}^\zeta\{\sigma_{-r} < \tau_D\},$$

then $q(r) \to 0$ as $r \to \infty$. Hence, if

$$p(r, s) = \sup_{|\zeta| = e^r} \{\sigma_{-s} < \tau_D\},$$

then for $|\zeta| = e^r$, and $s > r$,

$$\begin{aligned}\mathbb{P}^\zeta\{\sigma_{-s} < \tau_D\} &\le \mathbb{P}^\zeta\{\sigma_{-s} < \sigma_0\} + \mathbb{P}^\zeta\{\sigma_0 < \sigma_{-s}\}\,\mathbb{P}^\zeta\{\sigma_s < \tau_D \mid \sigma_0 < \sigma_{-s}\} \\ &\le \frac{r}{s} + q(r)\,p(r,s),\end{aligned}$$

which implies that

$$p(r,s) \le \frac{r}{s\,(1-q(r))}.$$

Hence,

$$\begin{aligned}\mathbb{P}^z\{\sigma_{-s} < \tau_D\} &= \mathbb{P}^z\{\sigma_{-r} < \tau_D\}\,\mathbb{P}^z\{\sigma_{-s} < \tau_D \mid \sigma_{-r} < \tau_D\} \\ &\le \mathbb{P}^z\{\sigma_{-r} < \tau_D\}\,\frac{r}{s\,(1-q(r))}.\end{aligned}$$

Therefore,

$$\limsup_{s\to\infty} s\,\mathbb{P}^z\{\sigma_{-s} < \tau_D\} \le \liminf_{r\to\infty} \frac{r}{1-q(r)}\,\mathbb{P}^z\{\sigma_{-r} < \tau_D\} = \theta.$$

(If this argument seems familiar, we are really just proving the existence of the "Green's function at infinity" which can be obtained from the Green's function at zero by the map $z \mapsto 1/z$.)

Since h_D is bounded on $\{|z| \le e^r\}$, we have

$$h_D(z) = \mathbb{E}^z[h_D(B_\tau); \tau < \sigma_{-r}] + \mathbb{E}^z[h_D(B_{\sigma_{-r}}); \sigma_{-r} < \tau].$$

Taking limits as $r \to \infty$ and using the monotone convergence theorem, we see that

$$h_D(z) = \mathbb{E}^z[h_D(B_\tau)] + \theta + G_D(\infty).$$

7. Monotonicity in D follows immediately from the definition.

8. Assume $D \in \mathcal{U}$. If $D_1 \subset D_2 \subset \cdots$ and $D = \cup D_n$, then monotonicity implies that the limit

$$G^*(z) := \lim_{n\to\infty} G_{D_n}(z)$$

exists and $G^*(z) \le G_D(z)$. Assume that $\mathbb{D}_s \subset D_n$. Then for $|w| < e^{-s}$,

$$G_{D_n}(w) \ge G_{\mathbb{D}_s}(w) = G_{\mathbb{D}}(e^s w) = -\log|w| - s.$$

Therefore, for $m \ge n$, $u \ge s$, and all $z \in D_m$,

$$G_{D_m}(z) \ge (u-s)\,\mathbb{P}^z\{\sigma_u < \tau_{D_m}\}.$$

Letting $m \to \infty$, we see that

$$G^*(z) \ge (u-s) \lim_{m\to\infty} \mathbb{P}^z\{\sigma_u < \tau_{D_m}\} = (u-s)\,\mathbb{P}^z\{\sigma_u < \tau_D\}.$$

Letting $u \to \infty$, we get $G^*(z) \ge G_D(z)$.

9. Note that it follows from the definition, that for any $u > 0$,

$$\lim_{s\to\infty} s\,\mathbb{P}^z\{\sigma_{s+u} < \tau\} = G(z). \tag{1.14}$$

Let $C_s = \{\zeta : |\zeta| = e^{-s}\}$ and $\hat{\sigma}_s = \inf\{t : B_t \in f(C_s)\}$. Conformal invariance of Brownian motion implies that for $z \in D, |z| > e^{-s}$,

$$\mathbb{P}^z\{\sigma_s < \tau_D\} = \mathbb{P}^{f(z)}\{\hat{\sigma}_s < \tau_{f(D)}\}.$$

Let $\theta = -\log|f'(0)|$. If $\epsilon > 0$, then for all s sufficiently large,

$$\sigma_{s+\theta+\epsilon} \le \hat{\sigma}_s \le \sigma_{s+\theta-\epsilon}.$$

Hence, using (1.14),

$$G_{f(D)}(f(z)) = \lim_{s\to\infty} s\,\mathbb{P}^{f(z)}\{\sigma_s < \tau_{f(D)}\} = G_D(z).$$

□

Definition If $D \in \mathcal{D}^*$, then the *Green's function* $G_D(z,w)$ is defined for distinct $z, w \in D$ by

$$G_D(z,w) = G_{D-z}(w-z).$$

In other words, if

$$\sigma_s(w) = \inf\{t : |B_t - w| \le e^{-s}\},$$

then

$$G_D(z,w) = \lim_{s\to\infty} s\,\mathbb{P}^z\{\sigma_s(w) < \tau_D\}.$$

Lemma 1.3.3. *If $\tau = \tau_{\mathbb{D}}$, and $z \in \mathbb{D}$,*

$$\mathbb{E}^z[\tau_D] = \frac{1}{2}\,[1 - |z|^2].$$

Proof. Itô's formula shows that

$$M_t = |B_t|^2 - 2t$$

is a martingale. Since $\mathbb{E}^z[M_\tau] = \mathbb{E}[M_0] = |z|^2$,

$$2\,\mathbb{E}^z[\tau] = \mathbb{E}^z[B^2_{\tau_D}] - |z|^2 = 1 - |z|^2.$$

□

Proposition 1.3.4. *If $D \in \mathcal{D}^*$, and $z, w \in D$ are distinct,*

$$\begin{aligned} G_D(z,w) &= \lim_{s\to\infty} e^{2s}\,\mathbb{E}^z\left[\int_0^{\tau_D} 1\{|B_t - w| \le e^{-s}\}\,dt\right] \\ &= \lim_{s\to\infty} e^{2s}\int_0^\infty \mathbb{P}^z\{|B_t - w| \le e^{-s}, B[0,t] \subset D\}\,dt. \end{aligned}$$

Proof. Let us first consider $D = \mathbb{D}_{-s}$, $w = 0$, and $|z| = 1$. Let $\tau_s = \tau_{\mathbb{D}_{-s}} = \inf\{t : |B_t| = e^s\}$ and

$$Y_s = \int_0^{\tau_s} 1\{|B_t - w| \le e^s\}\, dt, \quad F(s) = \mathbb{E}^z[Y_s].$$

Suppose $0 < r < s$ and note that if $|\zeta| = e^r$,

$$\mathbb{E}^\zeta[Y_s] = \mathbb{P}^\zeta\{\tau_0 < \tau_s\}\, \mathbb{E}^\zeta[Y_s \mid \tau_0 < \tau_s] = \frac{s-r}{s}\, F(s),$$

$$F(s) = \mathbb{E}^z\left[Y_s\right] = \mathbb{E}^z\left[Y_r\right] + \mathbb{E}^z\left[Y_s - Y_r\right] = F(r) + \frac{s-r}{s}\, F(s).$$

Therefore, there exists c_0 such that $F(s) = c_0 s$. Using scaling, we see that if $z \in \mathbb{D} \setminus \mathbb{D}_s$, then

$$e^{2s} \int_0^\infty \mathbb{P}^z\{|B_t| \le e^{-s}; B[0,t] \subset \mathbb{D}\}\, dt = c_0\, [-\log|z|] = c_0\, G_{\mathbb{D}}(z,0). \tag{1.15}$$

To find c_0, note that

$$\begin{aligned} \mathbb{P}^0\{|B_t - z| \le e^{-s}; B[0,t] \subset (1-e^{-s})\mathbb{D}\} &= \mathbb{P}^z\{|B_t| \le e^{-s}; B[0,t] \subset \mathbb{D}\} \\ &\le \mathbb{P}^0\{|B_t - z| \le e^{-s}; B[0,t] \subset (1+e^{-s})\mathbb{D}\} \end{aligned}$$

From this we see (we omit the details) that

$$\lim_{s\to\infty} \int_{e^{-s}\le|z|\le 1} e^{2s} \int_0^\infty \mathbb{P}^z\{|B_t| \le e^{-s}; B[0,t] \subset \mathbb{D}\}\, dt\, dA(z) = \pi\, \mathbb{E}^0[\tau_{\mathbb{D}}] = \pi/2.$$

Also,

$$\int_{\mathbb{D}} G(0,w)\, dA(w) = \int_{\mathbb{D}} \log|w|\, dA(w) = \int_0^{2\pi} \int_0^1 (r\log r) dr\, d\theta = \pi \int_0^1 r\, dr = \frac{\pi}{2}.$$

Therefore $c_0 = 1$.

We now choose $D \in \mathcal{U}$ and allow constants to depend on D. Let

$$q = \sup_{|\zeta|=1} \mathbb{P}^\zeta\{\sigma_s < \tau_D\},$$

and recall that $q \le c/s$. Then if $|\zeta| = 1$, the strong Markov property and (1.15) imply that

$$\mathbb{E}^z\left[\int_0^{\tau_D} 1\{|B_t - w| \le e^{-s}\}\, dt\right] \le \sum_{j=1}^\infty q^j\, s\, e^{-2s} \le c\, e^{-2s}.$$

Hence if $|z| > e^{-s}$,

$$\begin{aligned} \mathbb{E}^z\left[\int_0^{\tau_D} 1\{|B_t - w| \le e^{-s}\}\, dt\right] &= \mathbb{P}^z\{\sigma_{-s} < \tau_D\}\, s\, e^{-2s}\, [1 + O(s^{-1})] \\ &= e^{-2s}\, G_D(z,0)\, [1 + O(s^{-1})]. \end{aligned}$$

This gives the result for $w = 0, U \in \mathcal{U}$, and the more general case follows from scaling and translation.

□

Proposition 1.3.5. *If $D \in \mathcal{D}^*$ and $z, w \in D$,*

$$G_D(z, w) = G_D(w, z)$$

Proof. Let $D^\epsilon = \{\zeta \in D : \text{dist}(\zeta, \partial D) > \epsilon\}$. For a fixed t, let $W_r = w - B_t + B_{t-r}$, and note that $W_r, 0 \le r \le t$ is a Brownian motion starting at w. If $e^{-s} < \epsilon$, then

$$\mathbb{P}^z\{|B_t - w| \le e^{-s}; B[0,t] \subset \tau_D\} \ge \mathbb{P}^w\{|W_t - z| \le e^{-s}; W[0,t] \subset \tau_{D^\epsilon}\}.$$

By Proposition 1.3.4, we have $G_D(z, w) \ge G_{D^\epsilon}(w, z)$ and using (1.13) we get $G_D(z, w) \ge G_D(w, z)$. Similarly, $G_D(w, z) \ge G_D(z, w)$.

□

1.4 Riemann mapping theorem

We will prove one of the most important theorems in conformal mapping, the Riemann mapping theorem. We start with a lemma after which the proof will be rather short.

Lemma 1.4.1. *Suppose D is a domain containing the origin and $g : D \to \mathbb{D}$ is a holomorphic function satisfying $g(0) = 0$ with the following properties.*

1. *There exists $s > 0$ such for all $z \in \mathbb{D}_s$ there exists unique $w \in D$ with $g(w) = z$.*
2. *For each $s > 0$, there is a compact set $K_s \subset D$ such that $g^{-1}(\mathbb{D}_s) \subset K_s$.*

Then g is one-to-one and onto.

Proof. We recall a basic fact about holomorphic functions. If g is holomorphic at the origin, then

- If $g'(0) \neq 0$, then there is a neighborhood $\mathcal{N}$ of 0 for which g is one-to-one and onto the open set $g(\mathcal{N})$.
- If $g'(0) = 0$ and g is not a constant, then g is open and locally many-to-one, that is, there is neighborhood $\mathcal{N}$ of 0 such that $g(\mathcal{N})$ is open and each point in $g(\mathcal{N}) \setminus \{g(0)\}$ has at least two pre-images in $\mathcal{N}$.

Clearly g is not a constant function. We start by showing g is onto. Let r be the infimum of s such that there exists $z \in C_s \setminus g(D)$. The first condition shows that $r < \infty$. Suppose $r > 0$. Let $z \in C_r$. Then we can find a sequence $z_n \in \mathbb{D}_r$ with $z_n \to z$. There exist w_n with $g(w_n) = z_n$. Since $\{w_n\}$ lies in a compact set K_r, we can find a subsequence, which we also denote by w_n, that converges to $w \in K_r$. Continuity of g shows that $g(w) = z$. Hence $C_r \subset g(D)$. Around each $z \in C_r$ we can find open U_z contained in $g(D)$ and hence using compactness arguments, there exists $s < r$ with $D_s \subset g(D)$ which is a contradiction. Hence g is onto.

To show one-to-one, let r be the infimum of s such that there exists $z \in C_s$ with at least two preimages in D. The first condition shows that $r < \infty$. Suppose $r > 0$. Suppose first that there exists $z \in C_r$ and distinct $w_1, w_2 \in D$ with $g(w_1) = g(w_2) = z$. Let $\mathcal{N}_1, \mathcal{N}_2$ be nonintersecting neighborhoods of w_1, w_2 and let $U_j = g(\mathcal{N}_j)$. Then $U_1 \cap U_2$ is an open neighborhood of z such that all points have at least two preimages. This contradicts the value of r. Now suppose z_n is a sequence of points in $\mathbb{D}$ with $|z_n| \to e^{-r}$ and such that each z_n has two distinct preimages w_n and ζ_n.

By taking subsequences if necessary we can assume that $w_n \to w, \zeta_n \to \zeta$, $z_n \to z$ with $w, \zeta \in K_{r-1}$ and $z \in C_r$. Since z has only one preimage, we must have $w = \zeta$. In a small neighborhood about w, f must be locally one-to-one or locally many-to-one. Since points in $\mathbb{D}_r$ have only one preimage, it must be locally one-to-one. But this contradicts the definition of the sequences w_n, ζ_n.

□

Theorem 1.4.2 (Riemann mapping theorem)**.** *Suppose D is a simply connected strict subdomain of $\mathbb{C}$ containing the origin. Then there exists a unique conformal transformation $f : D \to \mathbb{D}$ with $f(0) = 0, f'(0) > 0$.*

Proof. Uniqueness in the case $D = \mathbb{D}$ follows as a consequence of the Schwarz lemma. More generally, if $f : D \to \mathbb{D}, g : D \to \mathbb{D}$ are two such transformation then $h := f \circ g^{-1}$ is a conformal transformation of $\mathbb{D}$ onto itself with $h(0) = 0, h'(0) > 0$, and hence h is the identity. The work is to show existence.

We will construct f using the Green's function $G_D(z) = G_D(z, 0)$. Recall that we can write

$$G_D(z) = -\log|z| + u(z),$$

where $u(z) = u_D(z)$ is a harmonic function in D, Since D is simply connected, there is a unique holomorphic function $h : D \to \mathbb{D}$ such that $\mathrm{Re}h = -u$ and $\mathrm{Im}h(0) = 0$. Let

$$f(z) = z\, e^{h(z)}.$$

Then f is holomorphic on D with $f(0) = 0, f'(0) = e^{h(0)} = e^{-u(0)} > 0$. Also $|f(z)| = e^{-G_D(z)}$. This will be the map f.

Since the Green's function goes to zero at the boundary we see that for all $r > 0$,

$$K_s := \{z \in D : G_D(z, 0) \geq s\}$$

is a compact set and $f^{-1}(\mathbb{D}_s) \subset K_s$. Also, since $f'(0) > 0$, there exist a neighborhood $\mathcal{N}$ of 0 such that f restricted to $\mathcal{N}$ is one-to-one and onto. If we choose s sufficiently large so that $G_D(z, 0) \leq s$ on $D \setminus \mathcal{N}$, we see that each $z \in \mathbb{D}_s$ has a unique preimage in D. Hence f satisfies the conditions of Lemma 1.4.1 and is one-to-one and onto.

□

The basic idea of this proof will be used for proving conformal equivalence of other domains. The idea is to assume that a transformation exists and try to construct the function. After a candidate is found, we then try to see if the candidate works.

The lemma can be considered a special case of what is known as the "argument principle" which is related to Rouché's theorem. Note that the lemma did not assume that D was simply connected; this comes as a consequence.

1.5 Analytic boundary points and arcs

While boundaries of domains can be very rough, there are times that we would like to restrict to nice "smooth" boundaries. It will suffice for our purposes to consider very smooth boundaries given by analytic arcs. For these we can do calculations in the upper half plane near zero with (an interval of) the real line as the boundary and make use of Schwarz reflection (see Proposition 1.2.7). If a conformal transformation is analytic near zero, then these calculations apply to the image as well.

Definition Let $\mathcal{K}_0$ denote the set of domains $D \subset \mathbb{H}$ such that $\operatorname{dist}(0, \mathbb{H} \setminus D) > 0$. Let $\mathcal{K}$ denote the set of domains $D \in \mathcal{K}_0$ with $\operatorname{dist}(0, \mathbb{H} \setminus D) > 1$.

Definition Suppose D is a domain.

- A point $z \in \partial D$ is called an *analytic (boundary) point* of D (or of ∂D) if there exists $D' \in \mathcal{K}$ and a conformal transformation $f : D' \to D$ with $f(0) = z$ that has an extension as an analytic function on $D' \cup \mathbb{D}$.
- A simple curve $\eta : (a, b) \to \partial D$ is called an *analytic arc* of D (or of ∂D) if $\eta(t)$ is an analytic point for each $a < t < b$.

In the definition of $\mathcal{K}_0$ and $\mathcal{K}$ it is not assumed that D is simply connected. The extension of f in the definition of analytic point must be an analytic function but it is not required to be one-to-one on $D' \cup \mathbb{D}$.

In the upper half plane we have (see Section 1.11.3)

$$H_{\mathbb{H}}(z, 0) = -\operatorname{Im}\left[\frac{1}{z}\right] = \frac{\operatorname{Im}(z)}{|z|^2}, \quad G_{\mathbb{H}}(z, w) = \log|z - \overline{w}| - \log|z - w|.$$

We can use the right-hand side to extend $w \mapsto G(z, w)$ to the disk of radius $|z|$ about the origin (or we could use Proposition 1.2.7), and direct calculation shows that

$$H_{\mathbb{H}}(z, 0) = \frac{1}{2}\, \partial_y\, G_{\mathbb{H}}(z, 0).$$

If $x \neq 0$ is real, we also have the boundary Poisson kernel

$$H_{\mathbb{H}}(0, x) = \partial_y H_{\mathbb{H}}(0, x),$$

where the derivative on the right can be taken with respect to either component. More generally, if $D \in \mathcal{K}$,

$$H_D(z, 0) = H_{\mathbb{H}}(z, 0) - \mathbb{E}^z[H_{\mathbb{H}}(B_\tau, 0)], \quad G_D(z, w) = G_{\mathbb{H}}(z, w) - \mathbb{E}^z[G_{\mathbb{H}}(B_\tau, w)],$$

where $\tau = \tau_D$. By integrating under the expectation we see that

$$H_D(z, 0) = \frac{1}{2}\, \partial_y G_D(z, 0). \tag{1.16}$$

Also if $\operatorname{dist}(x, \mathbb{H} \setminus D) > 0$, $H_D(x,0) = \partial_y H_D(x,0)$.

We really want to generalize this to consider "analytic prime ends". As an example, suppose that $D = \mathbb{C} \setminus [-1,1]$. Then D is a simply connected domain of the Riemann sphere $\hat{\mathbb{C}}$, and we can find a conformal transformation $f : \mathbb{H} \to D$ that sends 0 to what we will call 0^+, the "positive-y" side of 0 in D. (This is including ∞ in D; if we do not want to include ∞ we can consider f restricted to $\mathbb{H} \setminus f^{-1}(\infty)$. By scaling if necessary, we can assume that $|f^{-1}(\infty)| > 1$ and hence $\mathbb{H} \setminus f^{-1}(\infty) \in \mathcal{K}$.) This map can be extended to an analytic map in a neighborhood of 0 (this extension is not one-to-one on D). Hence 0^+ is an analytic boundary point (prime end). Note that 0^- is also an analytic boundary point, but it is considered as a different point. One can check that a point $z \in \partial D$ can correspond to at most two analytic prime ends and for convenience we will just use the term analytic point.

If z is an analytic boundary point, then there is a well-defined inward unit normal derivative $\mathbf{n} = \mathbf{n}(z, D)$ pointing into D. (If z is a "two-sided" point, then each prime end has a normal derivative. Hence we consider $\mathbf{n}(z, D)$ as a function of the prime end z.) If $f : D' \to D$ is a map as above, then we write

$$f(iy) = z + y\,|f'(0)|\,\mathbf{n} + O(|y|^2), \quad y \downarrow 0.$$

If ϕ is a harmonic function on D with boundary value 0 in a neighborhood of z, then we define $\tilde{\phi}$ on $D' = f^{-1}(D)$ by $\tilde{\phi}(w) = \phi(f^{-1}(w))$. Note that $\tilde{\phi}$ is harmonic with boundary value 0 in an interval $[-\delta, \delta]$ and hence

$$\partial_y \tilde{\phi}(0) = \lim_{y \downarrow 0} y^{-1}\, \tilde{\phi}(iy),$$

is well defined. We define

$$\partial_n \phi(z) = \lim_{y \downarrow 0} y^{-1}\, \phi(z + y\mathbf{n}) = \lim_{y \downarrow 0} y^{-1}\, \tilde{\phi}(|f'(0)|^{-1}\, yi + O(y^2)) = |f'(0)|^{-1}\, \partial_y \tilde{\phi}(z).$$

An immediate consequence of this and (1.16) is the following.

Proposition 1.5.1. *If w is an analytic boundary point of D and $z \in D$, then the Poisson kernel $H_D(z,w)$ exists and*

$$H_D(z,w) = \frac{1}{2}\, \partial_{\mathbf{n}_w} G_D(z,w),$$

where $\mathbf{n}_w$ denotes the inward unit normal. If w' is another analytic boundary point, then the boundary Poisson kernel $H_D(w,w')$ exists and

$$H_D(w,w') = \partial_{\mathbf{n}_w} H_D(w,w') = \partial_{\mathbf{n}_{w'}} H_D(w,w').$$

Suppose $f : D \to D'$ is a conformal transformation, $z \in D$ and w, w' are distinct analytic boundary points of D such that $f(w)$ and $f(w')$ are analytic boundary points of D'. Then

$$H_D(z,w) = |f'(w)|\, H_{f(D)}(f(z), f(w)), \quad H_D(w,w') = |f'(w)|\, |f'(w')|\, H_{f(D)}(f(w), f(w')).$$

As another check to see that the constant is correct, recall that we have normalized our quantities so that

$$H_{\mathbb{D}}(0,1) = \frac{1}{2}, \quad G_{\mathbb{D}}(0,x) = -\log x,$$

and hence $\partial_{\mathbf{n}} G_{\mathbb{D}}(0,1) = 1$.

Proposition 1.5.2. *Suppose $D \in \mathcal{K}$. If $z, w \in D$ with $|z|, |w| \le 1/2$,*

$$H_D(z,0) = H_{\mathbb{H}}(z,0)\ [1 + O(|z|)]\,.$$

$$G_D(z,w) = G_{\mathbb{H}}(z,w)\ [1 + O(|z|)]\,.$$

Proof. Note that

$$H_{\mathbb{D}_+}(z,0) \le H_D(z,0) \le H_{\mathbb{H}}(z,0), \quad G_{\mathbb{D}_+}(z,w) \le G_D(z,w) \le G_{\mathbb{H}}(z,w).$$

Using conformal transformation (see Section 1.11.4), we can show the estimates for $D = Disk_+$.

$$H_{\mathbb{D}_+}(z,0) = H_{\mathbb{H}}(z,0)\ [1 + O(|z|)]\,, \quad G_{\mathbb{D}_+}(z,w) = G_{\mathbb{H}}(z,w)\ [1 + O(|z|)]\,.$$

□

Proposition 1.5.3. *Suppose $D \in \mathcal{K}$ and $h : D \to \mathbb{R}$ is harmonic with $h \equiv 0$ on $[-x, x]$ for some $x > 1$. Then*

$$\partial_y h(0) = \frac{2}{\pi} \int_0^\pi h(e^{i\theta}) \sin\theta \, d\theta.$$

Proof. By Schwarz reflection, we can extend h to a harmonic function on $D \cup \{z : |z| < x\}$ and from this we see that h is bounded and continuous on $\{z : |z| \le 1\}$. The optional sampling theorem implies that if $z \in \mathbb{D}_+$, then

$$h(z) = \mathbb{E}^z \left[h(B_{\tau_{\mathbb{D}_+}}) \right] = \int_{\partial \mathbb{D}_+} h(w) \, \mathrm{hm}_{\mathbb{D}_+}(z, dw) = \frac{1}{\pi} \int_0^\pi h(e^{i\theta}) \, H_{\partial \mathbb{D}_+}(z, e^{i\theta}) \, d\theta.$$

Using conformal invariance (see Section 1.11.4) we can see that

$$H_{\partial \mathbb{D}_+}(iy, e^{i\theta}) = 2y \sin\theta \ [1 + O(y)]\,, \quad y \downarrow 0.$$

□

Proposition 1.5.4. *Suppose $D \in \mathcal{K}$. If $0 < \epsilon \le 1/2$, let $D_\epsilon = D \cap \{|z| > \epsilon\}$ and $\tau_\epsilon = \tau_{D_\epsilon}$. Then for $z \in D$ with $|z| \ge 1$,*

$$H_{D_\epsilon}(z, \epsilon e^{i\theta}) = \frac{\pi}{2} \, \mathbb{P}^z\{|B_{\tau_\epsilon}| = \epsilon\} \, \epsilon^{-1} \sin\theta \ [1 + O(\epsilon)]\,.$$

In other words, if $\psi(\theta; z, \epsilon, D)$ is the density of $\arg(B_{\tau_\epsilon})$ given $|B_{\tau_\epsilon}| = \epsilon$. Then,

$$\psi(\theta; z, \epsilon, D) = \frac{\sin\theta}{2} [1 + O(\epsilon)].$$

In particular, if ϕ is a nonnegative function defined on ∂D_ϵ that vanishes on ∂D, and $|z| \ge 1$,

$$\mathbb{E}^z \left[\phi(B_{\tau_\epsilon}) \right] = [1 + O(\epsilon)] \ \mathbb{P}^z\{|B_{\tau_\epsilon}| = \epsilon\} \int_0^\pi \phi(\epsilon e^{i\theta}) \, \frac{\sin\theta}{2} \, d\theta.$$

Proof. We fix $0 < \theta_1 < \theta_2 < \pi$, let $V_\epsilon = \{\epsilon e^{i\theta} : \theta_1 \le \theta \le \theta_2\}$, and let $p = (\cos\theta_1 - \cos\theta_2)/2$. Let $U_\epsilon = \{w \in \mathbb{H} : \epsilon < |w| < 1\}$ and let $\eta_\epsilon = \tau_{U_\epsilon}$. Using conformal invariance (see Section 1.11.2), we can see that if $\zeta \in U_\epsilon$ with $|\zeta| = 3/4$, then

$$\mathbb{P}^\zeta \{B_{\eta_\epsilon} \in V_\epsilon \mid |B_{\eta_\epsilon}| = \epsilon\} = p\,[1 + O(\epsilon)].$$

But $\mathbb{P}\{B_{\tau_\epsilon} \in V_\epsilon \mid |B_{\tau_\epsilon}|$ obviously lies between the infimum and the supremum of this quantity over $|\zeta| = 3/4$.

□

Proposition 1.5.5. *If $D \in \mathcal{K}$ and $|z| > 1$, then*

$$H_D(z, 0) = \frac{1}{\pi} \int_0^\pi G_D(e^{i\theta}, z)\, \sin\theta\, d\theta.$$

Proof. By the strong Markov property, we can see that

$$G_D(z, iy) = \frac{1}{\pi} \int_0^\pi G_D(e^{i\theta}, z)\, H_{\mathbb{D}_+}(iy, e^{i\theta})\, d\theta.$$

Letting $y \downarrow 0$, we get (see Section 1.11.4)

$$H_D(z, 0) = \frac{1}{2\pi} \int_0^\pi G_D(e^{i\theta}, z)\, H_{\mathbb{D}_+}(0, e^{i\theta})\, d\theta = \frac{1}{\pi} \int_0^\pi G_D(e^{i\theta}, z)\, \sin\theta\, d\theta.$$

□

1.5.1 Excursion measure

If D is a domain and z is an analytic boundary point, we define the *(point-to-set) excursion measure* $\mathcal{E}_D(z, \cdot)$ to be the derivative of the harmonic measure,

$$\mathcal{E}_D(z, V) = \partial_n \mathrm{hm}_D(z, V).$$

If D is an open set, not necessarily connected, we define $\mathcal{E}_D(z, V)$ to be $\mathcal{E}_{D'}(z, V)$ where D' is the connected component containing z on the boundary. The measure $\mathcal{E}_D(z, \cdot)$ is an infinite measure on ∂D, but if $\mathrm{dist}(z, V) > 0$, then $\mathcal{E}_D(z, V) < \infty$. If V is an analytic arc, we can write

$$\partial_n \mathrm{hm}_D(z, V) = \frac{1}{\pi} \int_V \partial_n H_D(z, w)\, |dw| = \frac{1}{\pi} \int_V H_{\partial D}(z, w)\, |dw|.$$

Note that if $f : D \to f(D)$ is a conformal transformation that is analytic in a neighborhood of z,

$$\mathcal{E}_D(z, V) = |f'(z)|\, \mathcal{E}_{f(D)}(f(z), f(V)).$$

If V_1, V_2 are two analytic arcs, we define the *(set-to-set) excursion measure*

$$\mathcal{E}_D(V_1, V_2) = \int_{V_1} \mathcal{E}_D(z, V_1)\, |dz| = \int_{V_1} \int_{V_2} H_{\partial D}(z, w)\, |dw|\, |dz|.$$

An important fact is that the set-to-set excursion measure is a conformal invariant.

Proposition 1.5.6. *If $f : D \to f(D)$ is a conformal transformation that is analytic on the arcs $V_1, V_2 \subset \partial D$, then*

$$\mathcal{E}_D(V_1, V_2) = \mathcal{E}_{f(D)}(f(V_1), f(V_2)).$$

The term (point-to-set) excursion measure is used both for a measure on the boundary $\mathcal{E}_D(z, \cdot)$ and also for the measure on paths starting at z, ending at ∂D, otherwise in D, corresponding to "Brownian motion conditioned to begin and end at the boundary". In this case $\mathcal{E}_D(z, V)$ is the total mass of curves that end at V. We can define a (set-to-set) excursion measure similarly.

We can also define point-to-point excursion measure which has total mass $H_{\partial D}(z, w)/\pi$.

The excursion measure viewed as a measure on paths is also conformally invariant provided that we change time as in the conformal invariance of Brownian motion.

Since the set-to-set excursion measure is a conformal invariant it is well defined even if the boundary is not analytic. For example, if V_1, V_2 are on the same connected component of the boundary, we can first map D to $\mathbb{H}$ mapping this component to the real line. If they are in different components K_1, K_2, then we can first map $\hat{\mathbb{C}} \setminus (K_1 \cup K_2)$ to an annulus.

Proposition 1.5.7. *Suppose D is a domain and z is a locally analytic point. Suppose $D' \subset D$ and D, D' agree in a neighborhood of z. Let $w \in D \setminus \overline{D}'$. Then*

$$H_D(w, z) = \frac{1}{2} \int_{\partial D'} G_D(\zeta, w)\, \mathcal{E}_{D'}(z, d\zeta).$$

If $\partial D' \cap D$ is analytic, we can write

$$H_D(w, z) = \frac{1}{2\pi} \int_{\partial D'} G_D(\zeta, w)\, H_{\partial D'}(z, \zeta)\, |d\zeta|.$$

As an example (and a check on the constants), suppose that $D = \mathbb{D}, D' = A_r = \{e^{-r} < |z| < 1\}$, $z = 1, w = 0$. Then $H_{\mathbb{D}}(0, 1) = 1/2$, and

$$\mathcal{E}_{A_r}(1, C_r) = \frac{1}{r}, \qquad G_{\mathbb{D}}(e^{-r+i\theta}, 0) = r, \qquad H_{A_r}(1, e^{-r+i\theta}) = \frac{e^r}{2r}\,[1 + o(1)].$$

Proof. We first consider the case with $z = 0$, $D, D' \in \mathcal{K}$. The function $h(\zeta) := G(\zeta, w)$ is a bounded harmonic function on D'. Therefore, for $y < 1$,

$$h(iy) = \mathbb{E}^{iy}\left[h(B_{\tau_{D'}})\right] = \int_{\partial D'} G_D(\zeta, w)\, \mathrm{hm}_{D'}(iy, d\zeta).$$

Letting $y \to 0$, we get

$$2H_D(0, w) = \partial_n G(0, w) = \partial_n \mathbb{E}^{iy}\left[h(B_{\tau_{D'}})\right] = \int_{\partial D'} G_D(\zeta, w)\, \mathcal{E}_{D'}(0, d\zeta).$$

For the more general case, since z is an analytic point we can find $\tilde{D} \in \mathcal{K}$ and conformal transformation $f : \tilde{D} \to D$ with $f(0) = z$. By scaling earlier, we can find such $\tilde{D}, F$ such that $\tilde{D}' := f^{-1}(D') \in \mathcal{K}$. We then use

$$H_D(w,z) = |f'(0)|\, H_{\tilde{D}}(f^{-1}(w),0), \quad \mathcal{E}_D(z,V) = |f'(0)|\, \mathcal{E}_{\tilde{D}}(0, f^{-1}(V)),$$
$$G_D(\zeta, w) = G_{\tilde{D}}(f^{-1}(\zeta), f^{-1}(w)).$$

□

Examples

- If $\mathbb{A}_r = \{e^{-r} < |z| < 1\}$ is the annulus with boundaries C, C_r, then $\mathcal{E}_{\mathbb{A}_r}(e^{i\theta}, C_r) = \frac{1}{r}$, and hence
$$\mathcal{E}_{\mathbb{A}_r}(C, C_r) = \int_0^{2\pi} \mathcal{E}_{\mathbb{A}_r}(e^{i\theta}, C_r)\, d\theta = \frac{2\pi}{r}.$$
In particular, we see that if $r \neq s$, then $\mathcal{E}_{\mathbb{A}_r}(C, C_r) \neq \mathcal{E}_{\mathbb{A}_s}(C, C_s)$. From this we can see that $\mathbb{A}_r$ and $\mathbb{A}_s$ are not conformally equivalent.

- Let $\mathcal{R}_L = \{x + iy : 0 < x < L, 0 < y < \pi\}$ and let $\partial_1 = [0, i\pi], \partial_2 = \partial_{2,L} = [L, L + i\pi]$ be the vertical boundaries. Let h be the harmonic function on $\mathcal{R}_L$ with boundary value 1 on ∂_2 and 0 on $\partial\mathcal{R}_L \setminus \partial_2$. This can be found by separation of variables,
$$h(x+iy) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sinh(nx)\, \sin(ny)}{n \sinh(nL)},$$
$$\partial_x h(iy) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(ny)}{\sinh(nL)} = \frac{8 \sin y}{\pi}\, e^{-L}\, \left[1 + O(e^{-L})\right], \ \ L \to \infty.$$
$$\mathcal{E}_{\mathcal{R}_L}(\partial_1, \partial_2) = \int_0^{\pi} \partial_x h(iy)\, dy = \frac{16}{\pi}\, e^{-L}\, \left[1 + O(e^{-L})\right], \ \ L \to \infty.$$
Although it is not immediately obvious from the last expression, one can use the definition to see that the function $L \mapsto \mathcal{E}_{\mathcal{R}_L}(\partial_1, \partial_2)$ is strictly decreasing.

- Let $D = D(a,b) = \{a < \mathrm{Im}(z) < b\}$ with boundaries $I_a = \{\mathrm{Im}(z) = a\}, I_b = \{\mathrm{Im}(z) = b\}$. Then, the gambler's ruin estimate implies that if $x + ia \in I_a$, then $\mathcal{E}_D(x + ia, I_b) = 1/(b-a)$, and hence if $V \subset I_a$,
$$\mathcal{E}_D(V, I_b) = \mathcal{E}_D(I_b, V) = \frac{1}{b-a}\, \ell(V),$$
where ℓ denotes Lebesgue measure. More generally, suppose that $D = \mathbb{H} \setminus K$ is a domain with $K \subset \{\mathrm{Im}(z) < a\}$. The strong Markov property, implies that if $b > a$, then
$$\mathcal{E}_D(I_b, V) = \frac{1}{b-a} \int_{-\infty}^{\infty} \mathrm{hm}_{D \cap \{\mathrm{Im}(z) < b\}}(x + ia, V)\, dx,$$
and hence,
$$\lim_{b \to \infty} b\, \mathcal{E}_D(I_b, V) = \int_{-\infty}^{\infty} \mathrm{hm}_D(x + ia, V)\, dx \tag{1.17}$$

Definition

- We say two domains D_1, D_2 are *conformally equivalent* if there exists a conformal transformation $f : D_1 \to D_2$.
- A domain D is a *conformal annulus* if $\hat{\mathbb{C}} \setminus D$ consists of two connected components each larger than a single point.

The Riemann mapping theorem states that any two simply connected domains other than the entire complex plane are conformally equivalent. We have used excursion measure to see that if $r \neq s$, then the annuli $\mathbb{A}_r$ and $\mathbb{A}_s$ are not conformally equivalent. The next proposition will show that every conformal annulus is conformally equivalent to $\mathbb{A}_s$ for some (necessarily unique) s.

Proposition 1.5.8. *If D is a conformal annulus with boundary components ∂_1, ∂_2, then D is conformally equivalent to $\mathbb{A}_r$ where $r = 2\pi/\mathcal{E}_D(\partial_1, \partial_2)$.*

Proof. Let V_1, V_2 denote the connected components of $\hat{\mathbb{C}} \setminus D$. Let $D' = \hat{\mathbb{C}} \setminus V_1$. By the Riemann mapping theorem, we can conformally transform D' onto the unit disk. For this reason, without loss of generality, we will assume that $D \subset \mathbb{D}$ and $\partial_1 = C$. Let

$$q(z) = q_D(z) = \mathbb{P}^z\{B_{\tau_D} \in \partial_2\}.$$

We let $\mathbf{n}$ denote inward normals in this proof.

Let $r > 0$, and suppose that $f : D \to A_r$ is a conformal transformation with $f(C) = C$. By conformal invariance,

$$\mathcal{E}_D(C, \partial_2) = \mathcal{E}_{A_r}(C, C_r) = \frac{2\pi}{r}.$$

Hence, $r = 2\pi/\mathcal{E}_D(C, \partial_2)$.

Note that $u(z) := rq_D(z)$ is a harmonic function on D with

$$\int_C \partial_{\mathbf{n}} u(z)\, |dz| = r\, \mathcal{E}_D(C, \partial_2) = 2\pi. \tag{1.18}$$

Suppose γ is a simple curve in D separating ∂_2 from C. Using the fact that u is harmonic, we see that (1.18) implies that

$$\int_\gamma \partial_{\mathbf{n}} u(z)\, |dz| = 2\pi. \tag{1.19}$$

We can find a harmonic function $h(z) = u(z) + iv(z)$ locally around each z, and let $f(z) = \exp\{-h(z)\}$. Using (1.19), we can see that f is well defined globally. This gives a map $f : D \to \mathbb{A}_r$. We need to show that f is one-to-one and onto.

As in the simply connected case, we can see that for each $0 < q < 1$, the sets $V_q = \{z : u(z) = q\}, \{z : u(z) > q\}, \{z : u(z) < q\}$ are connected, and using this we get that $f'(z) \neq 0$ for every z. To show global injectivity, consider the point smallest q for which f is not one-to-one on V_q.

□

We say two domains D_1, D_2 are *conformally equivalent* if there exists a conformal transformation $f : D_1 \to D_2$. Let us call D a *conformal annulus* if D is connected and ∂D consists of two connected components each larger than a single point. Suppose D is a domain and V is a connected component

of $\hat{\mathbb{C}} \setminus \mathbb{C}$ containing more than one point. Then, $\hat{\mathbb{C}} \setminus V$ is a simply connected subset of the Riemann sphere $\hat{\mathbb{C}}$ and hence can be mapped conformally onto the disk or conformally onto $\mathbb{H}$. For this reason, when we consider multiply connected domains it will suffice to consider subdomains of $\mathbb{D}$ (or $\mathbb{H}$) for which C (or $\mathbb{R}$) are contained in the boundary. Similarly, if V_1, V_2 are two connected components of $\hat{\mathbb{C}} \setminus \mathbb{C}$ containing more than one point, we can start by mapping $\hat{\mathbb{C}} \setminus (V_1 \cup V_2)$ to an annulus.

1.5.2 Poisson kernel

If D is a regular domain and z is an analytic boundary point, then the Poisson kernel is defined, up to a multiplicative constant, as a positive harmonic function f whose boundary value is zero everywhere except for z (here we are interpreting z in terms of a prime end. Suppose $D' \in \mathcal{K}$ and $f : D' \to D$ is a conformal transformation. If we define

$$h(w) = H_{\partial D'}(f^{-1}(w))$$

then h satisfies these conditions on D. Hence, we do not need a nice boundary point to have such a function.

If z, w are both analytic boundary points, then we define the boundary Poisson kernel $H_{\partial D}(z, w)$ by

$$H_{\partial D}(z, w) = \partial_{n_z} H_D(z, w) = 2\, \partial_{n_z}\, \partial_{z_w}\, G_D(z, w),$$

where we write n_z, z_w for the derivative at the inward normal at z, w, respectively. The second expression shows that $H_{\partial D}(z, w) = H_{\partial D}(w, z)$

1.6 Extremal length and reflecting Brownian motion

Suppose D is a domain and ∂_1, ∂_2 are disjoints subsets of ∂D. One conformally invariant way to measure the "distance" between ∂_1 and ∂_2 is in terms of Brownian excursion measure $\mathcal{E}_D(\partial_1, \partial_2)$. Roughly speaking, this gives the measure of the set of Brownian motions starting at ∂_1 that exit D at ∂_2. (Strictly speaking, this measure is zero, so the actual definition is in terms of the boundary Poisson kernel.)

There is a different conformally quantity, which we will denote by $\mathcal{E}^*_D(\partial_1, \partial_2)$, that is more commonly used in the complex variable literature. Its reciprocal is called *extremal length* or *extremal distance*. It is defined in the same way as $\mathcal{E}_D(\partial_1, \partial_2)$ except that the Brownian motions are not killed when they hit $\partial D \setminus (\partial_1 \cup \partial_2)$ rather, they are reflected orthogonally.

Defining reflected Brownian motion can be tricky for rough domains, but because it is a conformally invariant quantity we can restrict ourselves to a rather simple set of domains that will suffice for our purposes. Let $\mathcal{D}_{\rm ref}$ denote the set of domains $(D, \partial_1, \partial_2)$ such that: $D \subset \mathbb{H}$; ∂_1, ∂_2 are disjoint closed subsets of ∂D with a finite number of connected components each larger than a single point; and $I_D := \partial D \setminus (\partial_1 \cup \partial_2)$ consists of a finite or countable number of disjoint subsets of $\mathbb{R}$. If $(D, \partial_1, \partial_2) \in \mathcal{D}_{\rm ref}$, we let D^* be the reflected domain

$$D^* = D \cup I_D \cup \{\overline{z} : z \in D\}.$$

and $\partial_1^*, \partial_2^*$ be the corresponding closed subsets of ∂D^*. We let $\tilde{\mathcal{D}}_{\rm ref}$ be the set of conformal images of $(D, \partial_1, \partial_2) \in \mathcal{D}_{\rm ref}$.

Under these assumptions, we construct W_t, Brownian motion started at $z \in D \cup I_D$, reflected orthogonally on I_D, stopped when it reaches $\partial_1 \cup \partial_2$ by:

- Let $B_t = X_t + iY_t$ be a standard Brownian motion stopped at time τ, the first time it leaves D^*.
- Let $W_t = X_t + i|Y_t|$, $0 \le t \le \tau$.
- The reflected excursion measure is given by

$$\mathcal{E}^*_D(\partial_1, \partial_2) = \mathcal{E}_{D^*}(\partial_1, \partial_2^*) = \frac{1}{2}\,\mathcal{E}_{D^*}(\partial_1^*, \partial_2^*).$$

We extend $\mathcal{E}^*_D(\partial_1, \partial_2)$ to $D \in \tilde{\mathcal{D}}_{\rm ref}$ by conformal invariance. Let $f(z)$ denote the probability that the reflected Brownian motion starting at z in D reaches ∂_2 before reaching ∂_1. This is the same as the probability that the usual Brownian motion starting at z in D^* reaches ∂_2^* before reaching ∂_1^*. Then f is harmonic in D, $f \equiv 1_{\partial_2}$ on $\partial_1 \cup \partial_2$, and it satisfies the reflecting boundary condition

$$\partial_n f(x) = 0, \quad x \in I_D.$$

Here ∂_n denotes partial with respect to the interior normal derivative which for $x \in I_D$ is just the partial with respect to y. If ∂_1 is sufficiently smooth, we have

$$\mathcal{E}^*_D(\partial_1, \partial_2) = \int_{\partial_1} \partial_n f(z)\,|dz|.$$

Examples.

- Let $D = \mathcal{R}_L = (0, L) \times i(0, \pi)$ be the $L \times \pi$ rectangle and let $\partial_1 = i[0, \pi], \partial_2 = L + i[0, \pi]$ be the vertical edges. Then $(D, \partial_1, \partial_2) \in \tilde{\mathcal{D}}_{\rm ref}$. In this case

$$f(x + iy) = x/L, \quad \partial_n f(iy) = 1/L, \quad \mathcal{E}^*_D(\partial_1, \partial_2) = \frac{\pi}{L}.$$

  This is easy because the reflection only affects the imaginary part, and so the calculation boils down to the gambler's ruin estimate. Note that $\mathcal{E}_D(\partial_1, \partial_2) \asymp e^{-L}$ and hence decays much faster.

- Let D be the annulus $\mathbb{A}_s = \{e^{-s} < |z| < 1\}$ and let $\partial_1 = C_0, \partial_2 = C_s$ be the two boundary circles. Then $(D, \partial_1, \partial_2) \in \tilde{\mathcal{D}}_{\rm ref}$ (use a map that sends $\{|z| > e^{-s}\}$ on the Riemann sphere to $\mathbb{H}$). In this case there is no reflection, and hence

$$\mathcal{E}^*_D(\partial_1, \partial_2) = \mathcal{E}_D(\partial_1, \partial_2) = \frac{2\pi}{s}.$$

  Note that $f(z) = -s^{-1} \log |z|$.

- Let D be the half-annulus $\mathbb{A}_s^+ = \{z \in \mathbb{H} : e^{-s} < |z| < 1\}$ and let ∂_1, ∂_2 be as in the last example. Then $(D, \partial_1, \partial_2) \in \mathcal{D}_{\rm ref}$ with $I_D = (-1, -e^{-s}) \cup (e^{-s}, 1)$. Since $D \subset \mathbb{H}$ with reflection on the real axis, this is a domain of the type in the definition. Note that $D^* = \mathbb{A}_s, \partial_1^* = C_0, \partial_2^* = C_s$. This domain is also conformally equivalent to $\mathcal{R}_s$ with the circles being sent to the vertical edges of $\mathcal{R}_s$. Therefore,

$$\mathcal{E}^*_D(\partial_1, \partial_2) = \frac{\pi}{s}, \quad \mathcal{E}_D(\partial_1, \partial_2) \asymp e^{-s}.$$

Another common name for various analogs and generalizations of excursion measure and its reciprocal are *conductance* and *resistance*.

We now describe the more classical way of defining $\mathcal{E}_D^*(\partial_1, \partial_2)$. Let $\mathcal{K} = \mathcal{K}_D(\partial_1, \partial_2)$ denote the set of piecewise C^1 curves γ from ∂_1 to ∂_2 otherwise lying in D. We say that a positive function $\rho : D \to [0, \infty)$ is *admissible* (with respect to $\mathcal{K}$), if for every $\gamma \in \mathcal{K}$,

$$\int_\gamma \rho(z)\,|dz| \geq 1. \tag{1.20}$$

If $\mathcal{K}$ is a set of piecewise C^1 curves in a domain D (with endpoints perhaps on ∂D), we define the *modulus* of $\mathcal{K}$ by

$$\Lambda(\mathcal{K}) = \inf \int_D \rho(z)^2\,dA(z),$$

where the infimum is over all admissible functions ρ. The reciprocal of the modulus is called the *extremal length* or *extremal distance*. As an example, suppose that $\mathcal{K} = \mathcal{K}(D, \partial_1, \partial_2)$ where $D = \mathcal{R}_L$ and ∂_1, ∂_2 are the vertical edges. Then for any admissible ρ and $0 < y < \pi$,

$$\frac{1}{L}\int_0^L \rho(x+iy)^2\,dx \geq \left[\frac{1}{L}\int_0^L \rho(x+iy)\,dx\right]^2 = \frac{1}{L^2},$$

and hence,

$$\int \rho(z)^2\,dA(z) = \int_0^\pi \int_0^L \rho(x+iy)^2\,dx\,dy \geq \frac{\pi}{L}.$$

Since the function $\rho(x+iy) = 1/L$ is acceptable we can see that $\Lambda(\mathcal{K}) = \pi/L$ and the constant function $\rho(z) = 1/L$ is the minimizer.

Proposition 1.6.1. *The modulus is a conformal invariant. That is, if $\mathcal{K}$ is a collection of curves in D and $g : D \to f(D)$ is a conformal transformation, then $\Lambda(\mathcal{K}) = \Lambda(g \circ \mathcal{K})$ where $g \circ \mathcal{K} = \{g \circ \gamma : \gamma \in \mathcal{K}\}$.*

Proof. Suppose that ρ is admissible for D. Define ρ_g on $g(D)$ by $\rho_g(g(z)) = |g'(z)|^{-1}\,\rho(z)$. If $\gamma \in \mathcal{K}_D(\partial_1, \partial_2)$, let $g \circ \gamma \in \mathcal{K}_{f(D)}(f(\partial_1), f(\partial_2)))$ be the corresponding curve. (The parametrization of the curve is not important.) Then

$$\int_{g\circ\gamma} \rho_g(z)\,|dz| = \int_\gamma \rho_g(g(w))\,|g'(w)|\,|dw| = \int_\gamma \rho(w)\,|dw| \geq 1.$$

Also,

$$\int_{g(D)} \rho_g(z)^2\,dA(z) = \int_D [\rho(w)\,|g'(w)|^{-1}]^2\,|g'(w)|^2\,dA(w) = \int_D \rho(w)^2\,dA(w).$$

Taking infimums, we see that $\Lambda(g \circ \mathcal{K}) \leq \Lambda(\mathcal{K})$. Applying the same argument to g^{-1} gives the other direction.

□

Proposition 1.6.2. *If D is simply connected with $(D, \partial_1, \partial_2) \in \tilde{\mathcal{D}}_{\text{ref}}$ and ∂_1 is connected, then the minimizing ρ for $\mathcal{K}_D(\partial_1, \partial_2)$ is given by $|\nabla f(z)|$ where f is the unique harmonic function on D with boundary conditions $f \equiv 1_{\partial_2}$ on $\partial_1 \cup \partial_2$ and $\partial_n f \equiv 0$ on I_D. In particular,*

$$\Lambda[\mathcal{K}_D(\partial_1, \partial_2)] = \int_D |\nabla f(z)|^2 \, dA(z).$$

Proof. We will assume that $D = \mathcal{R}_L, \partial_1 = i[0, \pi]$ and

$$\partial_2 = A_1 \cup A_2 \cup \cdots \cup A_{k+1},$$

where A_j are disjoint nontrivial closed subintervals of the right vertical boundary $L + i[0, \pi]$, with the intervals ordered counterclockwise (imaginary parts increasing), and $L \in A_1, L + i\pi \in A_{k+1}$. We let $l_1, \ldots, l_k$ be the open intervals in between, so that

$$I_D = l_1 \cup \cdots \cup l_k \cup (0, L) \cup [(0, L) + i\pi].$$

Every $(D, \partial_1, \partial_2) \in \tilde{\mathcal{D}}_{\text{ref}}$ is conformally equivalent to such a domain, and the representation is unique. We let $f(z)$ be the probability that Brownian motion reflected off of I_D hits ∂_2 before ∂_1.

We will call $\hat{D}$ a "comb" domain if it is of the form

$$\hat{D} = \mathcal{R}_{L'} \setminus (l'_1 \cup \cdots \cup l'_k)$$

where $l'_1, \ldots, l'_k$ are disjoint intervals of the form

$$l'_j = \{x + iy_j : x_j \le x \le 1\},$$

where $0 < x_j < 1, 0 < y_j < L'$. We set $\hat{\partial}_1, \hat{\partial}_2$ to be the vertical boundaries. If $\hat{f}$ is the corresponding function, then $\hat{f}(x + iy) = x/L$, the same as for $\mathcal{R}_{L'}$, since the reflection is always in the y-direction and is independent of the real part. Using the same argument as for $\mathcal{R}_{L'}$, we see that $\Lambda[\mathcal{K}(\hat{D}, \partial_1, \partial_2)] = L'$. Similarly, $\mathcal{E}^*_{\hat{D}}(\partial_1, \partial_2) = L'$.

We claim that we can find a comb domain $\hat{D}$ and a conformal transformation $g : D \to \hat{D}$ so that $g \circ l_j = l'_j$. To see this, we will first determine what the parameters $L, x_1, \ldots, x_k, y_1, \ldots, y_k$ would need to be. Since we need $\mathcal{E}^*_D(\partial_1, \partial_2) = \mathcal{E}^*_{\hat{D}}(\hat{\partial}_1, \hat{\partial}_2)$, we choose L' satisfying

$$\pi/L' = \mathcal{E}^*_{\mathcal{R}_{L'}}(\partial_1, \partial_2) = \int_0^\pi \partial_x f(iy) \, dy.$$

We then choose $0 = y_0 < y_1 < \cdots < y_k < y_{k+1} = \pi$ uniquely so that

$$\mathcal{E}^*_{\mathcal{R}_{L'}}(\partial_1, L' + i[y_{k-1}, y_k]) = \mathcal{E}^*_D(\partial_1, A_k).$$

Finally, for $z = L + iy \in I_D \cap \partial_2$, we must have $f(z) = \hat{f}(g(z))$. This leads to the choice

$$x_j = \min\{f(z) : z \in l_j\}.$$

This defines $\hat{D}$ in terms of D, and given this we define g to be the unique holomorphic function on $\mathcal{R}_L$ with $g(0) = 0$ and $\text{Re}[g(z)] = L' f(z)$.

Once we have this transformation, we know that $\rho = |\nabla \hat{f}|$ is the minimizer in $\hat{D}$ and be conformal transformation, $|\nabla f|$ is the minimizer in D.

□

The next proposition is almost immediate using the definition of modulus but would be much harder use the definition coming from reflected Brownian motion.

Proposition 1.6.3 (Monotonicity)**.** *If $D' \subset D, \partial_1' \subset \partial_1, \partial_2' \subset \partial_2$, then $\mathcal{K}(D', \partial_1', \partial_2') \subset \mathcal{K}(D, \partial_1, \partial_2)$, and hence*

$$\Lambda[\mathcal{K}(D', \partial_1', \partial_2')] \leq \Lambda[\mathcal{K}(D, \partial_1, \partial_2)].$$

Proof. This follows immediately from the definition of modulus because if $\mathcal{K}' \subset \mathcal{K}$, then any ρ that is admissible for $\mathcal{K}$ is also admissible for $\mathcal{K}'$. □

1.7 Toolbox for conformal maps

Here we develop some of the classical tools for dealing with conformal transformations. One can get very far having three results in one's pocket: the Koebe-1/4 theorem, the distortion theorem, and the Beurling estimate. We will do these here. We call a function f on a domain D *univalent* if it is holomorphic and one-to-one.

1.7.1 Beurling estimate

The Beurling estimate is a uniform upper bound on the probability that a Brownian motion avoids a connected set. As an example suppose $K = [0, 1]$ and B_t is a Brownian motion starting at $-\epsilon$ and $D = \mathbb{D} \cap \mathbb{H}$. Then by considering the square root map that takes $\mathbb{D} \setminus K$ to D,

$$\begin{aligned} \mathbb{P}^{-\epsilon}\{B[0, \tau_{\mathbb{D}}] \cap K = \emptyset\} &= \mathbb{P}^{-i\sqrt{\epsilon}}\{B_{\tau_D} \notin \mathbb{R}\} \\ &= \frac{1}{\pi} \int_0^{\pi} H_D(i\sqrt{\epsilon}, e^{i\theta})\, d\theta \sim \frac{4\sqrt{\epsilon}}{\pi}, \quad \epsilon \downarrow 0 \end{aligned}$$

If we replace $[0, 1]$ with a different curve from 0 to the unit disk, we would expect that the probability for a Brownian motion to avoid the set would decrease. This statement is made precise in the Beurling projection theorem. From a practical perspective, what is used is the fact that the probability is bounded by $c\sqrt{\epsilon}$. This latter statement is often referred to as the Beurling estimate.

We will state and prove the Beurling projection theorem in this section. If $K \subset \mathbb{H}$ is a closed subset, we write

$$K_+ = K \cap \{\mathrm{Im}(w) \geq 0\}, \quad K_- = K \cap \{\mathrm{Im}(w) \leq 0\},$$

$$K^* = \{\overline{w} : w \in K\}, \quad K' = K_+ \cup K_-^*.$$

In other words, K' is obtained from K by reflecting the elements of K below the real line to the upper half plane. Note that $K \cap \mathbb{R} = K' \cap \mathbb{R} = (K \cap K^*) \cap \mathbb{R}$, and, more generally, $\mathrm{dist}(x, K) = \mathrm{dist}(x, K') = \mathrm{dist}(x, K \cup K^*)$ for all $x \in \mathbb{R}$.

Lemma 1.7.1. *Suppose $K \subset \mathbb{D}$ is closed and let K' be as above. Let $\tau = \tau_{\partial\mathbb{D}}$ and ρ, ρ' the first times to visit K, K' respectively. If $-1 < x < 1$, then*

$$\mathbb{P}^x\{\rho < \tau\} \geq \mathbb{P}^x\{\rho' < \tau\}.$$

Proof. We assume $x \notin K$ and write $\mathbb{P}$ for $\mathbb{P}^x$ throughout this proof. We will give an increasing sequence of stopping times. Let $\delta_0 = \operatorname{dist}(x, K \cup \partial\mathbb{D}) = \operatorname{dist}(x, K' \cup \partial\mathbb{D}) = \operatorname{dist}(x, K^* \cup \partial\mathbb{D})$, and

$$S_0 = \inf\{t : |B_t - x| = \delta_0\},$$

$$T_0 = \inf\{t \geq S_0 : B_t \in \mathbb{R}\}.$$

More generally, if $j \geq 1$, we set

$$\delta_j = \operatorname{dist}\left(B_{T_{j-1}}, K \cup \partial\mathbb{D}\right),$$

$$S_j = \inf\{t \geq T_{j-1} : |B_t - B_{T_{j-1}}| = \delta_j\},$$

$$T_j = \inf\{t \geq S_j : B_t \in \mathbb{R}\}.$$

It is possible that $B(T_j) \in K$ for some j in which case $S_k = T_k = T_j$ for $k \geq j$. However, if $B(T_j) \notin K$, then with probability one $T_j < S_{j+1} < T_{j+1}$. Note that on the event $\{B[0,\tau] \cap (K \cap \mathbb{R}) = \emptyset, B(\tau) \notin \mathbb{R}\}$, there exists j with $\{T_j < \tau < T_{j+1}\}$. Hence it suffices to show that for every $j \geq 0$,

$$\mathbb{P}\{T_j < \tau < T_{j+1}; B[0,\tau] \cap K = \emptyset\} \leq \mathbb{P}\{T_j < \tau < T_{j+1}; B[0,\tau] \cap K' = \emptyset\}. \tag{1.21}$$

It will be useful to add some randomness to the process. Let $J_0, J_1, \ldots$ be independent random variables, independent of the Brownian motion $B_t = B_t^1 + i\, B_t^2$ with $\mathbb{P}\{J_j = 1\} = \mathbb{P}\{J_j = -1\} = 1/2$. Define W_t by

$$W_t = B_t^1 + i\, J_j\, B_t^2 \quad T_{j-1} \leq t < T_j.$$

(Here $T_{-1} = 0$.) If $B_{T_j} \in \mathbb{R} \cap K$ so that $T_{j+1} = T_j$, we stop the process W_t at time T_j. Note that

$$\mathbb{P}\{T_j < \tau < T_{j+1}; B[0,\tau] \cap K = \emptyset\} = \mathbb{P}\{T_j < \tau < T_{j+1}; W[0,\tau] \cap K = \emptyset\},$$

and similarly with K' replacing K. Let $\mathcal{F}$ denote the σ-algebra generated by the Brownian motion B only, so that $\mathcal{F}$ is independent of the J_j. We claim that for each j,

$$\mathbb{P}\{T_j < \tau < T_{j+1}; W[0,\tau] \cap K = \emptyset \mid \mathcal{F}\} \leq \mathbb{P}\{T_j < \tau < T_{j+1}; W[0,\tau] \cap K' = \emptyset \mid \mathcal{F}\}.$$

Let us fix a j. The event $\{T_j < \tau < T_{j+1}\}$ is $\mathcal{F}$-measurable. On this event, we can write

$$\mathbb{P}\{T_j < \tau < T_{j+1}; W[0,\tau] \cap K = \emptyset \mid \mathcal{F}\} = I_0\, \mathbb{E}\left[I_1 \cdots I_j\, \hat{I}_{j+1} \mid \mathcal{F}\right],$$

where

$$I_k = 1\{J_k = 1\}\, 1\{B[S_k, T_k] \cap K = \emptyset\} + 1\{J_k = -1\}\, 1\{B[S_j, T_k] \cap K^* = \emptyset\},$$

$$\hat{I}_k = 1\{B[S_k, \tau] \cap K = \emptyset\} + 1\{J_k = -1\}\, 1\{B[S_k, \tau] \cap K^* = \emptyset\},$$

We get a similar expression for K' in terms of $I_k', \hat{I}_k'$, obtained by replacing K, K^* with $K', (K')^*$. The random variables $I_0, I_1, \ldots$ are conditionally independent given $\mathcal{F}$, and hence it suffices to show for every k,

$$\mathbb{P}\{I_k = 1 \mid \mathcal{F}\} \leq \mathbb{P}\{I_k' = 1 \mid \mathcal{F}\},$$

$$\mathbb{P}\{\hat{I}_k = 1 \mid \mathcal{F}\} \leq \mathbb{P}\{\hat{I}_k' = 1 \mid \mathcal{F}\},$$

We will show the first; the second is done in the same way. If $B(T_k) \in K$, then $I_k = 0$, so let us assume that $B(T_k) \notin K$. Consider the events

$$E_1 = \{B(S_k, T_k) \cap K_+ = \emptyset\}, \quad E_2 = \{B(S_k, T_k) \cap K_- = \emptyset\},$$

$$E_3 = \{B(S_k, T_k) \cap (K_+)^* = \emptyset\}, \quad E_4 = \{B(S_k, T_k) \cap (K_-)^* = \emptyset\}.$$

We can write

$$\mathbb{P}\{I_k = 1 \mid \mathcal{F}\} = \frac{1}{2}\, 1_{E_1 \cap E_2} + \frac{1}{2}\, 1_{E_3 \cap E_4},$$

$$\mathbb{P}\{I'_k = 1 \mid \mathcal{F}\} = \frac{1}{2}\, 1_{E_1 \cap E_4} + \frac{1}{2}\, 1_{E_3 \cap E_2}.$$

We therefore get

$$2\,\left[\mathbb{P}\{I'_k = 1 \mid \mathcal{F}\} - \mathbb{P}\{I_k = 1 \mid \mathcal{F}\}\right] = 1_{E_1 \cap (E_4 \setminus E_2)} + 1_{E_3 \cap (E_2 \setminus E_4)} - 1_{E_1 \cap (E_2 \setminus E_4)} - 1_{E_3 \cap (E_4 \setminus E_2)}.$$

Note that on the event $E_4 \setminus E_2$, $B(S_j, T_j)$ lies in the lower half-plane and hence $1_{E_1} = 1$. Similarly, on the event $E_2 \setminus E_4$, we have $1_{E_3} = 1$. Hence,

$$2\,\left[\mathbb{P}\{I'_k = 1 \mid \mathcal{F}\} - \mathbb{P}\{I_k = 1 \mid \mathcal{F}\}\right] = 1_{E_4 \setminus E_2} + 1_{E_2 \setminus E_4} - 1_{E_1 \cap (E_2 \setminus E_4)} - 1_{E_3 \cap (E_4 \setminus E_2)} \geq 0.$$

□

Proposition 1.7.2. *Under the assumptions above, if D, D' are the connected components of $\mathbb{C} \setminus K, \mathbb{C} \setminus K'$ containing the origin, and $x \in \mathbb{R} \setminus \{0\}$,*

$$G_D(x) \leq G_{D'}(x).$$

Proof. Since $\mathbb{R} \cap D = \mathbb{R} \cap D'$, the result is trivial for $x \in \mathbb{R} \setminus D$, so we assume $x \in D$. Let s be sufficiently small so that $\mathbb{D}_s \subset D$ and $|x| > e^{-s}$. Then we can follow the proof as above, to show that

$$\mathbb{P}^x\{\sigma_s < \tau_D\} \leq \mathbb{P}^x\{\sigma_s < \tau_{D'}\}.$$

Letting $s \to \infty$ gives the result. □

Theorem 1.7.3 (Beurling projection theorem)**.** *Suppose K is a connected, closed subset of $\overline{\mathbb{D}}$ such that for each $\epsilon \leq r \leq 1$*

$$K \cap \{|B_t| = r\} \neq \emptyset.$$

Then,

$$\mathbb{P}\{B[0, \tau_{\mathbb{D}}] \cap K = \emptyset\} \leq \mathbb{P}\{B[0, \tau_{\mathbb{D}}] \cap [\epsilon, 1] = \emptyset\}.$$

In particular,

$$\mathbb{P}\{B[0, \tau_{\mathbb{D}}] \cap K = \emptyset\} \leq 2\,\epsilon^{1/2}.$$

By conformal invariance, we can see that as $\epsilon \downarrow 0$,

$$\mathbb{P}\{B[0, \tau_{\mathbb{D}}] \cap [\epsilon, 1] = \emptyset\} = \frac{4}{\pi}\,\epsilon^{1/2} + O(\epsilon).$$

Although it is not optimal, we can write

$$\mathbb{P}\{B[0, \tau_{\mathbb{D}}] \cap [\epsilon, 1] = \emptyset\} \leq 2\,\epsilon^{1/2}, \quad 0 < \epsilon \leq 1.$$

Proof. Fix $\epsilon > 0$. For any K, let K' denote the set $K_+ \cup (K_-)^*$ as above. Then,

$$\mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap K = \emptyset\} \leq \mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap K' = \emptyset\}.$$

Similarly, we can reflect the negative real part of K' onto the positive real axis and increase the probabiilty. By doing this trick repeatedly and rotating, we see that for any $\delta > 0$ and any K we can find $K_\delta \subset \{0 \leq \arg(z) \leq \delta\}$ with

$$\mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap K = \emptyset\} \leq \mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap K_\delta = \emptyset\}.$$

For fixed ϵ, we can use connectivity of K_δ to see that

$$\lim_{\delta \downarrow 0} \mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap K_\delta = \emptyset \mid B[0,\tau_{\mathbb{D}}] \cap [\epsilon, 1] \neq \emptyset\} = 0.$$

Therefore,

$$\limsup_{\delta \downarrow 0} \mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap K_\delta = \emptyset\} \leq \mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap [\epsilon, 1] = \emptyset\}.$$

□

In applications one generally uses a corollary of the Beurling projection theorem often called the Beurling estimate.

Corollary 1.7.4 (Beurling estimate)**.** *There exists $c < \infty$ such that if D is domain with $0 \notin D$ and such that the connected component of $\mathbb{C} \setminus D$ containing the origin intersects the unit circle, then for all $|z| \geq 2$,*

$$\mathrm{hm}_D(z, \partial D \cap \{|\zeta| \leq \epsilon\}) \leq c\sqrt{\epsilon}.$$

Proof. By making D larger if necesssary, we can assume that $D = \mathbb{C} \setminus K$ where K is a compact connected subset of $\overline{\mathbb{D}}$ intersecting the unit circle. Let B_t be a Brownian motion starting at z and let $\rho = \inf\{t : |B_t| \leq \epsilon\}$. Then, $\mathrm{hm}_D(z, \partial D \cap \{|\zeta| \leq \epsilon\}) \leq \mathbb{P}^z\{\rho \leq \tau\}$. Now consider $f(w) = \epsilon/w$ and use

$$\mathbb{P}^z\{\rho \leq \tau\} = \mathbb{P}^{f(z)}\{\tau_{\mathbb{D}} \leq \tau_{f(D)}\}.$$

□

1.7.2 Koebe distortion theorems

The Riemann mapping theorem implies that there is a one-to-one correspondence between simply connected domains $D \subsetneq \mathbb{C}$ containing the origin and univalent functions f on the unit disk with $f(0) = 0, f'(0) > 0$. We let $\mathcal{S}^*$ denote the set of all such function and $\mathcal{S}$ the set of $f \in \mathcal{S}^*$ with $f'(0) = 1$. Any function $f \in \mathcal{S}$ can be written as a power series

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots .$$

One example of such a function is the *Koebe function* f_{Koebe},

$$f_{\text{Koebe}}(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4} = z + 2z^2 + 3z^3 + \cdots .$$

Using the second expressions, we can see that f_{Koebe} is a composition of conformal transformations, and hence is a conformal transformation, with $f_{\text{Koebe}}(\mathbb{D}) = \mathbb{C} \setminus (-\infty, -1/4]$. The Koebe function is an extremal function in $\mathcal{S}$, and a big problem in the twentieth century was to prove the Bieberbach conjecture:

- If $f \in \mathcal{S}$, then $|a_n| \le n$ for all n.

This was proved by de Branges. Fortunately, for most applications one does not need this result (and for this reason we do not need to go through the proof).

Lemma 1.7.5. *Suppose $f : \mathbb{D} \to D$ is a conformal transformation with $f(0) = 0, f'(0) > 0$ and $\operatorname{dist}(0, \partial D) \ge 1$. Then*

$$\log f'(0) = \frac{1}{2\pi} \int_0^{2\pi} G_D(e^{i\theta})\, d\theta. \tag{1.22}$$

Proof. Fix f, D and we allow constants and $O(\cdot)$ error terms to depend on f, D. Let k equal the right-hand side of (1.22). We know from (1.12) that

$$G_D(z) = -\log|z| + k + O(|z|), \quad z \to 0.$$

However, as $z \to 0$,

$$\begin{aligned} -\log|z| = G_{\mathbb{D}}(z) = G_D(f(z)) &= G_D(f'(0)\, z + O(|z|^2)) \\ &= -\log\left[f'(0)\, z + O(|z|^2)\right] + k + O(|z|) \\ &= -\log|z| - \log f'(0) + k + O(|z|). \end{aligned}$$

□

Lemma 1.7.6. *Suppose D is a regular domain containing the origin. Let $T = \inf\{t \ge 0 : B_t \in \mathbb{R}\}$. Then for every $z \in D$,*

$$G_D(z, 0) = \mathbb{E}^z \left[G_D(B_T); T < \tau_D\right].$$

Proof. We write $G(z) = G_D(z, 0)$. If $z \in \mathbb{R}$ the result is immediate. Assume that $\operatorname{Im}(z) > 0$ (the case $\operatorname{Im}(z) < 0$ is done similarly). We allow constants to depend on z, D. Let s be sufficiently large so that $\operatorname{dist}(0, \partial D) > e^{-s}$ and let $\xi_s = \tau_D \wedge T \wedge \sigma_s$. Since $M_t = G(B_{t \wedge \xi_s})$ is a continuous, bounded martingale,

$$\begin{aligned} G(z) &= \mathbb{E}^z \left[G(B_{\xi_s})\right] \\ &= \mathbb{E}^z \left[G(B_T); T < \tau_D \wedge \xi_s\right] + \mathbb{E}^z \left[G(B_{\sigma_s}); \sigma_s < \tau_D \wedge T\right]. \end{aligned}$$

We know that for $|\zeta| = e^{-s}$ that $G_D(\zeta) \le c\, s$. Also, using the Poisson kernel in $\mathbb{H}$, we see as $s \to \infty$,

$$\mathbb{P}^z\{\sigma_s < \tau_D \wedge T\} \le \mathbb{P}^z\{\sigma_s < T\} \le c\, e^{-s}.$$

Therefore,

$$\lim_{s \to \infty} \mathbb{E}^z \left[G(B_{\sigma_s}); \sigma_s < \tau_D \wedge T\right] = 0,$$

and, hence, by the monotone convergence theorem,

$$G(z) = \lim_{s \to \infty} \mathbb{E}^z \left[G(B_T); T < \tau_D \wedge \xi_s\right] = \mathbb{E}^z \left[G(B_T); T < \tau_D\right].$$

□

Proposition 1.7.7. *Suppose $f : \mathbb{D} \to D$ is a conformal transformation with $f(0) = 0, f'(0) > 0$ and $\operatorname{dist}(0, \partial D) = 1$. Then*

$$1 \le f'(0) \le 4. \tag{1.23}$$

Proof. The first inequality follows from (1.22) (or from the Schwarz lemma applied to f^{-1}, considered as a map from $\mathbb{D}$ into $\mathbb{D}$). To get the second inequality, we show that the right hand side of (1.22) is maximized if $D = \mathbb{C} \setminus [1, \infty)$. This is done similarly as in the Beurling inequality. Suppose that $D = \mathbb{C} \setminus K$, and as before we write

$$K_+ = \{z \in K : \operatorname{Im}(z) \ge 0\}, \quad K_- = \{z \in K : \operatorname{Im}(z) \le 0\},$$

$$(K_-)^* = \{z : \overline{z} \in K_-\}, \quad K' = K_+ \cup \overline{(K_-)^*}.$$

Let $D' = C \setminus K'$ and note that D is simply connected. We claim that

$$\frac{1}{2\pi} \int_0^{2\pi} G_D(e^{i\theta})\, d\theta \le \frac{1}{2\pi} \int_0^{2\pi} G_{D'}(e^{i\theta})\, d\theta. \tag{1.24}$$

To see this let $\sigma = \inf\{t : |B_t| = 1\}$ and $T = \inf\{t \ge \sigma : B_t \in \mathbb{R}\}$. Then, using Lemma 1.7.6, we see that

$$\frac{1}{2\pi} \int_0^{2\pi} G_D(e^{i\theta})\, d\theta = \mathbb{E}\left[G_D(B_T)\ 1\{B[0,T] \cap K = \emptyset\}\right],$$

and similarly for D', K'.

As in the proof of Lemma 1.7.1, we let $W_t = B_t^1 + i\, J\, B_t^2$ where J is a random variable independent of B with $\mathbb{P}\{J = \pm 1\} = 1/2$. Clearly W is a Brownian motion with $W_T = B_T$, and hence

$$\frac{1}{2\pi} \int_0^{2\pi} G_D(e^{i\theta})\, d\theta = \mathbb{E}\left[G_D(W_T)\ 1\{B[0,T] \cap K = \emptyset\}\right],$$

Let

$$E_1 = \{B[0,T] \cap K_+ = \emptyset\}, \quad E_2 = \{B[0,T] \cap K_- = \emptyset\},$$

$$E_3 = \{B[0,T] \cap (K_+)^* = \emptyset\}, \quad E_4 = \{B[0,T] \cap (K_-)^* = \emptyset\}.$$

Arguing as in that proof, conditioned on $B[0,T]$,

$$\mathbb{P}\{W[0,T] \cap K = \emptyset \mid B[0,T]\} \le \mathbb{P}\{W[0,T] \cap K' = \emptyset \mid B[0,T]\}.$$

Proposition 1.7.2 tells us that $G_D(x) \le G_{D'}(x)$ for $x \in \mathbb{R}$. This gives (1.24).

Given (1.24), we can do the argument in Theorem 1.7.3 to see that we can choose a maximizing D so that K lies in a wedge $\{0 \le \arg(w) \le \delta\}$ of arbitrarily small width, and then we argue as in the Beurling estiamate to see that the supremum is taken on by a slit domain $D = \mathbb{C} \setminus (-\infty, -1]$ for which $f(z) = 4\, f_{\text{Koebe}}(z)$.

□

Corollary 1.7.8 (Koebe (1/4)-theorem)**.** *Let $f \in \mathcal{S}^*$ and let $d = \operatorname{dist}(0, \partial f(\mathbb{D}))$. Then*

$$d \le f'(0) \le 4d.$$

In particular, if $f \in \mathcal{S}$, then $f(\mathbb{D})$ contains the open disk of radius $1/4$ about the origin.

The next proposition is a slightly weaker version of the "growth theorem".

Proposition 1.7.9. *There exists $c < \infty$ such that if $f \in \mathcal{S}$, then*

$$|f(z)| \le c\,[1-|z|]^{-2}.$$

Proof. All constants c in this proof are independent of f, that is, the estimates hold for all $f \in \mathcal{S}$. Let $D = f(\mathbb{D})$ and recall that $G_{\mathbb{D}}(z) = -\log|z| \ge 1-|z|$. By the Schwarz lemma applied to f^{-1}, we can see that there exists $z \in \partial\mathbb{D} \setminus f(\mathbb{D})$. We claim that there exists $c < \infty$ such that for all $f \in \mathcal{S}$ and all $z \in \partial\mathbb{D}$, $G_D(z) \le c_3$. Indeed, if one goes to the proof of Proposition 1.3.1, especially (1.9), (1.10) we see that $G_D(z) \le 1/\rho$ where $\rho = \rho_D$ is the infimum over $z \in \partial\mathbb{D}$ of the probability that a Brownian motion leaves D before reaching $C_1 := \{|w| = e^{-1}\}$. If $D = f(\mathbb{D})$ for $f \in \mathcal{S}$, this probability is greater than the probability that the Brownian motion makes a closed loop in $\{1 < |z| < e\}$ before reaching C_1.

Using Theorem 1.7.3, we can see that there exists c' such that if $|\zeta| > 1$, $\mathbb{P}^{\zeta}\{\sigma_0 < \tau_D\} \le c'\,|\zeta|^{-1/2}$. Hence,

$$G_D(\zeta) \le \mathbb{P}^{\zeta}\{\sigma_0 < \tau_D\} \sup_{|w|=1} G_D(w) \le c''\,|\zeta|^{-1/2},$$

and $|\zeta| \le c\,G_D(\zeta)^{-2}$ for some c. Hence, for all $z \in \mathbb{D}$ with $|f(z)| \ge 1$,

$$|f(z)| \le c\,G_D(f(z))^{-2} = c\,[G_{\mathbb{D}}(z)]^{-2} \le c\,[1-|z|]^{-2}.$$

□

We will now prove a form of the "distortion theorem". This is not as strong as the standard version, but this is easy to prove now and is that is needed for most arguments. The key fact is that the constants $c = C(D,V)$ can be chosen uniformly over $\mathcal{S}$.

Proposition 1.7.10 (Distortion Principle)**.** *Suppose D is a domain and $V \subset D$ is compact. Then there exists $c = c(D,V) < \infty$ such that if $f : D \to f(D)$ is a conformal transformation, then*

$$|f'(z)| \le c\,|f'(w)|, \quad z,w \in V.$$

Proof. We first assume that $D = \mathbb{D}$. Since f is uniformly bounded on $\{|z| \le 1/2\}$, the Cauchy integral formula gives a uniform bound on $|f''|$ for $|z| \le 1/4$, and this implies that there exists $c < \infty$ such that

$$|f'(z) - 1| \le c\,|z|, \quad |z| \le 1/4.$$

In particular, we can find δ such that $1/2 \le |f'(z)| \le 2$ for $|z| \le \delta$ and hence

$$|f'(w)| \le 4\,|f'(z)|, \quad |z|,|w| \le \delta. \tag{1.25}$$

Let us define a metric $\rho_D(z,w)$ on D to be the minimum integer k such that we can write down a sequence

$$z = \zeta_0, \zeta_1, \ldots, \zeta_k = w,$$

such that for $j = 1, \ldots, k$,

$$|\zeta_j - \zeta_{j-1}| < \delta \max\{\operatorname{dist}(\zeta_{j-1}, \partial D), \operatorname{dist}(\zeta_j, \partial D)\},$$

where δ is as in the last proof. Then, we have $|f'(z)| \le 4^{\rho_D(z,w)}\,|f'(w)|$. Arguing as in the proof of Proposition 1.2.6, we can see that for all compact V, $\max\{\rho_D(z,w) : z,w \in V\} < \infty$.

□

These arguments measure the "closeness" of z and w in D tp be the number of balls $\{\zeta : |\zeta - z_j| \le \operatorname{dist}(z_j, \partial D)\}$ are needed to "connect" z to w. This measure of distance is closely related to hyperbolic distance. This definition in the last proof is valid for all domains.

We end by stating the more precise distortion estimate. Usually we do not need the precision in this statement, but since the optimal constants are known, it is generally nicer to use them.

Theorem 1.7.11 (Distortion Theorem)**.** *If $f \in \mathcal{S}$ and $|z| < 1$, then*

$$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}.$$

$$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2}.$$

Corollary 1.7.12. *There exists $c_0 > 0$ such that if $f : \mathbb{H} \to f(\mathbb{H})$ is a conformal transformation and $x \in \mathbb{R}$, $r > 0$,*

$$\frac{|f'(ri)|}{c_0\,(x^4+1)} \le |f'(rx+ri)| \le c_0\,(x^4+1)\,|f'(ri)|, \tag{1.26}$$

$$|f(rx+ri) - f(ri)| \le c_0\, r\,(|x|^4+1)\,|f'(ri)|. \tag{1.27}$$

Proof. For (1.26), without loss of generality assume that $r = 1$, $f(i) = 0, f'(i) = -2i$. Let us also assume $|x| \ge 1$; otherwise we use the distortion principle immediately. Let $F(z) = (z-i)/(z+i)$ which maps $\mathbb{H}$ onto $\mathbb{D}$ with $F(i) = 0, F'(i) = -2i$, and let $g = f \circ F^{-1} \in \mathcal{S}$. Note that $|F(x+i)| \le 1 - c\,x^{-2}$, and hence the distortion theorem implies that

$$\frac{c}{x^2} \le |g'(F^{-1}(F(x+i)))| = \frac{|f'(x+i)|}{|F'(x+i))|}.$$

We check directly that $|F'(x+i)| \asymp x^{-2}$. Therefore $|f'(x+i)| \ge c\,x^{-4}$. This gives the first inequality in (1.26) and the second follows from the first applied to $\tilde{f}(z) = f(z-x)$. The estimate (1.27) follows from $|g'(F(z))| \le c\,(1-|F(z)|)^{-2}$.

$$|f(rx+ri) - f(ri)| \le \int_0^{rx} |f'(s+ri)|\,ds.$$

It is clear that by doing this proof slightly more carefully we could find an explicit c_0, but we will not need it. □

1.8 Loewner differential equation

We will give a number of versions of what are called Loewner differential equations. These equations describe the dynamics of conformal maps as a domain is perturbed. As a start we will describe one version of the half-plane equation. Suppose $\gamma : (0,\infty) \to \mathbb{H}$ is a simple curve with $\gamma(0+) = 0$. (For us curve means a continuous image of the real line and simple means that the function is

one-to-one. For each t, let $H_t = \mathbb{H} \setminus \gamma[0,t]$. The Riemann mapping theorem tells us that there exist conformal maps $g_t : H_t \to \mathbb{H}$. There are many such maps, but as we will see we can specify a unique one by requiring that

$$g_t(z) = z + o(1), \quad z \to \infty.$$

For fixed $z \in \mathbb{H}$, we can consider the flow $t \mapsto g_t(z)$. If $z \notin \gamma(0,\infty)$, then this flow exists for all times. If $z = \gamma(t)$ then the flow stops at time t at which $g_t(z) \in \mathbb{R}$.

The main result is that *if we reparametrize γ appropriately* then $g_t(z)$ is a a C^1 function of t that satisfies

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z,$$

where $U_t = g_t(\gamma(t))$.

We will first derive the equation in the case γ is a simple curve and show that $t \mapsto U_t$ is a continuous real-valued function. We then will consider the equation as an initial value problem for a given continuous U_t and discuss the solutions of the differential equation.

1.8.1 A class of transformations

We will consider simply connected subdomains of $\mathbb{H}$ of the form $\mathbb{H} \setminus K$ where K is a bounded set. We will be making estimates that are valid over all such domains so it useful to set up some notation. Recall that we write $\operatorname{rad}(K) = \operatorname{rad}(0,K) = \sup\{|z| : z \in K\}$.

Definition Let $\mathcal{J}_q$ denote the set of subdomains $D \subset \mathbb{H}$ with $\operatorname{rad}(\mathbb{H} \setminus D) \le q$, and let $\mathcal{J}'_q$ be the set of real translations $D = D + x, D \in \mathcal{J}_q, x \in \mathbb{R}$. Let $\mathcal{J} = \mathcal{J}_1, \mathcal{J}' = \mathcal{J}'_1$.

We allow multiply connected domains in $\mathcal{J}$. Note that $D \in \mathcal{J}$ if and only if $f(D) \in \mathcal{K}$ where $f(z) = -1/z$. Suppose $D \in \mathcal{J}_q, D' \in \mathcal{J}_{q'}$ and $g : D \to D'$ is a conformal transformation such that $g(\infty) = \infty$ (that is, if $z \to \infty$, then $g(z) \to \infty$) and such that for $x \in \mathbb{R} \setminus [x_1, x_2]$,

$$\lim_{y \downarrow 0} g(x + iy) \in \mathbb{R}.$$

Then we can use Schwartz reflection to extend g to a conformal transformation of

$$D^* := D \cup \{\overline{z} : z \in D\} \cup (-\infty, x_1) \cup (x_2, \infty).$$

The map

$$f(z) = \frac{1}{g(1/z)}$$

is a univalent function in a neighborhood $\mathcal{N}$ of the origin with $f(0) = 0$, and hence has a power series expansion

$$f(z) = a_1\, z + a_2\, z^2 + \cdots .$$

Since $f(\mathcal{N} \cap \mathbb{R}) \subset \mathbb{R}$, we can see that $a_j \in \mathbb{R}$, and since $f(\mathcal{N} \cap \mathbb{H}) \subset \mathbb{H}$, we can see that $a_1 > 0$. Using this we see that g has an expansion at infinity

$$g(z) = b_{-1}\, z + b_0 + b_1\, z^{-1} + b_2\, z^{-2} + \cdots , \quad b_{-1} > 0, \quad b_j \in \mathbb{R}.$$

We will write $g'(\infty) = 1$ if $b_{-1} = 1$. If g has an expansion as above, and $\tilde{g}(z) = (g(z) - b_0)/b_{-1}$, then $\tilde{g}(z) = z + o(1)$ as $z \to \infty$.

One might expect that we should define $g'(\infty) = b_{-1}$. However, for reasons that we will discuss later, there are good reasons to define $g'(\infty) = 1/b_{-1}$. We will avoid this issue for the moment by only defining "$g'(\infty) = 1$" which is the same under both definitions.

Definition Let $\mathcal{Q}_q$ denote the set of conformal transformations $g : D \longrightarrow g(D)$ where $D \in \mathcal{J}_q, g(D) \in \mathcal{J}'$ and such that

$$g(z) - z \to 0, \quad z \to \infty.$$

Let $\mathcal{Q} = \mathcal{Q}_1$. Transformations in $\mathcal{Q}$ are sometimes said to satisfy the *hydrodynamic normalization.*

Lemma 1.8.1. *Suppose $D \in \mathcal{J}_q$. Then there is a unique positive harmonic function v_D on D such that $v_D \equiv 0$ on ∂D and $v_D(z) = \operatorname{Im}(z) + O(1)$ as $z \to \infty$. It is given by*

$$v_D(z) = \operatorname{Im}(z) - \mathbb{E}^z\left[\operatorname{Im}(B_{\tau_D})\right] = \lim_{n\to\infty} n\mathbb{P}^z\{T_n < \tau_D\},$$

where $T_n = \inf\{t : \operatorname{Im}(B_t) = n\}$. Moreover, there exist $c = c_D < \infty$ such that for $|z| \geq 2q$,

$$|\operatorname{Im}(z) - v_D(z)| \leq \frac{c\operatorname{Im}(z)}{|z|^2}.$$

To be more precise, we mean that if we extend v_D to ∂D by setting $v_D(z) = 0$ for $z \in \partial D$, then v_D is continuous at the regular points of ∂D.

Proof. If v_D is such a function, then $\operatorname{Im}(z) - v_D(z)$ is a bounded harmonic function on D, and hence,

$$\operatorname{Im}(z) - v_D(z) = \mathbb{E}^z\left[h(B_{\tau_D})\right] = \mathbb{E}^z\left[\operatorname{Im}(B_{\tau_D})\right].$$

This gives existence and uniqueness of v_D. Since $\operatorname{Im}(w)$ is a bounded harmonic function on $\{0 < \operatorname{Im}(w) < n\}$, if $0 < \operatorname{Im}(z) < n$,

$$\operatorname{Im}(z) = \mathbb{E}^z\left[\operatorname{Im}(B_{\tau_D\wedge T_n})\right] = \mathbb{E}^z\left[\operatorname{Im}(B_{\tau_D}); \tau_D < T_n\right] + n\,\mathbb{P}^z\{T_n < \tau_D\}.$$

Using the monotone convergence theorem, we therefore see that

$$\begin{aligned}\lim_{n\to\infty} n\,\mathbb{P}^z\{T_n < \tau_D\} &= \operatorname{Im}(z) - \lim_{n\to\infty}\mathbb{E}^z\left[\operatorname{Im}(B_{\tau_D}); \tau_D < T_n\right] \\ &= \operatorname{Im}(z) - \mathbb{E}^z\left[\operatorname{Im}(B_{\tau_D})\right].\end{aligned}$$

To get the final assertion, note that

$$|\operatorname{Im}(z) - v_D(z)| \leq \mathbb{P}^z\{B_{\tau_D} \notin \mathbb{R}\}\,\sup\{\operatorname{Im}(w) : w \in \mathbb{H}\setminus D\} \leq c_D\,\mathbb{P}^z\{B[0,\tau_{\mathbb{H}}] \cap (q\mathbb{D}) \neq \emptyset\}.$$

The probability on the right can be computed by conformal invariance. We omit the details.

□

- One can consider $\mathcal{Q}$ as a half-plane analogue of the schlicht functions $\mathcal{S}$.
- If $g \in \mathcal{Q}_q$ with domain D, let $\tilde{D} = q^{-1}\,D$ and $\tilde{g}(z) = q^{-1}\,g(qz)$. Then $\tilde{g} \in \mathcal{Q}$ with $\tilde{g}'(z/q) = g'(z)$. We will focus on estimates for $g \in \mathcal{Q}$, but these immediately imply estimates for $g \in \mathcal{Q}_q$.

- Every $g \in \mathcal{Q}$ has an expansion at infinity

$$g(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots, \quad b_j \in \mathbb{R}.$$

- If we write $g(z) = u(z) + iv(z)$, then $h(z) := \mathrm{Im}(z) - v(z)$ is a bounded harmonic function on D such that $h(z) \to 0$ as $z \to \infty$. Note that h is a harmonic function on D^*.

- If $g : D \to g(D)$ is in $\mathcal{Q}$ then so is $g^{-1} : g(D) \to D$ with expansion

$$g^{-1}(z) = z - b_1\, z^{-1} + O(|z|^{-2}).$$

- If $D \in \mathcal{J}_q$ is simply connected, there exists unique $g \in \mathcal{Q}_q$ such that $g(D) = \mathbb{H}$. The existence of conformal transformations $f : D \to \mathbb{H}$ follows from the Riemann mapping theorem, and if $\tilde{f}$ is another such a transformation, then $\tilde{f} = T \circ f$ where T is a Möbius transfromation of $\mathbb{H}$. There is exactly one such T such that $T \circ f \in \mathcal{Q}$. In this case $\mathrm{Im} g = v_D$ where v_D is the function from Lemma 1.8.1.

- If $g = u + iv \in \mathcal{Q}$ and $z = x + iy$ with $|z| > 1$, we can use the Cauchy-Riemann equations to write

$$\begin{aligned} u(x,iy) &= \lim_{y_1 \to \infty} [u(x,iy_1) + u(x,iy) - u(x,iy_1)] \\ &= \lim_{y_1 \to \infty} \left[u(x,iy_1) - \int_y^{y_1} \partial_x v(x+it)\, dt \right] \\ &= x - \int_y^{\infty} \partial_x v(x+it)\, dt \\ &= x + \int_y^{\infty} \partial_x h(x+it)\, dt. \end{aligned}$$

  To see that the integral is well defined, note that for $|z| \geq 2$, h (extended to D^*) is a bounded harmonic function in the disk of radius $|z|/2$ about z and is bounded by $c\,|z|^{-1}$. Therefore by Proposition 1.2.5, $|\nabla h(z)| \leq c\,|z|^{-2}$.

Proposition 1.8.2. *Suppose $D \in \mathcal{J}$ and h is a positive harmonic function on D that is bounded on $\{|z| \geq 1\}$ and equals zero on $\{x \in \mathbb{R} : |x| \geq 1\}$. Then for $|z| \geq 2$,*

$$h(z) = H_{\mathbb{H}}(z,0)\, h_\infty\, [1 + O(|z|^{-1})] = -\mathrm{Im}(1/z)\, h_\infty\, [1 + O(|z|^{-1})],$$

where

$$h_\infty = \lim_{y \to \infty} y\, h(iy) = \frac{2}{\pi} \int_0^{\pi} h(e^{i\theta})\, \sin\theta\, d\theta. \tag{1.28}$$

Moreover, if $y > 1$,

$$h_\infty = \frac{1}{\pi} \int_{-\infty}^{\infty} h(x+iy)\, dx. \tag{1.29}$$

The condition $|z| \geq 2$ is put into the proposition to make the estimates uniform over z. We could replace it with $|z| \geq r$ for any $r > 1$, but then the implicit constant could depend on r.

As $|x| \to \infty$, $h(x+iy) \leq O(x^{-2})$ which shows that the integral in (1.29) is finite.

Proof. Let $O_+ = O \cap \mathbb{H} = \{z \in \mathbb{H} : |z| > 1\}$. Since h is a bounded harmonic function on O_+, the optional sampling theorem implies that

$$h(z) = \mathbb{E}^z \left[h(B_+)\right] = \frac{1}{\pi} \int_{\partial O_+} H(z,w)\, h(w)\, |dw|.$$

Using conformal invariance (see (1.61)), we can see that

$$H_{O_+}(z, e^{i\theta}) = 2\,\mathrm{Im}[-1/z]\, \sin\theta\, [1 + O(|z|^{-1})] = 2\, H_{\mathbb{H}}(z,0)\, \sin\theta\, [1 + O(|z|^{-1})].$$

Since $h(x) = 0$ for $x \in \mathbb{R} \cap \partial O_+$,

$$h(z) = \frac{2\, H_{\mathbb{H}}(z,0)}{\pi}\, \left[1 + O(|z|^{-1})\right] \int_0^\pi h(e^{i\theta})\, \sin\theta\, d\theta$$

This gives (1.28). Let $U_y = \{x + is : s > y\}$. Then, if $y' > y$,

$$y'\, h(iy') = \frac{y'}{\pi} \int_{\partial U_y} H_{U_y}(iy', x+iy)\, h(x+iy)\, dx = \frac{y'}{\pi(y'-y)} \int_{-\infty}^{\infty} \frac{(y'-y)^2}{x^2 + (y'-y)^2}\, h(x+iy)\, dx$$

Letting $y' \to \infty$, we get (1.29). □

We note that the Harnack principle implies that there exists c_1, c_2 such that for all such h,

$$c_1\, h_\infty \leq h(2i) \leq c_2\, h_\infty.$$

We can write

$$h_\infty = \frac{4}{\pi} \lim_{z \to \infty} \mathbb{E}^z \left[h(B_\tau) \mid |B_\tau| = 1\right].$$

We can then view the results as two separate estimates:

$$\mathbb{E}^z \left[h(B_\tau) \mid |B_\tau| = 1\right] = \frac{\pi}{4}\, h_\infty\, [1 + O(|z|^{-1})],$$

$$\mathbb{P}^z\{|B_\tau| = 1\} = \frac{4\mathrm{Im}(z)}{\pi |z|^2}\, \left[1 + O(|z|^{-1})\right].$$

Definition Suppose $K \subset \mathbb{H}$ is bounded such that $D = \mathbb{H} \setminus K$ is a domain. Then the *half-plane capacity* $\mathrm{hcap}(K)$ is defined by

$$\mathrm{hcap}(K) = \lim_{y \to \infty} y\, \mathbb{E}^{iy} \left[\mathrm{Im}[B(\tau_D)]\right].$$

Suppose $D = \mathbb{H} \setminus K, \tau = \tau_D$, and let $h(z) = \mathbb{E}^z [\mathrm{Im}(B_\tau)]$ be the bounded harmonic function on D with boundary value $\mathrm{Im}(z)$. If $D \in \mathcal{J}$, that is, if $K \subset \{|z| \le 1\}$, then in the notation of Proposition 1.8.2,

$$\mathrm{hcap}(K) = h_\infty = \frac{2}{\pi} \int_0^\pi \mathbb{E}^{e^{i\theta}} [\mathrm{Im}(B_\tau)] \sin\theta \, d\theta.$$

Also, for $|z| \ge 2$,

$$h(z) = -\mathrm{hcap}(K) \, \mathrm{Im}(1/z) \, [1 + O(|z|^{-1})].$$

In particular, $\mathrm{hcap}(K) \asymp \mathbb{E}^{2i} [\mathrm{Im}(B_\tau)]$.

Proposition 1.8.3. *Suppose $K \subset \mathbb{H}$ is bounded such that $D = \mathbb{H} \setminus K$ is a domain.*

1. *If $r > 0$, then $\mathrm{hcap}(rK) = r^2 \, \mathrm{hcap}(K)$.*
2. *If $x \in \mathbb{R}$, then $\mathrm{hcap}(x + K) = \mathrm{hcap}(K)$.*

Proof.

1. Let $D_r = \mathbb{H} \setminus rK$. Then, by conformal invariance

$$\mathbb{E}^{iry} \left[\mathrm{Im}(B_{\tau_{D_r}}) \right] = r \, \mathbb{E}^{iy} \left[\mathrm{Im}(B_{\tau_D}) \right].$$

   Therefore,

$$\mathrm{hcap}(rK) = \lim_{y \to \infty} ry \, \mathbb{E}^{iry} \left[\mathrm{Im}(B_{\tau_{D_r}}) \right] = r^2 \lim_{y \to \infty} y \, \mathbb{E}^{iy} \left[\mathrm{Im}(B_{\tau_D}) \right] = r^2 \, \mathrm{hcap}(K).$$

2. Let $D_x = \mathbb{H} \setminus (K + x) = D + x$. Then,

$$\mathbb{E}^{iy} \left[\mathrm{Im}(B_{\tau_{D_x}}) \right] = \mathbb{E}^{-x+iy} \left[\mathrm{Im}(B_{\tau_{D_x}}) \right].$$

   Using, for example, Proposition 1.8.2 (or just derivative estimates for harmonic functions), we can see that for fixed x, as $y \to \infty$,

$$\mathbb{E}^{-x+iy} \left[\mathrm{Im}(B_{\tau_{D_x}}) \right] \sim \mathbb{E}^{iy} \left[\mathrm{Im}(B_{\tau_{D_x}}) \right].$$

□

There is another notion of capacity that we will consider that scales differently from hcap.

Definition Suppose V is a compact subset of $\overline{\mathbb{H}}$. For a Brownian motion B_t and let $D = \mathbb{H} \setminus V$. Then

$$\mathrm{cap}_{\mathbb{H}}(V) = \lim_{y \to \infty} \mathbb{P}^{iy}\{B_\tau \in V\} = \lim_{y \to \infty} y \, \mathrm{hm}_{\mathbb{H} \setminus V}(iy, V).$$

Note that we allow V to be a subset of reals. The quantity $\mathrm{cap}_{\mathbb{H}}(V)$ is a normalized form of the point-to-set excursion measure as we now show. Let

$$f(z) = \frac{z - i}{z + i}$$

which is a conformal transformation of $\mathbb{H}$ onto $\mathbb{D}$. Then conformal invariance implies that

$$\mathrm{hm}_{\mathbb{H}\setminus V}(iy, V) = \mathrm{hm}_{\mathbb{D}\setminus f(V)}(f(iy), f(V)) = \mathrm{hm}_{\mathbb{D}\setminus f(V)}\left(\frac{y-1}{y+1}, f(V)\right).$$

Therefore,

$$\mathrm{cap}_{\mathbb{H}}(V) = 2\lim_{y\to\infty}\frac{y}{2}\,\mathrm{hm}_{\mathbb{D}\setminus f(V)}\left(1-\frac{2}{y+1}, f(V)\right) = 2\,\mathcal{E}_{\mathbb{D}\setminus f(V)}(1, f(V)).$$

- If $T = \inf\{t : B_t \in \mathbb{R}\}$, we can use the Poisson kernel in $\mathbb{H}$ to see that

$$\mathrm{cap}_{\mathbb{H}}([0,x]) = \lim_{y\to\infty} y\mathbb{P}^{iy}\{0 \le B_T \le x\} = \lim_{y\to\infty} y\int_0^x \frac{y\,dy}{\pi(t^2+y^2)}\,dt = \frac{x}{\pi}.$$

- If $V \subset \{|z| \le 1\}$ and $y \ge 2$,

$$\mathrm{cap}_{\mathbb{H}}(V) \asymp y\,\mathbb{P}^{iy}\{B[0,T]\cap V \ne \emptyset\} = y\,\mathrm{hm}_{\mathbb{H}\setminus V}(2i, V).$$

- Using conformal invariance, we get the following relations:

$$\mathrm{cap}_{\mathbb{H}}(V+x) = \mathrm{cap}_{\mathbb{H}}(V), \quad \mathrm{cap}_{\mathbb{H}}(rV) = r\,\mathrm{cap}_{\mathbb{H}}(V).$$

- Suppose V is the disk of radius ry about $z = x+iy$ where $0 < r < 1$. We claim that

$$\mathrm{cap}_{\mathbb{H}}(V) = 2y\,[\log(1/r) + O(r)]^{-1} \quad r \to 0. \tag{1.30}$$

  It suffices to prove this for $y = 1, x = 0$ for which it follows from

$$G_{\mathbb{H}}(iy, i) = \log\frac{y+1}{y-1} = 2y^{-1} + O(y^{-2}), \quad y \to \infty.$$

Since $\mathrm{cap}_{\mathbb{H}}$ scales linearly, one might expect that $\mathrm{cap}_{\mathbb{H}}$ of a connected set to be comparable to the diameter of a set. Indeed this is true if the set touches the boundary, but is not correct for "interior" sets.

Lemma 1.8.4. *There exists c_1, c_2 such that if $V \subset \mathbb{H}$ is compact and connected and $d = \mathrm{diam}(V)$, $y = \sup\{\mathrm{Im}(z); z \in V\}$, then*

$$c_1\,(d+y)\,[1+\log_+(y/d)]^{-1} \le \mathrm{cap}_{\mathbb{H}}(V) \le c_2\,(d+y)\,[1+\log_+(y/d)]^{-1}.$$

In particular, if both V and $V\cup\mathbb{R}$ are connected, then $\mathrm{cap}_{\mathbb{H}}(V) \asymp \mathrm{diam}(V)$.

Proof. By scaling and translation we may assume that $y = 1$ and that $\min\{\mathrm{Re}(z) : z \in V\} = 0$. Let D denote the unbounded connected component of $\mathbb{H}\setminus V$. As noted above,

$$\mathrm{cap}_{\mathbb{H}}(V) \asymp (d+1)\,\mathrm{hm}_D(2(d+1)i, V). \tag{1.31}$$

If $d \ge 4$, let $s = d-1 \ge 3$. The upper bound follows immediately from (1.31). For the lower bound, note that there exists $z \in V$ with $\mathrm{Re}(z) = s$. Consider the square $\{x+iy : 0 \le x \le s, 0 \le$

$y \le 1\}$. We know that V is connected and contains points on both vertical sides, $[0,i]$ and $[s,s+i]$, If a Brownian motion B_t starting at $2(d+1)i$ exits $\mathbb{H}$ at $(0,1)$ without hitting either of the vertical sides $[0,i]$ or $[s,s+i]$, then either the curve hits V or the curve disconnects V. Since we know that V is connected, the former must then hold. Let $\sigma = \inf\{t : \mathrm{Im}(B_t) = 1\}, x = \mathrm{Re}(B_\sigma)$. If $x \in [1, s-1]$, then there is a probability of $1/4$ that the continuation of the path will exit the square $[x-1, x+1] \times [x-1+2i, x+1-2i]$ at $[x-1, x+1]$. Therefore,

$$\mathrm{hm}_D(2(d+1)i, V) \ge \frac{1}{4}\,\mathbb{P}^{2(d+1)i}\{\mathrm{Re}(B_\sigma) \in [1, d-2]\}.$$

Using the exact form of the Poisson kernel in the upper half plane, we can see that

$$\inf_{d\ge 4} \mathbb{P}^{2(d+1)i}\{\mathrm{Re}(B_\sigma) \in [1, d-2]\} > 0.$$

If $d \le 1/2$, let $z = x+i$ be a point in V with maximal imaginary part and note that $0 \le x \le 1/2$. Let $\mathcal{B}_r$ denote the closed disk of radius r centered at z with boundary ∂_r. Since $\mathrm{cap}_{\mathbb{H}}(V) \le \mathrm{cap}_{\mathbb{H}}(\mathcal{B}_d)$, the upper bound follows from (1.30). The connected set V intersects $\partial_{d/2}$. Let $q > 0$ be the probability that a Brownian motion starting at $|z| = 1/2$ makes a closed loop about the origin before reaching the unit circle. Suppose that a Brownian motion starting at $2(d+1)i$ reaches $\partial_{d/4}$ before leaving $\mathbb{H}$. Then there is a probability q, that it will make a loop about z before reaching the circle of radius $d/2$. If that happens, the curve must hit V. From this we get the inequality

$$\mathrm{cap}_{\mathbb{H}}(V) \ge q\,\mathrm{hm}_{\mathbb{D}\setminus\mathcal{B}_{d/4}}(2(d+1)i, \mathcal{B}_{d/4}) \asymp q\,\mathrm{cap}_{\mathbb{H}}(\mathcal{B}_{d/4}).$$

This and (1.30) give the lower bound.

If $1/2 \le d \le 4$ we can use the $d = 1/2$ estimate for a lower bound and the $d = 4$ estimate for an upper bound.

□

The estimate above is useful in studying the boundary behavior of conformal maps. For future reference we state a disk version of the proposition that can be proved in the same way. We will only give the boundary version.

Proposition 1.8.5. *There exist $0 < c_1 < c_2 < \infty$ such that if $V \subset \overline{\mathbb{D}}$ is a connected compact set with $V \cap \partial\mathbb{D} \ne 0$, then*

$$c_1\,\mathrm{diam}(V) \le \mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap V \ne \emptyset\} \le c_2\,\mathrm{diam}(V).$$

Roughly speaking, the quantity $\mathrm{cap}_{\mathbb{H}}(V)$ is the normalized probability that a Brownian motion "starting at infinity" exits $\mathbb{H}\setminus V$ at V. It is a version of excursion measure. The quantity $\mathrm{hcap}(K)$ is a normalized probability that a Brownian motion "starting at infinity and conditioned to leave $\mathbb{H}$ at infinity" hits K. This is only nonzero if $K \subset \mathbb{H}$, and if K is very close to the real line it is near zero. It is analogous to what we will call boundary bubbles.

1.8.2 Compact hulls

Definition We call a compact $K \subset \overline{\mathbb{H}}$ a *compact* $\mathbb{H}$*-hull* if

- $K \cap \mathbb{R} \neq \emptyset$.
- $K \cup \mathbb{R}$ is connected.

We have chosen to make connectedness of $K \cup \mathbb{R}$ one of the conditions for being a hull. This is not always done. We choose this definition for convenience.

For any such K, let D_K denote the unbounded component of $\mathbb{H} \setminus K$ and note that D_K is simply connected. Let $x_-(K) = \min\{x : x \in K \cap \mathbb{R}\}, x_+(K) = \max\{x : x \in K \cap \mathbb{R}\}$. We define the *fill* of K by $\text{fill}(K) = (\mathbb{H} \setminus D_K) \cup [x_-(K), x_+(K)]$. Note that $\text{fill}(K)$ is a compact $\mathbb{H}$-hull. Let $R_K = \sup\{|z| : z \in K\} = \sup\{|z| : z \in \text{fill}(K)\}$ and

$$D^* = \mathbb{C} \setminus [\text{fill}(K) \cup \{\overline{z} : z \in \text{fill}(K)\}] .$$

Note that

$$\text{hcap}(K) = \text{hcap}(\text{fill}(K)).$$

Sometimes, in an abuse of notation, we will refer to a bounded, but not closed, $K \subset \mathbb{H}$ as a compact $\mathbb{H}$-hull. In this case the implicit hull is the union of $\overline{K}$ and the smallest closed line segment in $\mathbb{R}$ needed to make the union connected.

Proposition 1.8.6. *There exists $c_0 < \infty$ such that the following holds. Suppose that K is a compact $\mathbb{H}$-hull, $D = D_K$, $R = R_K$, $a = \text{hcap}(K)$.*

1. *There exists a unique conformal transformation $g = g_K : D \longrightarrow \mathbb{H}$ such that*

$$\lim_{z \to \infty} [g(z) - z] = 0.$$

It extends by Schwarz reflection to a conformal transformation $g : D^* \longrightarrow \mathbb{C} \setminus [x_1, x_2]$ for some $-\infty < x_1 < x_- \leq x_+ < x_2 < \infty$. For $z \in D$, $\text{Im} g(z)$ is the same as $v_D(z)$ from Lemma 1.56.

2. *The expansion of g at infinity is*

$$g(z) = z + \frac{a}{z} + \sum_{j=2}^{\infty} b_j \, z^{-j}, \quad b_j \in \mathbb{R}.$$

3.

$$x_1 = \lim_{y \to \infty} \pi \, y \left[\frac{1}{2} - \mathbb{P}^{iy}\{B_{\tau_D} \in (-\infty, x_-)\} \right],$$

$$x_2 = \lim_{y \to \infty} \pi \, y \left[\frac{1}{2} - \mathbb{P}^{iy}\{B_{\tau_D} \in (x_+, \infty)\} \right].$$

In particular, $x_1 \leq x_- \leq x_+ \leq x_2$.

4. *If $r > 0, x \in \mathbb{R}$, then*
$$g_{rK}(z) = r\, g(z/r), \qquad g'_{rK}(z) = g'(z/r),$$
$$g_{K+x}(z) = x + g_K(z - x).$$

5. *If $K \subset K'$, $g_{K'} = g_{g(K')} \circ g$. In particular,*
$$\operatorname{hcap}(K') = \operatorname{hcap}(K) + \operatorname{hcap}[g(K')]. \tag{1.32}$$
Here by $g(K')$ we mean the hull $\overline{g(K' \setminus K)}$.

6. *If $|z| \geq 2R$, then*
$$|g'(z) - 1| \leq c_0 \frac{a}{|z|^2}.$$

7. *If $|z| \geq 2R$, then*
$$\left| g_K(z) - z - \frac{a}{z} \right| \leq c_0 \frac{a\, R}{|z|}. \tag{1.33}$$

Proof.

1. The existence of the map was shown in the previous section.

2. Note that as $y \to \infty$,
$$g(iy) = i \left[y - \frac{b_1}{y} \right] + O(y^{-2}) = iv(iy) + O(y^{-2}).$$
Therefore, using Lemma 1.8.1 and the definition of hcap, we see that
$$b_1 = \lim_{y \to \infty} y\, [y - v(iy)] = \lim_{y \to \infty} y\, \mathbb{E}^{iy} [\operatorname{Im}(B_\tau)] = \operatorname{hcap}(K).$$

3. Using the Poisson kernel in $\mathbb{H}$, we see that
$$\lim_{y \to \infty} \pi\, y \left[\frac{1}{2} - \mathbb{P}^{iy}\{B_{\tau_{\mathbb{H}}} \in [x_+, \infty)\} \right] = x_+.$$
Conformal invariance implies that
$$\mathbb{P}^{iy}\{B_{\tau_D} \in [x_+, \infty)\} = \mathbb{P}^{g(iy)}\{B_{\tau_{\mathbb{H}}} \in [x_2, \infty)\}.$$
We know that $g(iy) = iy - iay^{-1} + O(y^{-2})$ and derivative estimates for harmonic function show that
$$\mathbb{P}^{g(iy)}\{B_{\tau_{\mathbb{H}}} \in [x_2, \infty)\} = \mathbb{P}^{i(y - ay^{-1})}\{B_{\tau_{\mathbb{H}}} \in [x_2, \infty)\} + O(y^{-2}).$$
Therefore,
$$\begin{aligned} \lim_{y \to \infty} \pi\, y \left[\frac{1}{2} - \mathbb{P}^{iy}\{B_{\tau_D} \in [x_+, \infty)\} \right] &= \lim_{y \to \infty} \pi\, y \left[\frac{1}{2} - \mathbb{P}^{i(y - ay^{-1})}\{B_{\tau_{\mathbb{H}}} \in [x_2, \infty)\} + O(y^{-2}) \right] \\ &= \lim_{y \to \infty} \pi\, (y - ay^{-1}) \left[\frac{1}{2} - \mathbb{P}^{i(y - ay^{-1})}\{B_{\tau_{\mathbb{H}}} \in [x_2, \infty)\} \right] \\ &= x_2. \end{aligned}$$
Since $\mathbb{P}^{iy}\{B_{\tau_D} \in [x_+, \infty)\} \leq \mathbb{P}^{iy}\{B_{\tau_{\mathbb{H}}} \in [x_+, \infty)\}$, we see that $x_2 \geq x_+$. The argument for x_1 is the same.

4. Note that $\tilde{g}(z) := r g_D(z/r)$ is a conformal transformation of $\mathbb{H} \setminus (rK)$ onto $\mathbb{H}$ satisfying $\tilde{g}(z) = z + o(1), \quad z \to \infty$. By uniqueness $\tilde{g} = g_{rK}$. We argue similarly for $\hat{g}(z) = x + g_K(z - x)$.

5. It is easy to see that $\tilde{g} := g_{g(K_2)} \circ g_{K_1}$ is a conformal transformation of D_{K_2} onto $\mathbb{H}$ satisfying $\tilde{g}(z) = z + o(1)$.

6. We first assume that $R = 1$. Let $h(z) = \mathrm{Im}[z - g_D(z)] = \mathrm{Im}(z) - v_D(z)$ which we consider as a harmonic function on $D^* \supset \{|z| > 1\}$. Using Lemma 1.8.2, we see there exists universal c such that
$$|h(z)| \le c\, a\, |z|^{-1}.$$
If $|z| \ge 2$, then $h(z)$ is a harmonic function defined on the disk of radius $|z|/2$ bounded by $caO(|z|^{-1})$. Hence using Proposition 1.2.5, we see that
$$|\nabla h(z)| \le c\, a\, |z|^{-2},$$
and hence
$$|1 - g_D'(z)| = \sqrt{[\partial_x v(z)]^2 + [1 - \partial_y v(z)]^2} \le c\, a\, |z|^{-2}.$$
For more general R, $g_{RD}'(z) = g_D'(z/R)$, and hence for $|z| \ge 2R$,
$$|1 - g_{RD}'(z)| = |1 - g_D'(z/R)| \le c\, R^2\, a\, |z|^{-2} = c\, \mathrm{hcap}(RK)\, |z|^{-2}.$$

7. Assume $R = 1$, let
$$f(z) = g(z) - z - \frac{a}{z}.$$
and let
$$v_f(z) = \mathrm{Im} f(z) = v(z) - \mathrm{Im}(z) - a\, \mathrm{Im}(1/z).$$
Using Proposition 1.8.2 with $h(z) = z - v(z)$, we see that
$$|v_f(z)| \le c\, \frac{a\, \mathrm{Im}(z)}{|z|^3}.$$
Using the fact that v_f (extended to D^*) is a harmonic function on $\{|w| \le |z|/2\}$ bounded by $c\, a\, |z|^{-2}$, we see that
$$|f'(z)| = |\nabla v_f(z)| \le c\, a\, |z|^{-3}.$$
Using $f(\infty) = 0$, we see that for $|z| \ge 2$, $|f(z)| \le c\, a\, |z|^{-2}$.
For more general R, recall that $g_{RK}(z) = R\, g_K(z/R)$ and hence
$$\begin{aligned} |g_{RK}(z) - z - \mathrm{hcap}(RK) z^{-1}| &= |R g_K(z/R) - z - R^2 a z^{-1}| \\ &= R |g_K(z/R) - (z/R) - a(z/R)^{-1}| \\ &\le c\, R\, a\, |z/R|^{-2} \\ &= c\, R\, \mathrm{hcap}(RK)\, |z|^{-2}. \end{aligned}$$

□

Examples.

- Let $K = \overline{\mathbb{D}_+}$. Then
$$g_K(z) = z + \frac{1}{z}.$$
In particular, $\operatorname{hcap}(\overline{\mathbb{D}_+}) = 1$.

- Let K be the vertical line segment $[0, i]$. Then
$$g_K(z) = \sqrt{z^2 + 1} = z + \frac{1}{2z} + \cdots .$$
To be more precise, note that if $z \in \mathbb{H} \setminus [0, i]$, then $z^2 + 1$ is not on the positive real line. Hence. we can take the branch of the square root with values in the positive half plane. This shows that $\operatorname{hcap}([0, i]) = 1/2$.

If K is a compact $\mathbb{H}$-hull, then $\operatorname{hcap}(K)$ is the coefficient of z^{-1} in the expansion of g_K from infinity. Indeed, that is how some people *define* the quantity. However, this definition does not work for compact K for which $K \cup \mathbb{R}$ is not connected.

As a slight abuse of notation, we write
$$g_D(x_-) = s \quad g_D(x_+) = t.$$
If K is disconnected it is possible that g_D can be extended to a slightly larger domain, but we will not need to consider this extension.

Lemma 1.8.7.
$$-2R \le g_D(x-) \le g_D(x_+) \le 2R.$$

Proof. We do the case $R = 1$; the other cases can be handled by scaling. Recall from Proposition 1.8.6 that
$$g_D(x_+) = \lim_{y \to \infty} \pi\, y \left[\frac{1}{2} - \mathbb{P}^{iy}\{B_{\tau_D} \in [x_+, \infty)\} \right].$$
The right-hand side is maximized (under the constraint $R = 1$) when $D = \mathbb{H} \setminus \overline{\mathbb{D}_+}$ in which case
$$g_D(z) = z + \frac{1}{z}, \quad g_D(1) = 2.$$
□

It follows that $g_D(x_+) - x_+ \le 3R$. However, we can get arbitrarily close to $3R$. If we let D be the maximizing domain for $R = 1$, then we can take
$$D_\epsilon = D \setminus \{x + iy : -1 < x \le 1 : 0 < y < \epsilon(x + 1)\}$$
for which $x_+ = -1$ and $g(x_+) \to 3$ as $\epsilon \to 0$.

1.8.3 Boundary behavior

The behavior of conformal transformations near the boundary is a delicate topic. We will consider here the case where K is a compact $\mathbb{H}$-hull contained in the closed unit disk, $D = \mathbb{H} \setminus K \in \mathcal{J}$, and $g = g_K$ is the unique conformal transformation $g : D \to \mathbb{H}$ with $g(z) - z \to 0$ as $z \to \infty$. We will write f for the inverse map $f = g^{-1} : \mathbb{H} \to D$. The question is whether or not f extends to a map on the $\overline{\mathbb{H}}$. If we only assume that D is the form above, then the situation can be difficult. As a bad example to consider as we go along, let

$$\hat{K} = \left[-\frac{1}{2}, \frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{1}{2} + \frac{i}{2}\right] \cup \left[-\frac{1}{2}, \frac{1}{2} + \frac{i}{2}\right] \cup \bigcup_{n=1}^{\infty} \left(\left[-\frac{1}{2} + \frac{i}{2^{2n-1}}, \frac{1}{4} + \frac{i}{2^{2n-1}}\right] \cup \left[-\frac{1}{4} + \frac{i}{2^{2n}}, \frac{1}{2} + \frac{i}{2^{2n}}\right] \right),$$

and $\hat{D} = \mathbb{H} \setminus \hat{K}$. Fortunately, such bad behavior will not arise if we assume K is the image of a curve.

Definition

- If D is a domain, then a *(simple) crosscut* is a simple curve $\eta : (0, t_0) \to D$ with such that the limits $\eta(0) = \eta(0+), \eta(1) = \eta(1-)$ exist and are on ∂D. (We allow $\eta(0) = \eta(1)$.)
- We say that a simple curve $\eta : [0, t_0] \to \mathbb{C}$ is an *accessing curve* for D if $\eta(0, t_0) \subset D$ and $\eta(0) \in \partial D$. We say that η accesses z if $\eta(0) = z$. The point $z \in \partial D$ is *accessible* if there exists at least one curve accessing η.

Note that under our definition, crosscuts (or their reversal) are accessing curves for both endpoints. In our pathological example $\hat{D}$, the origin is not an accessible point for $\hat{D}$. The Beurling estimate implies that following.

Proposition 1.8.8. *There exists $c < \infty$ such that if $D = \mathbb{H} \setminus K \in \mathcal{J}$, and η is a curve accessing $z \in \partial D$, then if* $\operatorname{diam}(\eta_t) \leq 1$,

$$\operatorname{diam}[g \circ \eta_t] \leq c \sqrt{\operatorname{diam}(\eta_t)}. \tag{1.34}$$

Here $\eta_t = \eta[0, t]$. In particular, the limit

$$\lim_{t \downarrow 0} g(\eta(t))$$

exists.

Proof. The proof is the same as that of Lemma 1.8.17. □

What makes the last proposition true is that if a curve in the upper half plane has a large diameter then there is a good chance that it will be hit by a Brownian motion. "Hit by Brownian motion", that is, harmonic measure, is a conformal invariant. However, we do not get a lower bound on $\operatorname{diam}[g \circ \eta_t]$ in terms of $\operatorname{diam}(\eta_t)$. If ∂D is very rough, or even it just has some protected "fjords", it is possible for $\operatorname{diam}(\eta)$ to be large but the harmonic measure of $c\eta$ to be small.

Proposition 1.8.9. *Suppose that η is a crosscut of $D = \mathbb{H} \setminus K \in \mathcal{J}$ whose endpoints are distinct. Then $g \circ \eta$ is a crosscut of $\mathbb{H}$ with distinct endpoints.*

Proof. The fact that $g \circ \eta$ is a crosscut follows from the previous proposition. To see that the endpoints are distinct, note that if $w \in D \setminus \eta$, then there is a positive probability that a Brownian motion starting at w hits $\mathbb{R}$ before hitting η and hence leaves D before hitting η. By conformal invariance this must hold for the image $g \circ \eta$. But if the endpoints of $g \circ \eta$ were the same, this would not be true for w in the bounded component of $\mathbb{H} \setminus (g \circ \eta)$. □

For each $z \in \partial D$, let $\mathbb{D}_s(z)$ denote the open ball of radius e^{-s} about z with boundary $C_s(z)$. The set $C_s(z) \cap D$ is the disjoint union of a finite or countably infinite number of *crosscuts* of D. The image of each crosscut under g is a crosscut of $\mathbb{H}$ and Proposition 1.8.8 implies that $g \circ l$ is a crosscut of $\mathbb{H}$ with $\operatorname{diam}[g \circ l] \le c\, r^{1/2}$ for some universal constant c. (One needs to be careful here; although the image of each crosscut is small, the images of different crosscuts may not be close to each other so the diameter of the union of the crosscuts can be large.) The last proposition implies that the endpoints of $g \circ l$ are distinct.

Let us fix z and assume that z is accessible. Let $\mathcal{B}_s = \mathbb{D}_s(z)$ and let $U_1^s, U_2^s, \ldots$ denote the connected components of $D \setminus C_s$ that contain z on its boundary. Accessibility implies that there is at least one such component. (In the example $\hat{D}$ above, there are no such components for $z = 0$; however, this point is not accessible.) Typically there will not be many such components, but it is possible for there to be a countable number. For each of these components U_j^s, there is a unique crosscut l_j^s of D such that $l_j^s \subset \partial U_j^s$ and the component of $D \setminus l_j^s$ containing U_j^s is a bounded component. (It is useful to draw pictures. The bounded component of $D \setminus l_j^s$ need not be contained in $\mathcal{B}_s$.) Let us call this bounded component V_j^s. It can be characterized as follows. Suppose η is a curve as in Proposition 1.8.8. Then for all t sufficiently small either $\eta(0,t) \subset V_j^s$ or $\eta(0,t) \cap V_j^s = \emptyset$. For each s we have an equivalence relations on η with $\eta^1 \equiv_s \eta^2$ if they end up in the same component V_j^s. Note that this is monotonic: if $\eta^1 \equiv_s \eta^2$ then $\eta^1 \equiv_r \eta^2$ for all $r < s$. Hence we can write $\eta^1 \equiv \eta^2$ if $\eta^1 \equiv_s \eta^s$ for some s.

Definition The equivalence classes of accessing curves for D are called the *prime ends*. The prime ends at $z \in \partial D$ are the equivalence classes of curves that access z.

We summarize our discussion in a proposition.

Proposition 1.8.10. *A boundary point $z \in D$ is accessible if and only if there is a prime end at z. If η^1, η^2 are two curves accessing z in D, then*

$$\lim_{t \downarrow 0} g(\eta^1(t)) = \lim_{t \downarrow 0} g(\eta^2(t)),$$

if and only if η^1, η^2 are equivalent as prime ends.

Proposition 1.8.11. *Suppose $\gamma : (0,1] \to \mathbb{H}$ is a simple curve with $\gamma(0+) = x \in \mathbb{H}$, and let $\eta(t) = f(\gamma(t))$. Suppose that*

$$\lim_{\epsilon \downarrow 0} \operatorname{diam}[\eta(0,\epsilon)] = 0.$$

Then

$$\lim_{t \downarrow 0} f(\gamma(t)) = z$$

exists and is in ∂D. The curve η accesses z in D. If $\tilde{\gamma} : (0,1] \to \mathbb{H}$ is another simple curve with $\gamma(0+) = x \in \mathbb{H}$, then

$$\lim_{t \downarrow 0} f(\tilde{\gamma}(t)) = z.$$

If l is a crosscut on $\mathcal{B}_{r,z}$, then $D \setminus l$ has two components, one bounded and one unbounded. If U is the bounded component, then we can see that

$$\operatorname{diam}[g(U)] \le c\, r^{1/2}.$$

However, even if r is very small, it is possible for $\operatorname{diam}[U]$ to be of order 1. As an example, consider the example $\hat{D}$ above. Let η_n be the crosscut formed by the vertical line segment from $2^{-n}i$ to $2^{-(n_1)i}$. Then $\operatorname{diam}(\eta_n) = 2^{-(n+1)}$. However, the diameter of the bounded component of $\hat{D} \setminus \eta_n$ is greater than 1 for each n. In order to prevent this from happening, we can require that $\mathbb{C} \setminus D$ be locally connected.

Definition The set V is *(uniformly) locally connected* if there exists a function $\epsilon(\delta)$ with $\epsilon(0+) = 0$ such that if $z, w \in V$ with $|z - w| \le \delta$, then there exists a closed connected set $V' \subset V$ containing z, w of diameter at most $\epsilon(\delta)$.

Indeed, suppose we knew that $\mathbb{H} \setminus D$ were locally connected with function $\epsilon(\cdot)$. Let η be a crosscut of D connecting boundary points z, w with $\delta = \operatorname{diam}(\eta)$, and let U be the bounded component of $D \setminus \eta$. Since $|z - w| \le \delta$, there exists closed $V \subset \mathbb{H} \setminus D$ containing z, w with $\operatorname{diam}(V) \le \epsilon(\delta)$. Note that U is contained in a bounded component of $\mathbb{C} \setminus (\eta \cup V)$, and hence

$$\operatorname{diam}(U) \le \operatorname{diam}(V \cup \eta) \le \delta + \epsilon(\delta).$$

The next topological lemma shows that the domains that we will be studying have locally connected complements.

Lemma 1.8.12. *If $\gamma = \gamma[0,1]$ is the image of a curve with $\gamma(0) = 0$, then γ and $\mathbb{R} \cup \operatorname{fill}[\gamma]$ are locally connected.*

Proof. Let $z \in \gamma$ and $\epsilon > 0$. let $T = \gamma^{-1}(z)$ which is a nonempty compact subset of $[0,1]$. For each $t \in T$, there exists an open interval I_t containing t such that $|\gamma(s) - z| < \epsilon/4$ for $s \in I_t$. By compactness, we can find $I_{t_1}, \ldots, I_{t_n}$ such that $I := I_{t_1} \cup \cdots \cup I_{t_n}$, covers T. Let $2\delta = \min\{|\gamma(s) - z| : s \in [0,1] \setminus I\} > 0$. If $w \in \gamma$ with $|w - z| < 2\delta$, then $w = \gamma(s)$ for some $s \in I_{t_j}$. Then $\gamma(I_{t_j})$ is a connected subset of γ containing w, z that has diameter at most $\epsilon/2$. Hence, for every $z \in \gamma$, there exists $\delta_z > 0$ such that if $|w - z| < \delta_z$, then for every w' with $|w' - w| < \delta_z$, we can find a connected subset of γ (in fact, the union of two subpaths each going through z) of diameter at most ϵ. Using compactness of γ, we can find $z_1, \ldots, z_m$ such that the open disks of radius δ_{z_j} cover γ. Let $\delta = \min \delta_{z_j}$. Then if $w, w' \in \gamma$ with $|w - w'| < \delta$, we find z_j with $|w - z_j| < \delta_j$. Since $|w - w'| < \delta \le \delta_j$, we can find a connected subset of γ including w, w' of diameter at most ϵ. Note that we made no assumptions about double points for the curve. Suppose $\operatorname{diam}\gamma \le R$. Then $[-2R, 2R] \cup \gamma$ is the image of a curve (start at $-2R$ go to $2R$ come back to 0 and then proceed along γ) and so $\gamma \cup [-2R, 2R]$ is locally connected. With this, showing that $am\gamma \cup \mathbb{R}$ is locally connected is easy.

Finally, suppose $w, w' \in \mathbb{R} \cup \operatorname{fill}[\gamma]$ with $|w - w'| < \delta$. If $\operatorname{dist}(w, \gamma \cap \mathbb{H}) \ge \delta$ or $\operatorname{dist}(w', \gamma \cap \mathbb{H}) \ge \delta$, then we can connect w, w' by the straight line segment of length $|w - w'|$. Otherwise, we connect w, w' to z, z' in $\mathbb{R} \cup \operatorname{fill}[\gamma]$ with line segments length less than δ. Therefore $|z - z'| < 3\delta$ and we can find a connected subset of $\mathbb{R} \cup \operatorname{fill}[\gamma]$ of diameter at most $\epsilon(3\delta)$ containing z, z'. The union of this subset and the two line segments is a connected subset of diameter at most $2\delta + \epsilon(3\delta)$ connecting w and w'.

□

Theorem 1.8.13. *Suppose $D = \mathbb{H} \setminus K \in \mathcal{J}$ and $g : D \to \mathbb{H}$ is a conformal transformation with $g(\infty) = \infty$. Suppose that $\mathbb{C} \setminus D$ is locally connected. Then g^{-1} can be extended to a continuous function from $\mathbb{H}$ to $\overline{D}$.*

Proof. Let $\epsilon(\delta)$ be the function as in the definition for $V = \mathbb{C} \setminus D$. Note that if η is a crosscut of D, then the bounded component of $D \setminus \eta$ must have diameter at most $\epsilon(\text{diam}(\eta))$. Let $f = g^{-1}$.

Let $l_{r,x}$ denote the crosscut in $\mathbb{H}$ given by the half-circle $l_{r,x}(t) = x + re^{it}, 0 \le t \le \pi$. We claim there exists $c < \infty$ such that for every $x \in \mathbb{R}$ and every $\rho < 1$ there exists $r = r(x, \rho)$ with $\rho \le r \le \sqrt{\rho}$ such that

$$\text{diam}(f \circ l_{r,x}) \le \frac{c}{\sqrt{\log(1/\rho)}}. \tag{1.35}$$

To see this, we first note that there exists $c_0 < \infty$ such that for all x, area$[f(\{z \in \mathbb{H} : |z - x| \le 1)] \le c_0$. Let $\Gamma = \Gamma_{\rho,x} = \{z \in \mathbb{H} : \rho \le |z - x| \le \sqrt{\rho}\}$. With aid of the Cauchy-Schwarz inequality, we see that

$$\begin{aligned}
c_0 \ge \text{area}[f(\Gamma)] &= \int_\Gamma |f'(z)|^2 \, dA(z) \\
&= \int_\rho^{\sqrt{\rho}} \left[\int_0^\pi |f'(re^{i\theta})|^2 \, d\theta \right] r \, dr \\
&\ge \int_\rho^{\sqrt{\rho}} \left[\frac{1}{\pi} \left(\int_0^\pi |f'(re^{i\theta})| \, d\theta \right)^2 \right] r \, dr \\
&\ge \int_\rho^{\sqrt{\rho}} \left[\frac{1}{\pi} \left(\int_0^\pi r\,|f'(re^{i\theta})| \, d\theta \right)^2 \right] r^{-1} \, dr \\
&\ge \frac{\log(1/\rho)}{2\pi} \inf_{\rho \le r \le \sqrt{\rho}} \left[\int_0^\pi r\,|f'(re^{i\theta})| \, d\theta \right]^2 \\
&\ge \frac{\log(1/\rho)}{2\pi} \inf_{\rho \le r \le \sqrt{\rho}} [\text{diam}(f \circ l_{r,x})]^2 .
\end{aligned}$$

This establishes the claim. This estimate was valid for all f (even if $\mathbb{C} \setminus D$ is not locally connected). If $|z - x| < r$, then $f(z)$ is in the bounded component of $f \circ l_{r,x}$. However, in our case we can conclude that diameter of this component is bounded above by

$$\epsilon\left(\frac{c}{\sqrt{\log(1/\rho)}} \right).$$

Therefore, for z, w in the bounded component of $\mathbb{H} \setminus l_{\rho,x}$,

$$|f(z) - f(w)| \le \frac{c}{\sqrt{\log(1/\rho)}} + \epsilon\left(\frac{c}{\sqrt{\log(1/\rho)}} \right),$$

which goes to zero as ρ goes to zero.

□

An important technical result was used in the last proof. From (1.35), we see that we can find half-circles l_r about the origin of radius r so that $\operatorname{diam}(f \circ l_r) \le c\sqrt{\log(1/r)}$. However, one must be careful with this. Although $\operatorname{diam}(f \circ l_r)$ is small, it is not always true that the image of the disk of radius r has small diameter.

We have restricted our consideration to domains in $\mathcal{J}$, but the argument for the last theorem is all local. Using the same argument we can get this more traditional statement of the theorem.

Theorem 1.8.14. *Suppose $f : \mathbb{D} \to D$ is a conformal transformation where D is a bounded domain with $\mathbb{C} \setminus D$ locally connected. Then f extends to a continuous function on $\overline{\mathbb{D}}$.*

Corollary 1.8.15. *Suppose that $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ and H_t is the unbounded component of $\mathbb{H} \setminus \gamma_t$. Then the inverse map $g_t^{-1} : \mathbb{H} \to D$ can be extended continuously to $\partial\mathbb{H}$. Moreover, all points of ∂H_t are accessible.*

Definition

- A curve $\gamma : [0, t_0] \to \mathbb{C}$ is called a Jordan curve, if $\gamma(0) = \gamma(t_0)$ and $\gamma(s) \neq \gamma(t)$ for $0 \le s < t < t_0$.
- A *Jordan domain* is a bounded domain D whose boundary is a Jordan curve.

The Jordan curve theorem which we will not prove here states that if γ is a Jordan curve, then $\mathbb{C} \setminus \gamma$ consists of two connected components. The bounded component is a Jordan domain.

If f in Theorem 1.8.14 is one-to-one on $\overline{\mathbb{D}}$, then $t \mapsto f(e^{it})$ gives a parameterization of ∂D as a Jordan curve. In this case f is a homeomophism of $\overline{\mathbb{D}}$ onto $\overline{D}$. (Continuity of $f^{-1} = g$ follows from the Beurling estimate as in Proposition 1.8.8.) Conversely, if we know that D is a Jordan domain, we can use Proposition 1.8.8 to see that f must be one-to-one on $\overline{\mathbb{D}}$. We end with a topological fact about domains generated by curves.

Proposition 1.8.16. *Suppose $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a curve with $\gamma(0) = 0$. Let H_t denote the unbounded component of $\mathbb{H} \setminus \gamma_t$, and*

$$H_{t-} = \bigcap_{s<t} H_s.$$

If $\gamma(t) \in H_{t-}$, then there is a single prime end of H_t associated to $\gamma(t)$.

Proof. Suppose $\gamma(t) \in H_{t-}$ and that there are at least two prime ends. We know that $\gamma(t)$ is an accessible point and hence there exists simple $\eta : (0, 1) \to H_t$ with with $\eta(0+) = \eta(1-) = \gamma(t)$. Since $\gamma(t) \in H_{t-}$, we see that $\eta \subset H_s$ for all $s < t$. Let V be the bounded component of $\mathbb{C} \setminus \eta$. Since H_s is simply connected for $s < t$, we see that $\gamma_s \cap V = \emptyset$ for $s < t$ and hence $\gamma_t \cap V = \emptyset$. Since V is connected we see that either $V \subset H_t$ or $V \cap H_t = \emptyset$. If $V \cap H_t = \emptyset$, then since H_t is open, $\overline{V} \cap H_t = \emptyset$. In particular $\eta \cap H_t = \emptyset$ which is a contradiction. Therefore, we know that $V \subset H_t$.

Since $V \subset H_t$, if $\zeta \in V$, a Brownian motion starting at ζ cannot reach ∂H_t without hitting η. This must also be true for $g_t(\zeta)$ and $g_t \circ \eta$ which implies that $g_t(\eta(0+)) = g_t(\eta(1-))$. Hence both endpoints give the same prime end.

□

1.8.4 Curves

In this section, we let $\gamma : (0, \infty) \to \mathbb{C}$ be a simple curve with $\gamma(0+) = 0$. For each t, let $\gamma_t = \gamma[0, t]$ which is a compact $\mathbb{H}$-hull with $D_t = \mathbb{H} \setminus \gamma_t$ simply connected. Let $g_t = g_{\gamma_t}$ be the corresponding map which has an expansion at infinity

$$g_t(z) = z + \frac{a_t}{z} + O(|z|^{-2}).$$

This expression defines a_t; in fact, as we have seen $a_t = \text{hcap}[\gamma_t]$. By (1.32), we see that a_t is strictly increasing in t. We will make the further assumption that

$$\lim_{t\to\infty} a_t = \infty.$$

This requires $\limsup_{t\to\infty} |\gamma(t)| = \infty$, although this last condition is not quite sufficient. Let $\tau_t = \tau_{D_t}$. The next proposition will show that a_t is a continuous function of t. It uses the Beurling estimate.

Lemma 1.8.17. *There exists $c < \infty$ such that for every γ, if $s < t$,*

$$\text{diam}\,(g_s[\gamma_t \setminus \gamma_s]) \le c\,\sqrt{\text{diam}(\gamma_t)}\,\sqrt{\text{diam}(\gamma[s,t])}.$$

Proof. Let $V = V_{s,t} = g_s[\gamma_t \setminus \gamma_s], u = \text{diam}[\gamma_t], r = \text{diam}(\gamma[s,t]) \le u$. By Lemma 1.8.4, $\text{cap}_{\mathbb{H}}(V) \asymp \text{diam}(V)$. By definition,

$$\text{cap}_{\mathbb{H}}(V) = \lim_{y\to\infty} y\,\mathbb{P}^{iy}\{B_{\tau_{\mathbb{H}\setminus V}} \in V\}.$$

Using the expansion of g_s at infinity and conformal invariance and the expansion $g_s(iy) = i[y - \text{hcap}(\gamma_s)y^{-1}] + O(y^{-2})$, we see that

$$\lim_{y\to\infty} y\,\mathbb{P}^{iy}\{B_{\tau_t} \in \gamma[s,t]\} = \lim_{y\to\infty} y\,\mathbb{P}^{g_s(iy)}\{B_{\tau_{\mathbb{H}\setminus V}} \in V\} = \lim_{y\to\infty} y\,\mathbb{P}^{iy}\{B_{\tau_{\mathbb{H}\setminus V}} \in V\} = \text{cap}_{\mathbb{H}}(V).$$

We will now estimate $\mathbb{P}^{iy}\{B_{\tau_t} \in \gamma[s,t]\}$ for large y. In order for $B_{\tau_t} \in \gamma[s,t]$, it is necessary for the Brownian motion starting at iy to reach the disk of radius $2u$ about the origin without leaving $\mathbb{H}$. The probability of this is $O(u/y)$. Given this, the Brownian motion must reach the disk of radius r about $\gamma(s)$ without leaving D_t. By the Beurling estimate, this probability is bounded by a constant times $\sqrt{r/u}$. Therefore

$$\lim_{y\to\infty} y\,\mathbb{P}^{iy}\{\tau_t < \tau_s\} \le c\,\sqrt{ru}.$$

□

It follows that we have an estimate

$$a_t - a_s \le c\,\text{diam}(\gamma_t)\,\text{diam}(\gamma[s,t]).$$

In particular, a_t is a continuous function of t and we can reparametrize the curve so that $\text{hcap}[\gamma_t] = 2t$.

Definition The curve γ has the *(standard) capacity parametrization* if $\text{hcap}[\gamma_t] = 2t$ for all t.

The choice of the constant 2 is somewhat arbitrary although we will see reasons later why this is a natural choice. More generally, we will say that γ is parametrized by capacity with rate a if $\operatorname{hcap}[\gamma_t] = at$. For now assume that we have the standard capacity parametrization so that

$$g_t(z) = z + \frac{2t}{z} + O(|z|^{-2}), \quad z \to \infty,$$

Proposition 1.8.16 tells us that there is only one prime end associated to the tip $\gamma(t)$, that is, if $z_n \in D_t$ with $z_n \to \gamma(t)$, then the limit

$$g(\gamma_t(t)) = \lim_{n\to\infty} g_t(z_n)$$

exists and the limit is independent of the sequence. We will denote the limit by U_t.

Theorem 1.8.18 (Half plane Loewner differential equation)**.** *Suppose γ is a simple curve as above parameterized so that* $\operatorname{hcap}[\gamma_t] = 2t$. *Then every $z \in \mathbb{H}$ the flow $t \mapsto g_t(z)$ satisfies*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad 0 \le t < T_z,$$

where $U_t = g_t(\gamma(t)), T_z = \inf\{s : \gamma(s) = z\}$. Moreover, the function $t \mapsto U_t$ is continuous.

Proposition 1.8.16 shows that there is one prime end of $\mathbb{H} \setminus \gamma_t$ at $\gamma(t)$ and hence $g_t(\gamma(t))$ is well defined. Our estimate will focus on the right time derivative. In order to convert the result to a usual derivative we will use this easy lemma.

Lemma 1.8.19. *Suppose $u : [0, \infty) \to \mathbb{R}^n$ is a continuous function whose right derivative*

$$u'_+(t) = \lim_{s\downarrow 0} \frac{u(s+t) - u(t)}{s},$$

exists for all t. Suppose also that $t \mapsto u'_+(t)$ is continuous. Then f is differentiable with $u'(t) = u'_+(t)$.

Proof. It suffices to prove the result when $u(0) = 0, u'_+ \equiv 0$ for then (using continuity of u'_+) we can consider

$$f(t) = u(t) - u(0) - \int_0^t u_+(s)\, ds.$$

Let $\epsilon > 0$ and let $\sigma = \sigma_\epsilon = \inf\{t : |u(t)| > \epsilon\, t\}$. Since $u'_+(0) = 0$, we can see that $\sigma > 0$. Suppose $\sigma < \infty$. By continuity of u, we can see that $|u(\sigma)| = \epsilon\, \sigma$. However, since $u'_+(\sigma) = 0$, there exists $\delta > 0$ such that $|u(\sigma + s) - u(\sigma)| < \epsilon s$ for $0 \le s < \delta$. This implies that $|u(\sigma + s)| \le \epsilon(\sigma + s)$ for $0 < s < \delta$ which contradicts the definition of σ. Therefore $\sigma = \infty$. Since this is true for every ϵ, $u \equiv 0$. □

Proof of Theorem 1.8.18. Using Lemma 1.8.7, we can see that $\operatorname{diam}[g_t(\gamma_t)] \le 4 \operatorname{diam}[\gamma_t]$ and hence $|U_t - U_0| \le 4 \operatorname{diam}(\gamma_t)$. More generally, if $s < t$,

$$|U_t - U_s| \le 4 \operatorname{diam}[g_s(\gamma_t \setminus \gamma_s)]).$$

Combining this with Lemma 1.8.17, we see that $t \mapsto U_t$ is continuous. Similarly, we see that for fixed z, $g_t(z)$ is continuous in t. Therefore, by Lemma 1.8.19, it suffices to establish the result for the right derivative. But this follows from (1.33).

□

1.8.5 Loewner differential equation

In the last section we started with a curve γ in the upper half plane which corresponded to a parametrized family of conformal maps. We then showed that the conformal maps satisfy a particular differential equation, In the next proposition, we start with a continuous function $t \mapsto U_t$ and find the appropriate maps. It will be useful to adopt the notation that dots refer to t-derivatives and primes refer to z-derivatives.

Theorem 1.8.20. *Suppose that $t \mapsto U_t$ is a continuous real valued function. For each $z \in \mathbb{C} \setminus \{0\}$, let $g_t(z)$ denote the solution to the initial value problem*

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z. \tag{1.36}$$

- *For each z, the solution exists up to time $T_z \in (0, \infty]$ defined to be the smallest t such that*

$$\inf\{s < t : |g_s(z) - U_s|\} = 0.$$

- *If $z \in \mathbb{H}$, then $\operatorname{Im}[g_t(z)]$ decreases with t and if $t < T_z$, $\operatorname{Im}[g_t(z)] > 0$.*
- *For all z, $T_z = T_{\overline{z}}$ and if $t < T_z$, $g_t(\overline{z}) = \overline{g_t(z)}$.*
- *Let $H_t = \{z \in \mathbb{H} : T_z > t\}$. Then g_t is the unique conformal transformation of H_t onto $\mathbb{H}$ satisfying $g_t(z) - z \to 0$ as $z \to \infty$. Moreover g_t has the expansion*

$$g_t(z) = z + \frac{2t}{z} + O(|z|^{-2}), \quad z \to \infty. \tag{1.37}$$

Proof. We write $g_t(z) = u_t(z) + iv_t(z)$ and note that (1.36) can be written as

$$\dot{u}_t(z) = \frac{2[u_t(z) - U_t]}{|g_t(z) - U_t|^2}, \quad \dot{v}_t(z) = -\frac{2\, v_t(z)}{|g_t(z) - U_t|^2}.$$

In particular, $|v_t(z)|$ decreases with t. Since $|g_t(z) - U_t|^2 \geq v_t(z)^2$, these equations have solutions up to time T_z and we can write

$$g_t(z) = \int_0^t \frac{2\, ds}{g_s(z) - U_s},$$

Differentiating with respect to z gives

$$\dot{g}'_t(z) = -\frac{2\, g'_t(z)}{(g_t(z) - U_t)^2}, \quad g'_t(z) = \exp\left\{ -\int_0^t \frac{2\, ds}{(g_s(z) - U_s)^2} \right\}.$$

This shows that g_t is holomorphic on $\{T_z > t\}$ and since

$$\partial_t \left[g_t(z) - g_t(w) \right] = \frac{2[g_t(w) - g_t(z)]}{[g_t(z) - U_t]\, [g_t(w) - U_t]},$$

we get

$$g_t(z) - g_t(w) = (w - z)\, \exp\left\{ \int_0^t \frac{2\, ds}{[g_s(z) - U_s]\, [g_s(w) - U_s]} \right\},$$

from which we can deduce that g_t is one-to-one on $\{T_z > t\}$. If $z \in H_t$ and we define $\tilde{g}_t(\overline{z}) = \overline{g_t(z)}$, it is immediate that $\tilde{g}_t$ satisfies Loewner and hence $\tilde{g}_t(\overline{z}) = g_t(\overline{z})$.

To show that $g_t(H_t) = \mathbb{H}$ we "reverse the flow". For fixed t_0 and $z \in \mathbb{H}$, consider the differential equation

$$\dot{h}_t(z) = -\frac{2}{h_t(z) - V_t}, \quad h_0(0) = z.$$

where $V_t = U_{t_0-t}, 0 \le t \le t_0$. Note that $\mathrm{Im}[h_t(z)]$ increases with t so this solution exists for all times t with

$$h_t(z) = z - \int_0^t \frac{2}{h_s(z) - V_s}\, ds.$$

Also $\phi_t(z) := h_{t_0-t}(z)$ satisfies

$$\dot{\phi}_t(z) = \frac{2}{h_{t_0-t}(z) - V_{t_0-t}} = \frac{2}{\phi_t(z) - U_t},$$

with $\phi_0(z) = h_{t_0}(z), \phi_{t_0}(z)$. In other words, $g_{t_0}(h_{t_0}(z)) = z$.

To get the expansion at infinity note that

$$\dot{g}_t(z) = \frac{2}{g_t(z)} \left[1 + O(|z|^{-1})\right],$$

where the $O(\cdot)$ term depends on $U_s, 0 \le s \le t$. If we fix t and let $z \to \infty$ we get (1.37).

□

In the proof there was another construction which turns out to be useful, the *reverse Loewner equation*

$$\dot{h}_t(z) = -\frac{2}{h_t(z) - V_t}, \quad h_0(0) = z. \tag{1.38}$$

Theorem 1.8.21 (Reverse Loewner flow)**.** *Suppose $t \mapsto V_t$ is a continuous real-valued function and h_t is the solution to* (1.38)*. Then if $z \in \mathbb{H}$, the solution exists for all times t. Moreover, for each t, h_t is a conformal transformation $h_t : \mathbb{H} \to h_t(\mathbb{H})$ where $h_t(\mathbb{H}) \subset \mathbb{H}$ with $\mathbb{H} \setminus h_t(\mathbb{H})$ bounded. It satisfies*

$$h_t(z) = z - \frac{2t}{z} + O(|z|^{-2}), \quad z \to \infty.$$

Moreover, if $t_0 < \infty$, $U_t = V_{t_0} - t, 0 \le t \le t_0$, and g_t is the solution to (1.36)*, then $h_{t_0} = g_{t_0}^{-1}$.*

The proof of this is essentially in the last proof. We remark that if $x \in \mathbb{R}$, the solution of (1.38) exists up to some time T_x but it is possible (and usually true) that $T_x < \infty$. Note that we have found a way to get $g_{t_0}^{-1}$ using the reverse flow. However, t_0 was used in the definition of V_t, and h_t for other values of t does not equal g_t^{-1}.

There is another way to get the inverse function of g_t which we now demonstrate. If we let $f_t = g_t^{-1}$ and use the chain rule to differentiate the equation

$$f_t(g_t(z)) = z$$

with respect to t, we get

$$\dot{f}_t(g_t(z)) + f_t'(g_t(z))\, \dot{g}_t(z) = 0.$$

Here we are writing dots for t-derivatives and primes for z-derivatives. If g_t satisfies (1.36) and we write $w = g_t(z)$, then we get

$$\dot f_t(w) = -f_t'(w)\,\dot g_t(z) = -f_t'(w)\,\frac{2}{w-U_t}, \quad f_0(w) = w. \tag{1.39}$$

We call this the *inverse Loewner equation*. At each time t, f_t is a conformal transformation of $\mathbb{H}$ onto a domain $f(\mathbb{H})$.

For future use, we prove the following proposition.

Lemma 1.8.22. *There exists $c < \infty$ such that if f is the solution to the inverse Loewner equation, then for all t, and all $w = x + iy \in \mathbb{H}$,*

$$e^{-10s/y^2}\,|f_t'(w)| \le |f_{t+s}'(w)| \le e^{10s/y^2}\,|f_t'(w)|.$$

$$|f_{s+t}(w) - f_t(w)| \le \frac{\mathrm{Im}(w)\,[e^{10s/y^2}-1]}{5}\,\left[|f_{s+t}'(w)| \wedge |f_t'(w)|\right].$$

Proof. By differentiating both sides of (1.39), we see that

$$\dot f_t'(w) = f_t'(w)\,\frac{2}{(w-U_t)^2} - f_t''(w)\,\frac{2}{w-U_t}.$$

The Bieberbach estimate on the second coefficient of schlicht functions, shows that if h is a univalent function on $\mathbb{D}$, then $|h''(0)| \le 4\,|h'(0)|$. Applied to the disk of radius y about w, we see that $|f''(w)| \le 4\,y^{-1}|f'(w)|$, and hence

$$|\partial_t \log|f_t'(w)|| \le 10\,y^{-2}, \quad e^{-10s/y^2}\,|f_t'(w)| \le |f_{t+s}'(w)| \le e^{10s/y^2}\,|f_t'(w)|.$$

$$|\dot f_{t+s}(w)| \le \frac{2\,|f_{t+s}'(w)|}{|U_{t+s}-w|} \le 2\,e^{10s/y^2}\,y^{-1}\,|f_t'(w)|$$

$$\begin{aligned} |f_{t+s}(w) - f_t(w)| &\le \int_0^s |\dot f_{t+r}(w)|\,dr \\ &\le 2\,\left(|f_t'(w)| \wedge |f_{t+s}'(w)|\right)\,y^{-1}\int_0^s e^{10r/y^2}\,dr \\ &= \left(|f_t'(w)| \wedge |f_{t+s}'(w)|\right)\,\frac{y\,[e^{10s/y^2}-1]}{5}. \end{aligned}$$

□

1.8.6 Loewner chains generated by a curve

Definition Suppose $U_t, 0 \le t \le T$ is a continuous real valued function.

- The collection of conformal maps g_t obtained from (1.36) is called a *Loewner chain*.
- The function U_t is called the *driving function* for the chain.

- We will say that t is an *accessible time* for the driving function U if the limit

$$\gamma(t) = \lim_{y \downarrow 0} g_t^{-1}(U_t + iy). \tag{1.40}$$

  exists. We say that a driving function is *(everywhere) accessible* if all times are accessible.

- We say that z is a *pioneer point* for the chain at time $t > 0$ if $z \in H_s$ for all $s < t$ and $z \in \partial H_t$. The *pioneer set* γ_t at time t is the set of pioneer points for all $0 \leq s \leq t$.

- We say that g_t is *generated by a curve* or U_t *generates a curve* if U_t is everywhere accessible and $\gamma(t), 0 \leq t \leq T$ is a continuous function of t. Equivalently, $\gamma_t = \gamma[0,t]$.

- We will call a curve γ a *pioneer curve* if all $\gamma(t)$ are pioneer points at time t. This is equivalent to saying that γ is simple and $\gamma(0,T] \subset \mathbb{H}$.

The term "pioneer curve" is not standard but we do not want to have to say the phase "γ is simple and $\gamma(0,T] \subset \mathbb{H}$."

Recall that (1.40) holds if and only if there exists some simple curve $\eta : (0,1] \to \mathbb{H}$ with $\eta(0+) = U_t$ such that

$$\gamma(t) = \lim_{s \downarrow 0} g_t^{-1}(\eta(s)).$$

It is not true that the limit on the right-hand side of (1.40) exists for all t for all continuous functions U_t. It is also possible that the limit is not a continuous function of t. However, we will show that under some regularity assumptions on U_t this does hold. We will state sufficient, but not necessary, conditions. A necessary and sufficient condition for a curve to be a pioneer curve is that for each s, $g_s(\gamma(s,T]) \subset \mathbb{H}$. Indeed, if $r < s$ and $\gamma(r) = \gamma(s)$ and $r < t < s$, then $g_t(\gamma(t,T]) \cap \mathbb{R} \neq \emptyset$,

Theorem 1.8.23. *Suppose $c_0 < 4$ and for all s, t,*

$$|U_s - U_t| \leq c_0 \, |t - s|^{1/2}. \tag{1.41}$$

Then the limit in (1.40) *exists for all t, and γ is a pioneer curve.*

To understand why the condition $|U_s - U_t| \asymp \sqrt{|t-s|}$ should be critical for the Loewner equation, consider the case $U_t \equiv 0$ for which $g_t(z) = \sqrt{z^2 + 4t}$. This is generated by the vertical curve $\gamma(t) = \sqrt{4t}\, i$. In time t, the curve moves distance $O(\sqrt{t})$ from the origin. Now suppose that U_t is not constant. If U_t grows slower than $\sqrt{t}$, then the horizontal effect will not be enough to bring the curve down to the real line. If $U_t \gg \sqrt{t}$, then there may be problems. This is why $\Delta(r)$ as defined in Proposition 1.8.24 is a natural quantity for driving functions of the Loewner equation.

Note that it suffices to assume that (1.41) holds for all s, t with $|t-s|$ sufficiently small. This theorem is not true for all values of c_0. One can find c_0 and driving function U_t satisfying (1.41) for which the limit (1.40) does not exists for all t. We will prove the theorem in a series of propositions. The first two will show that the chain is generated by a curve and the last will show that the curve is a pioneer curve. The The first proposition is stronger than we need for this section; however, the stronger version will be used when we consider the Schramm-Loewner evolution so we prove it now.

Proposition 1.8.24. *There exists $c < \infty$ such that the following holds. Suppose $U_s, 0 \le s \le 1$, satisfies* (1.41) *for all $0 \le s, t \le 1$, and let*

$$\Delta(r) = 1 + \max\left\{ \frac{|U_t - U_s|}{\sqrt{t-s}} : 0 \le s < t \le 1,\ t - s \ge r \right\},$$

$$I(y) = \sup_{0 \le t \le 1} \int_0^y |f_t'(U_t + ir)|\, dr.$$

If $I(y) < \infty$, then the limit (1.40) *exists for all $0 \le t \le 1$ and*

$$|\gamma(t) - \gamma(s)| \le c_1\, I(\sqrt{t-s})\, \Delta(t-s)^4, \quad 0 \le s < t \le 1.$$

In particular, if

$$\lim_{r \downarrow 0} I(\sqrt{r})\, \Delta(r)^4 = 0,$$

then γ is a curve.

Note that if U_t satisfies (1.41), then $\Delta(r)$ is uniformly bounded. Another important case for use will be when U_t is a Brownian motion path for which $\Delta(r) \le O(\sqrt{\log(1/r)})$ as $r \downarrow 0$.

Proof. Let $\hat{f}_t(z) = f_t(U_t + z)$. The existence of the limit (1.40) follows immediately from finiteness of $I(y)$ with

$$|\gamma(t) - \hat{f}_t(iy)| \le I(y).$$

The distortion theorem implies that $|\hat{f}_t'(iy')| \asymp |\hat{f}_t'(iy)|$ for $y/2 \le y' \le 2y$, and hence

$$I(y) \ge \int_{y/2}^{y} |\hat{f}_t'(ir)|\, dr \ge c_2\, y\, |\hat{f}_t'(iy)|.$$

Suppose $0 \le s \le t \le s + \delta^2 \le 1 + \delta^2$. The triangle inequality implies that $|\gamma(s) - \gamma(t)|$ is bounded above by

$$|\gamma(s) - \hat{f}_s(i\delta)| + |\gamma(t) - \hat{f}_t(i\delta)| + |\hat{f}_s(i\delta) - \hat{f}_t(i\delta)| \le 2I(\delta) + |\hat{f}_s(i\delta) - \hat{f}_t(i\delta)|,$$

$$|\hat{f}_s(i\delta) - \hat{f}_t(i\delta)| \le |f_s(U_s + i\delta) - f_s(U_t + i\delta)| + |f_s(U_t + i\delta) - f_t(U_t + i\delta)|.$$

Lemma 1.8.22 shows that $|f_s(U_t + i\delta) - f_t(U_t + i\delta)| \le c\,\delta\, |f_t'(U_t + i\delta)| \le c\, I(\delta)$. Using the distortion theorem as in (1.27) and $|s - t| \le \delta^2$, we see that

$$|f_s(U_s + i\delta) - f_s(U_t + i\delta)| \le c\,\delta \left[1 + \frac{|U_t - U_s|^4}{\delta^4}\right] |f_s'(U_s + i\delta)| \le c\, \Delta(\delta^2)^4\, I(\delta).$$

□

We will consider the reverse Loewner flow

$$\dot{h}_t(z) = -\frac{2}{h_t(z) - U_t}$$

where U_t satisfies (1.41). If $z \in \mathbb{H}$, let $Z_t = Z_t(z) = h_t(z) - U_t$, $X_t = \mathrm{Re}[Z_t]$, $Y_t = \mathrm{Im}[Z_t]$. Then we can write the equation as

$$\partial_t \left[X_t + U_t\right] = -\frac{2\,X_t}{X_t^2 + Y_t^2}, \quad \partial_t Y_t = \frac{2\,Y_t}{X_t^2 + Y_t^2}. \tag{1.42}$$

Note that

$$\partial_t[\log h_t'(z)] = \frac{2}{Z_t^2}, \quad \partial_t[\log|h_t'(z)|] = \frac{2\,(X_t^2 - Y_t^2)}{|Z_t|^4}, \quad \log|h_t'(z)| = \int_0^t \frac{2\,(X_s^2 - Y_s^2)}{(X_s^2 + Y_s^2)^2}\,ds. \tag{1.43}$$

Example Example. If $U_t = 2b\sqrt{t}$ with $0 \le b < 2$, then the solution of (1.42) satisfying $X_0 = 0, Y_0 = 0$ is

$$X_t = -b\sqrt{t}, \quad Y_t = \sqrt{4 - b^2}\,\sqrt{t}.$$

Hence $h_\epsilon(0) = z_\epsilon + U_\epsilon := \sqrt{\epsilon}\,[b + i\sqrt{4 - b^2}]$. Let us write $h_t = h_{t,\epsilon} \circ h_\epsilon$. and hence if $t > \epsilon$,

$$|h_{t,\epsilon}'(z_\epsilon)| = \exp\left\{\int_\epsilon^t \frac{2\,(X_s^2 - Y_s^2)}{(X_s^2 + Y_s^2)^2}\,ds\right\} = \exp\left\{\int_\epsilon^t \frac{2\,(2b^2 - 4)}{16\,(s + \epsilon)}\,ds\right\} = \left(\frac{t}{\epsilon}\right)^{\frac{b^2-2}{4}}.$$

Using the reverse Loewner equation and the distortion principle, we can see that $|h_\epsilon'(\sqrt{\epsilon}i)| \asymp 1$ and $|h_{t,\epsilon}'(h_\epsilon(\sqrt{\epsilon}i))| \asymp |h_{t,\epsilon}'(z_\epsilon)|$, and therefore, $|h_t'(\sqrt{\epsilon}i)| \asymp |h_{t,\epsilon}'(z_\epsilon)|$. Ig g_t is a solution of the Loewner equation (1.36) with $U_t = 2b\sqrt{1-t}, 0 \le t \le 1$, then the distribution of $f_1 := g_1^{-1}$ is the same as that of h_1 above. In particular, using the distortion principle, we can see that

$$|f_1'(iy)| \asymp y^{\frac{2-b^2}{2}}, \quad y \downarrow 0.$$

Proposition 1.8.25. *For each $c_0 < 4$, there exists $\theta < 1$ and $c < \infty$ such that if U_t satisfies* (1.41)*, then for all $0 \le t \le 1$ and all $y \le 1$,*

$$|f_t'(iy)| \le c\,y^{-\theta}, \quad I(y) \le c\,y^{1-\theta}, \quad \theta = 1 - \frac{c_0^2}{16}.$$

Proof. We write $c_0 = 2b$. Consider the equation,

$$\partial_t X_t = -\frac{2(b^2/4)}{X_t} - \partial + tU_t,$$

under the constraint $U_t \le 2b\sqrt{t}$. To maximize $|X_t|$ under these constraints, we choose U_t with constant sign and maximal absolute value. If we choose $U_t = 2b\sqrt{t}$, and let $R_t = X_t + U_t$, then the solution is $X_t = -b\sqrt{t}$. Hence for any U_t satisfying the condition, we have $X_t^2 \le b^2\,t$. If we assume that $X_0 = 0, Y_0 > 0$, then by induction, we see that for all t, $Y_t^2 \ge \frac{4-b^2}{b^2}\,X^2$, and hence $Y_t^2 \ge (4 - b^2)\,t$. The derivative estimate then follows as in (1.43) □

Proposition 1.8.26. *Suppose U_t satisfies* (1.41) *with $c_0 < 4$ and $U_0 = 0$, and u_t satisfies*

$$\partial_t u_t = \frac{2}{u_t - U_t}, \quad 0 \le t \le r^2,$$

with $u_0 > 0$. Then $u_t > U_t$ for $0 \le t \le r^2$.

Proof. If $U_t, 0 \le t \le r^2$ satisfies (1.41) and $\tilde{u}_t = r^{-1} u_{r^2 t}(z/r)$, then

$$\partial_t \tilde{u}_t = \frac{2}{\tilde{u}_t - \tilde{U}_t},$$

where $\tilde{U}_t = r^{-1} U_{r^2 t}$. Since, $\tilde{U}_t$ satisfies (1.40), it suffices to prove the result for $r^2 = 1$.

If $U_t \le 0$ for $0 \le t \le \delta$, then we can see that $u_t - U_t$ is locally strictly increasing; hence we can wait until it returns to value u_0 again. More generally, we can see that we may assume that U_t is nondecreasing. In order to minimize $u_t - U_t$ subject to $U_1 = \beta c_0$, we need to choose U_t minimal under the constraints of (1.40) and monotonicity. Therefore, the minimizer is given by a function

$$U_t = \begin{cases} 0, & t \le 1 - \beta^2 \\ c_0 \left[\beta - \sqrt{1-t}\right], & 1 - \beta^2 \le t \le t. \end{cases}.$$

for some $0 < \beta \le 1$. Then $u_{1-\beta^2} = \sqrt{u_0^2 + 1 - \beta^2}$. If $X_t = u_t - U_t$, then

$$\partial_t X_t = \frac{2}{X_t} - \frac{(c_0/2)}{\sqrt{1-t}}, \quad 1 - \beta^2 \le t \le 1.$$

So we need to see that solutions to this with $X_{1-\beta^2} 0$ satisfy $X_1 > 0$. Let $\phi(t) = X_t/\sqrt{1-t}$ which satisfies

$$\partial_t \phi(t) = \frac{1}{1-t}\left[\frac{2}{\phi(t)} - \frac{c_0}{2} + \frac{\phi(t)}{2(1-t)}\right] \ge \frac{2 - \frac{c_0}{2}}{1-t}.$$

Since $c_0 < 4$, $\phi(1-) = \infty$ and we can find t with $X_t \ge 2c_0\sqrt{1-t}$. Hence

$$X_1 \ge X_t - [U_t - U_t] \ge c_0 \sqrt{1-t} > 0.$$

□

Non-crossing curves

In this section, we assume that $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a continuous curve. We allow self-intersections and intersections with the boundary. As before, we write $\gamma_t = \gamma[0, t]$ and H_t for the unbounded component of $\mathbb{H} \setminus H_t$. Let $a(t) = \mathrm{hcap}(H_t)$ which is a continuous function of t. Let $g_t : H_t \to \mathbb{H}$ be the unique conformal transformation with $g_t(z) - z \to 0$ as $z \to \infty$.

- **Assumption 1.** The function $t \mapsto a(t)$ is strictly increasing with $a(t) \to \infty$ as $t \to \infty$.

If follows from this assumption that for all t and all δ, $\gamma(t, t+\delta] \cap H_t \ne \emptyset$. This assumption prevents the path from going into the complement of H_t and reappearing somewhere else. An example of a curve in the upper half plane that does not satisfy Assumption 1 is a Brownian

excursion from 0 to ∞. If γ satisfies Assumption 1, then we can reparametrize γ so that it satisfies $a(t) = 2t$ for all t.

We also do not want the curve to jump from one side of a domain to another. This condition is expressed most easily in terms of prime ends. Since $\mathbb{C} \setminus H_t$ is locally connected, the point $\gamma(t)$ is accessible from H_t. (This essentially also follows from the fact that $\gamma(t, \infty)$ accesses $\gamma(t)$; however, since $\gamma(t, \infty)$ is not simple and may hit ∂H_t, we need a little more argument to prove accessibility.) Each prime end of H_t with endpoint $\gamma(t)$ is associated to a point on the real line by the map g_t.

- **Assumption 2.** For each t there is a prime end of H_t at $\gamma(t)$ which we associate to $U_t \in \mathbb{R}$ such that the following holds. For every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < s < \delta$ and $\gamma(t + s) \in H_t$, then $|U_t - g_t(\gamma(t + s))| < \epsilon$. (We write this as $g_t(\gamma(t+)) = U_t$.) Moreover, U_t is a continuous function of t.

Definition A function $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is called a *non-crossing* curve if it satisfies Assumptions 1 and 2.

If in Assumption 2 we had also put the condition that $\gamma(t, t + \delta) \cap H_t \neq \emptyset$ for all δ, then we would have $a(t)$ is strictly increasing. However, we would need to separately include the condition $a(t) \to \infty$, so we have made Assumption 1 as an assumption.

The following is proved in exactly the same way as Theorem 1.8.18. We do emphasize one difference. For a simple curve, the continuity of U_t was not assumed but rather was proved. For the theorem below, continuity of U_t is one of the assumptions.

Theorem 1.8.27. *Suppose $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a non-crossing curve as above parameterized so that $\operatorname{hcap}[\gamma_t] = 2t$. Then every $z \in \mathbb{H}$ the flow $t \mapsto g_t(z)$ satisfies*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad 0 \leq t < T_z,$$

where $U_t = g_t(\gamma(t+)), T_z = \inf\{s : \gamma(s) = z\}$.

The converse is the following.

Theorem 1.8.28. *Suppose U_t is a continuous real-valued function of t and g_t is the solution to the Loewner equation (1.36). Suppose there exists a continuous function $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ such that for all t, H_t is the unbounded component of $\mathbb{H} \setminus \gamma_t$. Then γ is a non-crossing curve.*

1.8.7 Perturbation of Maps

In this section we will assume that $U_t, 0 \leq t \leq t_0$ is a continuous real valued function with $U_0 = 0$ and g_t is the solution to the Loewner equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Let H_t denote the corresponding domains and let $K_t = \overline{\mathbb{H} \setminus H_t}$ be the hulls. Let $\mathcal{N}$ be domain symmetric about the real axis containing K_{t_0} and suppose that $\Phi : \mathcal{N} \to \Phi(\mathcal{N})$ conformal transformation with $\Phi(\mathbb{R} \cap \mathcal{N}) \subset \mathbb{R}, \Phi(\mathbb{H} \cap \mathcal{N}) \subset \mathbb{H}$.

Let $K_t^* = \Phi(K_t)$ and $H_t^* = \mathbb{H} \setminus K_t^*$. (It might be tempting to write $H_t^* = \Phi(H_t)$ but we are not assuming that Φ is defined on all of H_t.) Let $g_t^* : H_t^* \to \mathbb{H}$ be the unique conformal transformation with $g_t^*(z) = z + o(1)$ as $z \to \infty$. Then,

$$g_t^*(z) = z + \frac{a^*(t)}{z} + O(|z|^{-2}),$$

where $a^*(t) = \text{hcap}[K_t^*]$. As we will see below, it is not the case that $a^*(t) = 2t$. We define

$$\Phi_t(z) = g_t^* \circ \Phi \circ g_t^{-1}(z).$$

Using the crosscuts as in (1.35), we can see that there exists $\epsilon > 0$ such that for each $t \le t_0$, the map Φ_t is a conformal transformation of $\{z \in \mathbb{H} : |z - U_t| < \epsilon\}$ with $\lim_{y \downarrow 0} \text{Im}[\Phi_t(x + iy)] = 0$. Hence Φ_t extends to a conformal transformation of $\{z \in \mathbb{C} : |z - u_t| < \epsilon\}$. Let $U_t^* = \Phi_t(U_t)$.

It may be worth stopping and thinking about this. There exists a sequence of half circles η_n of smaller and smaller radius around U_t such that $\text{diam}[g_t^{-1}(\eta_n)]$ goes to zero. Using the Beurling estimate, say, we can then see that

$$\text{diam}\left[g_t^* \circ \Phi \circ g_t^{-1}(\eta_n)\right] \to 0,$$

and hence $g_t^* \circ \Phi \circ g_t^{-1}$ is well defined near U_t. However, the diameter of $g_t^{-1}\{z \in \mathbb{H} : |z - U_t| < r\}$ does not necessary go to zero as r goes to zero.

The Cauchy integral formula implies that for s sufficiently small,

$$\Phi'_{t+s}(U_t) = \frac{1}{\pi \epsilon i} \int_C \frac{\Phi_{t+s}(z)}{z - U_t}\, dz,$$

where C denotes the circle of radius $\epsilon/2$ about U_t. Using this, we can see that $s \mapsto \Phi'_{t+s}(U_t)$ is continuous (it is, in fact, differentiable), and from this we can see that $t \mapsto \Phi'_t(U_t)$ is continuous. (This latter function is not necessarily differentiable since the map $t \mapsto U_t$ does not have to be differentiable.)

Proposition 1.8.29. *Under the assumptions above,*

$$\partial_t a^*(t) = 2\, \Phi'_t(U_t)^2,$$

Proof. Since the right-hand side is continuous, it suffices to show that the right-derivative of $a^*(t)$ equals $2\, \Phi'_t(U_t)^2$. The argument is the same for all t, so for ease let us compute the right-derivative at the origin. By scaling and translation, we may assume that $\Phi = \Phi_0$ is defined in the unit disk with $\Phi(0) = 0$, $\Phi'_0(0) = 1$. This then reduces to the next lemma. □

Lemma 1.8.30. *There exist $c < \infty$ such that the following holds. Let $\Phi : \mathbb{D} \to \Phi(\mathbb{D})$ be a conformal transformation with $\Phi(\mathbb{D} \cap \mathbb{H}) = \mathbb{H} \cap \Phi(\mathbb{D})$, $\Phi(0) = 0, \Phi'(0) = 1$. Then for every compact hull K with $\operatorname{rad}(K) \le 1/2$,*

$$|\operatorname{hcap}[\Phi(K)] - \operatorname{hcap}(K)| \le c \sqrt{\operatorname{rad}(K)}\, \operatorname{hcap}(K).$$

Proof. Let $h = \operatorname{hcap}(K), r = \operatorname{rad}(K), \tilde{K} = \Phi(K)$, $q = \sqrt{r}$. Distortion estimates tell us that there exists $c < \infty$ (uniform over all such Φ) such that for $|z| \le 1/2$,

$$|\Phi(z) - z| \le c\,|z|^2, \qquad |\operatorname{Im}[\Phi(z)] - \operatorname{Im}(z)| \le c\,|z|\operatorname{Im}(z). \tag{1.44}$$

In particular, $\operatorname{rad}(\tilde{K}) = r + O(r^2)$.

Let τ (resp., $\tilde{\tau}$) be the first time that a Brownian motion B_t hits $K \cup \mathbb{R}$ (resp., $\tilde{K} \cup \mathbb{R}$). We know that

$$\operatorname{hcap}(K) = \frac{2r}{\pi} \int_0^\pi \mathbb{E}^{qe^{i\theta}} [\operatorname{Im}(B_\tau)] \sin\theta \, d\theta,$$

$$\operatorname{hcap}(\tilde{K}) = \frac{2r}{\pi} \int_0^\pi \mathbb{E}^{qe^{i\theta}} [\operatorname{Im}(B_{\tilde{\tau}})] \sin\theta \, d\theta.$$

Let $\sigma = \tau_\wedge \inf\{s : |B_s| = 1/10\}$ and similarly for $\tilde{\sigma}$. Note that for $|z| = q$,

$$\mathbb{E}^z [\operatorname{Im}(B_\tau)] = \mathbb{E}^z [\operatorname{Im}(B_\tau); \tau = \sigma] \, [1 + O(q)].$$

$$\mathbb{E}^z [\operatorname{Im}(B_{\tilde{\tau}})] = \mathbb{E}^z [\operatorname{Im}(B_{\tilde{\tau}}); \tilde{\tau} = \tilde{\sigma}] \, [1 + O(q)].$$

Here $O(q)$ is an upper bound for the probability that a Brownian motion starting on the circle of radius $1/10$ reaches the circle of radius q without hitting $\mathbb{R}$.

Using (1.44), we have

$$\operatorname{Im}[\Phi(z)] = \operatorname{Im}[z] \, [1 + O(r)], \quad z \in K.$$

Hence we can write

$$\mathbb{E}^z [\operatorname{Im}(B_{\tilde{\tau}}); \tilde{\tau} = \tilde{\sigma}] = \mathbb{E}^z \left[\operatorname{Im}(\Phi^{-1}(B_{\tilde{\sigma}})); \tilde{\tau} = \tilde{\sigma}\right] \, [1 + O(q)].$$

Using (1.44) again, we see that $\Phi(z) = z + O(q^2)$ for $|z| = q$. Hence derivative estimates for harmonic functions give us for $|z| = q$,

$$\mathbb{E}^z \left[\operatorname{Im}(\Phi^{-1}(B_{\tilde{\sigma}})); \tilde{\tau} = \tilde{\sigma}\right] = \mathbb{E}^{\Phi^{-1}(z)} \left[\operatorname{Im}(\Phi^{-1}(B_{\tilde{\sigma}})); \tilde{\tau} = \tilde{\sigma}\right] \, [1 + O(q)].$$

Finally, conformal invariance of Brownian motion shows us that

$$\mathbb{E}^{\Phi^{-1}(z)} \left[\operatorname{Im}(\Phi^{-1}(B_{\tilde{\sigma}})); \tilde{\tau} = \tilde{\sigma}\right] = \mathbb{E}^{z)} [\operatorname{Im}(B_\sigma); \tau = \sigma] \, [1 + O(q)].$$

□

With this result, we see that g_t^* satisfies the Loewner equation

$$\partial_t g_t^*(z) = \frac{2\,\Phi_t'(U_t)^2}{g_t^*(z) - U_t^*}. \tag{1.45}$$

Proposition 1.8.31. *Under the assumptions above, if $0 \le t < t_0$ and $|z - U_t| < \epsilon$, then*

$$\dot{\Phi}_t(z) = 2 \left[\frac{\Phi_t'(U_t)^2}{\Phi_t(z) - \Phi(U_t)} - \frac{\Phi_t'(z)}{z - U_t} \right],$$

$$\dot{\Phi}_t'(z) = 2 \left[\frac{\Phi_t'(U_t)^2 \, \Phi_t'(z)}{(\Phi_t(z) - \Phi_t(U_t))^2} + \frac{\Phi_t'(z)}{(z - U_t)^2} - \frac{\Phi_t''(z)}{z - U_t} \right].$$

In particular,

$$\dot{\Phi}_t(U_t) = -3 \, \Phi_t''(U_t), \tag{1.46}$$

$$\dot{\Phi}_t'(U_t) = \frac{\Phi_t''(U_t)^2}{2 \, \Phi_t'(U_t)} - \frac{4 \, \Phi_t'''(U_t)}{3}. \tag{1.47}$$

Here we are writing $\dot{\Phi}_t(U_t)$ for $\dot{\Phi}_t(z)$ evaluated at $z = U_t$ and similarly for $\dot{\Phi}_t'(U_t)$.

Proof. We have noted that $t \mapsto \Phi_t'(U_t)$ is continuous so the right hand side is continuous in z, t. The first equation is an exercise in the chain rule, writing $\Phi_t = g_t^* \circ \Phi \circ g_t^{-1}$ and using (1.39) and (1.45). The second is obtained by differentiating with both sides with respect to z.

The limits are straightforward.

□

1.8.8 Radial Loewner equation

The Loewner equation in the upper half-plane is used to describe a curve connecting two boundary points in a simply connected domain. There is a similar equation called the *radial Loewner equation* that describes curves connecting a boundary point to an interior point. For ease, we choose the domain to be the unit disk and the interior point to be the origin. The proofs are similar to the upper half plane case, so we will not give all the details.

Suppose that $D = \mathbb{D} \setminus K$ is a simply connected subdomain of $\mathbb{D}$ containing the origin. The Riemann mapping theorem implies that there is a unique conformal transformation $g_D : D \to \mathbb{D}$ with $g_D(0) = 0, g_D'(0)0$. The construction of the map (see the proof of Theorem 1.4.2) shows that we can write $g(z) = z \, e^{f(z)}$ where $f(z) = \phi(z) + i\theta(z)$ is the holomorphic extension of the harmonic function

$$\phi(z) = \mathbb{E}^z \left[-\log |B_\tau| \right],$$

with $\theta(0) = 0$. Here B is a complex Brownian motion and $\tau = \inf\{t : B_t \notin D\}$. For $z \in D \setminus \{0\}$, we can write

$$\log g(z) = \log z + f(z),$$

provided that we interpret this correctly. We must either interpret g as a multi-valued function, or if we restrict to a simply connected neighborhood of z in $D \setminus \{0\}$ we can take a particular branch. Recall that

$$\frac{H_{\mathbb{D}}(z, w)}{H_{\mathbb{D}}(0, w)} = \frac{1 - |z|^2}{|z - w|}, \; ; \; |z| < 1, |w| = 1.$$

Proposition 1.8.32. *There exists $c < \infty$ such that if $D = \mathbb{D} \setminus K$ is a domain with $K \subset \{z : |z - w| \le r\}$ and $z \in \mathbb{D}$ with $|z - w| \ge 2r$,*

$$\left| \phi(z) - \frac{1 - |z|^2}{|z - w|} \, \phi(0) \right| \le \frac{c\phi(0)(1 - |z|^2)}{|z - w|}.$$

We can write this as

$$\phi(z) = \frac{1-|z|^2}{|z-w|}\,\phi(0)\,\left[1+O\left(\frac{r}{|z-w|}\right)\right].$$

Sketch of proof. Let $U = U_{w,r} = \mathbb{D} \setminus \{\zeta : |w-\zeta| \le r\}$. Let $\xi = \xi_{r,w} = \inf\{t : |w - B_t| \le r\}$. Then for $|z-w| > r$,

$$\phi(z) = \mathbb{P}^z\{\xi < \tau\}\,\mathbb{E}^z\left[-\log|B_\tau| \mid \xi < \tau\right] = \frac{1}{2}\int_{\partial U} H_U(z,\zeta)\,\phi(\zeta)\,|d\zeta|.$$

Hence, it suffices to show that

$$H_U(z,\zeta) = H_U(0,\zeta)\,\frac{1-|z|^2}{|z-w|}\,\left[1+O\left(\frac{r}{|z-w|}\right)\right].$$

This can be done either explicitly by mapping U to the $\mathbb{D}$ or by an argument similar to the chordal case. We leave the details to the reader.

□

Proposition 1.8.33. *Suppose $|w| = 1$ and $D_t = \mathbb{D} \setminus K_t$ is a decreasing sequence of domains with corresponding maps $g_t = g_{D_t}$ satisfying $g_t'(0) = e^{2at}$ and $\operatorname{rad}(K_t - w) \to 0$ as $t \downarrow 0$. Then,*

$$\lim_{t\downarrow 0}\frac{\phi_t(z)}{t} = 2a\,\frac{1-|z|^2}{|z-w|}.$$

Theorem 1.8.34. *Suppose $a > 0$, $\gamma : (0,t] \to \mathbb{D}\setminus\{0\}$ is a simple curve with $\gamma(0+) = w \in \partial\mathbb{D}$. Let $D_t = \mathbb{D}\setminus\gamma_t$ and let g_t be the corresponding conformal transformation Suppose g_t is parametrized so that $g_t'(0) = e^{2at}$. Then g_t satisfies the radial Loewner equation*

$$\dot g_t(z) = 2a\,g_t(z)\,\frac{w_t + g_t(z)}{w_t - g_t(z)}$$

where $w_t = g_t(\gamma(t)) \in \partial\mathbb{D}$. Moreover, the function $t \mapsto w_t$ is continuous. If we define h_t by

$$g_t(e^{2iz}) = \exp\{2ih_t(z)\},$$

then

$$\dot h_t(z) = a\,\cot(h_t(z) - X_t),$$

where X_t is a continuous process with $e^{2iX_t} = w_t$.

Proof. The full proof is similar to that of Theorem 1.8.20 so we do not give the details. Without loss of generality, assume that $2a = 1$. Note that this is clearly true at $z = 0$ and for $z \neq 0$, we can write

$$g_t(z) = z\,\exp\{\phi_t(z) + i\theta_t(z)\}.$$

Proposition 1.8.32 implies that

$$\partial_t[\log g_t(z)] = \frac{w_t + g_t(z)}{w_t - g_t(z)}, \tag{1.48}$$

$$\dot{\phi}_t(z) = \operatorname{Re}\left[\frac{w_t + g_t(z)}{w_t - g_t(z)}\right] = \frac{1 - |g_t(z)^2|}{|g_t(z) - w_t|}.$$

Since $\theta_t(0) = 1$ by the definition of g_t, we have $\dot{\theta}_t(0) = 0$. Hence we get (1.48). The chain rule gives

$$\dot{h}_t(z) = \frac{1}{2i}\,\frac{e^{2iX_t} + e^{2ih_t(z)}}{e^{2iX_t} - e^{2ih_t(z)}} = \frac{i}{2}\,\frac{e^{2i(h_t(z)-X_t)} + 1}{e^{2i(h_t(z)-X_t)} - 1} = \frac{1}{2}\,\cot(h_t(z) - X_t).$$

□

Writing points on the unit circle as $e^{2i\theta}$ rather than $e^{i\theta}$ makes some formulas nicer. For example $H_{\partial\mathbb{D}}(1, e^{2i\theta}) = (1/4)\,\sin^{-2}\theta$ and the sine of the argument of 0 with respect to 1 and $e^{2i\theta}$ is $\sin\theta$.

1.8.9 Whole plane Loewner equation

There is a similar equation describing curves or hulls starting "at infinity at time $-\infty$". Recall that $\mathbb{D}_s$ is the open disk of radius e^{-s} about the origin.

Theorem 1.8.35. *Suppose $a > 0$, $\gamma : (-\infty, \infty) \to \mathbb{C} \setminus \{0\}$ is a simple curve with $\gamma(-\infty) = \infty, \gamma(\infty) = 0$. Let $D_t = \mathbb{C} \setminus \gamma_t$ and let $g_t : D_t \to \mathbb{D}_{a(t)}$ be the unique conformal tranformation of D_t onto a multiple of the unit disk such that $g_t(0) = 0, g_t'(0) = 1$. This description defines $a(t)$ and now assume that $a(t) = -t$. Then g_t satisfies the whole plane Loewner equation*

$$\dot{g}_t(z) = \frac{2g_t(z)^2}{w_t - g_t(z)}, \quad g_0(z) = z, \tag{1.49}$$

where $w_t = g_t(\gamma(t)) \in \partial\mathbb{D}_{-t}$. Moreover, the function $t \mapsto w_t$ is continuous.

Proof. For fixed T define $\gamma^*(t) = e^T\, g_T(\gamma(T + t))$. Then γ^* is a curve satisfying the condition of Theorem 1.8.34. Let $g_t^* : \mathbb{D}\setminus\gamma_t^* \to \mathbb{D}$ be the unique transformation satisfying $g_t^*(0) = 0, (g_t^*)'(0) = e^t$, and note that g_t^* satisfies

$$\dot{g}_t^*(z) = g_t^*(z)\,\frac{w_t^* + g_t^*(z)}{w_t^* - g_t^*(z)},$$

with $w_t^* = g_t^*(\gamma(t))$. Also, note that

$$g_{t+T}(z) = e^{-(T+t)}\, g_t^*(e^T\, g_T(z)),$$
$$w_{t+T} = e^{-(t+T)}\, w_t^*.$$

$$\begin{aligned}
\dot{g}_{t+T}(z) &= g_{t+T}(z)\,\left[\frac{w_t^* + g_t^*(g_T(z))}{w_t^* - g_t^*(g_T(z))} - 1\right] \\
&= g_{t+T}(z)\,\frac{2\,g_t^*(g_T(z))}{w_t^* - g_t^*(g_T(z))} \\
&= 2\,\frac{e^{T+t}\,g_{t+T}(z)^2}{w_t^* - e^{T+t}\,g_{t+T}(z)} \\
&= 2\,\frac{g_{t+T}(z)^2}{w_{t+T} - g_{t+T}(z)}
\end{aligned}$$

□

Similarly to the radial equation we can start with $X_t : (-\infty, \infty) \to \mathbb{R}$ let $w_t = e^{2iX_t}$ and define g_t to the solution to (1.49) and we get the maps g_t. (It takes a little argument to show that it is well defined and unique, but we omit this.) The key fact is that once the whole plane curve is "started", the remainder follows a scaled version of the radial equation.

Proposition 1.8.36. *Let g_t, w_t be as in Theorem 1.8.35 and let $f_t = g_t^{-1}$. Then f_t satisfies the equation*

$$\dot{f}_t(\zeta) = f_t'(\zeta)\,\frac{2\zeta^2}{w_t + \zeta}, \quad f_{-\infty}(\zeta) = \zeta. \tag{1.50}$$

Proof. Differentiating both sides of the equation $f_t(g_t(z)) = z$ with respect to t we get

$$\dot{f}_t(g_t(z)) + f_t'(g_t(z))\,\dot{g}_t(z) = 0.$$

Setting $\zeta = g_t(z)$, we get (1.50). □

As an example suppose that $w_t = -e^{-t}$, and this becomes

$$\dot{f}_t(\zeta) = f_t'(\zeta)\,\frac{2\zeta^2}{e^{-t} + \zeta} \tag{1.51}$$

If we assume that $f_t(\zeta) = e^{-t}\, f(e^t\,\zeta)$, we have

$$\dot{f}_t(\zeta) = -e^{-t}\, f(e^t\,\zeta) + \zeta\, f'(e^t\zeta) = e^t\,\left[w\, f'(w) - f(w)\right], \quad w = e^{-t}\zeta.$$

Also, (1.51) gives

$$\dot{f}_t(\zeta) = f_t'(\zeta)\,\frac{2\zeta^2}{e^{-t} + \zeta} = f'(w)\,\frac{2\,e^t\,w^2}{1+w}.$$

Solving with the addtional condition $f'(0) = 1$, we see that f is the Koebe function.

We have defined the whole plane curve from infinity to the origin. It is generally defined from the origin to infinity but this can be obtained under the map $z \mapsto 1/z$. Let

$$\phi_t(z) = \frac{1}{g_t(1/z)}.$$

in which case

$$\dot{\phi}_t(z) = -\frac{\dot{g}_t(1/z)}{g_t(1/z)^2} = -2\,\frac{1}{w_t - g_t(1/z)} = \phi_t(z)\,\frac{2\,v_t}{v_t - \phi_t(z)}. \quad v_t = w_t^{-1}$$

The map ϕ_t is normalized to have derivative one at infinity and its image is the complement of $\overline{\mathbb{D}_t}$. If $h_t(z) = e^{-t}\,\phi_t(z)$ so that the image is the complement of $\overline{\mathbb{D}}$ we have

$$\dot{h}_t(z) = h_t(z)\,\left[-1 + \frac{2v_t}{v_t - e^{-t}\,\phi_t}\right] = h_t(z)\,\left[-1 + \frac{2\tilde{v}_t}{\tilde{v}_t - h_t(z)}\right] = h_t(z)\,\frac{\tilde{v}_t + h_t(z)}{\tilde{v}_t - h_t(z)},$$

where $\tilde{v}_t = e^{-t}\,v_t \in \partial\mathbb{D}$.

1.9 Properties of the Loewner equation

In this section we suppose that U_t is a continuous real-valued function with $U_0 = 0$ and that g_t is the solution to the Loewner equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{1.52}$$

which for $z \in \mathbb{C} \setminus \{0\}$ is valid up to time $T_z \in (0, \infty]$. It is immediate that the flow is symmetric about the real axis, that is, $g_t(\overline{z}) = \overline{g_t(z)}$; in particular, $T_{\overline{z}} = T_z$. We let

$$H_t = \{z \in \mathbb{H} : T_z < \infty\}, \quad H_t^* = \{z \in \mathbb{C} \setminus \{0\} : T_z < \infty\}, \quad K_t = \mathbb{H} \setminus H_t.$$

If $z = x \in \mathbb{R}$, then (1.52) is a real differential equation for which it is easy to check that if $x < x'$ and $T_x, T_{x'} > t$, then $g_t(x) < g_t(x')$. From this we can see $\{x \in \mathbb{R} : T_x \le t\}$ is a closed interval which we write as $[x_t^-, x_t^+]$ with $x_t^- \le 0 \le x_t^+$. Here the trivial interval $x_t^- = x_t^+ = 0$ is a possibility. Then we can write

$$H_t^* = H_t \cup \{z : \overline{z} \in H_t\} \cup (-\infty, x_t^-) \cup (x_t^+, \infty).$$

Recall that $g_t(H_t) = \mathbb{H}$. We define

$$D_t^* = g_t(H_t^*), \quad D_t^+ = g_t(H_t \cup \{z : \overline{z} \in H_t\} \cup (x_t^+, \infty)),$$

and we define D_t^- similarly. We can write

$$D_t^* = \mathbb{C} \setminus [g_t(x_t^-), g_t(x_t^+)],$$

where $g_t(x_t^-) = \sup\{g_t(y) : y < x_t^-\}, g_t(x_t^+) = \inf\{g_t(y); y > x_t^+\}$, and similarly for $D_t^{\pm}$. Note that $g_t(x_t^-) \le U_t \le g_t(x_t^+)$.

Lemma 1.9.1. *Suppose g_t satisfies (1.52), $r > 0$, and $t \mapsto a(t)$ is a strictly increasing C^1 function. Define $\tilde{g}_t$ by*

$$\tilde{g}_t(z) = r\, g_{a(t)/2}(z/r).$$

Then $\tilde{g}_t$ satisfies

$$\partial_t \tilde{g}_t(z) = \frac{r^2 \dot{a}(t)}{\tilde{g}_t(z) - \tilde{U}_t}, \qquad \text{where} \quad \tilde{U}_t = r\, U_{a(t)/2}.$$

In particular, if $U_t = \sqrt{\kappa}\, B_t$ where B_t is a standard Brownian motion, then

- *If $a(t) = t/r^2$, then $\tilde{U}_t$ is a Brownian motion with variance parameter κ.*
- *If $r = 1$ and $a(t) = (2/\kappa)\, t$, then $\tilde{U}_t$ is a standard Brownian motion.*

Proof. The first assertion follows from chain rule and the second uses standard scaling properties of Brownian motion. □

Proposition 1.9.2. *Suppose $g_t = u_t + iv_t$ satisfies (1.52).*

1. *For all $z = x + iy \in \mathbb{H}$ and all $t < T_z$,*

$$v_t(z) \ge \sqrt{y^2 - 4t}.$$

In particular $T_z \ge y^2/4$.

2. *If $r > 0$, $z \in \mathbb{H}$ and $\|U\|_\infty \le r$, then*

$$|g_t(z)|^2 \ge |z|^2 - 8t, \quad \text{if} \quad 0 \le t \le \frac{|z|^2 - 4r^2}{8}.$$

In particular, $|g_t(z)| \ge r/\sqrt{2}$ for $|z|^2 \ge 8t + 4r^2$, and

$$K_t \subset \{z : |z| < 2\sqrt{2t + r^2}, \ \operatorname{Im}(z) \le 2\sqrt{t}\}. \tag{1.53}$$

Proof.

1. Let $Y_t = \operatorname{Im}[g_t(z)]$ and note that

$$\partial_t Y_t^2 = -\frac{4Y_t^2}{|Z_t|^2} \ge -4.$$

Therefore, $Y_t^2 \ge y^2 - 4t$.

2. Let $Q_t = |g_t(z)|$ and $\sigma = \inf\{t : Q_t = 2r\}$. Then for $t \le \sigma$, $Q_t - r \ge Q_t/2$, and hence,

$$\partial_t Q_t \ge -\frac{4}{Q_t}, \quad \partial_t[Q_t^2] \ge -8.$$

and hence

$$Q_t^2 \ge Q_0^2 - 8t, \quad t \le \sigma,$$

and $\sigma \ge (|z|^2 - 4r^2)/8$.

□

With a little more care in the proof, we could improve the constants in (1.53), but this result suffices for our purposes.

1.10 Multiply connected domains

We will classify finitely connected domains up to conformal equivalence. Suppose $D \subset \hat{\mathbb{C}}$ is a domain whose complement consists of a finite number of connected components $\partial_0, \partial_1, \ldots, \partial_k$, each consisting of more than a single point. Let $\mathcal{O}_k$ denote the set of such domains with $\partial_0 = \hat{\mathbb{C}} \setminus \mathbb{H}$, that is, $D \subset \mathbb{H}$ of the form

$$D = \mathbb{H} \setminus (\partial_1 \cup \cdots \cup \partial_k),$$

where $\partial_1, \ldots, \partial_k$ are disjoint compact sets with $\operatorname{dist}(\partial_j, \mathbb{R}) > 0$. It suffices to classify domains in $\mathcal{O}_k$ since we can map $\hat{\mathbb{C}} \setminus \partial_0$ to the upper half plane. We let $\mathcal{O}^*$ denote the set of such domains such that each ∂_j is a horizontal line segment,

$$\partial_j = [x_{j,-} + iy_j, x_{j,+} + iy_j].$$

Theorem 1.10.1. *Every domain $D \in \mathcal{O}_k$ is conformally equivalent to a domain $D^* \in \mathcal{O}_k^*$. Moreover, there exists a unique conformal map $f : D \to D^*$ to a domain $D^* \in \mathcal{O}$ that*

$$f(z) = z + O(|z|^{-1}), \quad |z| \to \infty.$$

Remarks.

- If $D \in \mathcal{O}_k$ and $f : D \to D^* \in \mathcal{O}^*$ is a conformal transformation sending the real line to the real line with $f(\infty) = \infty$, then f can be extended to

  $$\mathbb{R} \cup D \cup \{\overline{z} : z \in D\}.$$

  by Schwarz reflection. By considering the function

  $$g(z) = \frac{1}{f(1/z)},$$

  which is holomorphic in a neighborhood of the origin sending a neighbhorhood of the real line to the real line, we can see that f has an expansion at infinity,

  $$f(z) = b_{-1}z + b_0 + b_1 z^{-1} + \cdots$$

  where $b_{-1} > 0$ and $b_j \in \mathbb{R}$ for $j \geq 0$. If we set $\hat{f}(z) = [f(z) - b_0]/b_{-1}$, then

  $$\hat{f}(z) = z + O(|z|^{-1}).$$

- The "dimension" of the set of conformally equivalent domains is $3k - 2$. There are $3k$ choices for the $\{x_{j,-}, x_{j,+}, y_j\}$ but the equivalence classes are invariant under the map $z \mapsto rz + t$.

Proof. Let D be given and suppose $D^* \in \mathcal{O}^*$ given by $\{(x_{j,-}, x_{j,+}, y_j)\}$. Let $y(D) = \max\{\mathrm{Im}(z) : z \in \partial D\}, y(D^*) = \max\{\mathrm{Im}(z) : z \in \partial D^*\} = \max\{y_j\}, R = R(D) = \sup\{|z| : z \in \mathbb{H} \setminus D\}$. Suppose that $f : D \to D^*$ is a conformal transformation with $f(\infty) = \infty, f'(\infty) = 1$. Let us write $f(z) = u(z) + iv(z)$. Note that v is a harmonic function on D with boundary value 0 on $\mathbb{R}$ and y_j on ∂_j and satisfying

$$v(z) = \mathrm{Im}(z) + O(1), \quad \mathrm{Im}(z) \to \infty.$$

Since $h(z) := v(z) - \mathrm{Im}(z)$ is a bounded harmonic function, the function v is given on D by

$$v(z) = v_D(z) + \mathbb{E}^z\left[v(B_{\tau_D})\right] = \mathrm{Im}(z) + \mathbb{E}^z\left[h(B_{\tau_D})\right]$$

where v_D is the function from Lemma 1.8.1. Note that

$$|v(z) - \mathrm{Im}(z)| \leq \mathbb{P}^z\{B_{\tau_D} \notin \mathbb{R}\}\,[y(D) + y(D^*)].$$

Using explicit forms of Poisson kernels, we can see that $\mathbb{P}^z\{B_{\tau_D} \notin \mathbb{R}\} \leq cR/|z|$, and hence

$$|h(z)| \leq \frac{c}{|z|}.$$

Here, and for the rest of this proof, we allow constants to depend on D. Using derivative estimates for harmonic function, we see that for $|z| \geq 2R$,

$$|\nabla h(z)| \leq \frac{c}{|z|^2}. \tag{1.54}$$

We have found the candidate for v without making any restriction on the target domain D^*. If $f = u + iv$ is a holomorphic extension, then we know that it is given for $y > y(D)$ by

$$u(iy) = -\int_y^\infty \partial_x v(it)\, dt. \tag{1.55}$$

The existence of the integral follows from (1.54) as well as the estimate $|u(iy)| \leq c/y$.

We will now give a criterion under which we can find a conjugate harmonic function u such that $f = u + iv$ is holomorphic. For each $j = 1, \ldots, k$, let F_j be a confomal map

$$F_j : \{|z| > 1\} \to \mathbb{C} \setminus \partial_j$$

and let $\gamma_r(t) = \gamma_{r,j}(t) = F(e^{r+it}), 0 \leq t \leq 2\pi$. For sufficiently small r, the curve $\gamma_{r,j}$ separates ∂_j from the other parts of ∂D. We only consider such r here. Let $\phi_j(z) = v(F_j(z))$ which is a harmonic function on $\{1 < |z| < e^r\}$. The condition that we need satisfied is

$$\int_{\gamma_{r,j}} \partial_n v(z) = 0, \tag{1.56}$$

which is the same as

$$\int_{C_{-r}} \partial_n \phi_j(z)\, |dz| = 0. \tag{1.57}$$

Since ϕ is harmonic on $\{1 < |z| < e^r\}$, Proposition 1.2.14 shows that

$$M_r = \frac{1}{2\pi} \int_0^{2\pi} \phi_j(e^{r+i\theta})\, d\theta,$$

is a linear function of r and (1.57) holds if and only if M_r is constant. Note that if $O_r = \{1 < |z| < e^r\}$ and $\mathcal{E}^{\#}_{O_r}(C_0, dw)$ denotes the probability measure on C_r,

$$\mathcal{E}^{\#}_{O_r}(C_0, \cdot) = \frac{\mathcal{E}_{O_r}(C_0, \cdot)}{\mathcal{E}_{O_r}(C_0, C_r)},$$

then

$$M_r = \int_{C_r} \phi_j(w)\, \mathcal{E}^{\#}_{O_r}(C_0, dw).$$

Using conformal invariance, we see that this translates to the condition

$$\frac{1}{\mathcal{E}_D(\partial_j, \gamma_{r,j})} \int_{\gamma_{r,j}} v(w)\, d\mathcal{E}_D(\partial_j, dw) = v(\partial_j) = y_j.$$

Using Proposition 1.2.14, we can see that if γ is any simple curve γ that wraps around ∂_j but no other part of the boundary,

$$\frac{1}{\mathcal{E}_D(\partial_j, \gamma)} \int_{\gamma} v(w)\, d\mathcal{E}_D(\partial_j, dw) = y_j. \tag{1.58}$$

In fact, if this holds for one such γ, then it holds for all such γ. This is the necesssary and sufficient condition so that u as defined in (1.55) gives a holomorphic function.

This condition can be expressed in terms of a process that is called *excursion reflected Brownian motion*. This is a process X_t whose state space is $D \cup \{\partial_0, \partial_1, \ldots, \partial_k\}$. In other words, we identify the "holes" ∂_j into single points. To describe the process we give the properties.

- When the process reaches ∂_0 the process stops.
- When the process is in D it acts like usual Brownian motion.
- Suppose $j \geq 1$ and γ is a curve as above that separates ∂_j from the rest of the boundary. Let $T = \inf\{t : B_t = \partial_j\}$ and $S = \inf\{t \geq T : B_t \in \gamma\}$. Then for any z, the conditional distribution on B_S given $T < \infty$ is that of normalized excursion measure $\mathcal{E}_C^{\#}(\partial_j, \cdot)$.

It is not difficult to define Brownian motion "excursion reflected" off of the unit circle. Roughly speaking, everytime the process hits the boundary it chooses an angle randomly. (Since the number of visits to the boundary is uncountable, one needs to take a little care here, but it is not a problem.) For other domains, we can use conformal invariance to define the process near holes, and away from holes it acts like Brownian motion. The equation (1.58) can be viewed as a mean value property for the function v with respect to excursion reflected Brownian motion, that is, the required condition on v is that v is excursion reflected harmonic.

We now ask: can we find y_j so that v is excursion reflected harmonic? Let us view the excursion reflected Brownian motion at the times it visits the boundary points $\{\partial_0, \partial_1, \ldots, \partial_k\}$. This gives a discrete time, discrete space Markov chain Y_n with absorbing state ∂_0. The transition probabilities $q(j, l)$ are given by

$$\mathbb{P}\{Y_{n+1} = l \mid Y_n = j\} = \frac{\mathcal{E}_D(\partial_j, \partial_l)}{\mathcal{E}_D(\partial_j, \partial D \setminus \partial_j)}.$$

Let N be sufficiently large so that $\mathbb{H} \cap \{\mathrm{Im}(w) \geq N\} \subset D$. Let X_t be the excursion reflected Brownian motion starting at ∂_j, let $A_j = \{\partial_0, \ldots, \partial_k\} \setminus \{\partial_j\}$, and let

$$T_N = T_{N,j} = \inf\{t : \mathrm{Im}(X_t) = N \text{ or } X_t \in A_j\}.$$

If v is excursion reflected harmonic, then

$$y_j = \mathbb{E}\left[v(X_{T_N})\right] = \mathbb{P}\{\mathrm{Im}(X_T) = N\}\, \mathbb{E}[v(X_T) \mid \mathrm{Im}(X_T) = N\} + \sum_{l \neq j} \mathbb{P}\{X_T = \partial_l\}\, y_l.$$

Note that

$$\lim_{N \to \infty} \sum_{l \neq j} \mathbb{P}\{X_T = \partial_l\}\, y_l = \sum_{l \neq j} q(j, l)\, y_l.$$

Since $v(z) = \mathrm{Im}(z) + O(1)$ as $z \to \infty$,

$$\lim_{N \to \infty} \mathbb{P}\{\mathrm{Im}(X_T) = N\}\, \mathbb{E}[v(X_T) \mid \mathrm{Im}(X_T) = N\} = \lim_{N \to \infty} N \mathbb{P}\{\mathrm{Im}(X_T) = N\}.$$

Also,

$$\lim_{N \to \infty} N\, \mathbb{P}\{\mathrm{Im}(X_T) = N\} = \frac{\mathcal{E}_D(\infty, \partial_j)}{\mathcal{E}_D(\partial_j, \partial D \setminus \partial_j)},$$

where

$$\mathcal{E}_D(\infty, \partial_j) = \lim_{N\to\infty} N\mathcal{E}_D(I_N, p_j) = \alpha_j := \int_{-\infty}^{\infty} \mathrm{hm}_D(x + ib, \partial_j)\, dx.$$

The integral on the right is the same for all $b > y(D)$.

The parameters $q(j,l)$ and α_j can be determined from the domain D. What we have seen is that we need y_j to satisfies

$$y_j = \alpha_j + \sum_{l\neq j} q(j,l)\, \alpha_l,$$

where $y_0 = 0$. This has a unique solution that can be given in terms of the finite Markov chain,

$$y_j = \mathbb{E}^{\partial_j}\left[\sum_{n=0}^{\infty} \alpha_{Y_n}\right] = \sum_{l=1}^{k} \alpha_l\, g(j,l).$$

Here

$$g(j,l) = \sum_{n=0}^{\infty} \mathbb{P}\{Y_n = y_l \mid Y_0 = y_j\}$$

is the Green's function for the discrete Markov chain with absorbing state ∂_0.

At this point we have shown that for every $D \in \mathcal{O}$, there is a unique choice of $\{y_j\}$ for which we can find a function v on $\mathbb{H}$ with the following properties:

- v is positive on D with
$$v(z) = \mathrm{Im}(z) + O(1), \quad z \to \infty.$$
- $v \equiv 0$ on $\mathbb{R}$.
- v is continuous on $\mathbb{H}$ with $v(z) = y_j$ for $z \in \partial_j, j \geq 1$.
- v is excursion reflected harmonic. Equivalently for every closed curve γ in v,
$$\int_\gamma \partial_n v(w)\, |dw| = 0. \tag{1.59}$$

Let $z_0 \in D$. We define u by

$$u(z) = \int_\gamma \partial_n v(w)\, |dw|.$$

Here γ is any curve from z_0 to z. Since v satisfies (1.59) the value is independent of the curve γ. This gives a holomorphic function $f : D \to D^*$. We need to show that f is one-to-one and onto. As in the proof of the Riemann mapping theorem, we consider the level sets of v. Let

$$V_s = \{z : v(z) = s\}, \quad V_s^+ = \{z : v(s) > s\}, \quad V_s^- = \{z : v(s) < s\}.$$

Let $\mathbb{H}_b = \{z \in \mathbb{H} : \mathrm{Im}(z) < b\}$. There exists c such that $v(z) \leq \mathrm{Im}(z) + c$. We claim that V_s^+ is connected. Indeed there is one connected component, say U, of V_s^+ that contains $\mathbb{H} \setminus \mathbb{H}_{s+c}$ for some c. Since v is bounded on $\mathbb{H}_{s+c}$, we can see that v is bounded on any other component U_1,

and its maximum value on that component must be bounded by the maximum of v on ∂U_1 which is s. For V_s^-, note that there exists $c_D < \infty$ such that for all z,

$$v(z) \le \operatorname{Im}(z) + \|y_j\|_\infty \, \mathbb{P}^z\{B_\tau \notin \mathbb{R}\} \le c_D \operatorname{Im}(z).$$

In particular, $\mathbb{H}_\epsilon \subset V_s^-$ for all ϵ sufficiently small, and there is a unique component of V_s^- that contains $\mathbb{R}$ on its boundary. Suppose there were another component of V_s^-, say U'. Then the value of v on $\partial[D \cap U']$ is either s or in $\{y_1, \ldots, y_k\}$. Since v is a bounded harmonic function on $D \cap U'$,

$$v(z) = \mathbb{E}^z \left[v(B_{\tau'}) \right], \quad \tau' = \tau_{D \cap U'}.$$

Since $v(z) < s$, there must be at least one j such that $\partial_j \subset U'$. Let us choose ∂_j such that y_j is minimal (if there is a tie, choose any one). Then by the maximal (minimal) principle, $v(z) \ge y_j$ for all $z \in U'$. However, if γ is a curve in D surrounding ∂_j and close enough to ∂_j, we know that the average value of v on γ (with respect to normalized excursion measure) is y_j. This means either v is constant (which is clearly not true in our case), or v takes on values smaller than v_j. This is a contradiction, and hence U' does not exist.

We fix D and allow constants to depend on D. Let $z = x + iy$ and consider $|z|$ large. Note that

$$v(z) = y + \mathbb{E}^z[h(B_\tau)],$$

where $\phi(z) = \mathbb{E}^z[h(B_\tau)]$ and h is a bounded function that vanishes on the real line. Using the form of the Poisson kernel, we can see that

$$|\phi(x + iy)| \le \frac{c\, y}{|z|^2}.$$

Using the fact that ϕ is a harmonic function in the disk of radius $|z|/2$ about z (if x is large, but y is not, use Schwarz reflection about the origin to extend the function), we see that

$$|\nabla \phi(z)| \le \frac{c\, y}{|z|^3},$$

and hence

$$|\partial_x v(x + iy)| + |\partial_y v(x + iy) - 1| \le \frac{c\, y}{|z|^3}.$$

We will define (at least for large y)

$$u(iy) = -\int_y^\infty \partial_x v(iy')\, dy'.$$

Since $|\partial_x v(y')| = O(|y|^{-2})$, we see that $u(iy)$ is well defined and $|u(iy)| = O(|y|^{-1})$. We then define

$$u(x + iy) = u(iy) + \int_0^x \partial_y v(x' + iy)\, dx',$$

and, similarly we see that

$$u(x + iy) = x + O(|y|^{-1}).$$

From this we can see that

$$\lim_{y \to \infty} u(x + iy) = x,$$

and this allows us to see that we can also write

$$u(x+iy) = x - \int_y^\infty \partial_x v(x+iy')\,dy'.$$

This allows to improve the error to

$$u(z) = \mathrm{Re}(z) + O(|z|^{-1}),$$

and

$$f(z) = z + O(|z|^{-1}).$$

This guarantees injectivity outside a compact subset. Inside we do an argument as in the Riemann mapping theorem.

□

The following is proved similarly. We only sketch the proof. Let $\mathcal{O}'_k$ denote the set of domains $U \subset \mathbb{D}$ of the form

$$U = \mathbb{D} \setminus (\partial'_1 \cup \cdots \cup \partial'_k)$$

where the ∂_j are disjoint circular arcs of the form

$$\partial_j = \{e^{-r_j+i\theta} : \theta_{j,-} \le \theta \le \theta_{j,+}\},$$

with $r_j > 0$ and $0 \le \theta_{j,-} < 2\pi, \theta_{j,-} < \theta_{j,+} < \theta_{j,-} + 2\pi$.

Proposition 1.10.2. *Suppose $D = \mathcal{O}_k$ and $\zeta \in D$. Then there exists a unique $U \in \mathcal{O}'_k$ such that there exists a (unique) conformal map $f : D \to U$ with $f(\partial_0) = \partial\mathbb{D}$, $f(\zeta) = 0$, $f'(\zeta) > 0$.*

Proof. Without loss of generality we will assume that $\zeta = 0$. Suppose such a domain U and function f exists with parameters $(r_j, \theta_{j,-}, \theta_{j,+})$. Consider the harmonic function on $D \setminus \{\zeta\}$ given by

$$h(z) = -\log|f(z)|.$$

If h is known, then $\theta(z) := \arg f(z)$ can be determined from the Cauchy-Riemann equations. As in Theorem 1.10.1, in order for this to be well defined, we need the condition that $h(z)$ must be a harmonic function for excursion reflected Brownian motion. Note that as $z \to 0$,

$$h(z) = -\log|z| - \log|f'(0)| + o(1).$$

Consider excursion reflected Brownian motion started on ∂_j stopped when it reaches the either $\partial D \setminus \partial_j$ or $\mathcal{B}_s := e^{-s}\mathbb{D}$. For fixed s, h is a bounded excursion reflected Brownian motion on $D \setminus \mathcal{B}_s$. Arguing as above and letting $s \to \infty$, we see that

$$r_j = G_D^{ER}(z,0) + \sum_{l \ne j} q(j,l) r_l,$$

where $G_D^{ER}(z,0)$ is the Green's function for excursion reflected Brownian motion defined in the natural way, and $q(j,l)$ are the probabilities as before. This determines the parameters r_j and hence it determines $|f(z)|$. The function $f(z)$ is obtained by fixing some $z \in D \setminus \{0\}$ and arbitrarily choosing $\theta(z)$. This will define f up to a rotation and exactly one of those rotations will have $f'(0) > 0$.

□

1.11 Poisson kernels and Green's functions for standard domains

Here we will give the Poisson kernels and Green's functions for a number of standard domains. Recall that we have chosen the constants in our definitions so that

$$H_{\mathbb{D}}(0, e^{i\theta}) = \frac{1}{2}, \quad G_{\mathbb{D}}(0, z) = -\log|z|.$$

If $n_z = n_{z,D}$ represents the inward normal at z, then if $z \in D$ and $w, \zeta \in \partial D$,

$$H_{\mathbb{D}}(z, w) = \frac{1}{2}\,\partial_{n_w} G_D(z, w),$$

$$H_{\partial D}(\zeta, w) = \partial_{n_\zeta} H_D(\zeta, w) = \partial_{n_w} H_D(w, \zeta) = \frac{1}{2}\,\partial_{n_\zeta}\partial_{n_w}\, G_D(\zeta, w).$$

We will do straightforward computations using the scaling rules if $f : D \to f(D)$ is a conformal transformation, then

$$G_D(z, z') = G_{f(D)}(f(z), f(z')), \quad H_D(z, w) = |f'(w)|\, H_{f(D)}(f(z), f(w)),$$

$$H_{f(D)}(f(z), f(w)) = |f'(\zeta)|\,|f'(w)|\, H_{f(D)}(f(\zeta)), f(w)).$$

1.11.1 Disk

Proposition 1.11.1. *For the unit disk $\mathbb{D} = \{|z| < 1\}$,*

$$H_{\mathbb{D}}(w, e^{i\theta}) = H_{\mathbb{D}}(we^{-i\theta}, 1) = \frac{1}{2}\,\frac{1 - |w|^2}{|e^{i\theta} - w|^2}$$

$$G_{\mathbb{D}}(z, w) = -\log\left|\frac{z - w}{1 - z\overline{w}}\right|,$$

$$H_{\partial\mathbb{D}}(e^{i2\theta}, e^{i2\psi}) = \frac{1}{4\,\sin^2(\theta - \psi)}.$$

Proof. Using the Möbius transformation

$$T_w(z) = \frac{z - w}{1 - z\overline{w}}, \quad T_w'(z) = \frac{1 - |w|^2}{(1 - z\overline{w})^2}$$

$$G_{\mathbb{D}}(z, w) = G_{\mathbb{D}}(T_w(z), 0) = -\log\left|\frac{z - w}{1 - z\overline{w}}\right|,$$

$$H_{\mathbb{D}}(w, 1) = |T_w'(1)|\, H_{\mathbb{D}}(T_w(w), T_w(1)) = \frac{1}{2}\,\frac{1 - |w|^2}{|1 - \overline{w}|^2} = \frac{1}{2}\,\frac{1 - |w|^2}{|1 - w|^2}.$$

$$H_{\mathbb{D}}(w, e^{i\theta}) = H_{\mathbb{D}}(we^{-i\theta}, 1) = \frac{1}{2}\,\frac{1 - |w|^2}{|e^{i\theta} - w|^2}$$

$$\begin{aligned} H_{\partial\mathbb{D}}(e^{i\theta},1) &= \lim_{r\downarrow 0} r^{-1}\, H_{\mathbb{D}}((1-r)e^{i\theta},1) \\ &= \lim_{r\downarrow 0} \frac{1}{r}\left[\frac{1}{2}\,\frac{1-(1-r)^2}{|1-(1-r)e^{i\theta}|^2}\right] \\ &= \frac{1}{|1-e^{i\theta}|^2} = \frac{1}{4\sin^2(\theta/2)}, \end{aligned}$$

and

$$H_{\partial\mathbb{D}}(e^{i2\theta},e^{2i\psi}) = H_{\partial\mathbb{D}}(e^{i2(\theta-\psi)}) = \frac{1}{4\,\sin^2(\theta-\psi)}.$$

□

We start with the upper half plane,

$$H_{\mathbb{H}}(x+iy,x') = \frac{y}{(x-x')^2+y^2}, \quad H_{\mathbb{H}}(x,x') = (x-x')^{-2}.$$

1.11.2 Rectangle

Let $\mathcal{R}_L$ be the rectangle $\{x+iy : 0<x<L, 0<y<\pi\}$ and let ∂_L denote the vertical line segment $[L, L+i\pi]$.

Using separation of variables, we can see that

$$f(x+iy) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sinh(nx)\,\sin(ny)\,\sin(ny'n)}{\sinh(nL)},$$

is a harmonic function in $\mathcal{R}_L$ that vanishes on $\partial\mathcal{R}\setminus\partial_L$ and equals (in the sense of distributions) $\delta_{y'}$ on ∂_L. Therefore,

$$H_{\mathcal{R}_L}(x+iy, L+iy') = 2\sum_{n=1}^{\infty} \frac{\sinh(nx)\,\sin(ny)\,\sin(ny')}{\sinh(nL)},$$

$$H_{\mathcal{R}_L}(iy, L+iy') = 2\sum_{n=1}^{\infty} \frac{n\,\sin(ny)\,\sin(ny')}{\sinh(nL)}.$$

The sums are absolutely convergent for $|x|<L$. Note that for $0<x<1$,

$$H_{\mathcal{R}_L}(x+iy, L+iy') = 2\sinh x\,\sin y\,\sin y'\,[1+O(e^{-L})].$$

$$H_{\mathcal{R}_L}(iy, L+iy') = 2\sin y\,\sin y'\,[1+O(e^{-L})].$$

1.11.3 Upper half plane

Proposition 1.11.2. *For the upper half plane $\mathbb{H} = \{x+iy : Y>0\}$,*

$$G_{\mathbb{H}}(z,w) = \log|z-\overline{w}| - \log|z-w|$$

$$H_{\mathbb{H}}(x+iy,x') = \frac{y}{(x-x')^2+y^2}, \quad H_{\mathbb{H}}(x,x') = \frac{1}{(x-x')^2},$$

Proof. The map

$$f(z) = \frac{z-i}{z+i}, \quad f'(z) = \frac{2i}{(z+i)^2},$$

takes the upper half plane $\mathbb{H}$ onto $\mathbb{D}$ with $f(i) = 0$. Therefore,

$$G_{\mathbb{H}}(z,i) = G_{\mathbb{D}}(f(z), f(i)) = -\log\left|\frac{z-i}{z+i}\right| = \log\frac{|z+i|}{|z-i|},$$

$$G_{\mathbb{H}}(z, x+iy) = G_{\mathbb{H}}\left(\frac{z-x}{y}, i\right) = \log\frac{|\frac{z-x}{y}+i|}{|\frac{z-x}{y}-i|} = \log\frac{|z-\overline{(x+iy)}|}{|z-(x+iy)|},$$

$$H_{\mathbb{H}}(i,x) = |f'(x)|\, H_{\mathbb{H}}(0, f(x)) = \frac{1}{2}\,\frac{2}{|x+i|^2} = \frac{1}{x^2+1},$$

$$H_{\mathbb{H}}(x+iy, x') = y^{-1}\, H_{\mathbb{H}}\left(i, \frac{x'-x}{y}\right) = \frac{y}{(x-x')^2+y^2}.$$

$$H_{\mathbb{H}}(x,x') = \lim_{y\downarrow 0} y^{-1}\, H_{\mathbb{H}}(x, x'+iy) = \frac{1}{(x-x')^2}.$$

Another way to see that the Green's function is

$$G_{\mathbb{H}}(z,w) = \log|z-\overline{w}| - \log|z-w|.$$

is to note that for fixed z, the right-hand side is a harmonic function of w that vanishes on the real line and looks like $-\log|z-w| + O(1)$ as $w \to z$.

□

If $z = re^{i\theta}$, then

$$G_{\mathbb{H}}(re^{i\theta}, i) = \frac{1}{2}\,\log\left|\frac{r^2\cos^2\theta + (r\sin\theta+1)^2}{r^2\cos^2\theta + (r\sin\theta-1)^2}\right|.$$

As $r \to \infty$,

$$G_{\mathbb{H}}(re^{i\theta}, i) = \frac{1}{2}\,\log\left|1 + \frac{4r\,\sin\theta}{r^2\cos^2\theta + (r\sin\theta-1)^2}\right| = 2r^{-1}\sin\theta\,[1+O(r^{-1})]. \tag{1.60}$$

1.11.4 Half Disk

Proposition 1.11.3. *Let $\mathbb{D}_+ = \mathbb{H}\cap\mathbb{D}$ be the upper half disk. Then,*

$$H_{\partial\mathbb{D}_+}(e^{i\theta}, e^{i\psi}) = \frac{\sin\theta\,\sin\psi}{(\cos\theta-\cos\psi)^2},$$

$$H_{\partial\mathbb{D}_+} = (x, e^{i\theta})\frac{2(1-x^2)}{[x^2-2x\cos\theta+1]^2}\,\sin\theta.$$

In particular, $H_{\partial\mathbb{D}_+}(0, e^{i\theta}) = 2\,\sin\theta$. More generally, for z near the origin,

$$H_{\mathbb{D}_+}(z, e^{i\theta}) = 2\,\mathrm{Im}(z)\,\sin\theta\,\left[1+O(|z|)\right]. \tag{1.61}$$

Proof. The function

$$f(z) = \frac{2z}{z^2+1}, \quad f'(z) = \frac{2\,(1-z^2)}{(z^2+1)^2},$$

is a conformal transformation of $\mathbb{D}_+$ onto $\mathbb{H}$ with $f(0) = 0, f(i) = \infty, f(1) = 1, f(-1) = -1$ and

$$f(e^{i\theta}) = \frac{2e^{i\theta}}{e^{2i\theta}+1} = \frac{2}{e^{i\theta}+e^{-i\theta}} = \frac{1}{\cos\theta}.$$

Note that

$$|f'(e^{i\theta})| = \left|\frac{2(1-e^{2i\theta})}{(e^{2i\theta}+1)^2}\right| = \frac{\sin\theta}{\cos^2\theta}.$$

Therefore,

$$\begin{aligned} H_{\partial\mathbb{D}_+}(e^{i\theta}, e^{i\psi}) &= |f'(e^{i\theta})|\,|f'(e^{i\psi})|\,H_{\mathbb{H}}(f(e^{i\theta}), f(e^{i\psi})) \\ &= \frac{\sin\theta\,\sin\psi}{\cos^2\theta\,\cos^2\psi}\,\left[\frac{1}{\cos\theta} - \frac{1}{\cos\psi}\right]^{-2} \\ &= \frac{\sin\theta\,\sin\psi}{(\cos\theta-\cos\psi)^2}. \end{aligned}$$

$$H_{\partial\mathbb{D}_+}(x, e^{i\theta}) = \frac{2(1-x^2)}{(x^2+1)^2}\,\frac{\sin\theta}{\cos^2\theta}\,\left[\frac{2x}{x^2+1} - \frac{1}{\cos\theta}\right]^{-2} = \frac{2(1-x^2)}{[x^2-2x\cos\theta+1]^2}\,\sin\theta.$$

One could derive (1.61) from the exact formulas. Alternatively, note that the function $h(z) = H_{\mathbb{D}_+}(z, e^{i\theta})$ can be extended by Schwarz reflection to a harmonic function on $\mathbb{D}$ that vanishes on the real line. Note that $\partial_y h(0) = 2\,\sin\theta$ and $\partial_x h$ vanishes on the real line. Also, $|h(z)| \le c\,\sin\theta$ for $|z| \le 1/2$, and hence we can see that that for $|z| \le 1/4$ all the second partials are bounded by a universal constant times $\sin\theta$. Hence

$$\partial_y h(x) = 2\,\sin\theta\,[1 + O(|x|)],$$

and

$$h(x+iy) = y\,\partial_y h(x) + O((\sin\theta)\,y^2) = 2\,y\sin\theta\,[1 + O(|x|) + O(y)].$$

□

1.11.5 Infinite Strip

Proposition 1.11.4. *Let $S_r = \{x+iy \in \mathbb{H} : y < r\}$ denote the half-infinite strip. Then*

$$H_{\partial S_r}(0, x) = \frac{\pi^2}{4r^2}\left[\sinh\left(\frac{\pi x}{2r}\right)\right]^{-2},$$

$$H_{\partial S_r}(0, x+ir) = \frac{\pi^2}{4r^2}\left[\cosh\left(\frac{\pi x}{2r}\right)\right]^{-2},$$

Proof. Note that $f(z) = e^z$ is a conformal transformation of S_π onto $\mathbb{H}$ and hence

$$\begin{aligned} H_{\partial S_\pi}(0,x) &= |f'(0)|\,|f'(x)|\,H_{\partial\mathbb{H}}(f(0),f(x+i\pi)) \\ &= e^x\,H_{\mathbb{H}}(1,e^x) \\ &= \frac{e^x}{(e^x-1)^2} = \frac{1}{(e^{x/2}-e^{-x/2})^2} = \frac{1}{4\sinh^2(x/2)}. \end{aligned}$$

$$\begin{aligned} H_{\partial S_\pi}(0,x+i\pi) &= |f'(0)|\,|f'(x)|\,H_{\partial\mathbb{H}}(f(0),f(x+i\pi)) \\ &= e^x\,H_{\partial\mathbb{H}}(1,-e^x) \\ &= \frac{e^x}{(e^x+1)^2} = \frac{1}{4\cosh^2(x/2)}. \end{aligned}$$

By using the conformal transformation $z \mapsto (\pi/r)z$, we get

$$H_{\partial S_r}(0,x) = (\pi/r)^2\,H_{\partial S_\pi}(0,\pi x/r) = \frac{\pi^2}{4r^2}\left[\sinh\left(\frac{\pi x}{2r}\right)\right]^{-2},$$

$$H_{\partial S_r}(0,x+ir) = (\pi/r)^2\,H_{\partial S_\pi}(0,\pi x/r+i\pi) = \frac{\pi^2}{4r^2}\left[\cosh\left(\frac{\pi x}{2r}\right)\right]^{-2}.$$

□

We will compute this another way using Fourier series. This will be useful when comparing to simple random walk. We will let $S = S_1$, and we first consider the domain $R_m = \{x+iy : 0 < x < 2m, 0 < y < 1\}$. Consider

$$F(x+iy) = \sum_{j=1}^{\infty} b_j\,\sin(j\pi x/2m)\,\sinh(j\pi y/2m).$$

For any choice of constants (decaying sufficiently fast), this gives a harmonic function. If we choose

$$b_j = \frac{\sin(mj\pi/2m)}{2m\,\sinh(j\pi/2m)},$$

we see that the boundary condition on ∂R_m is the delta function at $m+ni$. Therefore,

$$\frac{1}{\pi}\,H_{R_m}((m+u)+iy,m+i) = \sum_{j=1}^{\infty}\frac{\sin(mj\pi/2m)}{2m\,\sinh(j\pi/2m)}\,\sin(j\pi(m+u)/2m)\,\sinh(j\pi y/2m).$$

Note that

$$\sin(mj\pi/2)\,\sin(j\pi(m+u)/2mm) = \begin{cases} \cos(j\pi u/2m), & j \text{ odd}, \\ 0 & j \text{ even}, \end{cases}$$

Therefore,

$$\frac{1}{\pi}\,H_{R_m}((m+u)+iy,m+in) = \sum_{j=1}^{\infty}\frac{\cos((2j-1)\pi u/2m)\,\sinh((2j-1)\pi y/2m)}{2m\,\sinh((2j-1)\pi/2m)}.$$

This is a Riemann sum approximation of an integral and hence we get

$$\lim_{m\to\infty} H_{R_m}((m+u)+iy, m+i) = \pi \int_0^\infty \frac{\cos(t\pi u)\,\sinh(t\pi y)}{\sinh(\pi t)}\,dt = \int_0^\infty \frac{\cos(su)\,\sinh(sy)}{\sinh s}\,ds,$$

$$\begin{aligned} H_{\partial S}(u,i) &= \partial_y H_S(z,i)\mid_{z=u} \\ &= \int_0^\infty \frac{s\,\cos(su)}{\sinh s}\,ds \\ &= \partial_u \left[\int_0^\infty \frac{\sin(su)}{\sinh s}\,ds\right] \\ &= \partial_u \left[\frac{\pi}{2}\,\tanh(u\pi/2)\right] \\ &= \frac{\pi^2}{4\cosh^2(u\pi/2)}. \end{aligned}$$

The penultimate inequality is identity 711 in the CRC table of integrals.

Chapter 2

Bessel process

2.1 Introduction

This is an adaptation of one of the chapters of a long, far from finished, project about conformally invariant processes and the Schramm-Loewner evolution (SLE). Although the motivation and the choice of topics for these notes come from applications to SLE, the topic is the one-dimensional Bessel process. No SLE is assumed or discussed. It is assumed that the reader has a background in stochastic calculus including Itô's formula, the product rule, Girsanov's theorem, and time changes of diffusions. The Girsanov perspective is taken from the beginning.

The Bessel process is one of the most important one-dimensional diffusion processes. There are many ways that if arises. We will start by viewing the Bessel process as a Brownian motion "tilted" by a function of the current value.

We will give a summary of what is contained here as well as some discussion of the relevance to the Schramm-Loewner evolution (SLE). For an introduction to SLE, see [1]. The discussion here is for people who know about SLE; we emphasize that knowledge of SLE is not required for this paper.

The chordal Schramm-Loewner evolution with parameter $\kappa = 2/a > 0$ (SLE_κ) is the random path $\gamma(t)$ from 0 to infinity in the upper half-plane $\mathbb{H}$ defined as follows. Let H_t denote the unbounded component of $\mathbb{H} \setminus \gamma(0,t]$ and let $g_t : H_t \to \mathbb{H}$ be the unique conformal transformation with $g(x) = z = o(1)$ as $z \to \infty$. Then

$$\partial_t g_t(z) = \frac{a}{g_t(z) + B_t}, \quad g_0(z) = z,$$

where B_t is a standard Brownian motion. In fact, g_t can be extended by Schwarz reflection to a conformal transformation on H_t^*, defined to be the unbounded component of the complement of $\{z : z \in \gamma[0,t] \text{ or } \overline{z} \in \gamma[0,t]\}$. Note that $H_0^* = \mathbb{C} \setminus \{0\}$ and for fixed $z \in H_0^*$, the solution of the equation above exists for $t < T_z \in (0, \infty]$. In our parametrization,

$$g_t(z) = z + \frac{at}{|z|} + O(|z|^{-2}), \quad z \to \infty.$$

If $x > 0$ and $X_t = X_t^x = g_t(z) + B_t$, then X_t satisfies

$$dX_t = \frac{a}{X_t}\, dt + dB_t, \quad X_0 = x.$$

This is the first place that the Bessel process arises; although only $a > 0$ is relevant here, other applications in SLE require understanding the equation for negative values as well, so we define the Bessel process for all $a \in \mathbb{R}$. We use a as our parameter throughout because it is easiest; however there two other more common choices: the index $\nu = a - \frac{1}{2}$ or the dimension $d = 2a + 1$. The latter terminology comes from the fact that the absolute value of a d-dimensional Brownian motion is a Bessel process with parameter $a = (d-1)/2$.

In the first section we consider the process only for $t < T_0$, that is, killed when it reaches the origin. While the process can be defined by the equation, we derive the equation by starting with X_t as a Brownian motion and then "weighting locally" by X_t^a. The equation comes from the Girsanov theorem. We start with some basic properties: scaling, the Radon-Nikodym derivative with respect to Brownian motion, and the phase transition at $a = 1/2$ for recurrence and transience (which corresponds to the phase transition at $\kappa = 4$ between simple paths and self-intersecting paths for SLE). Here we also consider the logarithm of the Bessel process and demonstrate the technique of using random time changes to help understand the process.

The transition density for the killed process is given in Section 2.2.2. It involves a special function; rather than writing it in terms of the modified Bessel function, we choose to write it in terms of the "entire" part of the special function that we label as h_a. The Radon-Nikodym derivative gives the duality between the process with parameters a and $1 - a$ and allow us to compute the density only for $a \geq 1/2$. From this the density for the hitting time T_0 for $a < 1/2$ follows easily. We then consider the Green's function and also the process on geometric time scales which is natural when looking at the large time behavior of the process.

In Section 2.2.5, the Bessel process X_t^x is viewed as a function of its starting position x; indeed, this is how it appears in the Schramm-Loewner evolution. Here we prove when it is possible for the process starting at different points to reach the origin at the same time. For SLE, this corresponds to the phase transition at $\kappa = 2/a = 8$ between plane filling curves and non-plane filling curves. in the following subsection, we show how to give expectations of a certain functional for Brownian motion and Bessel process; this functional appears as a power of the spatial derivative of X_t^x. There is a basic technique to finding the asymptotic value of these functionals: use a (local) martingale of the form of the functional times a function of the current value of the process; find the limiting distribution under this martingale; then compute the expected value of the reciprocal of the function in the invariant distribution.

The next section discusses the reflecting Bessel process for $-1/2 < a < 1/2$. If $a \geq 1/2$, the process does not reach the origin and if $a \leq -1/2$ the pull towards the origin is too strong to allowed a reflected process to be defined. There are various ways to define the process that is characterized by:

- Away from the origin it acts like a Bessel process.
- The Lebesgue measure of the amount of time spent at the origin is zero.

We start by just stating what the transition density is. Proving that it is a valid transition density requires only computing some integrals (we make use of integral tables [2, 3] here) and this can be used to construct the process by defining on dyadics and establishing continuity of the paths. The form of the density also shows that for each t, the probability of being at the origin at time t is zero. We then show how one could derive this density from the knowledge of the density starting at the origin — if we start away from the origin we proceed as in the Bessel process stopped at time 0 and then we continue.

In the following two subsections we consider two other constructions of the reflecting process both of which are useful for applications:

- First construct the times at which the process is at the origin (a random Cantor set) and then define the process at other times using Bessel excursions. For $a = 0$, this is the Itô excursion construction for reflecting Brownian motion. Constructing Bessel excursions is an application of the Girsanov theorem.
- Consider the flow of particles $\{X_t^x : x > 0\}$ doing coupled Bessel processes stopped at the origin and define the reflected process by

$$\inf\{X_t^x : T_x > t\}.$$

The Bessel process relates to the chordal, or half-plane, Loewner equation. For the radial equation for which curves approach an interior point, a similar equation, the radial Bessel equation arises. It is an equation restricted to $(0, \pi)$ and the value often represents one-half times the argument of a process on the unit circle. We generalize some to consider processes that locally look like Bessel processes near 0 and π (the more general processes also arise in SLE.) The basic assumption is that the drift of the process looks like the Bessel process up to the first *two* terms of the expansion. In this case, the process can be defined in terms of the Radon-Nikodym derivative (locally) to a Bessel process. This approach also allows us to define a reflected process for $-1/2 < a < 1/2$. A key fact about the radial Bessel process is the exponentially fast convergence of the distribution to the invariant distribution. This is used to estimate functionals of the process and these are important in the study of radial SLE and related processes.

In the final section, we discuss the necessary facts about special functions that we use, again relying on some integral tables in [2, 3].

2.2 The Bessel process (up to first visit to zero)

Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space on which is defined a standard one-dimensional Brownian motion X_t with filtration $\{\mathcal{F}_t\}$ with $X_0 > 0$. Let $T_x = \inf\{t : X_t = x\}$. Let $a \in \mathbb{R}$, and let $Z_t = X_t^a$. The Bessel process with parameter a will be the Brownian motion "weighted locally by X_t^a". If $a > 0$, then the Bessel process will favor larger values while for $a < 0$ it will favor smaller values. The value $a = 0$ will correspond to the usual Brownian motion. In this section we will stop the process at the time T_0 when it reaches the origin.

Itô's formula shows that if $f(x) = x^a$, then

$$\begin{aligned} dZ_t &= f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)\, dt \\ &= Z_t \left[\frac{a}{X_t}\, dX_t + \frac{a(a-1)}{2\, X_t^2}\, dt \right], \quad t < T_0. \end{aligned}$$

For the moment, we will not consider $t \geq T_0$. Let

$$N_t = N_{t,a} = \left[\frac{X_t}{X_0} \right]^a \exp\left\{ -\frac{(a-1)a}{2} \int_0^t \frac{ds}{X_s^2} \right\}, \quad t < T_0. \tag{2.1}$$

Then the product rule combined with the Itô calculation above shows that N_t is a local martingale for $t < T_0$ satisfying

$$dN_t = \frac{a}{X_t} N_t \, dX_t, \quad N_0 = 1.$$

Suppose $0 < \epsilon < X_0 < R$ and let $\tau = \tau_{\epsilon,R} = T_\epsilon \wedge T_R$. For fixed ϵ, R and $t < \infty$, we can see that $N_{s\wedge\tau}$ is uniformly bounded for $s \le t$ satisfying

$$dN_{t\wedge\tau} = \frac{a}{X_{t\wedge\tau}} 1\{t < \tau\} \, N_{t\wedge\tau} \, dX_t.$$

In particular, $N_{t\wedge\tau}$ is a positive continuous martingale.

We will use Girsanov's theorem and we assume the reader is familiar with this. This is a theorem about positive martingales that can also be used to study positive local martingales. We discuss now in some detail how to do this, but later on we will not say as much. For each t, ϵ, R, we define the probability measure $\hat{\mathbb{P}}_{a,\epsilon,R,t}$ on $\mathcal{F}_{t\wedge\tau}$ by

$$\frac{d\hat{\mathbb{P}}_{a,\epsilon,R,t}}{d\mathbb{P}} = N_{t\wedge\tau}, \quad \tau = \tau_{\epsilon,R}.$$

Since $N_{t\wedge\tau}$ is a martingale, we can see that if $s < t$, then $\hat{\mathbb{P}}_{a,\epsilon,R,t}$ restricted to $\mathcal{F}_{s\wedge\tau}$ is $\hat{\mathbb{P}}_{a,\epsilon,R,s}$. For this reason, we write just $\hat{\mathbb{P}}_{a,\epsilon,R}$. Girsanov's theorem states that if

$$B_t = B_{t,a} = X_t - \int_0^t \frac{a \, ds}{X_s}, \quad t < \tau,$$

then B_t is a standard Brownian motion with respect to $\hat{\mathbb{P}}_{a,\epsilon,R}$, stopped at time τ. In other words,

$$dX_t = \frac{a}{X_t} \, dt + dB_t, \quad t \le \tau.$$

Note that as X_t gets large, the drift term a/X_t gets smaller. By comparison with a Brownian motion with drift, we can therefore see that as $R \to \infty$,

$$\lim_{R\to\infty} \hat{\mathbb{P}}_{a,\epsilon,R}\{t \wedge \tau = T_R\} \to 0,$$

uniformly in ϵ. Hence we can write $\hat{\mathbb{P}}_{a,\epsilon}$, and see that

$$dX_t = \frac{a}{X_t} \, dt + dB_t, \quad t \le T_\epsilon.$$

Note that this equation does not depend on ϵ except in the specification of the allowed values of t. For this reason, we can write $\hat{\mathbb{P}}_a$, and let $\epsilon \downarrow 0$ and state that

$$dX_t = \frac{a}{X_t} \, dt + dB_t, \quad t < T_0,$$

where in this case we define $T_0 = \lim_{\epsilon\downarrow 0} T_\epsilon$. This leads to the definition.

Definition X_t is a Bessel process starting at $x_0 > 0$ with parameter a stopped when it reaches the origin, if it satisfies

$$dX_t = \frac{a}{X_t} \, dt + dB_t, \quad t < T_0, \quad X_0 = x_0, \tag{2.2}$$

where B_t is a standard Brownian motion.

We have written the parameter a to make the equation as simple as possible. However, there are two more common ways to parametrize the family of Bessel process.

- The *dimension* d,
$$a = \frac{d-1}{2}, \quad d_a = 2a + 1.$$
This terminology comes from the fact that if W_t is a standard d-dimensional Brownian motion, then the process $X_t = |W_t|$ satisfies the Bessel process with $a = (d-1)/2$. We sketch the argument here. First note that
$$dX_t^2 = \sum_{j=1}^{d} d[(W_t^j)^2] = 2\sum_{j-1}^{d} W_t^j \, dW_t^j + d\, dt,$$
which we can write as
$$dX_t^2 = d\, dt + 2\, X_t \, dZ_t,$$
for the standard Brownian motion
$$Z_t = \sum_{j=1}^{d} \int_0^t \frac{|W_s^j|}{X_s} \, dW_s^j.$$
(To see that Z_t is a standard Brownian motion, check that Z_t is a continuous martingale with $\langle Z \rangle_t = t$.) We can also use Itô's formula to write
$$dX_t^2 = 2\, X_t \, dX_t + d\langle X \rangle_t.$$
Equating the two expressions for dX_t^2, we see that
$$dX_t = \frac{\frac{d-1}{2}}{X_t} \, dt + dZ_t.$$
The process X_t^2 is called the *d-dimensional squared-Bessel process.*

- The *index* ν,
$$a = \frac{2\nu + 1}{2}, \quad \nu_a = \frac{2a-1}{2}.$$
Note that $\nu_{1-a} = -\nu_a$. We will see below that there is a strong relationship between Bessel processes of parameter a and parameter $1-a$.

As we have seen, to construct a Bessel process, we can start with a standard Brownian motion X_t on $(\Omega, \mathbb{P}, \mathcal{F})$, and then consider the probability measure $\hat{\mathbb{P}}_a$. Equivalently, we can start with a Brownian motion B_t on $(\Omega, \mathbb{P}, \mathcal{F})$ and define X_t to be the solution of the equation (2.2). There is a technical issue that the next proposition handles. The measure $\hat{\mathbb{P}}_a$ is defined only up to time T_{0+}. The next proposition shows that we can replace this with T_0 and get continuity at time T_0.

Proposition 2.2.1. *Let* $T_{0+} = \lim_{r \downarrow 0} T_r$.

1. *If* $a \geq 1/2$, *then for each* $x > 0, t > 0$.
$$\hat{\mathbb{P}}_a^x\{T_{0+} \leq t\} = 0.$$

2. If $a < 1/2$, then for each $t > 0$,

$$\hat{\mathbb{P}}_a^x\{T_{0+} \le t\} > 0.$$

Moreover, on the event $\{T_{0+} \le t\}$, except on an event of $\hat{\mathbb{P}}_a$-probability zero,

$$\lim_{t \uparrow T_0} X_t = 0.$$

Proof. If $0 < r < x < R < \infty$, let $\tau = T_r \wedge T_R$. Define $\phi(x) = \phi(x; r, R, a)$ by

$$\phi(x) = \frac{x^{1-2a} - r^{1-2a}}{R^{1-2a} - r^{1-2a}}, \quad a \neq 1/2,$$

$$\phi(x) = \frac{\log x - \log r}{\log R - \log r}, \quad a = 1/2.$$

This is the unique function on $[r, R]$ satisfying the boundary value problem

$$x\,\phi''(x) = -2a\,\phi'(x), \quad \phi(r) = 0, \;\; \phi(R) = 1. \tag{2.3}$$

Using Itô's formula and (2.3), we can see that $\phi(X_{t\wedge\tau})$ is a bounded continuous $\hat{\mathbb{P}}_a$-martingale, and hence by the optional sampling theorem

$$\phi(x) = \hat{\mathbb{E}}_a^x\left[\phi(X_\tau)\right] = \hat{\mathbb{P}}_a^x\{T_R < T_r\}.$$

Therefore,

$$\hat{\mathbb{P}}_a^x\{T_R < T_r\} = \frac{x^{1-2a} - r^{1-2a}}{R^{1-2a} - r^{1-2a}}, \quad a \neq 1/2, \tag{2.4}$$

$$\hat{\mathbb{P}}_a^x\{T_R < T_r\} = \frac{\log x - \log r}{\log R - \log r}, \quad a = 1/2.$$

In particular, if $x > r > 0$,

$$\hat{\mathbb{P}}_a^x\{T_r < \infty\} = \lim_{R\to\infty} \hat{\mathbb{P}}_a^x\{T_r < T_R\} = \begin{cases} (r/x)^{2a-1}, & a > 1/2 \\ 1, & a \le 1/2, \end{cases} \tag{2.5}$$

and if $x < R$,

$$\hat{\mathbb{P}}_a^x\{T_R < T_{0+}\} = \lim_{r\downarrow 0} \hat{\mathbb{P}}_a^x\{T_R < T_r\} = \begin{cases} (x/R)^{1-2a}, & a < 1/2 \\ 1, & a \ge 1/2. \end{cases} \tag{2.6}$$

If $a \ge 1/2$, then (2.5) implies that for each $t, R < \infty$,

$$\hat{\mathbb{P}}_a^x\{T_{0+} \le t\} \le \hat{\mathbb{P}}_a^x\{T_{0+} < T_R\} + \hat{\mathbb{P}}_a^x\{T_R < T_{0+}; T_R < t\}.$$

Letting $R \to \infty$ (and doing an easy comparison with Brownian motion for the second term on the right), shows that for all t,

$$\hat{\mathbb{P}}_a^x\{T_{0+} \le t\} = 0,$$

and hence, $\hat{\mathbb{P}}_a^x\{T_{0+} < \infty\} = 0$.

If $a < 1/2$, let $\tau_n = T_{2^{-2n}}$ and $\sigma_n = \inf\{t > \tau_n : X_t = 2^{-n}\}$. Then if $x > 2^{-2n}$, (2.6) implies that

$$\hat{\mathbb{P}}_a^x\{\sigma_n < T_{0+}\} = 2^{n(2a-1)}.$$

In particular,

$$\sum_{n=0}^{\infty} \hat{\mathbb{P}}_a^x\{\sigma_n < T_{0+}\} < \infty,$$

and by the Borel-Cantelli lemma, with $\mathbb{P}_a^x$-probability one, $T_{0+} < \sigma_n$ for all n sufficiently large. On the event that this happens, we see that

$$\lim_{t \uparrow T_{0+}} X_t = 0,$$

and hence $T_0 = T_{0+}$. On this event, we have $\max_{0 \le t \le T_0} X_t < \infty$, and hence

$$\hat{\mathbb{P}}_a^x\{T_{0+} = \infty\} \le \hat{\mathbb{P}}_a^x \left\{ \sup_{0 \le t < T_{0+}} X_t = \infty \right\} \le \lim_{R \to \infty} \hat{\mathbb{P}}_a^x\{T_{0+} < T_R\} = 0.$$

□

With this proposition, we can view $\hat{\mathbb{P}}_a^x$ for each t as a probability measure on continuous paths $X_s, 0 \le s \le t \wedge T_0$.

Proposition 2.2.2. *For each $x, t > 0$ and $a \in \mathbb{R}$, the measures $\mathbb{P}^x$ and $\hat{\mathbb{P}}_a^x$, considered as measures on paths $X_s, 0 \le s \le t$, restricted to the event $\{T_0 > t\}$ are mutually absolutely continuous with Radon-Nikodym derivative*

$$\frac{d\hat{\mathbb{P}}_a^x}{d\mathbb{P}^x} = \left[\frac{X_t}{x}\right]^a \exp\left\{-\frac{(a-1)a}{2} \int_0^t \frac{ds}{X_s^2}\right\}. \tag{2.7}$$

In particular,

$$\frac{d\hat{\mathbb{P}}_a^x}{d\mathbb{P}_{1-a}^x} = \left[\frac{X_t}{x}\right]^{2a-1}. \tag{2.8}$$

Proof. The first equality is a restatement of what we have already done. and the second follows by noting that the exponential term in (2.7) is not changed if we replace a with $1 - a$. □

Corollary 2.2.3. *Suppose $x < y$ and $a \ge 1/2$. Consider the measure $\tilde{\mathbb{P}}_a^y$ conditioned on the event $\{T_x < T_y\}$, considered as a measure on paths $X_t, 0 \le t \le T_x$. Then $\tilde{\mathbb{P}}_a^y$ is the same as $\tilde{\mathbb{P}}_{1-a}^y$, again considered as a measure on paths $X_t, 0 \le t \le T_x$.*

Proof. Using (2.7), we can see that the Radon-Nikodym derivative of the conditioned measure is proportional to the exponential term (the other term is the same for all paths). We also see from (2.8), a rederivation of the fact that $\tilde{\mathbb{P}}_a^y\{T_x < \infty\} = (x/y)^{2a-1}$. □

For fixed t, on the event $\{T_0 > t\}$, the measures $\mathbb{P}^x$ and $\hat{\mathbb{P}}_a^x$ are mutually absolutely continuous. Indeed, if $0 < r < x < R$, and $\tau = T_r \wedge T_R$, then $\mathbb{P}^x$ and $\hat{\mathbb{P}}^x$ are mutually absolutely continuous on $\mathcal{F}_\tau$ with

$$\frac{d\hat{\mathbb{P}}^x}{d\mathbb{P}^x} = N_{\tau,a} \in (0, \infty).$$

However, if $a < b < 1/2$, the measures $\hat{\mathbb{P}}_a$ and $\hat{\mathbb{P}}_b$ viewed as measure on on curves $X_t, 0 \le t \le T_0$, can be shown to be singular with respect to each other.

Proposition 2.2.4 (Brownian scaling)**.** *Suppose X_t is a Bessel process satisfying* (2.2)*, $r > 0$, and*

$$Y_t = r^{-1} X_{r^2 t}, \quad t \le r^{-2} T_0.$$

Then Y_t is a Bessel process with parameter a stopped at the origin.

Proof. This follows from the fact that Y_t satisfies

$$dY_t = \frac{a}{Y_t}\, dt + dW_t,$$

where $W_t = r^{-1} B_{r^2 t}$ is a standard Brownian motion. □

2.2.1 Logarithm

When one takes the logarithm of the Bessel process, the parameter $\nu = a - \frac{1}{2}$ becomes the natural one to use. Suppose X_t satisfies

$$dX_t = \frac{\nu + \frac{1}{2}}{X_t}\, dt + dB_t, \quad X_0 = x_0 > 0.$$

If $L_t = \log X_t$, then Itô's formula shows that

$$dL_t = \frac{1}{X_t}\, dX_t - \frac{1}{2X_t^2}\, dt = \frac{\nu}{X_t^2}\, dt + \frac{1}{X_t}\, dB_t.$$

Let

$$\sigma(s) = \inf\left\{ t : \int_0^t \frac{dr}{X_r^2} = s \right\}, \quad \hat{L}_s := L_{\sigma(s)}.$$

Then $\hat{L}_s$ satisfies

$$d\hat{L}_s = \nu\, ds + dW_s, \quad W_s := \int_0^{\sigma(s)} \frac{dB_r}{X_r},$$

Here W_s is a standard Brownian motion. In other words the logarithm of the Bessel process is a time change of a Brownian motion with constant drift. Note that $\sigma(\infty) = T_0$. If $\nu < 0$, it takes an infinite amount of time for the logarithm to reach $-\infty$ in the new parametrization, but it only takes a finite amount of time in the original parametrization.

2.2.2 Density

For most of the remainder of this paper, we will now drop the bulky notation $\hat{\mathbb{P}}_a^x$ and use $\mathbb{P}$ or $\mathbb{P}^x$. We assume that B_t is a standard Brownian motion and X_t satisfies

$$dX_t = \frac{a}{X_t}\, dt + dB_t, \quad X_0 = x > 0$$

This is valid until time $T = T_0 = \inf\{t : X_t = 0\}$. When a is fixed, we will write L, L^* for the generator and its adjoint, that is, the operators

$$Lf(x) = \frac{a}{x}\, f'(x) + \frac{1}{2}\, f''(x),$$

$$L^* f(x) = -\left[\frac{af(x)}{x}\right]' + \frac{1}{2} f''(x) = \frac{a}{x^2} f(x) - \frac{a}{x} f'(x) + \frac{1}{2} f''(x).$$

For $x, y > 0$, let $q_t(x, y; a)$ denote the transition density for the Bessel process stopped when it reaches the origin. In other words, if $I \subset (0, \infty)$ is an interval,

$$\mathbb{P}^x\{T > t; X_t \in I\} = \int_I q_t(x, y; a)\, dy.$$

In particular,

$$\int_0^\infty q_t(x, y; a)\, dy = \mathbb{P}^x\{T > t\} \begin{cases} = 1, & a \geq 1/2, \\ < 1, & a < 1/2 \end{cases}.$$

If $a = 0$, this is the density of Brownian motion killed at the origin for which we know that $q_t(x, y; 0) = q_t(y, x; 0)$. We can give an explicit form of the density by solving either of the "heat equations"

$$\partial_t q_t(x, y; a) = L_x q_t(x, y; a), \quad \partial_t q_t(x, y; a) = L_y^* q_t(x, y; a),$$

with appropriate initial and boundary conditions. The subscripts on L, L^* denote which variable is being differentiated with the other variables remaining fixed. The solution is given in terms of a special function which arises as a solution to a second order linear ODE. We will leave some of the computations to an appendix, but we include the important facts in the next proposition. We will use the following elementary fact: if a nonnegative random variable T has density $f(t)$ and $r > 0$, then the density of rT is

$$r^{-1} f(t/r). \tag{2.9}$$

Proposition 2.2.5. *Let $q_t(x, y; a)$ denote the density of the Bessel process with parameter a stopped when it reaches the origin. Then for all $x, y, t, r > 0$,*

$$q_t(x, y; 1 - a) = (y/x)^{1-2a}\, q_t(x, y; a), \tag{2.10}$$

$$q_t(x, y; a) = q_t(y, x; a)\, (y/x)^{2a}, \tag{2.11}$$

$$q_{r^2 t}(rx, ry; a) = r^{-1}\, q_t(x, y; a). \tag{2.12}$$

Moreover, if $a \geq 1/2$,

$$q_1(x, y; a) = y^{2a} \exp\left\{-\frac{x^2 + y^2}{2}\right\} h_a(xy), \tag{2.13}$$

where h_a is the entire function

$$h_a(z) = \sum_{k=0}^\infty \frac{z^{2k}}{2^{a+2k-\frac{1}{2}}\, k!\, \Gamma(k + a + \frac{1}{2})}. \tag{2.14}$$

In particular,

$$q_1(0, y; a) := q_1(0+, y, a) = \frac{2^{\frac{1}{2}-a}}{\Gamma(a + \frac{1}{2})}\, y^{2a}\, e^{-y^2/2}, \quad a \geq 1/2. \tag{2.15}$$

In the proposition we defined h_a by its series expansion (2.14), but it can also be defined as the solution of a particular boundary value problem.

Lemma 2.2.6. *h_a is the unique solution of the second order linear differential equation*

$$z\,h''(z) + 2a\,h'(z) - z\,h(z) = 0. \tag{2.16}$$

with $h(0) = 2^{\frac{1}{2}-a}/\Gamma(a+\frac{1}{2}), h'(0) = 0$.

Proof. See Proposition 2.5.1. □

Remarks.

- By combining (2.12) and (2.13) we get for $a \geq 1/2$,

$$q_t(x,y;a) = \frac{1}{\sqrt{t}}\,q_1\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}; a\right) = \frac{y^{2a}}{t^{a+\frac{1}{2}}}\,\exp\left\{-\frac{x^2+y^2}{2t}\right\}\,h_a\left(\frac{xy}{t}\right). \tag{2.17}$$

- The density is often written in terms of the modified Bessel function. If $\nu = a - \frac{1}{2}$, then $I_\nu(x) := x^\nu\,h_{\nu+\frac{1}{2}}(x)$ is the *modified Bessel function of the first kind of index* ν. This function satisfies the *modified Bessel equation*

$$x^2\,I''(x) + x\,I'(x) - [\nu^2 + x^2]\,I(x) = 0.$$

- The expressions (2.13) and (2.17) hold only for $a \geq 1/2$. For $a < 1/2$, we can use (2.10) to get

$$\begin{aligned} q_t(x,y;a) &= \frac{1}{\sqrt{t}}\,q_1\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}; a\right) \\ &= (y/x)^{2a-1}\,\frac{1}{\sqrt{t}}\,q_1\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}; 1-a\right) \\ &= \frac{y}{x^{2a-1}\,t^{\frac{3}{2}-a}}\,\exp\left\{-\frac{x^2+y^2}{2t}\right\}\,h_{1-a}\left(\frac{xy}{t}\right). \end{aligned}$$

- In the case $a = 1/2$, we can use the fact that the radial part of a two-dimensional Brownian motion is a Bessel process and write

$$q_1(x,y;1/2) = \frac{y}{2\pi}\int_0^{2\pi}\exp\left\{\frac{(x-y\cos\theta)^2 + (y\sin\theta)^2}{2}\right\}\,d\theta = y\,e^{-(x^2+y^2)/2}\,h_{1/2}(xy),$$

  where

$$h_{1/2}(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{r\cos\theta}\,d\theta = I_0(r).$$

  The last equality is found by consulting an integral table. Note that $h_{1/2}(0) = 1, h'_{1/2}(0) = 0$.

- We can write (2.15) as

$$c_d\,y^{d-1}\,e^{-y^2/2} \qquad d = 2a+1,$$

  which for positive integer d can readily be seen to be the density of the radial part of a random vector in $\mathbb{R}^d$ with density proportional to $\exp\{-|x|^2/2\}$. This makes this the natural guess for all d for which c_d can be chosen to make this a probability density, that is, $a > -1/2$. For $-1/2 < a < 1/2$, we will see that this is the density of the Bessel process reflected (rather than killed) at the origin.

- Given the theorem, we can determine the asymptotics of $h_a(x)$ as $x \to \infty$. Note that

$$\lim_{x\to\infty} q_1(x,x;a) = \frac{1}{\sqrt{2\pi}},$$

since for large x and small time, the Bessel process looks like a standard Brownian motion. Therefore,

$$\lim_{x\to\infty} x^{2a}\, e^{-x^2}\, h_a(x^2) = \frac{1}{\sqrt{2\pi}},$$

and hence

$$h_a(x) \sim \frac{1}{\sqrt{2\pi}}\, x^{-a}\, e^x, \quad x \to \infty.$$

In fact, there is an entire function u_a with $u_a(0) = 1/\sqrt{2\pi}$ such that for all $x > 0$,

$$h_a(x) = u(1/x)\, x^{-a}\, e^x.$$

See Proposition 2.5.3 for details. This is equivalent to a well known asymptotic behavior for I_ν,

$$I_\nu(x) = x^\nu\, h_{\nu+\frac{1}{2}}(x) \sim \frac{e^x}{\sqrt{2\pi x}}.$$

- The asymptotics implies that for each $a \geq 1/2$, there exist $0 < c_1 < c_2 < \infty$ such that for all $0 < x, y, t < \infty$,

$$c_1 \left[\frac{y}{x}\right]^{2a} \frac{1}{\sqrt{t}}\, e^{-(x-y)^2/2t} \leq q_t(x,y;a) \leq c_2 \left[\frac{y}{x}\right]^{2a} \frac{1}{\sqrt{t}}\, e^{-(x-y)^2/2t}, \quad t \leq xy, \tag{2.18}$$

$$c_1 \left[\frac{y}{\sqrt{t}}\right]^{2a} \frac{1}{\sqrt{t}}\, e^{-(x^2+y^2)/2t} \leq q_t(x,y;a) \leq c_2 \left[\frac{y}{\sqrt{t}}\right]^{2a} \frac{1}{\sqrt{t}}\, e^{-(x^2+y^2)/2t}, \quad t \geq xy, \tag{2.19}$$

Proof. The relation (2.10) follows from (2.8). The relation (2.11) follows from the fact that the exponential term in the compensator $N_{t,a}$ for a reversed path is the same as for the path. The scaling rule (2.12) follows from Proposition 2.2.4 and (2.9).

For the remainder, we fix $a \geq 1/2$ and let $q_t(x,y) = q_t(x,y;a)$. The heat equations give the equations

$$\partial_t q_t(x,y;a) = \frac{a}{x}\, \partial_x\, q_t(x,y;a) + \frac{1}{2}\, \partial_{xx}\, q_t(x,y;a), \tag{2.20}$$

$$\partial_t q_t(x,y;a) = \frac{a}{y^2}\, q_t(x,y;a) - \frac{a}{y}\, \partial_y\, q_t(x,y;a) + \frac{1}{2}\, \partial_{yy}\, q_t(x,y;a), \tag{2.21}$$

In our case, direct computation (See Proposition 2.5.2) shows that if h_a satisfies (2.16) and q_t is defined as in (2.15), then q_t satisfies (2.20) and (2.21). We need to establish the boundary conditions on h_a. Let

$$q_t(0,y;a) = \lim_{x\downarrow 0} q_t(x,y;a)$$

Since

$$1 = \int_0^\infty q_1(0+,y;a)\, dy = \int_0^\infty h_a(0)\, y^{2a}\, e^{-y^2/2}\, dy,$$

we see that

$$\frac{1}{h_a(0)} = \int_0^\infty y^{2a}\, e^{-y^2/2}\, dy = \int_0^\infty (2u)^a\, e^{-u}\, \frac{du}{\sqrt{2u}} = 2^{a-\frac{1}{2}}\, \Gamma\left(a+\frac{1}{2}\right).$$

Note that

$$q_1(x,1;a) = q_1(0,1;a) + x\, h_a'(0)\, e^{-1/2} + O(x^2), \quad x \downarrow 0.$$

Hence to show that $h_a'(0) = 0$, it suffices to show that $q_1(x,1;0) = q_1(0,1;a) + o(x)$. Suppose $0 < r < x$, and we start the Bessel process at r. Let τ_x be the the first time that the process reaches x. Then by the strong Markov property we have

$$q_1(r,1;a) = \int_0^1 q_{1-s}(x;1;0)\, dF(s),$$

where F is the distribution function for τ_x. Using (2.20), we see that $q_{1-s}(x;1;0) = q_1(x,1;0)+O(s)$. Therefore,

$$|q_1(x,1;a) - q_1(r,1;a)| \le c \int_0^1 s\, dF(s) \le c\, \mathbb{E}^r[\tau_x].$$

Using the scaling rule, we can see that $\mathbb{E}^r[\tau_x] = O(x^2)$.

□

We note that one standard way to solve a heat equation

$$\partial_t f(t,x) = L_x f(t,x),$$

with zero boundary conditions and given initial conditions, is to find a complete set of eigenfunctions for L

$$L\phi_j(x) = -\lambda_j\, \phi_j(x),$$

and to write the general solution as

$$\sum_{j=1}^\infty c_j\, e^{-\lambda_j t}\, \phi_j(x).$$

The coefficients c_j are found using the initial condition. This gives a series solution. This could be done for the Bessel process, but we had the advantage of the scaling rule (2.12), which allows the solution to be given in terms of a single special function h_a.

There are other interesting examples that will not have this exact scaling. Some of these, such as the radial Bessel process below, look like the Bessel process near the endpoints. We will use facts about the density of the Bessel process to conclude facts about the density of these other processes. The next proposition gives a useful estimate — it shows that the density of a Bessel process at $y \in (0, \pi/2]$ is comparable to that one would get by killing the process when it reaches level $7\pi/8$. (The numbers $3\pi/4 < 7\pi/8$ can be replaced with other values, of course, but the constants depend on the values chosen. These values will be used in discussion of the radial Bessel process.)

Proposition 2.2.7. *Let $\hat{q}_t(x,y;a)$ be the density of the Bessel process stopped at time $T = T_0 \wedge T_{7\pi/8}$. If $a \ge 1/2$, then for every $0 < t_1 < t_2 < \infty$, there exist $0 < c_1 < c_2 < \infty$ such that if $t_1 \le t \le t_2$ and $0 < x, y \le 3\pi/4$, then*

$$c_1\, y^{2a} \le \hat{q}_t(x,y;a) \le q_t(x,y;a) \le c_2\, y^{2a}.$$

This is an immediate corollary of the following.

Proposition 2.2.8. *Let $\hat{q}_t(x,y;a)$ be the density of the Bessel process stopped at time $T = T_0 \wedge T_{7\pi/8}$. For every $a \geq 1/2$ and $t_0 > 0$, there exists $0 < c_1 < c_2 < \infty$ such that for all $0 \leq x, y \leq 3\pi/4$ and $t \geq t_0$,*

$$\hat{q}_t(x,y;a) \geq c\, e^{-\beta t}\, q_t(x,y;a).$$

$$c_1 \mathbb{P}^x\{T > t - t_0; X_t \leq 3\pi/4\} \leq y^{-2a}\, \hat{q}_t(x,y;a) \leq c_2\, \mathbb{P}^x\{T > t - t_0\}.$$

Proof. It suffices to prove this for t_0 sufficiently small. Note that the difference $q_t(x,y;a) - \hat{q}_t(x,y;a)$ represents the contribution to $q_t(x,y;a)$ by paths that visit $7\pi/8$ some time before t. Therefore, using the strong Markov property, we can see that

$$q_t(x,y;a) - \hat{q}_t(x,y;a) \leq \sup_{0\leq s\leq t} [q_s(x,y;a) - \hat{q}_s(x,y;a)] \leq \sup_{0\leq s\leq t} q_s(3\pi/4, y; a).$$

Using the explicit form of $q_t(x,y;a)$ (actually it suffices to use the up-to-constants bounds (2.18) and (2.19)), we can find $t' > 0$ such that

$$c_1\, y^{2a} \leq \hat{q}_t(x,y;a) \leq c_2\, y^{2a}, \quad t' \leq t \leq 2t', \quad 0 < x, y \leq \pi/2.$$

If $s \geq 0$, and $t' \leq t \leq 2t'$,

$$\begin{aligned} \hat{q}_{s+t}(x,y;a) &= \int_0^{7\pi/8} \hat{q}_s(x,z;a)\, \hat{q}_t(z,y;a)\, dy \\ &\leq c\, \mathbb{P}^x\{T > s\} \sup_{0\leq z\leq 7\pi/8} q_t(z,y;a) \\ &\leq c\, \mathbb{P}^x\{T > s\}\, y^{2a}, \end{aligned}$$

$$\begin{aligned} \hat{q}_{s+t}(x,y;a) &\geq \int_0^{3\pi/4} \hat{q}_s(x,z;a)\, \hat{q}_t(z,y;a)\, dy \\ &\geq c\, \mathbb{P}^x\{T > s, X_s \leq 3\pi/4\} \inf_{0\leq z\leq 1} \hat{q}_t(z,y;a) \\ &\geq c\, \mathbb{P}^x\{T > s, X_s \leq 3\pi/4\}\, y^{2a}. \end{aligned}$$

□

Proposition 2.2.9. *Suppose X_t is a Bessel process with parameter $a < 1/2$ with $X_0 = x$, then the density of T_0 is*

$$\frac{2^{a-\frac{1}{2}}}{\Gamma(\frac{1}{2} - a)}\, x^{1-2a}\, t^{a-\frac{3}{2}}\, \exp\{-x^2/2t\}, \tag{2.22}$$

Proof. The distribution of T_0 given that $X_0 = x$ is the same as the distribution of $x^2\, T_0$ given that $X_0 = 1$. Hence by (2.9), we may assume $x = 1$. Let

$$F(t) = \mathbb{P}\{T_0 \leq t \mid X_0 = 1\} = \mathbb{P}\{T_0 \leq tx^2 \mid X_0 = x\}.$$

Then the strong Markov property implies that

$$\begin{aligned} s^{-1}\,\mathbb{P}\{t < T_0 \le t+s\} &= s^{-1}\int_0^\infty q_t(1,y;a)\,F(s/y^2)\,dy \\ &= \int_0^\infty [x\sqrt{s}]^{-1} q_t(1,\sqrt{s}\,x;a)\,x\,F(1/x)\,dx. \end{aligned}$$

Hence the density is

$$\lim_{s\downarrow 0}\int_0^\infty [x\sqrt{s}]^{-1} q_t(1,\sqrt{s}\,x;a)\,x\,F(1/x)\,dx.$$

We write $y_t = y/\sqrt{t}$ Using Proposition 2.2.5, we see that

$$\begin{aligned} y^{-1}\,q_t(1,y;a) &= t^{-1/2}\,y^{-1}\,q_1(t^{-1/2},y_t;a) \\ &= t^{-1/2}\,y^{2a-1}\,q_1(y_t,t^{-1/2};a) \\ &= t^{-1/2}\,q_1(y_t,t^{-1/2};1-a). \end{aligned}$$

Therefore,

$$\lim_{y\downarrow 0} y^{-1}\,q_t(1,y;a) = t^{-1/2}\lim_{z\downarrow 0} q_1(z,t^{-1/2};1-a) = h_{1-a}(0)\,t^{a-\frac{3}{2}}\,e^{-1/2t}.$$

This establishes the density up to a constant which is determined by

$$\int_0^\infty t^{a-\frac{3}{2}}\,e^{-1/2t}\,dt = 2^{\frac{1}{2}-a}\int_0^\infty u^{-\frac{1}{2}-a}\,e^{-u}\,du = 2^{\frac{1}{2}-a}\,\Gamma\left(\frac{1}{2}-a\right).$$

□

2.2.3 Geometric time scale

It is often instructive to consider the scaled Bessel process at geometric times (this is sometimes called the Ornstein-Uhlenbeck scaling). For this section we will assume $a \ge 1/2$ although much of what we say is valid for $a < 1/2$ up to the time that the process reaches the origin and for $-1/2 < a < 1/2$ for the reflected process.

Suppose X_t satisfies

$$dX_t = \frac{a}{X_t}\,dt + dB_t,$$

and let

$$Y_t = e^{-t/2}\,X_{e^t} \quad W_t = \int_0^{e^t} e^{-s/2}\,dB_s.$$

Note that

$$dX_{e^t} = \frac{a\,e^t}{X_{e^t}}\,dt + e^{t/2}\,dB_{e^t} = e^{t/2}\,\left[\frac{a}{Y_t}\,dt + dB_{e^t}\right],$$

or

$$dY_t = \left[\frac{a}{Y_t} - \frac{Y_t}{2}\right]\,dt + dW_t. \tag{2.23}$$

This process looks like the usual Bessel process near the origin, and it is not hard to see that processes satisfying (2.23) with $a \geq 1/2$, never reaches the origin. Of course, we knew this fact from the definition of Y_t and the fact that X_t does not reach the origin.

Not as obvious is the fact that Y_t is a recurrent process, in fact, a positive recurrent process with an invariant density. Let us show this in two ways. First, note that the process Y_t is the same as a process obtained by starting with a Brownian motion Y_t and weighting locally by the function

$$\phi(x) = x^a \, e^{-x^2/4}.$$

Indeed, using Itô's formula and the calculations,

$$\phi'(x) = \left[\frac{a}{x} - \frac{x}{2}\right] \, \phi(x),$$

$$\phi''(x) = \left(\left[\frac{a}{x} - \frac{x}{2}\right]^2 - \frac{a}{x^2} - \frac{1}{2}\right) \, \phi(x) = \left[\frac{a^2 - a}{x^2} - (a + \frac{1}{2}) + \frac{x^2}{4}\right] \, \phi(x),$$

we see that if

$$M_t = \phi(X_t) \, \exp\left\{ \int_0^t \left(\frac{a - a^2}{Y_s^2} + (a + \frac{1}{2}) - \frac{Y_s^2}{4} \right) \, ds \right\},$$

then M_t is a local martingale satisfying

$$dM_t = \left[\frac{a}{Y_t} - \frac{Y_t}{2}\right] \, M_t \, dY_t.$$

The invariant probability density for this process is given by

$$f(x) = c \, \phi(x)^2 = \frac{2^{\frac{1}{2}-a}}{\Gamma(a + \frac{1}{2})} \, x^{2a} e^{-x^2/2}. \tag{2.24}$$

where the constant is chosen to make this a probability density. The equation for an invariant density is $L^* f(x) = 0$, where L^* is the adjoint of the generator

$$\begin{aligned} L^* f(x) &= -\left(\left[\frac{a}{x} - \frac{x}{2}\right] \, f(x) \right)' + \frac{1}{2} \, f''(x) \\ &= \left(\frac{a}{x^2} + \frac{1}{2} \right) \, f(x) + \left[\frac{x}{2} - \frac{a}{x}\right] \, f'(x) + \frac{1}{2} \, f''(x). \end{aligned}$$

Direct differentiation of (2.24) gives

$$f'(x) = \left[\frac{2a}{x} - x\right] \, f(x),$$

$$f''(x) = \left(\left[\frac{2a}{x} - x\right]^2 - \frac{2a}{x^2} - 1 \right) \, f(x) = \left[\frac{4a^2 - 2a}{x^2} - 4a - 1 + x^2 \right] \, f(x),$$

so the equation $L^* f(x) = 0$ comes down to

$$\frac{a}{x^2} + \frac{1}{2} + \left[\frac{x}{2} - \frac{a}{x}\right] \, \left[\frac{2a}{x} - x\right] + \frac{1}{2} \, \left[\frac{4a^2 - 2a}{x^2} - 4a - 1 + x^2 \right] = 0,$$

which is readily checked.

Let $\phi_t(x,y)$ denote the density of a process satisfying (2.23). Then

$$\phi_t(x,y) = e^{t/2}\, q_{e^t}\left(x, e^{t/2}\, y; a\right) = y^{2a}\, \exp\left\{-\frac{e^{-t}\, x^2 + y^2}{2}\right\}\, h_a\left(e^{-t/2}\, xy\right).$$

In particular,

$$\lim_{t\to\infty} \phi_t(x,y) = q_1(0,y;a) = \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})}\, y^{2a}\, e^{-y^2/2} = f(y).$$

2.2.4 Green's function

We define the Green's function (with Dirichlet boundary conditions) for the Bessel process by

$$G(x,y;a) = \int_0^\infty q_t(x,y;a)\, dt.$$

If $a \geq 1/2$, then

$$G(x,y;a) = \int_0^\infty q_t(x,y;a)\, dt = \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})} \int_0^\infty \frac{y^{2a}}{t^{a+\frac{1}{2}}}\, \exp\left\{-\frac{x^2+y^2}{2t}\right\}\, h_a\left(\frac{xy}{t}\right),$$

and

$$G_{1-a}(x,y;a) = \int_0^\infty q_t(x,y;1-a)\, dt = (y/x)^{1-2a} \int_0^\infty q_t(x,y;a)\, dt = (y/x)^{1-2a}\, G(x,y;a).$$

In particular,

$$\begin{aligned} G(1,1;a) = G(1,1;1-a) &= \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})} \int_0^\infty \frac{1}{t^{a+\frac{1}{2}}}\, e^{-1/t}\, h_a(1/t)\, dt \\ &= \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})} \int_0^\infty \frac{1}{u^{\frac{3}{2}-a}}\, e^{-u}\, h_a(u)\, du. \end{aligned}$$

Proposition 2.2.10. *For all a, x, y,*

$$G(x,y;a) = (x/y)^{1-2a}\, G(x,y;1-a).$$

$$G(x,y;a) = (y/x)^{2a}\, G(y,x;a),$$

If $a = 1/2$, $G(x,y;a) = \infty$ for all x, y. If $a > 1/2$, then

$$G(r,ry;a) = C_a\, r\, \left[1 \wedge y^{1-2a}\right],$$

where

$$C_a = \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})} \int_0^\infty \frac{1}{u^{\frac{3}{2}-a}}\, e^{-u}\, h_a(u)\, du < \infty.$$

Proof.

$$\begin{aligned} G(x,y;1-a) &= \int_0^\infty q_t(x,y;1-a)\,dt \\ &= (y/x)^{1-2a}\int_0^\infty q_t(x,y;a)\,dt = (y/x)^{1-2a}\,G(x,y;a). \end{aligned}$$

$$G(x,y;a) = \int_0^\infty q_t(x,y;a)\,dt = (y/x)^{2a}\int_0^\infty q_t(y,x;a)\,dt = (y/x)^{2a}\,G(y,x;a).$$

$$\begin{aligned} G(rx,ry;a) &= \int_0^\infty q_t(rx,ry;a)\,dt \\ &= \int_0^\infty \frac{1}{r}\,q_{t/r^2}(x,y;a)\,dt \\ &= \int_0^\infty r\,q_s(x,y;a)\,ds = r\,G(x,y;a). \end{aligned}$$

We assume that $a \geq 1/2$ and note that the strong Markov property implies that

$$G(x,1;a) = \mathbb{P}^x\{T_1 < \infty\}\; G(1,1;a) = \left[1 \wedge x^{1-2a}\right]\; G(1,1;a).$$

$$G(1,1;a) = \int_0^\infty \frac{1}{t^{(1-a)+\frac{1}{2}}}\,e^{-1/t}\,h_{1-a}(1/t)\,dt.$$

□

2.2.5 Another viewpoint

The Bessel equation is

$$dX_t = \frac{a}{X_t}\,dt + dB_t,$$

where B_t is a Brownian motion and $a \in \mathbb{R}$. In this section we fix a and the Brownian motion B_t but vary the initial condition x. In other words, let X_t^x be the solution to

$$dX_t^x = \frac{a}{X_t^x}\,dt + dB_t, \quad X_0 = x, \tag{2.25}$$

which is valid until time $T_0^x = \inf\{t : X_t^x = 0\}$. The collection $\{X_t^x\}$ is an example of a stochastic flow. If $t < T_0^x$, we can write

$$X_t^x = x + B_t + \int_0^t \frac{a}{X_s^x}\,ds.$$

If $x < y$, then

$$X_t^y - X_t^x = y - x + \int_0^t \left[\frac{a}{X_s^y} - \frac{a}{X_s^y}\right]\,ds = y - x - \int_0^t \left[\frac{a(X_s^y - X_s^x)}{X_s^x\,X_s^x}\right]\,ds. \tag{2.26}$$

In other words, if $t < T_0^x \wedge T_0^y$, then $X_t^y - X_t^x$ is differentiable in t with

$$\partial_t\,[X_t^y - X_t^x] = -[X_t^y - X_t^x]\,\frac{a}{X_t^x\,X_t^y},$$

which implies that

$$X_t^y - X_t^x = (y-x)\,\exp\left\{-a\int_0^t \frac{ds}{X_s^x\,X_s^y}\right\}.$$

From this we see that $X_t^y > X_t^x$ for all $t < T_0^x$ and hence $T_0^x \le T_0^y$. By letting $y \to x$ we see that

$$\partial_x X_t^x = \exp\left\{-a\int_0^t \frac{ds}{(X_s^x)^2}\right\}. \tag{2.27}$$

Although $X_t^x < X_t^y$ for all $t > T_0^x$, as we will see, it is possible for $T_0^x = T_0^y$.

Proposition 2.2.11. *Suppose $0 < x < y < \infty$ and X_t^x, X_t^y satisfy (2.25) with $X_0^x = x, X_0^y = y$.*

1. *If $a \ge 1/2$, then $\mathbb{P}\{T_0^x = \infty$ for all $x\} = 1$.*
2. *If $1/4 < a < 1/2$ and $x < y$, then*

$$\mathbb{P}\{T_0^x = T_0^y\} > 0.$$

3. *If $a \le 1/4$, then with probability one for all $x < y$, $T_0^x < T_0^y$.*

Proof. If $a \ge 1/2$, then Proposition 2.2.1 implies that for each x, $\mathbb{P}\{T_0^x = \infty\} = 1$ and hence $\mathbb{P}\{T_0^x = \infty$ for all rational $x\} = 1$. Since $T_0^x \le T_0^y$ for $x \le y$, we get the first assertion.

For the remainder we assume that $a < 1/2$. Let us write $X_t = X_t^x, Y_t = X_t^y, T^x = T_0^x, T^y = T_0^y$. Let $h(x,y) = h(x,y;a) = \mathbb{P}\{T^x = T^y\}$. By scaling we see that $h(x,y) = h(x/y) := h(x/y,1)$. Hence, we may assume $y = 1$. We claim that $h(0+,1) = 0$. Indeed, T^r has the same distribution as $r^2\,T^1$ and hence for every $\epsilon > 0$ we can find r, δ such that $\mathbb{P}\{T^r \ge \delta\} \le \epsilon/2, \mathbb{P}\{T^1 \le \delta\} \le \epsilon/2$, and hence $\mathbb{P}\{T^1 = T^r\} \le \epsilon$.

Let $u = \sup_{t<T^x} Y_t/X_t$. We claim that

$$\mathbb{P}\{T^x < T^1; u < \infty\} = 0.$$

$$\mathbb{P}\{T^x = T^1; u = \infty\} = 0.$$

The first equality is immediate; indeed, if $Y_t \le cX_t$ for all t, then $T^1 = T^x$. For the second equality, let $\sigma_N = \inf\{t : Y_t/X_t = N\}$. Then,

$$\mathbb{P}\{u \ge N; T^1 = T^x\} \le \mathbb{P}\{T^1 = T^x \mid \sigma_N < \infty\} = h(1/N) \longrightarrow 0, \quad N \to \infty.$$

Let

$$L_t = \log\left(\frac{Y_t}{X_t} - 1\right) = \log(Y_t - X_t) - \log X_t.$$

Note that

$$d\log(Y_t - X_t) = -\frac{a}{X_t\,Y_t}\,dt,$$

$$d\log X_t = \frac{1}{X_t}\,dX_t - \frac{1}{2X_t^2}\,dt = \frac{a-\frac{1}{2}}{X_t^2}\,dt + \frac{1}{X_t}\,dB_t,$$

and hence

$$\begin{aligned} dL_t &= \left[\frac{\frac{1}{2}-a}{X_t^2} - \frac{a}{X_t\,Y_t}\right] dt - \frac{1}{X_t}\,dB_t \\ &= \frac{1}{X_t^2}\left[\frac{1}{2} - a - \frac{a}{e^{L_t}+1}\right] dt - \frac{1}{X_t}\,dB_t. \end{aligned} \tag{2.28}$$

In order to understand this equation, let us change the time parametrization so that the Brownian term has variance one. More precisely, define $\sigma(t), W_t$ by

$$\int_0^{\sigma(t)} \frac{ds}{X_s^2} = t, \quad W_t = -\int_0^{\sigma(t)} \frac{1}{X_s}\,dB_s.$$

Then W_t is a standard Brownian motion and $\tilde{L}_t := L_{\sigma(t)}$ satisfies

$$d\tilde{L}_t = \left[\frac{1}{2} - a - \frac{aX_{\sigma(t)}}{\tilde{Y}_{\sigma(t)}}\right] dt + dW_t = \left[\frac{1}{2} - a - \frac{a}{e^{\tilde{L}_t}+1}\right] dt + dW_t.$$

For every $a < 1/2$, there exists $u > 0$ and $K < \infty$ such that if $\tilde{L}_t \geq K$, then

$$\frac{1}{2} - a - \frac{a}{e^{\tilde{L}_t}+1} > u.$$

Hence, by comparison with a Brownian motion with drift u, we can see that if $\tilde{L}_t \geq K+1$, then with positive probability, $\tilde{L}_t \to \infty$ and hence $Y_t/X_t \to \infty$. Hence starting at any initial value $\tilde{L}_t = l$ there is a positive probability (depending on l) that $\tilde{L}_t \to \infty$.

If $a > 1/4$, then there exists $u > 0$ and $K < \infty$ such that if $\tilde{L}_t \leq -K$, then

$$\frac{1}{2} - a - \frac{a}{e^{\tilde{L}_t}+1} < -u.$$

Hence by comparison with a Brownian motion with drift $-u$, we can see that if $\tilde{L}_t \leq -(K+1)$, then with positive probability, $\tilde{L}_t \to -\infty$. Hence starting at any initial value $\tilde{L}_t = l$ there is a positive probability (depending on l) that $\tilde{L}_t \to -\infty$.

If $a \leq 1/4$, then

$$\frac{1}{2} - a - \frac{a}{e^{\tilde{L}_t}+1} > 0,$$

and hence by comparison with driftless Brownian motion, we see that

$$\limsup \tilde{L}_t \to \infty. \tag{2.29}$$

But as mentioned before, if $\tilde{L}_t \geq K+1$ for some K there is a positive probability that $\tilde{L}_t \to \infty$. Since (2.29) shows that we get an "infinite number of tries" we see that $\tilde{L}_t \to \infty$ with probability one.

Using this argument, we also see that for $1/4 < a < 1/2$, then with probability one either $\tilde{L}_t \to -\infty$ or $\tilde{L}_t \to \infty$. Therefore,

$$\lim_{t \uparrow T^x} \frac{X_t}{Y_t} \in \{0, 1\}. \tag{2.30}$$

We note that by closer examination, we can see that if $1/4 < a < 1/2$, then

$$\lim_{y \downarrow x} \mathbb{P}\{T_0^x = T_0^y\} = 1.$$

□

Proposition 2.2.12. *In the notation of the previous proposition, if* $1/4 < a < 1/2$ *and* $0 < x < y = 1$. *Then,*

$$\psi(x) := \mathbb{P}\{T_0^x \neq T_0^1\} = \frac{\Gamma(2a)}{\Gamma(4a-1)\,\Gamma(1-2a)} \int_0^x \frac{ds}{(1-s)^{2-4a}\, s^{2a}}.$$

Proof. We note that $\psi(x)$ is the solution of the boundary value problem

$$\frac{1}{2}\,\psi''(x) + \left[\frac{1-2a}{1-x} - \frac{a}{x}\right]\,\psi'(x) = 0, \quad \psi(0) = 0, \psi(1) = 1. \tag{2.31}$$

In the notation of the previous proof, let $R_t = X_t/Y_t$,. Itô's formula and the product rule give

$$dX_t = X_t \left[\frac{a}{X_t^2}\,dt + \frac{1}{X_t}\,dB_t\right],$$

$$\begin{aligned} d\left[\frac{1}{Y_t}\right] &= -\frac{1}{Y_t^2}\,dY_t + \frac{1}{Y_t^3}\,d\langle Y\rangle_t \\ &= \frac{1}{Y_t}\left[\frac{1-a}{Y_t^2}\,dt - \frac{1}{Y_t}\,dB_t\right]. \end{aligned}$$

$$\begin{aligned} dR_t &= R_t\left[\frac{1}{X_t^2}\left(a + (1-a)\,R_t^2 - R_t\right)\,dt + \frac{1}{X_t}\,(1-R_t)\,dB_t\right] \\ &= R_t\left[\frac{1}{X_t^2}\left((1-a)R_t - a)(R_t - 1\right)\,dt + \frac{1}{X_t}\,(1-R_t)\,dB_t\right] \end{aligned}$$

After a suitable time change, we see that $\hat{R}_t := R_{\sigma(t)}$ satisfies

$$d\hat{R}_t = \frac{(1-a)\hat{R}_t - a}{\hat{R}_t\,(1-\hat{R}_t)}\,dt + dW_t = \left[\frac{1-2a}{1-\hat{R}_t} - \frac{a}{\hat{R}_t}\right]\,dt + dW_t,$$

where W_t is a standard Brownian motion. Using (2.31), we see that $\psi(\hat{R}_t)$ is a bounded martingale, and using the optional sampling theorem and (2.30) we get the result.

□

2.2.6 Functionals of Brownian motion and Bessel process

In the analysis of the Schramm-Loewner evolution, one often has to evaluate or estimate expectations of certain functionals of Brownian motion or the Bessel process. One of the most important functionals is the one that arises as the compensator in the change-of-measure formulas for the Bessel process.

Suppose X_t is a Brownian motion with $X_t = x > 0$ and let

$$J_t = \int_0^t \frac{ds}{X_s^2}, \quad K_t = e^{-J_t} = \exp\left\{ - \int_0^t \frac{ds}{X_s^2} \right\},$$

which are positive and finite for $0 < t < T_0$. We have seen K_t in (2.27). Let I_t denote the indicator function of the event $\{T_0 > t\}$. The local martingale from (2.1) is

$$N_{t,a} = (X_t/X_0)^a \, K_t^{\lambda_a}, \qquad \text{where} \quad \lambda_a = \frac{a(a-1)}{2}.$$

Note that

$$a = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 8\lambda_a} \geq \frac{1}{2}. \tag{2.32}$$

In this section, we write $\mathbb{E}$ for Brownian expectations and $\hat{\mathbb{E}}_a$ for the corresponding expectation with respect to the Bessel process with parameter a. In particular, if Y is an $\mathcal{F}_t$-measurable random variable,

$$\hat{\mathbb{E}}_a^x \left[V \, I_t \right] = \mathbb{E}^x \left[V \, I_t \, N_{t,a} \right].$$

Proposition 2.2.13. *Suppose $\lambda \geq -1/8$ and*

$$a = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 8\lambda} \geq \frac{1}{2},$$

is the larger root of the polynomial $a^2 - a - 2\lambda$. If X_t is a Brownian motion with $X_0 = x > 0$, then

$$\mathbb{E}^x[K_t^\lambda \, I_t] = x^a \, \hat{\mathbb{E}}_a^x \left[X_t^{-a} \, I_t \right] = x^a \int_0^\infty q_t(x, y; a) \, y^{-a} \, dy.$$

In particular, if $x = 1$, as $t \to \infty$,

$$\mathbb{E}^1[K_t^\lambda \, I_t] = t^{-\frac{a}{2}} \, \frac{\Gamma(\frac{a}{2} + \frac{1}{2})}{2^{a/2} \, \Gamma(a + \frac{1}{2})} \, [1 + O(t^{-1})].$$

Proof.

$$\begin{aligned} \mathbb{E}^x \left[K_t^\lambda \, I_t \right] &= \mathbb{E}^x \left[N_{t,a} \, (X_t/X_0)^{-a} \, I_t \right] \\ &= x^a \, \hat{\mathbb{E}}_a^x \left[X_t^{-a} \, I_t \right] \\ &= x^a \int_0^\infty q_t(x, y; a) \, y^{-a} \, dy. \end{aligned}$$

Since $q_t(x, y; a) \asymp y^{2a}$ as $y \downarrow 0$, the integral is finite.

We now set $x = 1$ and use Proposition 2.2.5 to get the asymptotics as $t \to \infty$. Note that for $a \geq 1/2$,

$$\begin{aligned}
\int_0^\infty q_t(1,y;a)\, y^{-a}\, dy &= t^{-1/2} \int_0^\infty q_1(1/\sqrt{t}, y/\sqrt{t}; a)\, y^{-a}\, dy \\
&= t^{-(a+\frac{1}{2})}\, e^{-1/2t} \int_0^\infty y^a\, e^{-y^2/2t}\, h\left(\frac{y}{t}\right) dy \\
&= t^{-(a+\frac{1}{2})}\, e^{-1/2t} \int_0^\infty (\sqrt{t}u)^a\, e^{-u^2/2} h\left(\frac{u}{\sqrt{t}}\right) \sqrt{t}\, du \\
&= t^{-\frac{a}{2}}\, [1 + O(t^{-1})] \int_0^\infty \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})}\, u^a\, e^{-u^2/2}\, [1 + O(u^2/t)]\, du \\
&= t^{-\frac{a}{2}}\, \frac{\Gamma(\frac{a}{2}+\frac{1}{2})}{2^{a/2}\, \Gamma(a+\frac{1}{2})}\, [1 + O(t^{-1})].
\end{aligned}$$

□

The next proposition is similar computing the same expectation for a Bessel process.

Proposition 2.2.14. *Suppose $b \in \mathbb{R}$ and*

$$\lambda + \lambda_b \geq -\frac{1}{8}. \tag{2.33}$$

Let

$$a = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 8(\lambda + \lambda_b)} \geq \frac{1}{2}. \tag{2.34}$$

and assume that $a + b > -1$. Then, if X_t is a Bessel process with parameter b starting at $x > 0$,

$$\hat{\mathbb{E}}_b^x[K_t^\lambda\, I_t] = x^{a-b}\, \hat{\mathbb{E}}_a^x\left[X_t^{b-a}\right] = x^{a-b} \int_0^\infty y^{b-a}\, q_t(x,y;a)\, dy.$$

Note that if $b > -3/2$, then the condition $a + b > -1$ is automatically satisfied. If $b \leq -3/2$, then the condition $a+b > -1$ can be considered a stronger condition on λ than (2.33). If $b \leq -3/2$, then the condition on λ is

$$\lambda > 1 + 2b.$$

Proof. By comparing (2.32) and (2.34), we can see that

$$\begin{aligned}
\hat{\mathbb{E}}_b^x[K_t^\lambda\, I_t] &= x^{-b}\, \mathbb{E}^x\left[K_t^{\lambda_b+\lambda}\, X_t^b\, I_t\right] \\
&= x^{a-b}\, \mathbb{E}^x\left[N_{t,a}\, X_t^{b-a}\, I_t\right] \\
&= x^{a-b}\, \hat{\mathbb{E}}_a^x\left[X_t^{b-a}\right] \\
&= x^{a-b} \int_0^\infty y^{b-a}\, q_t(x,y;a)\, dy.
\end{aligned}$$

In the third equation, we drop the I_t since $I_t = 1$ with $\hat{\mathbb{P}}_a$-probability one. The condition $a+b > -1$ is needed to make the integral finite. □

Proposition 2.2.15. *Let $\lambda > -1/8$ and let*

$$a = \frac{1}{2} - \frac{1}{2}\sqrt{1+8\lambda} \leq \frac{1}{2},$$

be the smaller root of the polynomial $a^2 - a - 2\lambda$. Then if X_t is a Brownian motion starting at $x > 0$ and $0 < y < x$,

$$\mathbb{E}^x \left[K^\lambda_{T_y} \right] = (x/y)^a.$$

Proof. Let $n > x$ and let $\tau_n = T_y \wedge T_n$. Note that

$$\mathbb{E}^x \left[K^\lambda_{\tau_n} \right] = x^a \, \mathbb{E}^x \left[N_{\tau_n, a} \, X^{-a}_{\tau_n} \right] = x^a \, \hat{\mathbb{E}}^x_a \left[X^{-a}_{\tau_n} \right]$$

and similarly,

$$\mathbb{E}^x \left[K^\lambda_{\tau_n}; T_y < T_n \right] = x^a \, \hat{\mathbb{E}}^x_a \left[X^{-a}_{T_y}; T_y < T_n \right] = (x/y)^a \, \hat{\mathbb{P}}^x_a \left\{ T_y < T_n \right\},$$

$$\mathbb{E}^x \left[K^\lambda_{\tau_n}; T_y > T_n \right] = x^a \, \hat{\mathbb{E}}^x_a \left[X^{-a}_{T_n}; T_y > T_n \right] = (x/n)^a \, \hat{\mathbb{P}}^x_a \left\{ T_y > T_n \right\}.$$

Using (2.4), we see that

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}^x \left[K^\lambda_{t\wedge\tau_n}; T_y > T_n \right] &= \lim_{n\to\infty} (x/n)^a \, \hat{\mathbb{P}}^x_a \left\{ T_y > T_n \right\} \\ &= \lim_{n\to\infty} (x/n)^a \, \frac{x^{1-2a} - y^{1-2a}}{n^{1-2a} - y^{1-2a}} = 0. \end{aligned}$$

Therefore,

$$\mathbb{E}^x \left[K^\lambda_{T_y} \right] = \lim_{n\to\infty} \mathbb{E}^x \left[K^\lambda_{T_y}; T_y < T_n \right] = (x/y)^a \lim_{n\to\infty} \hat{\mathbb{P}}^x_a \left\{ T_y < T_n \right\} = (x/y)^a.$$

The first equality uses the monotone convergence theorem and the last equality uses $a \leq 1/2$. □

Proposition 2.2.16. *Suppose $b \in \mathbb{R}$ and $\lambda + \lambda_b \geq -1/8$. Let*

$$a = \frac{1}{2} - \frac{1}{2}\sqrt{1+8(\lambda+\lambda_b)} \leq \frac{1}{2},$$

the smaller root of the polynomial $a^2 - a - 2(\lambda + \lambda_b)$. Then if X_t is a Bessel process with parameter b starting at $x > 0$ and $0 < y < x$,

$$\hat{\mathbb{E}}^x_b \left[K^\lambda_{T_y}; T_y < \infty \right] = (x/y)^{a-b}.$$

A special case of this proposition occurs when $b \geq 1/2$, $\lambda = 0$. Then $a = 1 - b$ and

$$\hat{\mathbb{E}}^x_b \left[K^\lambda_{T_y}; T_y < \infty \right] = \mathbb{P}^x_b \{ T_y < \infty \} = (x/y)^{a-b} = (y/x)^{2b-1},$$

which is (2.5).

Proof.

$$\hat{\mathbb{E}}_b^x \left[K_{T_y}^\lambda ; T_y < \infty \right] = x^{-b}\, \mathbb{E}^x \left[K_{T_y}^\lambda \, K_{T_y}^{\lambda_b} \, X_{T_y}^b ; T_y < \infty \right] = (y/x)^b\, \mathbb{E}^x \left[K_{T_y}^{\lambda_a} \right] = (x/y)^{a-b}.$$

□

It is convenient to view the random variable J_t on geometric scales. Let us assume that $X_0 = 1$ and let

$$\hat{J}_t = J_{e^{-t}}.$$

Then if n is a positive integer, we can write

$$\hat{J}_n = \sum_{j=1}^{n} [\hat{J}_j - \hat{J}_{j-1}].$$

Scaling shows that the random variables $\hat{J}_j - \hat{J}_{j-1}$ are independent and identically distributed. More generally, we see that $\hat{J}_t$ is an increasing Lévy process, that is, it has independent, stationary increments. We will assume that $a \le 1/2$ and write $a = \frac{1}{2} - b$ with $b = -\nu \ge 0$. Let Ψ_a denote the *characteristic exponent* for this Lévy process, which is defined by

$$\mathbb{E}[e^{i\lambda \hat{J}_t}] = \exp\{t\Psi_a(\lambda)\}.$$

It turns out that $\nu = a - \frac{1}{2}$ is a nicer parametrization for the next proposition so we will use it.

Proposition 2.2.17. *Suppose $b \ge 0$ and X_t satisfies*

$$dX_t = \frac{\frac{1}{2} - b}{X_t}\, dt + dB_t, \quad X_0 = 1.$$

Then if $\lambda \in \mathbb{R}$,

$$\mathbb{E}^x \left[\exp\left\{ i\lambda \int_0^{T_y} \frac{ds}{X_s^2} \right\} \right] = y^{-r},$$

where

$$r = b - \sqrt{b^2 - 2i\lambda}.$$

is the root of the polynomial $r^2 - 2br + 2i\lambda$ with smaller real part. In other words, if $a \le 1/2$

$$\Psi_{\frac{1}{2}-b}(\lambda) = b - \sqrt{b^2 - 2i\lambda},$$

where the square root denotes the root with positive real part. In particular,

$$\mathbb{E}\left[\hat{J}_t \right] = \frac{t}{b}. \tag{2.35}$$

Proof. We will assume $\lambda \neq 0$ since the $\lambda = 0$ case is trivial. If r_-, r_+ denote the two roots of the polynomial ordered by their real part, then $\mathrm{Re}(r_-) < b, \mathrm{Re}(r_+) > 2b$; we have chosen $r = r_-$.

Let $\tau_k = T_y \wedge T_k$. Using Itô's formula, we see that $M_{t\wedge\tau_k}$ is a bounded martingale where

$$M_t = \exp\left\{ i\lambda \int_0^t \frac{ds}{X_s^2} \right\} X_t^r.$$

Therefore,

$$\mathbb{E}[M_{\tau_k}] = 1.$$

If $b > 0$.

$$\mathbb{E}\left[|M_{\tau_k}|; T_k < T_y\right] \le k^{\mathrm{Re}(r)}\, \mathbb{P}\{T_k < T_y\} \le c(y)\, k^{\mathrm{Re}(r)}\, k^{2a-1} = c(r)\, k^{\mathrm{Re}(r)}\, k^{-2b},$$

and hence,

$$\lim_{k\to\infty} \mathbb{E}\left[|M_{\tau_k}|; T_k < T_y\right] = 0.$$

(One may note that if $\lambda \neq 0$ and we had used r_+, then $\mathrm{Re}(r_+) > 2b$ and this term does not go to zero.) Similarly, if $b = 0$,

$$\lim_{k\to\infty} \mathbb{E}\left[|M_{\tau_k}|; T_k < T_y\right] = 0.$$

Therefore,

$$1 = \lim_{k\to\infty} \mathbb{E}\left[M_{\tau_k}; T_k > T_y\right] = \mathbb{E}[M_{T_y}] = y^r\, \mathbb{E}\left[\exp\left\{i\lambda \int_0^{T_y} \frac{ds}{X_s^2}\right\}\right].$$

The last assertion (2.35) follows by differentiating the characteristic function of $\hat{J}_t$ at the origin. □

The moment generating function case is similar but we have to be a little more careful because the martingale is not bounded for $\lambda > 0$.

Proposition 2.2.18. *Suppose $b > 0$, X_t satisfies*

$$dX_t = \frac{\frac{1}{2} - b}{X_t}\, dt + dB_t, \quad X_0 = 1,$$

and $2\lambda < b^2$. Then, if $0 < y < 1$,

$$\mathbb{E}^x\left[\exp\left\{\lambda \int_0^{T_y} \frac{ds}{X_s^2}\right\}\right] = y^{-r},$$

where

$$r = b - \sqrt{b^2 - 2\lambda}.$$

is the smaller root of the polynomial $r^2 - 2br + 2\lambda$.

Proof. By scaling, it suffices to prove this result when $y = 1$. Let $\tau = T_1$ and let

$$K_t = \exp\left\{\lambda \int_0^t \frac{ds}{X_s^2}\right\}, \qquad M_t = K_t\, X_t^r.$$

By Itô's formula, we can see that M_t is a local martingale for $t < \tau$ satisfying

$$dM_t = \frac{r}{X_t}\, M_t\, dt, \quad M_0 = 1.$$

If we use Girsanov and weight by the local martingale M_t, we see that

$$dX_t = \frac{r + \nu + \frac{1}{2}}{X_t}\, dt + dW_t, \quad t < \tau$$

where W_t is a standard Brownian motion in the new measure which we denote by $\hat{\mathbb{P}}$ with expectations $\hat{\mathbb{E}}$. Since $r + \nu < 0$, then with probability one in the new measure $\hat{\mathbb{P}}^x\{\tau < \infty\} = 1$, and hence

$$\mathbb{E}^x\left[K_\tau; \tau < \infty\right] = x^r\, \mathbb{E}^x\left[M_\tau; \tau < \infty\right] = x^r\, \hat{\mathbb{E}}^x[1\{\tau < \infty\}] = x^r.$$

□

We can do some "multifractal" or "large deviation" analysis. We start with the moment generating function calculation

$$\mathbb{E}\left[e^{\lambda \hat{J}_t}\right] = e^{k\xi(\lambda)},$$

where

$$\xi(\lambda) = \xi_b(\lambda) = b - \sqrt{b^2 - 2\lambda}, \quad \xi'(\lambda) = \frac{1}{\sqrt{b^2 - 2\lambda}}, \quad \xi''(\lambda) = \frac{1}{(b^2 - 2\lambda)^{3/2}}.$$

This is valid provided that $\lambda < b^2/2$. Recall that $\mathbb{E}[\hat{J}_t] = t/b$. If $\theta > 1/b$, then

$$\mathbb{P}\{\hat{J}_t \geq \theta t\} \leq e^{-\lambda\theta t}\, \mathbb{E}[e^{\lambda \hat{J}_t}] = \exp\left\{t[\xi(\lambda) - \lambda\theta]\right\}, \quad .$$

This estimate is most useful for the value λ that minimizes the right-hand side, that is, at the value λ_θ satisfying $\xi'(\lambda_\theta) = \theta$, that is,

$$\lambda_\theta = \frac{1}{2}\left[b^2 - \theta^{-2}\right], \quad \xi(\lambda_\theta) = b - \frac{1}{\theta}$$

Therefore,

$$\mathbb{P}\{\hat{J}_t \geq \theta t\} \leq \exp\left\{t\rho(\theta)\right\}, \qquad \text{where} \qquad \rho(\theta) = b - \frac{1}{2\theta} - \frac{\theta b^2}{2}$$

While this is only an inequality, one can show (using the fact that ξ is C^2 and strictly concave in a neighborhood of λ_θ),

$$\mathbb{P}\{\hat{J}_t \geq \theta t\} \asymp \mathbb{P}\{\theta t \leq \hat{J}_t \leq \theta t + 1\} \asymp t^{-1/2}\, \exp\left\{t\rho(\theta)\right\}.$$

Similarly, if $\theta < 1/b$,

$$\mathbb{P}\{\hat{J}_t \leq \theta t\} \leq e^{\lambda\theta t}\, \mathbb{E}[e^{-\lambda\, \hat{J}_t}] = \exp\left\{k[\xi(-\lambda) + \lambda\theta]\right\}, \quad .$$

The right-hand side is minimized when $\xi'(-\lambda) = \theta$, that is, when

$$\lambda_\theta = \frac{1}{2}\left[\theta^{-2} - b^2\right], \quad \xi(-\lambda_\theta) = b - \sqrt{2b^2 - \theta^{-2}}$$

$$\mathbb{P}\{\hat{J}_t \leq \theta t\} \leq \exp\left\{t\rho(\theta)\right\}, \qquad \text{where} \qquad \rho(\theta) = \frac{1}{2\theta} - \frac{\theta b^2}{2} + b - \sqrt{2b^2 - \theta^{-2}}$$

2.3 The reflected Bessel process for $-1/2 < a < 1/2$

The Bessel process can be defined with reflection at the origin in this range. Before defining the process formally, let us describe some of of the properties. In this section, we assume that $-1/2 < a < 1/2$.

- The reflected Bessel process X_t is a strong Markov process with continuous paths taking values in $[0, \infty)$. It has transition density

$$\psi_t(x, y; a) = \frac{1}{\sqrt{t}}\, \psi_1\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}; a\right) = \frac{y^{2a}}{t^{a+\frac{1}{2}}}\, \exp\left\{-\frac{x^2+y^2}{2t}\right\}\, h_a\left(\frac{xy}{t}\right). \tag{2.36}$$

  Note that this is exactly the same formula as for $q_t(x, y; a)$ when $a \geq 1/2$. We use a new notation in order not to conflict with our use of $q_t(x, y; a)$ for the density of the Bessel process killed when it reaches the origin. We have already done the calculations that show that

$$\partial_t \psi_t(x, y; a) = L_x \psi_t(x, y; a),$$

  and

$$\partial_x \psi_t(x, y; a) \mid_{x=0} = 0.$$

  However, if $a \leq 1/2$, it is not the case that

$$\partial_y \psi_t(x, y; a) \mid_{y=0} = 0.$$

  In fact, for $a < 1/2$, the derivative does not exists at $y = 0$.

- Note that

$$\psi_1(0, y; a) = \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})}\, y^{2a}\, e^{-y^2/2},$$

  and hence

$$\psi_t(0, y; a) = t^{-1/2}\, \psi_1(0, y/\sqrt{t}; a) = \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})}\, t^{-\frac{1}{2}-a}\, y^{2a}\, e^{-y^2/2t}. \tag{2.37}$$

- We have the time reversal formula for $x, y > 0$.

$$\psi_t(x, y; a) = (y/x)^{2a}\, \psi_t(y, x; a). \tag{2.38}$$

  Because of the singularity of the density at the origin we do not write $\psi_t(x, 0; a)$.

- A calculation (see Proposition 2.5.4) shows that for $x > 0$,

$$\int_0^\infty \psi_t(x, y; a)\, dy = 1. \tag{2.39}$$

  We can view the process as being "reflected at 0" in a way so that the the total mass on $(0, \infty)$ is always 1.

- Another calculation (see Proposition 2.5.6) shows that the ψ_t give transition probabilities for a Markov chain on $[0, \infty)$.

$$\psi_{t+s}(x, y; a) = \int_0^\infty \psi_t(x, z; a)\, \psi_s(z, y; a)\, dz.$$

  Note that this calculation only needs to consider values of $\psi_t(x, y; z)$ with $y > 0$.

- With probability one, the amount of time spent at the origin is zero, that is,

$$\int_0^\infty 1\{X_t = 0\}\, dt = 0.$$

  This follows from (2.39) which implies that

$$\int_0^k 1\{X_t > 0\}\, dt = \int_0^k \int_0^\infty \psi_t(x, y; a)\, dy\, dt = k.$$

- For each $t, x > 0$, if $\sigma = \inf\{s \geq t : X_s = 0\}$, the distribution of $X_s, t \leq s \leq \sigma$, given X_t, is that of a Bessel process with parameter a starting at X_t stopped when it reaches the origin.

- The process satisfies the Brownian scaling rule: if X_t is the reflected Bessel process started at x and $r > 0$, then $Y_t = r^{-1} X_{r^2 t}$ is a reflected Bessel process started at x/r.

- To construct the process, we can first restrict to dyadic rational times and use standard methods to show the existence of such a process. With probability one, this process is not at the origin for any dyadic rational t (except maybe the starting point). Then, as for Brownian motion, one can show that with probability one, the paths are uniformly continuous on every compact interval and hence can be extended to $t \in [0, \infty)$ by continuity. (If one is away from the origin, one can argue continuity as for Brownian motion. If one is "stuck" near the origin, then the path is continuous since it is near zero.) The continuous extensions do hit the origin although at a measure zero set of times.

Here we explain why we need the condition $a > -1/2$. Assume that we have such a process for $a < 1/2$. Let $e(x) = e(x; a) = \mathbb{E}^0[T_x]$ and $j(x) = j(x; a) = \mathbb{E}^x[T_0 \wedge T_{2x}]$. We first note that $e(1) < \infty$; indeed, it is obvious that there exists $\delta > 0, s < \infty$ such that $\mathbb{P}^0\{T_1 < s\} \geq \delta$ and hence $\mathbb{P}^x\{T_x < s\} \geq \delta$ for every $0 \leq x < 1$. By iterating this, we see that $\mathbb{P}^0\{T_1 \geq ns\} \leq (1-\delta)^n$, and hence $\mathbb{E}^0[T_1] < \infty$. The scaling rule implies that $e(2x) = 4e(x), j(2x) = 4j(x)$. Also, the Markov property implies that

$$e(2x) = e(x) + j(x) + \mathbb{P}^x\{T_0 < T_{2x}\} e(2x),$$

which gives

$$4e(x) = e(2x) = \frac{e(x) + j(x)}{\mathbb{P}^x\{T_0 \geq T_{2x}\}}.$$

By (2.6), we know that

$$\mathbb{P}^x\{T_0 \geq T_{2x}\} = \min\{2^{2a-1}, 1\}.$$

If $a \leq -1/2$, then $\mathbb{P}^x\{T_0 \geq T_{2x}\} \leq 1/4$, which is a contradiction since $j(x) > 0$.

There are several ways to construct this process. In the bullets above we outline one which starts with the transition probabilities and then constructs a process with these transitions. In the next subsection, we will do another one which constructs the process in terms of excursions. In this section, we will not worry about the construction, but rather we will give the properties. We will write the measure as $\hat{\mathbb{P}}_a^x$ (this is the same notation as for the Bessel process killed at the origin — indeed, it is the same process just continued onward in time).

If $x > 0$, the scaling rule will imply

$$\psi_t(x, y; a) = t^{-1/2}\, \psi_1(x/\sqrt{t}, y/\sqrt{t}; a),$$

so we need only give $\psi_1(x, y; a)$. What we will show now is that if we assume that (2.37) holds and gives $\psi_t(0; y; a)$, then the value $\psi_t(x, y; a)$ must hold for all x. We will use T_0, the first time that the process reaches the origin and write

$$\begin{aligned} \psi_1(x, y; a) &= \tilde{\psi}_1(x, y; a) + q_1(x, y; a) \\ &= \tilde{\psi}_1(x, y; a) + (y/x)^{2a-1}\, q_1(x, y; 1-a) \end{aligned}$$

where

$$\tilde{\psi}_1(x, y; a) = \int_0^1 \psi_{1-s}(0, y; a)\, d\mathbb{P}^x\{T_0 = s\}. \tag{2.40}$$

The term $q_1(x, y; a)$ gives the contribution from paths that do not visit the origin before time 1, and $\tilde{\psi}_1(x, y; a)$ gives the contribution of those that do visit. The next proposition is a calculation. The purpose is to show that our formula for $\psi_1(x, y; a)$ must be valid for $x > 0$ provided that it is true for $x = 0$.

Proposition 2.3.1. *If $-\frac{1}{2} < a < \frac{1}{2}$, then*

$$\tilde{\psi}_1(x, y; a) = y^{2a}\, e^{-(x^2+y^2)/2}\, \left[h_a(xy) - (xy)^{1-2a}\, h_{1-a}(xy)\right].$$

Proof. Using (2.22), we see that

$$d\mathbb{P}^x\{T_0 = s\} = \frac{2^{a-\frac{1}{2}}}{\Gamma(\frac{1}{2} - a)}\, x^{1-2a}\, s^{a-\frac{3}{2}}\, \exp\{-x^2/2s\}\, ds,$$

and hence if (2.37) holds. Using equation 2.3.16 #1 of [2] (see also the top of page 790), and a well known identity for the Gamma function, we see that

$$\begin{aligned} \int_0^\infty r^{-\nu-1} e^{-rz/2}\, e^{-z/2r}\, dr &= \frac{\pi}{\sin(-\pi\nu)}\, [I_\nu(z) - I_{-\nu}(z)] \\ &= \Gamma(\nu)\, \Gamma(1+\nu)\, [I_\nu(z) - I_{-\nu}(z)] \\ &= \Gamma\left(a - \frac{1}{2}\right)\, \Gamma\left(\frac{1}{2} - a\right)\, \left[z^{a-\frac{1}{2}}\, h_a(z) - z^{\frac{1}{2}-a}\, h_{1-a}(z)\right]. \end{aligned}$$

Hence,

$$\begin{aligned}
&\tilde{\psi}_1(x,y;a)\\
&= \frac{1}{\Gamma(\frac{1}{2}-a)\,\Gamma(\frac{1}{2}+a)} \int_0^1 s^{a-\frac{3}{2}}\,(1-s)^{-a-\frac{1}{2}}\,x^{1-2a}\,y^{2a}\,e^{-x^2/2s}\,e^{-y^2/2(1-s)}\,ds\\
&= \frac{x^{1-2a}\,y^{2a}\,e^{-(x^2+y^2)/2}}{\Gamma(\frac{1}{2}-a)\,\Gamma(\frac{1}{2}+a)} \int_0^1 \left(\frac{1-s}{s}\right)^{-a-\frac{1}{2}} \exp\left\{-\frac{x^2}{2}\,\frac{1-s}{s}\right\} \exp\left\{-\frac{y^2}{2}\,\frac{s}{1-s}\right\} s^{-2}\,ds\\
&= \frac{x^{1-2a}\,y^{2a}\,e^{-(x^2+y^2)/2}}{\Gamma(\frac{1}{2}-a)\,\Gamma(\frac{1}{2}+a)} \int_0^\infty u^{-a-\frac{1}{2}} \exp\left\{-\frac{x^2u}{2}\right\} \exp\left\{-\frac{y^2}{2u}\right\} du\\
&= \frac{x^{\frac{1}{2}-a}\,y^{a+\frac{1}{2}}\,e^{-(x^2+y^2)/2}}{\Gamma(\frac{1}{2}-a)\,\Gamma(\frac{1}{2}+a)} \int_0^\infty r^{-a-\frac{1}{2}} \exp\left\{-\frac{xyr}{2}\right\} \exp\left\{-\frac{xy}{2r}\right\} dr\\
&= y^{2a}\,e^{-(x^2+y^2)/2}\,\left[h_a(xy) - (xy)^{1-2a}\,h_{1-a}(xy)\right].
\end{aligned}$$

□

From the expression, we see that $\tilde{\psi}_1(x,y;a)$ is a decreasing function of x. Indeed, we could give another argument for this fact using coupling. Let us start two independent copies of the process at $x_1 < x_2$. We let the processes run until they collide at which time they run together. By the time the process starting at x_2 has reached the origin, the processes must have collided.

Recall that we are assuming $-1/2 < a < 1/2$. We will describe the measure on paths that we will write as $\hat{\mathbb{P}}_a^x$. Let $\psi_t(x,y;a), x \geq 0, y > 0, t > 0$ denote the transition probability for the process. We will derive a formula for this using properties we expect the process to have. First, the reversibility rule (2.11) will hold: if $x, y > 0$, then

$$\psi_t(x,y;a) = (y/x)^{2a}\,\psi_t(y,x;a).$$

In particular, we expect that $\psi_t(1,x;a) \asymp x^{2a}$ as $x \downarrow 0$. Suppose that $X_t = 0$ for some $1-\epsilon \leq t \leq \epsilon$. By Brownian scaling, we would expect the maximum value of X_t on that interval to be of order $\sqrt{\epsilon}$ and hence

$$\int_{1-\epsilon}^1 1\{|X_t| \leq \sqrt{\epsilon}\}\,dt \asymp \epsilon.$$

But,

$$\mathbb{E}\left[\int_{1-\epsilon}^1 1\{|X_t| \leq \sqrt{\epsilon}\}\,dt\right] = \int_{1-\epsilon}^1 \int_0^{\sqrt{\epsilon}} \psi_t(0,x;a)\,dx\,dt \sim c\,\epsilon^{a+\frac{3}{2}}.$$

Hence, we see that we should expect $\hat{\mathbb{P}}_a^x\{X_t = 0 \text{ for some } 1-\epsilon \leq t \leq 1 \sim c'\,\epsilon^{\frac{1}{2}+a}$. Brownian scaling implies that $\hat{\mathbb{P}}_a^0\{X_t = 0 \text{ for some } ru \leq t \leq r\}$ is independent of r and from this we see that there should be a constant $c = c(a)$ such that

$$\hat{\mathbb{P}}_a^x\{X_t = 0 \text{ for some } 1-\epsilon \leq t \leq 1\} \sim c\,\epsilon^{\frac{1}{2}+a}.$$

In fact, our construction will show that we can define a *local time* at the origin. In other words, there is a process L_t that is a normalized version of "amount of time spent at the origin by time t" with the following properties. Let $Z = \{s : X_s = 0\}$ be the zero set for the process.

- L_t is continuous, nondecreasing, and has derivative zero on $[0,\infty) \setminus Z$.
- As $\epsilon \downarrow 0$,
$$\mathbb{P}^0\{Z \cap [1-\epsilon, 1] \neq \emptyset\} = \mathbb{P}^0\{L_1 > L_{1-\epsilon}\} \asymp \epsilon^{\frac{1}{2}+a}.$$
- The Hausdorff dimension of Z is $\frac{1}{2} - a$.
- $$\mathbb{E}[L_t] = c \int_0^t s^{-\frac{1}{2}-a}\, ds = \frac{c}{\frac{1}{2}-a}\, t^{\frac{1}{2}-a}.$$

We will use a "last-exit decomposition" to derive the formula for $\psi_t(0,x;a)$.

Proposition 2.3.2. *If $y > 0$, then*

$$\psi_1(0,y;a) = \frac{y}{\Gamma(\frac{1}{2}-a)\,\Gamma(\frac{1}{2}+a)} \int_0^1 s^{-\frac{1}{2}-a}(1-s)^{a-\frac{3}{2}}\, e^{-y^2/2(1-s)}\, ds. \tag{2.41}$$

The proof of this proposition is a simple calculation,

$$\begin{aligned}
\int_0^1 s^{-\frac{1}{2}-a}(1-s)^{a-\frac{3}{2}}\, e^{-y^2/2(1-s)}\, ds &= \int_0^1 (1-s)^{-\frac{1}{2}-a} s^{a-\frac{3}{2}}\, e^{-y^2/2s}\, ds \\
&= e^{-y^2/2} \int_0^1 \left[\frac{1-s}{s}\right]^{-\frac{1}{2}-a} \exp\left\{-\frac{y^2}{2}\frac{1-s}{s}\right\} s^{-2}\, ds \\
&= e^{-y^2/2} \int_0^\infty u^{-\frac{1}{2}-a}\, e^{-u\,y^2/2}\, du \\
&= 2^{a+\frac{1}{2}}\, y^{2a}\, e^{-y^2/2} \int_0^\infty (uy^2/2)^{-\frac{1}{2}-a}\, e^{-uy^2/2}\, d(uy^2/2) \\
&= 2^{\frac{1}{2}-a}\, y^{2a-1} \int_0^\infty v^{-\frac{1}{2}-a}\, e^{-v}\, dv \\
&= 2^{\frac{1}{2}-a}\, y^{2a-1}\, \Gamma\left(\frac{1}{2}-a\right).
\end{aligned}$$

We would like to interpret the formula (2.41) in terms of a "last-exit" decomposition. What we have done is to split paths from 0 to t at the largest time $s < t$ at which $X_s = 0$. We think of $s^{-\frac{1}{2}-a}$ as being a normalized version of $\psi_s(0,0)$ and then $t^{a-\frac{3}{2}}\, e^{-y^2/2t}$ represents the normalized probability of getting to y at time t with no later return to the origin. To be more precise, let

$$q_t^*(y;a) = \lim_{x \downarrow 0} x^{2a-1}\, q_t(x,y;a),$$

and note that

$$\begin{aligned}
q_1^*(y;a) &= \lim_{x\downarrow 0} x^{2a-1}\, q_1(x,y;a) \\
&= \lim_{x\downarrow 0} x^{2a-1}\, (y/x)^{2a-1}\, q_1(x,y;1-a) \\
&= y^{2a-1}\, q_1(0,y;1-a) = c\, y\, e^{-y^2/2}.
\end{aligned}$$

$$\begin{aligned} q_t^*(y;a) &= \lim_{x\downarrow 0} x^{2a-1}\, q_t(x,y;a) \\ &= t^{-\frac{1}{2}} \lim_{x\downarrow 0} x^{2a-1} q_1(x/\sqrt{t}, y/\sqrt{t}; a) \\ &= t^{a-1} \lim_{z\downarrow 0} z^{2a-1} q_1(z, y/\sqrt{t}; a) \\ &= t^{a-1}\, q_1^*(y/\sqrt{t}; a) \\ &= c\, t^{a-\frac{3}{2}}\, y\, e^{-y^2/(2t)}. \end{aligned}$$

Proposition 2.3.3. *For every $0 < t_1 < t_2 < \infty$ and $y_0 < \infty$, there exists c such that if $t_1 \le t \le t_2$ and $0 \le x, y \le y_0$, then*

$$c_1\, y^{2a} \le \psi_t(x,y;a) \le c_2\, y^{2a}.$$

Proof. Fix t_1, t_2 and y_0 and allow constants to depend on these parameter. It follows immediately from (2.37) that there exist $0 < c_1 < c_2 < \infty$ such that if $t_1/2 \le t \le t_2$ and $y \le y_0$,

$$c_1\, y^{2a} \le \psi_t(0,y;a) \le c_2\, y^{2a}.$$

We also know that

$$\psi_t(x,y;a) = \tilde{\psi}_t(x,y;a) + q_t(x,y;a) \le \tilde{\psi}_t(0,y;a) + q_t(x,y;a).$$

Using (2.10) and Proposition 2.2.7, we see that

$$q_t(x,y;a) = (y/x)^{2a-1}\, q_t(x,y;1-a) \le c\, y^{2a-1}\, y^{2(1-a)} = cy \le c\, y^{2a}.$$

Also,

$$\tilde{\psi}_t(x,y;a) \ge \mathbb{P}^x\{T_0 \le t_1/2\} \inf_{t_1/2 \le s \le t_2} \psi_s(0,y;a) \ge c\, y^{2a} \ge c\, y^{2a}\, \mathbb{P}^{y_0}\{T_0 \le t_1/2\} \ge c\, y^{2a}.$$

□

For later reference, we prove the following.

Proposition 2.3.4. *There exists $c < \infty$ such that if $x \ge 3\pi/4$ and $y \le \pi/2$, then for all $t \ge 0$,*

$$\psi_t(x,y;a) \le c\, y^{2a}. \tag{2.42}$$

Proof. Let $z = 3\pi/4$. It suffices to prove the estimate for $x = z$. By (2.38),

$$(z/y)^{2a}\, \psi_t(z,y;a) = \psi_t(y,z;a) \le q_t(y,z;a) + \inf_{0 \le s < \infty} \psi_t(0,z;a) \le c.$$

□

2.3.1 Excursion construction of reflected Bessel process

In this section we show how we can construct the reflected Bessel process using excursions. In the case $a = 0$ this is the Itô construction of the reflected Brownian motion in terms of local time and Brownian excursions. Let $0 < r = a + \frac{1}{2} < 1$ and let $\mathcal{C}$ denote a Poisson point process from measure

$$(r\, t^{-r-1}\, dt) \times \text{ Lebesgue }.$$

Note that the expected number of pairs (t, x) with $0 \le x \le x_0$ and $2^{-n} \le t \le 2^{-n+1}$ is

$$x_0 \int_{2^{-n}}^{2^{-n+1}} r\, t^{-r-1}\, dr = x_0\, (1 - 2^{-r})\, 2^{rn},$$

which goes to infinity as $n \to \infty$. However,

$$\mathbb{E}\left[\sum_{(t,x)\in\mathcal{C}; x\le x_0, t\le 1} t \right] = x_0 \int_0^1 r\, t^{-r}\, dr = \frac{r\, x_0}{1-r} < \infty.$$

In other words, the expected number of excursions in $\mathcal{C}$ by time one is infinite (and a simple argument shows, in fact, that the number is infinite with probability one), but the expected number by time one of time duration at least $\epsilon > 0$ is finite. Also, the expected amount of time spent in excursions by time 1 of time duration at most one is finite. Let

$$T_x = \sum_{(t,x')\in\mathcal{C}; x'\le x} t.$$

Then with probability one, $T_x < \infty$. Note that T_x is increasing, right continuous, and has left limits. It is discontinuous at x such that $(t, x) \in \mathcal{C}$ for some t. In this case $T_x = T_{x-} + t$. Indeed, the expected number of pairs (t, x') with $x' \le x, t \ge 1$ is finite and hence with probability one the number of loops of time duration at least 1 is finite. We define L_t to be the "inverse" of T_x in the sense that

$$L_t = x \text{ if } T_{x-} \le t \le T_x.$$

Then L_t is a continuous, increasing function whose derivative is zero almost everywhere.

The density rt^{-r-1} is not a probability density because the integral diverges near zero. However we can still consider the conditional distribution of a random variable conditioned that it is at least k. Indeed we write

$$\mathbb{P}\{T \le t \mid T \ge k\} = \frac{\int_k^t r s^{-r-1}}{\int_k^\infty r s^{-r-1}\, ds} = 1 - \left(\frac{k}{t}\right)^r,$$

which means that the "hazard function" is r/k,

$$\mathbb{P}\{T \le k + dt \mid T \ge k\} = (r/k)\, dt + o(dt).$$

2.3.2 Excursions and bridges

Here we study the Bessel process with parameter $a < 1/2$ started at $x > 0$ "conditioned so that $T_0 = t$". We write

$$dX_t = \frac{a}{X_t}\, dt + dB_t, \quad t < T.$$

where B_t is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, and $T = T_0$ is the first hitting time of the origin. This is conditioning on an event of measure zero, but we can make sense of it using the Girsanov formula. Let

$$F(x,t) = x^{1-2a}\, t^{a-\frac{3}{2}}\, \exp\{-x^2/2t\}.$$

Up to a multiplicative constant, $F(x,\cdot)$ is the density of T_0 given $X_0 = x$ (see (2.22)). Let $M_s = F(X_s, t-s)$; heuristically, we think of M_s as the probability that $T = t$ given $\mathcal{F}_s$. Given this interpretation, it is reasonable to expect that M_s is a local martingale for $s < t$. Indeed, if we let $E_t = E_{t,\epsilon}$ be the event $E_t = \{t \le T_0 \le t + \epsilon\}$, and we weight by

$$F_\epsilon(x,t) = c\, \hat{\mathbb{P}}_a^x(E_t),$$

then $F_\epsilon(X_s, t-s) = c\, \hat{\mathbb{P}}_a^x(E_t \mid \mathcal{F}_s)$ which is a martingale. We can also verify this using Itô's formula which gives

$$dM_s = M_s \left[\frac{1-2a}{X_s} - \frac{X_s}{t-s} \right] dB_s.$$

Hence, if we tilt by the local martingale M_s, we see that

$$dX_s = \left[\frac{1-a}{X_s} - \frac{X_s}{t-s} \right] ds + dW_t, \tag{2.43}$$

where W_t is a Brownian motion in the new measure $\mathbb{P}^*$.

One may note that the SDE (2.43) gives the same process that one obtains by starting with a Bessel process X_t with parameter $1-a > 1/2$ and weighting locally by $J_s := \exp\{-X_t^2/2(t-s)\}$. Itô's formula shows that if X_s satisfies

$$dX_s = \frac{1-a}{X_s}\, ds + dB_s,$$

then

$$dJ_s = J_s \left[-\frac{X_s}{t-s}\, dB_s + \frac{a-\frac{3}{2}}{t-s}\, ds \right],$$

which shows that

$$N_s = \left(\frac{t}{t-s} \right)^{\frac{3}{2}-a} J_s,$$

is a local martingale for $s < t$ satisfying

$$dN_s = -\frac{X_s}{t-s}\, N_s\, dB_s.$$

There is no problem defining this process with initial condition $X_0 = 0$, and hence we have the distribution of a Bessel excursion from 0 to 0.

We can see from this that if $a < 1/2$, then the distribution of an excursion X_s with $X_0 = X_t = 0$ and $X_s > 0$ for $0 < s < t$ is the same as the distribution of a Bessel process with parameter $1-a$ "conditioned to be at the origin at time t". More precisely, if we consider the paths up to time $t - \delta$, then the Radon-Nikodym derivative of the excursion with respect to a Bessel with parameter $1-a$ is proportional to $\exp\{-X_{t-\delta}^2/2(t-\delta)\}$.

There are several equivalent ways of viewing the excursion measure. Above we have described the probability measure associated to excursions starting and ending at the origin of time duration t. Let us write $\mu^{\#}(t;a)$ for this measure. Then the excursion measure can be given by

$$c \int_0^\infty \mu^{\#}(t,a)\, t^{a-\frac{3}{2}}\, dt.$$

The constant c is arbitary. This is an infinite measure on paths but can be viewed as the limit of the measure on paths of time duration at least s,

$$c \int_s^\infty \mu^{\#}(t,a)\, t^{a-\frac{3}{2}}\, dt,$$

which has total mass

$$c \int_s^\infty t^{a-\frac{3}{2}}\, dt = \frac{c}{\frac{1}{2}-a}\, s^{a-\frac{1}{2}}.$$

Another way to get this measure is to consider the usual Bessel process started at $\epsilon > 0$ stopped when it reaches the origin. This is a probability measure on paths that we will denote by $\tilde{\mu}^{\#}(\epsilon;a)$. The density of the hitting time T is a constant times $\epsilon^{1-2a}\, t^{a-\frac{3}{2}}\, \exp\{-\epsilon^2/2t\}$. Then the excursion measure can be obtained as

$$\lim_{\epsilon \downarrow 0} \epsilon^{2a-1}\, \tilde{\mu}^{\#}(\epsilon;a).$$

From this perspective it is easier to see that in the excursion measure has the following property: the distribution of the remainder of an excursion given that the time duration is at least s and $X_s = y$ is that of a Bessel process with parameter a started at y stopped when it reaches the origin.

We can also consider m_t which is the excursion measure restricted to paths with $T > t$ viewed as a measure on the paths $0 \le X_s \le t, 0 < s \le t$. For each t this is a finite measure on paths, The density of the endpoint at time t (up to an arbitrary multiplicative constant) is given by

$$\psi_t(x) = \lim_{\epsilon \downarrow 0} \epsilon^{2a-1} q_t(\epsilon, x; a) = \lim_{\epsilon \downarrow 0} x^{2a-1}\, q_t(\epsilon, x; 1-a) = x^{2a-1}\, q_t(0, x; 1-a) = x\, t^{\frac{1}{2}-a}\, e^{-x^2/2t}.$$

Note that ψ_t is not a probability density; indeed,

$$\int_0^\infty \psi_t(x)\, dx = \int_0^\infty x\, t^{\frac{1}{2}-a}\, e^{-x^2/2t}\, dx = t^{\frac{3}{2}-a}.$$

Note that ψ_t satisfies the Chapman-Kolomogorov equations

$$\psi_{t+s}(x) = \int_0^\infty \psi_t(y)\, q_s(y, x; a)\, dx.$$

For example if $s = 1 - t$, then this identity is the same as

$$\begin{aligned} x\, e^{-x^2/2} &= \int_0^\infty y\, t^{\frac{1}{2}-a}\, (1-t)\, e^{-x^2/2t}\, (y/x)^{2a-1}\, q_{1-t}(x, y; 1-a)\, dy \\ &= \int_0^\infty y\, t^{\frac{1}{2}-a}\, e^{-x^2/2t}\, (y/x)^{2a-1}\, \frac{y^{2-2a}}{(1-t)^{\frac{3}{2}-a}} \exp\left\{-\frac{x^2+y^2}{1-t}\right\} h_a(xy/1-t). \end{aligned}$$

2.3.3 Another construction

Let us give another description of the reflected Bessel process using a single Brownian motion B_t. Suppose $a \in \mathbb{R}$, B_t is a standard Brownian motion, and for $x > 0$ let X_t^x satisfy

$$dX_t^x = \frac{a}{X_t^x}\,dt + dB_t, \quad X_0^x = x. \tag{2.44}$$

For a given x, this is valid up to time $T^x = \inf\{t : X_t^x > 0\}$. We define

$$Y_t = \inf\{X_t^y : y > 0, t < T^y\}, \quad .$$

We state the main result of the section here.

Theorem 2.3.5.

- *If $a \geq 1/2$, then Y_t has the distribution of the Bessel process with parameter a starting at the origin.*
- *If $-1/2 < a < 1/2$, then Y_t has the distribution of the reflected Bessel process starting at the origin.*
- *If $a \leq -1/2$, then with probability one $Y_t = 0$ for all t.*

For $a \geq 1/2$, this result is easy. By (2.26), if $x > 0$, then

$$X_t^x - x \leq Y_t \leq X_t^x.$$

Hence, for every $t > 0$, the distribution of $Y_s, s \geq t$, is that of the Bessel process starting at Y_t. It is not hard to see that $\mathbb{P}\{Y_t > 0\} = 1$ for each $t > 0$; indeed the density of Y_t is given by (2.15).

For the remainder of this section, we assume that $a < 1/2$. Note that this process is coalescing in the sense that if

$$Y_t^x = \inf\{X_t^y : y > x, t < T^y\},$$

then

$$Y_t^x = \begin{cases} X_t^x & t \leq T^x \\ Y_t & t \geq T^x \end{cases}$$

As an example, let us consider the case $a = 0$ for which the reflected Bessel process is the same as reflected Brownian motion. In this case $X_t^x = x + B_t$, and

$$T^x = \inf\{t : B_t = -x\}.$$

The set of times $\{T^x : x > 0\}$ are exactly the same as the set of times t at which the Brownian motion obtains a minimum, that is, $B_t < B_s, 0 \leq s < t$. This is also the set of times t at which $B_t \leq B_s, 0 \leq s < t$ (this is not obvious). The distribution of this set is the same as the distribution of the zero set of Brownian motion and is a topological Cantor set of Hausdorff dimension $1/2$.

We will set up some notation. If B_t is a standard Brownian motion, and $t \geq 0$, we let $B_{s,t} = B_{t+s} - B_t$. Let $\mathcal{G}_t$ be the "future" σ-algebra at time t, that is, the σ-algebra generated by $\{B_{s,t} :$

$s \geq 0\}$. Note that $\mathcal{G}_t$ is independent of $\mathcal{F}_t$, the σ-algebra generated by $\{B_s : s \leq t\}$. If $x > 0$ we write $X^x_{s,t}$ for the ($\mathcal{G}_t$-measurable) solution of

$$dX^x_{s,t} = \frac{a}{X^x_s}\, ds + dB_{s,t}, \quad X^x_{0,t} = x.$$

This is valid up to time $T^x_{0,t} = \inf\{s : X^x_{s,t} = 0\}$ and for $s < T^x_{0,t}$, we have

$$X^x_{s,t} = B_{s,t} + a \int_0^s \frac{dr}{X^x_{r,t}}.$$

The Markov property can be written as

$$X^x_{s+r,t} = X^{X^x_{s,t}}_{r,t+s}, \quad s + r < T^x_{0,t}.$$

If $\tau > 0$, we will say that t is a *τ-escape time* if for all $x > 0$,

$$X^x_{s,t} > 0, \quad 0 \leq s \leq \tau.$$

We say that t is an *escape time* if it is a τ-escape time for some $\tau > 0$. Note that if $Y_t > 0$ and

$$z = \inf\{x : X^x_t > 0\},$$

then $T_z < t < \inf_{z<z'} T_{z'}$. In particular, T_z is an escape time. The next proposition proves our main theorem in the $a \leq -1/2$ case.

Proposition 2.3.6. *If $a \leq -1/2$, then with probability one there are no τ-escape times for any $\tau > 0$.*

Proof. By scaling it suffices to prove that there are no 1-escape times t with $t \leq 1$. For each integer n, let $J_{k,n}$ be the indicator function that $T^{2^{-n}}_{0,k2^{-2n}} \geq 1/2$. Note that $J_{k,n}$is $\mathcal{G}_{k2^{-2n}}$-measurable. Using (2.22) we can see that that $\mathbb{E}[J_{k,n}] = \mathbb{E}[J_{0,n}] \asymp 2^{(2a-1)n}$, and if

$$J_n = \sum_{k=0}^{2^{2n}} J(k,n),$$

then $\mathbb{E}[J_n] \asymp 2^{(2a+1)n}$. If $a < -1/2$, this immediately implies that

$$\mathbb{P}\{J_n \geq 1\} \leq \mathbb{E}[J_n] \leq c\, 2^{(2a+1)n}.$$

We claim that

$$\mathbb{P}\{J_n \geq 1\} \leq \frac{c}{n}, \quad a = -\frac{1}{2}. \tag{2.45}$$

To see this, on the event $J_n \geq 1$, let $q = q_n$ be the largest index $k \leq 2^{2n}$ such that $J(k,n) = 1$ Note that

$$\mathbb{P}\{J_n \geq 1; q \leq 2^n\} \leq \sum_{k=1}^{2^n} \mathbb{E}[J(k,n)] \leq c\, 2^{-n}.$$

Now consider the $\mathcal{G}_{k2^{-2n}}$-measurable event $E(k,n) = \{J_n \geq 1, q = k\}$. Using the fact that $E(k,n)$ is independent of $\mathcal{F}_{k2^{-2n}}$ and using (2.22), we get

$$\begin{aligned}\mathbb{P}\{J(k-j,n) = 1 \mid E(k,n)\} &\geq \mathbb{P}^{2^{-n}}\{T_0 \geq (k-j)\,2^{-2n}; T_{(k-j)\,2^{-2n}} \geq 2^{-n}\} \\ &= \mathbb{P}^1\{T_0 \geq (k-j); T_{k-j} \geq 1\} \\ &\geq c/j.\end{aligned}$$

By summing over j, we see that

$$\mathbb{E}[J_n \mid J_n \geq 1, q > 2^n\} \geq c\,n,$$

and this gives (2.45).

We now consider the event that there exists a $t \leq 1$ that is a 1-escape time. It is not hard to see that the set of such t is closed and hence we can define σ to be the largest t. Note that σ is a backwards stopping time, that is, the event $\{\sigma \geq t\}$ is $\mathcal{G}_t$-measurable. If we take the largest dyadics smaller than σ for a given n, then we can see that given σ there is a positive probability of $J_n > 0$ (uniform in n for n large). But this contradicts the previous paragraph.

□

Proposition 2.3.7. *If $-1/2 < a < 1/2$, then with probability one, the set of escape times is a dense set of Hausdorff dimension $\frac{1}{2} + a$. In particular, it is a non-empty set of Lebesgue measure zero.*

Proof. We will only consider $t \in [0,1]$ and let R_τ denote the set of τ-escape times in $[0,1]$. If R is the set of escape times in $[0,1]$, then

$$R = \bigcup_{n=1}^{\infty} R_{1/n}.$$

Let us first fix τ. Let $\mathcal{Q}_n$ denote the set of dyadic rationals in $(0,1]$ with denominator 2^n,

$$\mathcal{Q}_n = \left\{ \frac{k}{2^n} : k = 1, 2, \ldots, 2^n \right\}.$$

We write $I(k,n)$ for the interval $[(k-1)2^{-n}, k2^n]$. We say that the interval $I(k,n)$ is *good* if there exists a time $t \in I(k,n)$ such that $X_{s,t}^{2^{-n/2}} > 0$ for $0 \leq s \leq 1$. Let

$$I_n = \bigcup_{I(k,n) \text{ good}} I(k,n), \quad I = \bigcap_{n=1}^{\infty} I_n.$$

Note that $I_1 \supset I_2 \supset \cdots$, and for each n, $R_1 \subset I_n$. We also claim that $I \subset R_{1/2}$. Indeed, suppose that $t \notin R_{1/2}$. Then there exists $x > 0$ such that $T_{0,t}^x \leq 1/2$, and hence $T_{0,t}^y \leq 1/2$ for $0 < y \leq x$. Using continuity of the Brownian motion, we see that there exists $y > 0$ and $\epsilon > 0$ (depending on the realization of the Brownian motion B_t), such that $T_{0,s}^y \leq 3/4$ for $|t - s| \leq \epsilon$. (The argument is slightly different for $s < t$ and $s > t$.) Therefore, $t \notin I_n$ if $2^{-n} < \epsilon$.

Let $J(k,n)$ denote the corresponding indicator function of the event $\{I(k,n) \text{ good}\}$. We will show that there exist $0 < c_1 < c_2 < \infty$ such that

$$c_1\, 2^{n(a-\frac{1}{2})} \leq \mathbb{E}\,[J(j,n)] \leq c_2\, 2^{n(a-\frac{1}{2})}, \tag{2.46}$$

$$\mathbb{E}[[J(j,n)\,J(k,n)] \le c_3\, 2^{n(a-\frac{1}{2})}\,[|j-k]+1]^{a-\frac{1}{2}}. \tag{2.47}$$

Using standard techniques (see, e.g., [1, Section A.3]), (2.46) implies that $\mathbb{P}\{\dim_h(R_1) \le a+\frac{1}{2}\} = 1$ and (2.46) and (2.47) imply that there exists $\rho = \rho(c_1, c_2, c_3, a) > 0$ such that

$$\mathbb{P}\left\{\dim_h(R_{1/2}) \ge a + \frac{1}{2}\right\} \ge \mathbb{P}\left\{\dim_h(I) \ge a + \frac{1}{2}\right\} \ge \rho.$$

Using stationarity of Brownian increments, it suffices to prove (2.46) and (2.47) for $j = 1$ and $k \ge 3$. Let us fix n and write $x = x_n = 2^{-n/2}, t = t_n = 2^{-n}$. The lower bound in (2.46) follows from

$$\mathbb{E}[J(1,n)] \ge \mathbb{P}\{T^x_{0,0} \ge 1\} \asymp 2^{n(a-\frac{1}{2})}.$$

Using $a \le 1/2$, we can see that $0 \le s \le t$,

$$\bar{X} := \max_{0\le s\le t} X^x_{t-s,s} \le x + \frac{x}{2} + \max_{0\le s\le t}[B_t - B_s].$$

Note that (for $n \ge 1$)

$$\mathbb{E}[J(1,n) \mid \bar{X} = z] \le c\,\mathbb{P}\{T^z_{0,t} \ge 1-t\} \le c\,(z \wedge 1)^{1-2a}.$$

We then get the upper bound in (2.46) using a standard estimate (say, using the reflection principle) for the distribution of the maximum of a Brownian path.

For the second moment, let us consider the event that $I(1,n)$ and $I(k,n)$ are both good. Let V_k denote the event that there exists $0 \le s \le 2^{-n}$ such that $T^x_{s,(k-1)t-s} > 0$. Then V_k is independent of the event $\{I(k,n) \text{ good}\}$ and

$$\{I(1,n) \text{ good}, I(k,n) \text{ good}\} \subset V_k \cap \{I(k,n) \text{ good}\}\}.$$

Using the argument for the upper bound in the previous paragraph and scaling, we see that

$$\mathbb{P}(V_k) \le c\,k^{a-\frac{1}{2}}.$$

Using Brownian scaling, we see that the upper bound implies that for all $\tau > 0$,

$$\mathbb{P}\{\dim_h(R_\tau) \le a + \frac{1}{2}\} = 1,$$

and hence with probability one $\dim_h(R) \le a + \frac{1}{2}$.. We claim that

$$\mathbb{P}\left\{\dim_h(R) = a + \frac{1}{2}\right\} = 1.$$

Indeed, if we consider the events

$$E_{j,n} = \left\{\dim_h[R_{2^{-(n+1)/2}} \cap I(j,n)] \ge a + \frac{1}{2}\right\}, \quad j = 1, 2, 3, \ldots, 2^{n-1},$$

then these are independent events each with probability at least ρ. Therefore,

$$\mathbb{P}\left\{E_{1,n} \cup E_{3,n} \cup \cdots \cup E_{2^n-1,n}\right\} \ge 1 - (1-\rho)^{2^{n-1}}.$$

Using this and scaling we see that with probability one for all rationals $0 \le p < q \le 1$, $\dim(R \cap [p,q]) = a + \frac{1}{2}$.

□

Proof of Proposition 2.3.5. We follow the same outline as the previous proof, except that we define $I(k,n)$ to be β-good if if there exists $t \in I(k,n)$ such that $X_{s,t}^{2^{-n/2}} > 0$ for $0 \le s \le 1$ and

$$X_{s,t}^{2^{-n/2}} \ge 2\beta, \quad \frac{1}{4} \le s \le 1.$$

Arguing as before, we get the estimates (2.46) and (2.47), although the constant c_1 now depends on β. Let $R_{1/2,\beta}$ be the set of $t \in R_{1/2}$ such that

$$\lim_{x \downarrow 0} X_{1/2,t}^x \ge \beta.$$

Then $R_{1/2,\beta} \subset I^\beta$ where

$$I_n^\beta = \bigcup_{I(k,n)\ \beta\text{-good}} I(K,n), \quad I^\beta = \bigcap_{n=1}^\infty I_n^\beta.$$

There exists $\rho_\beta > 0$ such that

$$\mathbb{P}\left\{\dim_h(R_{1/2,\beta}) = \frac{1}{2} + a\right\} \ge \rho_\beta.$$

For each time $t \in R$, we define

$$X_{s,t}^0 = \inf\{X_{s,t}^x, x > 0\}$$

where the right-hand side is defined to be zero if $T_{0,t}^x \le s$ for some $x > 0$. Recall that

$$\tilde{X}_t = \inf\{X_t^x : T_0^x > t\}.$$

Note that for every $0 \le t \le 1$,

$$\tilde{X}_1 \ge X_{1-t,t}^0.$$

We claim: with probability one, there exists $t < 1$ such that $X_{1-t,t}^0 > 0$. To see this, consider the events V_n defined by

$$V_n = \{\exists t \in I(2^n - 1, n) \text{ with } X_{1-t,0} > 2^{-n/2}\}.$$

The argument above combined with scaling shows that $\mathbb{P}(V_n)$ is the same and positive for each n. Also if we choose a sequence $n_1 < n_2 < n_3 < \cdots$ going to infinity sufficiently quickly, the events V_{n_j} are almost independent. To be more precise, Let

$$V^j = \{\exists t \in I(2^{n_j} - 1, n_j) \text{ with } X_{1-t-2^{-(n+1)},t} > 2 \cdot 2^{-n_j/2}\}.$$

Then the events $V^1, V^2, \ldots$ are independent and there exists $\rho > 0$ with $\mathbb{P}(V^j) > 0$. Hence $\mathbb{P}\{V^j \text{ i.o.}\} = 1$. If we choose the sequence n_j to grow fast enough we can see that

$$\sum_{j=1}^\infty \mathbb{P}(V^j \setminus V_{n_j}) < \infty,$$

and hence, $\mathbb{P}\{V_{n_j} \text{ i.o.}\} > 0$.

□

2.4 Radial Bessel and similar processes

We will now consider similar processes that live on the bounded interval $[0, \pi]$ and arise in the study of the radial Schramm-Loewner evolution. These processes look like the Bessel process near the boundaries. One main example is the radial Bessel process. We will first consider the process restricted to the open interval $(0, \pi)$ and then discuss possible reflections on the boundary. As in the case of the Bessel process, we will define our process by starting with a Brownian motion and then weighting by a particular function.

2.4.1 Weighted Brownian motion on $[0, \pi]$

We will consider Brownian motion on the interval $[0, \pi]$ "weighted locally" by a positive function Φ. Suppose $m : (0, \pi) \to \mathbb{R}$ is a C^1 function and let $\Phi : (0, \pi) \to (0, \infty)$ be the C^2 function

$$\Phi(x) = c \exp\left\{ - \int_x^{\pi/2} m(y)\, dy \right\}.$$

Here c is any positive constant. Everything we do will be independent of the choice of c so we can choose $c = 1$ for convenience. Also, $\pi/2$ is chosen for convenience; choosing a different limit for the integral will change Φ by a constant. Note that

$$\Phi'(x) = m(x)\, \Phi(x), \quad \Phi''(x) = [m(x)^2 + m'(x)]\, \Phi(x).$$

Let X_t be a standard Brownian motion with $0 < X_0 < \pi$, $T_y = \inf\{t : X_t = y\}$ and $T = T_0 \wedge T_\pi = \inf\{t : X_t = 0 \text{ or } X_t = \pi\}$. For $t < T$, let

$$M_t = M_{t,\Phi} = \frac{\Phi(X_t)}{\Phi(X_0)}\, K_t, \quad K_t = K_{t,\Phi} = \exp\left\{ -\frac{1}{2} \int_0^t \left[m(X_s)^2 + m'(X_s) \right]\, ds \right\}. \tag{2.48}$$

Then Itô's formula shows that M_t is a local martingale for $t < T$ satisfying

$$dM_t = m(X_t)\, M_t\, dX_t, \quad M_0 = 1.$$

Using the Girsanov theorem (being a little careful since this is only a local martingale), we get a probability measure on paths $X_t, 0 \le t < T$ which we denote by $\mathbb{P}_\Phi$. To be precise, if $0 < \epsilon < \pi/2$, $\tau = \tau_\epsilon = \inf\{t : X_t \le \epsilon \text{ or } X_t \ge \pi - \epsilon\}$, then $M_{t \wedge \tau}$ is a positive martingale with $M_0 = 1$. Moreover, if V is a random variable depending only on $X_s, 0 \le s \le t \wedge \tau$, then

$$\mathbb{E}^x_\Phi [V] = \mathbb{E}^x \left[M_{t \wedge \tau}\, V \right].$$

The Girsanov theorem implies that

$$dX_t = m(X_t)\, dt + dB_t, \quad t < T,$$

where B_t is a standard Brownian motion with respect to $\mathbb{P}_\Phi$.

Examples

- If
$$\Phi(x) = x^a, \quad m(x) = \frac{a}{x},$$
then X_t is the Bessel process with parameter a.

- If
$$\Phi(x) = (\sin x)^a, \quad m(x) = a \cot x,$$
then X_t is called the *radial Bessel process with parameter a*.

Note that the Bessel process and the radial Bessel process with the same parameter are very similar near the origin. The next definition makes this idea precise.

Definition

- We say that Φ (or the process generated by Φ) is *asymptotically Bessel-a at the origin* if there exists $c < \infty$ such that for $0 < x \le \pi/2$,
$$\left| m(x) - \frac{a}{x} \right| \le c\, x, \quad \left| m'(x) + \frac{a}{x^2} \right| \le c.$$
Similarly, we say that Φ is *asymptotically Bessel-a at* π if $\tilde{\Phi}(x) := \Phi(\pi - x)$ is asymptotically Bessel-a at the origin.

- We let $\mathcal{X}(a, b)$ be the set of Φ that are asymptotically Bessel-a at the origin and Asymptotically Bessel-b at π.

If $\Phi \in \mathcal{X}(a, b)$, then as $x \downarrow 0$
$$\int_x^{\pi/2} m(y)\, dy = -a \log x + C + O(x^2), \quad C = \int_0^{\pi/2} \left[m(y) - \frac{a}{y} \right] dy,$$
and hence
$$\Phi(x) = e^{-C}\, x^a \, \left[1 + O(x^2) \right].$$
In particular, if $0 < r < 1$,
$$\Phi(x) = r^a\, \Phi(rx) \, \left[1 + O(x^2) \right].$$

Examples

- The radial Bessel-a process is in $\mathcal{X}(a, a)$.

- If
$$\Phi(x) = [\sin x]^a\, [1 - \cos x]^v, \quad m(x) = (a + v) \cot x + \frac{v}{\sin x},$$
then Φ is in $\mathcal{X}(a, a + 2v)$.

Lemma 2.4.1. *Suppose $\Phi \in \mathcal{X}(a, b)$ with martingale $M_t = M_{t,\Phi}$ and let $\tilde{\Phi}(x) = a/x$ with corresponding martingale $\tilde{M}_t$. There exists $c < \infty$ such that the following holds. Suppose $0 < x < y \le 7\pi/8$, $X_0 = x$, $\tau = t \wedge T_0 \wedge T_y$. Then,*
$$|\log M_\tau - \log \tilde{M}_\tau| \le c\, [t + y^2].$$

Proof. This follows from

$$\left| \int_0^\tau \left[m(X_s)^2 - \frac{a^2}{X_s^2} \right] ds \right| \le c\, t,$$

$$\left| \int_0^\tau \left[m'(X_s) + \frac{a}{X_s^2} \right] ds \right| \le ct$$

$$\frac{\Phi(X_\tau)}{\Phi(X_0)} = \frac{X_\tau^a}{X_0^a} \left[1 + O(y^2) \right].$$

□

Lemma 2.4.2. *For every $\Phi \in \mathcal{X}(a,b)$ with $a \ge 1/2$, there exists $c < \infty$ such that the following holds. Suppose $0 < x < y \le 7\pi/8$ and let μ denote the measure on paths $X_t, 0 \le t \le T_y$. Let $\tilde{\mu}$ be the measure obtained by replacing X_t with a Bessel process $\tilde{X}_t$ with parameter a. Then the variation distance between μ and $\tilde{\mu}$ is less than $c\, y^2$.*

Proof. For the Bessel process there exists $\rho > 0$ such that for $z < y$, $\mathbb{P}^z\{T_y \le y^2\} \ge \rho$. By Iterating this, we see that for every positive integer k

$$\mathbb{P}^z\{T_y \ge k y^2\} \le \rho^k.$$

On the event $\{(k-1)\, y^2 < T_y \le y^2\}$, we have

$$M_\tau = \tilde{M}_\tau \, [1 + O(k y^2)].$$

and hence the variation distance between μ and $\tilde{\mu}$ on the event $\{(k-1)y^2 < T_y \le ky\}$ is less than $c\, \rho^k\, k y^2$. Summing over k gives the result.

□

It is sometimes more convenient to compare the asymptotically Bessel process to a Bessel process rather than to a Brownian motion. Suppose $a, b \ge 1/2$ and let us define the Bessel-(a,b) process on $(0,\pi)$ as follows. Let

$$\sigma_0 = 0, \quad \tau_0 = \inf\{t \ge 0 : X_t \ge 7\pi/8\},$$

and recursively,

$$\sigma_k = \inf\{t > \tau_{k-1} : X_t = \pi/8\}, \quad \tau_k = \inf\{t > \sigma_k : X_t = 7\pi/8\}.$$

Then the Bessel-(a,b) process on $(0,\pi)$ is defined to be the process X_t such that

- if $\sigma_j \le t < \tau_j$, then X_t evolves as a Bessel process with parameter a;
- if $\tau_j \le t < \sigma_j$, then $\pi - X_t$ evolves as a Bessel with parameter b.

If $\Phi \in \mathcal{X}(a,b)$ with corresponding m, then we define the martingale M_t by $M_0 = 1$ and

$$M_t = M_{\sigma_j} \frac{\Phi(X_t)/\Phi(X_{\sigma_j})}{(X_t/X_{\sigma_j})^a} \exp\left\{ -\frac{1}{2} \int_{\sigma_j}^t \left[m(X_s)^2 + m'(X_s) - \frac{a(a-1)}{X_s^2} \right] ds \right\},$$

if $\sigma_j \le t \le \tau_j$, and if $\tau_j \le t \le \sigma_{j+1}$,

$$M_t = M_{\tau_j} \frac{\Phi(X_t)/\Phi(X_{\tau_j})}{(X_t/X_{\tau_j})^b} \exp\left\{ -\frac{1}{2} \int_{\tau_j}^t \left[m(X_s)^2 + m'(X_s) - \frac{b(b-1)}{(\pi - X_s)^2} \right] ds \right\},$$

Note that there exists β with $e^{-\beta t} \le M_t \le e^{\beta t}$, so this is a martingale. We can say that the process titled by Φ is mutually absolutely continuous with the Bessel-(a,b) process with Radon-Nikodym derivative M_t. If $a < 1/2$ or $b < 1/2$ we can similarly define the Φ-process up to the first time that it leaves $[0, \pi]$.

Let

$$F(x) = F_\Phi(x) = \int_{\pi/2}^x \frac{dy}{\Phi(y)^2},$$

and note that

$$F'(x) = \frac{1}{\Phi(x)^2}, \quad F''(x) = -\frac{2\,\Phi'(x)}{\Phi(x)^3} = -\frac{2\,m(x)}{\Phi(x)^2}.$$

Using this and Itô's formula we see that $F(X_t)$ is a $\mathbb{P}^x_\Phi$ local martingale for $t < T$.

Proposition 2.4.3. *If $\Phi \in \mathcal{X}(a,b)$, $0 < x < z < \pi$, then*

$$\lim_{\epsilon \downarrow 0} \mathbb{P}^x_\Phi\{T_\epsilon < T_z\} = 0$$

if and only if $a \ge 1/2$. Similarly

$$\lim_{\epsilon \downarrow 0} \mathbb{P}^z_\Phi\{T_{\pi-\epsilon} < T_x\} = 0$$

if and only if $b \ge 1/2$.

Proof. We will prove the first; the second follows similarly. If $\Phi \in \mathcal{X}(a,b)$,

$$\Phi(x)^{-2} \sim c_1\, x^{-2a}\,[1 + O(x^2)],$$

Note that F is strictly increasing on $(0,\pi)$ with $F(\pi/2) = 0$ and $F(0) = -\infty$ if and only if $a \ge 1/2$. Let $\tau = T_\epsilon \wedge T_z$. Since $F(X_{t\wedge\tau})$ is a bounded martingale, the optional sampling theorem implies that.

$$F(x) = F(z)\,\mathbb{P}\{T_z < T_\epsilon\} + F(\epsilon)\,\mathbb{P}\{T_\epsilon < T_z\} = F(\epsilon)\,\mathbb{P}\{T_\epsilon < T_z\},$$

$$\lim_{\epsilon \downarrow 0} \mathbb{P}\{T_\epsilon < T_z\} = \lim_{\epsilon \downarrow 0} \frac{F(z) - F(x)}{F(z) - F(\epsilon)} = \frac{F(z) - F(x)}{F(z) - F(0)}.$$

□

2.4.2 $a, b > 1/2$

In this section we consider $\Phi \in \mathcal{X}(a,b)$ with $a, b \ge 1/2$ so that the process does not hit the origin. Let

$$f(x) = c\,\Phi(x)^2, \quad 0 < x < \pi$$

where c is chosen so that f is a probability density. We will show that f is the invariant density for the process and the convergence to equilibrium is exponentially fast uniformly over the starting position.

The form of the invariant density follows almost immediately from the fact that the process is obtained from Brownian motion by tilting by Φ. Let $\tilde{p}_t(x,y)$ denote the density of a Brownian motion killed when it reaches $\{0,\pi\}$ and let $q_t(x,y)$ denote the transition density for X_t. Then

$$q_t(x,y) = \frac{\Phi(y)}{\Phi(x)}\,\tilde{p}_t(x,y)\,\mathbb{E}^*[K_t],$$

where K_t is as above, and $\mathbb{E}^*$ is the process corresponding to Brownian motion starting at x conditioned to be at y at time t and having not left the interval $(0,\pi)$ by that time. Using reversibility of Brownian motion, we see that

$$f(x)\,q_t(x,y) = f(y)\,q_t(y,x),$$

and hence

$$\int_0^\pi f(x)\,p_t(x,y)\,dx = \int_0^\pi f(y)\,p_t(y,x)\,dx = f(y).$$

The key to exponentially fast convergence to equilibrium is the following lemma.

Proposition 2.4.4. *If $\Phi \in \mathcal{X}(a,b)$ with $a,b \ge 1/2$ and $t_0 > 0$, then there exist $0 < c_1 < c_2 < \infty$ such that for all $x,y \in (0,\pi)$, and $t \ge t_0$,*

$$c_1\,f(y) \le q_t(x,y) \le c_2\,f(y). \tag{2.49}$$

Proof. Let

$$I_1 = \left[\frac{\pi}{4}, \frac{7\pi}{4}\right], \quad I_2 = \left[\frac{\pi}{8}, \frac{7\pi}{8}\right], \quad I_3 = \left[\frac{\pi}{16}, \frac{15\pi}{16}\right].$$

For $x,y \in I_3$, let $\tilde{q}_t(x,y)$ be the density for the process killed when it leaves I_3. We claim that there exist c_1, c_2 such that

$$c_1 \le \tilde{q}_t(x,y) \le c_2, \quad \frac{1}{4} \le t \le 1, \quad x,y \in I_2, \tag{2.50}$$

$$\tilde{q}_t(x,y) \le c_2, \quad t > 0, \quad x \in \partial I_2, \ \ y \in I_1. \tag{2.51}$$

Indeed this is standard for Brownian motion killed when it leaves I_3 and the martingale is bounded uniformly away from 0 and ∞. To get an upper bound for $q_t(x,y)$ we split into excursions. Let

$$\sigma_1 = \inf\{t : X_t \in \partial I_3\}, \quad \tau_1 = \inf\{t > \sigma_1 : X_t \in I_2\},$$

and recursively,

$$\sigma_j = \inf\{t > \tau_{j-1} : X_t \in \partial I_3\}, \quad \tau_j = \inf\{t > \sigma_j : X_t \in I_2\}.$$

Then if $t \le 1$, $x,y \in I_2$,

$$q_t(x,y) = \tilde{q}_t(x,y) + C\sum_{j=1}^{\infty} \mathbb{P}^x\{\tau_j < 1\},$$

where

$$C = \max\{\tilde{q}_t(z,w) : t \ge 1, z \in \partial I_2, w \in I_1\}.$$

Since the process starting on ∂I_3 has positive probability of not reaching I_2 by time 1, we see there exists $\rho < 1$ such that $\mathbb{P}^x\{\tau_j < 1\} \le \rho^j$. Hence we get (2.50) and (2.51) with $\tilde{q}_t(x,y)$ replaced by $q_t(x,y)$ with a different value of c_2.

Since there exists uniform $\delta > 0$ such that for $x \in (0,\pi) \setminus I_3$, $\mathbb{P}\{\sigma_1 \le 1/4\} > \delta$, we can use the strong Markov property to conclude that

$$q_{1/2}(x,y) \ge c_3, \quad x \in (0,\pi), y \in I_1$$

and hence also, $q_{1/2}(y,x) \ge c_4\, f(x)$ for these x, y. More generally, if $x, y \in (0,\pi)$,

$$q_1(x,y) \ge \int_{I_1} q_{1/2}(x,z)\, q_{1/2}(z,y)\, dz \ge c\, f(y).$$

We have upper bounds if one of x or y is in I_1 and if $x \ge \pi/4, y \ge 3\pi/4$, we can use the strong Markov property stopping the process at time $T_{\pi/4}$. The last case is the upper bound for $x, y \le \pi/4$ (or $x, y \ge 7\pi/4$ that is done similarly). For this we compare to the Bessel process with parameter a using the estimates in Proposition 2.2.7.

□

Proposition 2.4.5. *If $\Phi \in \mathcal{X}(a,b)$ with $a, b \ge 1/2$, then there exist c, β such that for all $t \ge 1$ and $0 < x, y < \pi$,*

$$[1 - ce^{-\beta t}]\, f(y) \le q_t(x,y) \le [1 + ce^{-\beta t}]\, f(y).$$

In particular, if $g : (0,\pi) \to [0,\infty)$ with

$$\bar{g} := \int_0^\pi g(x)\, f(x)\, dx < \infty,$$

then

$$\bar{g}\,[1 - ce^{-\beta t}] \le \mathbb{E}^x[g(X_t)] \le \bar{g}\,[1 + ce^{-\beta t}].$$

Proof. It suffices to prove the result for positive integer t. Let us write the $1 - c_1$ in the last proposition with $t_0 = 1$ as $e^{-\beta}$. Let us fix x and write $f_t(y) = q_t(x,y)$. Then (2.49) implies that we can write

$$f_1(y) = [1 - e^{-\beta}]\, f(y) + e^{-\beta}\, g_1(y),$$

for some probability density g_1. By iterating this and using the fact that f is invariant, we see that we we can write for integer t

$$f_t(y) = [1 - e^{-\beta t}]\, f(y) + e^{-\beta t}\, g_t(y)$$

for a probability density g_t. Note at this point we have used only the lower bound in (2.49). From this equation we can conclude that the variation distance between the distribution at time t and the invariant measure decays exponentially. However, our claim is stronger. We appeal to the upper bound to get

$$[1 - e^{-\beta t}]\, f(y) \le f_{t+1}(y) \le [1 - e^{-\beta t} + c_2 e^{-\beta t}]\, f(y).$$

□

Proposition 2.4.6. *Suppose $\Phi \in \mathcal{X}(a,b)$ and $M_t = M_{t,\Phi}, K_t = K_{t,\Phi}$ is as defined in (2.48). There exists $\beta > 0$ such that if X_t is a standard Brownian motion, then*

$$\mathbb{E}^x[K_{t,\Phi}; T > t] = c_* \, \Phi(x) \, [1 + O(e^{-\beta t})], \quad c_* = \frac{\int_0^\pi \Phi(y)\, dy}{\int_0^\pi \Phi(y)^2 \, dy}.$$

Proof.

$$\begin{aligned} \mathbb{E}^x[K_{t,\Phi}; T > t] &= \Phi(x) \, \mathbb{E}^x \left[\Phi(X_t)^{-1} \, M_{t,\Phi}; T > t \right] \\ &= \Phi(x) \, \mathbb{E}^x_\Phi \left[\Phi(X_t)^{-1}; T > t \right] \\ &= \Phi(x) \, \mathbb{E}^x_\Phi \left[\Phi(X_t)^{-1} \right] \\ &= c_* \, \Phi(x) \, [1 + O(e^{-\beta t})] \end{aligned}$$

The third equality uses $\mathbb{P}^x_\Phi\{T > t\} = 1$. □

Example. Suppose $\Phi(x) = (\sin x)^a$ with $a \geq 1/2$. Then,

$$m(x)^2 + m'(x) = a^2 \, \cot^2 x - \frac{a}{\sin^2 x} = \frac{a(a-1)}{\sin^2 x} - a^2,$$

$$K_t = e^{a^2 t/2} \, \exp\left\{ \frac{a(1-a)}{2} \int_0^t \frac{ds}{\sin^2 X_s} \right\}.$$

We therefore get

$$\begin{aligned} \mathbb{E}^x \left[\exp\left\{ \frac{a(1-a)}{2} \int_0^t \frac{ds}{\sin^2 X_s} \right\}; T > t \right] &= e^{-a^2 t/2} \, \mathbb{E}^x[K_t; T > t] \\ &= e^{-a^2 t/2} \, c_* \, [\sin x]^a \, [1 + O(e^{-\beta t})], \end{aligned}$$

where

$$c_* = \frac{\int_0^\pi [\sin y]^a \, dy}{\int_0^\pi [\sin y]^{2a} \, dy}.$$

2.4.3 Reflected process for $a, b > -1/2$

For future reference, we note that the unique cubic polynomial $g(x)$ satisfying $g(0) = 0, g'(0) = 0, g(\epsilon) = \epsilon\,\gamma, g'(\epsilon) = \theta$ is

$$g(x) = \epsilon \, [\theta - 2\gamma] \, (x/\epsilon)^3 + \epsilon[3\gamma - \theta] \, (x/\epsilon)^2.$$

Note that for $|x| \leq \epsilon$,

$$|g(x)| + \epsilon \, |g'(x)| \leq \epsilon \, [17|\gamma| + 7|\theta|]. \tag{2.52}$$

Here we will discuss how to define the reflecting Φ-process for $\Phi \in \mathcal{X}(a,b)$ with $a, b > -1/2$. As our definition we will give the Radon-Nikodym derivative with respect to the reflecting Bessel process. We start by defining the reflecting Bessel-(a,b) process X_t on $(0,\pi)$ in the same way that the Bessel-(a,b) was defined in Section 2.4.1 where the "Bessel process with parameter a (or

b)" is defined to the the reflecting process. We then define the Φ-process to be the process with Radon-Nikodym derivative given by the martingale defined by $M_0 = 1$ and

$$M_t = M_{\sigma_j} \frac{\Phi(X_t)/\Phi(X_{\sigma_j})}{(X_t/X_{\sigma_j})^a} \exp\left\{ -\frac{1}{2} \int_{\sigma_j}^t \left[m(X_s)^2 + m'(X_s) - \frac{a(a-1)}{X_s^2} \right] ds \right\},$$

if $\sigma_j \le t \le \tau_j$, and if $\tau_j \le t \le \sigma_{j+1}$,

$$M_t = M_{\tau_j} \frac{\Phi(X_t)/\Phi(X_{\tau_j})}{(X_t/X_{\tau_j})^b} \exp\left\{ -\frac{1}{2} \int_{\tau_j}^t \left[m(X_s)^2 + m'(X_s) - \frac{b(b-1)}{(\pi - X_s)^2} \right] ds \right\},$$

Note that there exists β with $e^{-\beta t} \le M_t \le e^{\beta t}$, so this is a martingale. We can see that the process tilted by Φ is mutually absolutely continuous with the Bessel-(a,b) process with Radon-Nikodym derivative M_t.

A technical issue here that might concern us is the fact that the proof that M_t is a martingale uses the Girsanov theorem. This is valid locally away from the boundary, but it may not be clear that it works at the boundary. If it were the case that for some $\epsilon > 0$

$$m(x) = \frac{a}{x}, \quad m(\pi - x) = -\frac{b}{\pi - x}, \quad 0 < x < \epsilon,$$

then this would not be a problem, since the process M_t would not change when $X_t < \epsilon$ or $X_t > \pi - \epsilon$. More generally, we can find a sequence m_n such that

$$m_n(x) = \frac{a}{x}, \quad m_n(\pi - x) = -\frac{b}{\pi - x}, \quad 0 < x < 2^{-n},$$

$$m_n(x) = m(x), \quad 2^{-n+1} \le x \le \pi - 2^{-n+1},$$

and such that for all $x \le 2^{-n+1}$,

$$|m_n(x) - m(x)| + |m_n(\pi - x) - m(\pi - x)| \le c\, x,$$

$$|m_n'(x) - m'(x)| + |m_n'(\pi - x) - m'(\pi - x)| \le c.$$

The comment above about cubic polynomials is useful in constructing a particular example.

One can check that the proof required more than the fact that $x \sim \sin x$ near the origin. We needed that

$$\sin x = x \left[1 - O(x^2) \right].$$

An error term of $O(x)$ would have not been good enough.

We can now describe how to construct the reflecting radial Bessel process.

- Run the paths until time $\tau = T_{3\pi/4}$. Then

$$\frac{d\mathbb{P}_a}{d\hat{\mathbb{P}}_a} = K_\tau.$$

- Let $\sigma = \inf\{t \geq \tau : X_t = \pi/4\}$. The measure on $X_t, \tau \leq t \leq \sigma$ is defined to be the measure obtained in the first step by reflecting the paths around $x = \pi/2$.

- Continue in the same way.

Let $\phi_t(x, y; a)$ denote the transition probability for the reflected process which is defined for $0 < x, y < \pi$ and $t > 0$. This is also defined for $x = 0$ and $x = \pi$ by taking the limit, but we restrict to $0 < y < \pi$. We will will use $p_t(x, y; a)$ for the transition probability for the process killed at 0.

If μ_0 is any probability distributition on $[0, \pi]$, let $\Phi_t\mu$ denote the distribution of X_t given X_0 has distribution μ_0.

Lemma 2.4.7. *If $-1/2 < a < 1/2$, there exists c, β and a probability distribution μ such that if μ_0 is any initial distribution and $\mu_t = \Phi_t\mu_0$, then*

$$\|\mu - \mu_t\| \leq c\, e^{-\beta t}. \tag{2.53}$$

Proof. This uses a standard coupling argument. The key fact is that there exists $\rho > 0$ such that for every $x \in [0, \pi]$, the probability that the process starting at x visits 0 by time 1 is at least ρ.

Suppose μ^1, μ^2 are two different initial distributions. We start processes X_1, X_2 independently with distributions μ^1, μ^2. When the particles meet we coalesce the particles and they run together. If $X_1 \leq X_2$, then the coalescence time will be smaller than the time for X_2 to reach the origin. If $X_1 \geq X_2$, the time will be smaller than the time for X_1 to reach the origin. Hence the coalescence time is stochastically bounded by the time to reach the origin. Using the strong Markov property and the previous paragraph, the probability that $T > n$ is bounded above by $(1 - \rho)^n = e^{-\beta n}$ and $\|\mu^1_t - \mu^2_t\|$ is bounded above by the probability that the paths have not coalesced by time t. If $s > t$, we can apply the same argument using initial probability distributions μ^1_{s-t}, μ^2_0 to see that

$$\|\mu^1_s - \mu^2_t\| \leq c\, e^{-\beta t}, \quad s \geq t.$$

Using completeness, we see that the limit measure

$$\mu = \lim_{n\to\infty} \mu^1_n$$

exists and satisfies (2.53).

□

The construction of the reflected process shows that $\{t : \sin X_t = 0\}$ has zero measure which shows that the limiting measure must be carried on $(0, \pi)$.

We claim that the invariant density is given again by $f_a = C_{2a}\,(\sin x)^{2a}$. As mentioned before, it satisfies the adjoint equation

$$-[m(x)f_a(x)]' + \frac{1}{2}\, f''_a(x) = 0. \qquad \text{where } m(x) = a\, \cot x.$$

Another way to see that the invariant density is proportional to $(\sin x)^{2a}$ is to consider the process reflected at $\pi/2$. Let $p_t(z,x) = p_t(z,x) + p_t(z,\pi - x)$ be the probability density for this reflected process. Suppose that $0 < x < y < \pi/2$ and consider the relative amounts of time spent at x and y during an excursion from zero. If an excursion is to visit either x or y, it must start by visiting x. Given that it is x, the amount of time spent at x before the excursion ends is

$$\int_0^\infty \bar{p}_t(x,x;a)\,dt,$$

and the amount of time spent at y before the excursion ends is

$$\int_0^\infty \bar{p}_t(x,y;a)\,dt = \left[\frac{\sin y}{\sin x}\right]^{2a} \int_0^\infty \bar{p}_t(y,x;a)\,dt.$$

The integral on the right-hand side gives the expected amount of time spent at x before reaching zero for the process starting at y. However, if it starts at y it must hit x before reaching the origin. Hence by the strong Markov property,

$$\int_0^\infty \bar{p}_t(y,x;a)\,dt = \int_0^\infty \bar{p}_t(x,x;a)\,dt,$$

and hence,

$$\int_0^\infty \bar{p}_t(x,y;a)\,dt = \left[\frac{\sin y}{\sin x}\right]^{2a} \int_0^\infty \bar{p}_t(x,x;a)\,dt.$$

An important property of the radial Bessel process is the exponential rate of convergence to the equilibrium density. The next proposition gives a Harnack-type inequality that states within time one that one is within a multiplicative constant of the invariant density.

Proposition 2.4.8. *For every $-1/2 < a < 1/2$ and $t_0 > 0$, there exists $c = c < \infty$ such that for every $0 < x, y < 2\pi$ and every $t \geq t_0$,*

$$c^{-1}\,[\sin x]^{2a} \leq \phi_t(y,x;a) \leq c\,[\sin x]^{2a}.$$

Proof. By the Markov property it suffices to show that for each $s > 0$ there exists $c = c(s) < \infty$ such that

$$c^{-1}\,[\sin x]^{2a} \leq \phi_s(y,x;a) \leq c\,[\sin x]^{2a}.$$

We fix s and allow constants to depend on s. We write $z = \pi/2$. By symmetry, we may assume that $x \leq \pi/2$. for which $\sin x \asymp x$.

By comparison with Brownian motion, it is easy to see that

$$\inf\{\phi_t(z,y) : s/3 \leq t \leq s, \pi/4 \leq y \leq 3\pi/4\} > 0.$$

Therefore, for any $0 \leq x \leq \pi/2$, $2s/3 \leq t \leq s$, $\pi/4 \leq y \leq 3\pi/4$,

$$\phi_t(x,y;a) \geq \mathbb{P}_a^x\{T_z \leq s/3\}\,\inf\{\phi_r(z,y) : s/3 \leq r \leq s, \pi/4 \leq y \leq 3\pi/4\} \geq c,$$

and hence for such t, using ???,

$$\phi_t(z,x;a) = (x/z)^{2a}\,\phi_t(x,z;a) \geq c\,x^{2a}.$$

Hence, for every $0 \le y \le \pi$,

$$x^{-2a}\,\phi_s(y,x;a) \ge \mathbb{P}_a^y\{T_z \le s/3\}\,\inf\{x^{-2a}\,\phi_r(z,x) : s/3 \le r \le s, 0 \le y \le \pi\} \ge c.$$

This gives the lower bound.

Our next claim is if $w = 3\pi/4$ and

$$\theta_1 := \sup\{x^{-2a}\,\phi_t(w,x) : 0 \le t \le s, 0 \le x \le \pi/2\},$$

then $\theta_1 < \infty$. To see this let $\phi_t^*(y,x)$ be the density of the process $X_{t\wedge T_{7\pi/8}}$. Using (2.42) and absolute continuity, we can see that

$$\phi_t^*(w,x) \le c\,x^{2a}.$$

However, by the strong Markov property, we can see that

$$x^{-2a}\,\phi_t(y,x) \le x^{-2a}\,\phi_t^*(y,x) + \mathbb{P}^{7\pi/8}\{T_w \le s\}\,\theta_1 v, \le x^{-2a}\,\phi_t^*(w,x) + (1-\rho)\,\theta_1,$$

for some $\rho > 1$. Hence $\theta_1 \le x^{-2a}\,\phi_t^*(w,x) \le c/\rho < \infty$.

We now invoke Proposition 2.3.3 and absolute continuity,to see that for all $0 \le y \le 3\pi/4$ $\phi_s^*(y,x) \le c\,x^{2a}$. Hence, by the Markov property,

$$\phi_s(y,x) \le \phi_s^*(y,x) + \sup\{\phi_t(w,x) : 0 \le t \le s\} \le c\,x^{2a}.$$

□

Proposition 2.4.9. *For every $-1/2 < a < 1/2$, there exists $\beta > 0$ such that for all $t \ge 1$ and all $0 < x, y < \pi$,*

$$\phi_t(x,y;a) = f_a(y)\,\left[1 + O(e^{-t\beta})\right].$$

More precisely, for every $t_0 > 0$, there exists $c < \infty$ such that for all x, y and all $t \ge t_0$,

$$f_a(y)\,[1 - ce^{-\beta t}] \le \phi_t(x,y;a) \le f_a(y)\,[1 + ce^{-\beta t}].$$

Proof. Exactly as in Proposition 2.4.5. □

2.4.4 Functionals of Brownian motion

Now suppose X_t is a Brownian motion with $X_0 = x \in (0,\pi)$. Let $T = \inf\{t : \Theta_t = 0 \text{ or } \pi\}$ and let I_t denote the indicator function of the event $\{T > t\}$. Suppose that $\lambda > -1/8$ and let

$$a = \frac{1}{2} + \frac{1}{2}\sqrt{1+8\lambda} \ge \frac{1}{2},$$

be the larger root of the polynomial $a^2 - a - 2\lambda$. Let

$$J_t = \exp\left\{-\int_0^t \frac{ds}{S_s^2}\right\}.$$

If $M_{t,a}$ denotes the martingale in (2.48), then we can write

$$M_{t,a} = \left[\frac{S_t}{S_0}\right]^a\,J^{\lambda_a}, \quad \text{where } \lambda_a = \frac{a(a-1)}{2}$$

Proposition 2.4.10. *Suppose $\lambda \geq -1/8$. Then there exists $\beta = \beta(\lambda) > 0$ such that*

$$\mathbb{E}^x[J_t^\lambda \, I_t] = [C_{2a}/C_a] \, (\sin x)^a \, e^{-at} \, [1 + O(e^{-\beta t})],$$

where

$$a = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 8\lambda} \geq \frac{1}{2}. \tag{2.54}$$

Proof. Let a be defined as in (2.54). Then,

$$\begin{aligned}
\mathbb{E}^x[J_t^\lambda \, I_t] &= (\sin x)^a \, e^{-at} \, \mathbb{E}^x \left[M_{t,a} \, I_t \, S_t^{-a} \right] \\
&= (\sin x)^a \, e^{-at} \, \mathbb{E}_a^x \left[I_t \, S_t^{-a} \right] \\
&= (\sin x)^a \, e^{-at} \int_0^\pi p_t(x, y; a) \, [\sin y]^{-a} \, dy. \\
&= c' \, (\sin x)^a \, e^{-at} \, [1 + O(e^{-\beta t})].
\end{aligned}$$

Here $\beta = \beta_a$ is the exponent from Proposition 2.4.5 and

$$c' = \int_0^\pi f_a(y) \, [\sin y]^{-a} \, dy = C_{2a}/C_a.$$

Note that in the third line we could drop the I_t term since $\mathbb{P}_a^x\{I_t = 1\} = 1$. □

Proposition 2.4.11. *Suppose $b \in \mathbb{R}$ and*

$$\lambda + \lambda_b \geq -\frac{1}{8}. \tag{2.55}$$

Let

$$a = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 8(\lambda + \lambda_b)} \geq \frac{1}{2}. \tag{2.56}$$

and assume that $a + b > -1$. Then, there exists $\beta = \beta(\lambda, b) > 0$ such that

$$\mathbb{E}_b^x[J_t^\lambda \, I_t] = [C_{2a}/C_{a+b}] \, (\sin x)^{a-b} \, e^{(b-a)t} \, [1 + O(e^{-\beta t})].$$

Proof. Let a be as in (2.56) and note that $\lambda_a = \lambda + \lambda_b$.

$$\begin{aligned}
\mathbb{E}_b^x[J_t^\lambda \, I_t] &= \mathbb{E}^x[M_{t,b} \, J_t^\lambda \, I_t] \\
&= (\sin x)^{-b} \, e^{bt} \, \mathbb{E}^x[S_t^b \, J_t^{\lambda_b + \lambda} \, I_t] \\
&= (\sin x)^{a-b} \, e^{(b-a)t} \, \mathbb{E}^x[S_t^{b-a} \, M_{t,a} \, I_t] \\
&= (\sin x)^{a-b} \, e^{(b-a)t} \, \mathbb{E}_a^x[S_t^{b-a}] \\
&= (\sin x)^{a-b} \, e^{(b-a)t} \int_0^\pi p_t(x, y; a) \, [\sin y]^{b-a} \, dy \\
&= c' \, (\sin x)^{a-b} \, e^{(b-a)t} \, [1 + O(e^{-\beta t})].
\end{aligned}$$

Here $\beta = \beta_a$ is the exponent from Proposition 2.4.5 and

$$c' = \int_0^\pi f_a(y) \, [\sin y]^{b-a} \, dy = C_{2a}/C_{a+b}.$$

The fourth equality uses the fact that $\mathbb{P}_a^x\{I_t = 1\} = 1$.

□

2.4.5 Example

We consider Brownian motion on $(0,\pi)$ weighted locally by

$$\Phi(x) = [\sin x]^u\,[1-\cos x]^v, \quad m(x) = \frac{v}{\sin x} + (u+v)\,\cot x.$$

When we tilt by the appropriate local martingale, we get

$$dX_t = \left[\frac{v}{\sin X_t} + (u+v)\,\cot X_t\right]\,dt + dB_t. \tag{2.57}$$

This process is asymptotically Bessel-$(u+2v)$ at the origin and asymptotically Bessel-u at π. we will assume that $u > -1/2$ and if $1/2 < u < 1/2$, we will consider the reflected process.

If $u, u+v > -1/2$, we have the invariant density

$$f_{u,v}(x) = c_{u,v}\,\Phi(x)^2 = c_{u,v}\,[\sin x]^{2u}\,[1-\cos x]^{2v},$$

where

$$c_{u,v} = \left[\int_0^\pi [\sin x]^{2u}\,[1-\cos x]^{2v}\,dx\right]^{-1} = \frac{\Gamma(2u+2v+1)}{2^{2u+2v}\,\Gamma(u+2v+\frac{1}{2})\,\Gamma(u+\frac{1}{2})}.$$

We have used an integral identity. By first substituting $\theta = x/2$ and then $y = \sin^2\theta$ we see that

$$\begin{aligned}
\int_0^\pi [\sin x]^{2r}\,[1-\cos x]^s\,dx &= 2\int_0^{\pi/2} [\sin 2\theta]^{2r}\,[1-\cos 2\theta]^s\,d\theta \\
&= 2\int_0^{\pi/2} [2\,\sin\theta\,\cos\theta]^{2r}\,[2\,\sin^2\theta]^s\,d\theta \\
&= 2^{2r+s}\int_0^{\pi/2} [\sin^2\theta]^{r+s-\frac{1}{2}}\,[\cos^2\theta]^{r-\frac{1}{2}}\,[2\,\sin\theta\,\cos\theta\,d\theta] \\
&= 2^{2r+s}\int_0^1 y^{r+s-\frac{1}{2}}\,(1-y)^{r-\frac{1}{2}}\,dy \\
&= 2^{2r+s}\,\text{Beta}\left(r+s+\frac{1}{2}, r+\frac{1}{2}\right) \\
&= 2^{2r+s}\,\frac{\Gamma(r+s+\frac{1}{2})\,\Gamma(r+\frac{1}{2})}{\Gamma(2r+s+1)}.
\end{aligned}$$

Proposition 2.4.12. *Suppose $u, u+v > -1/2$, and X_t satisfies (2.57). There exists $\alpha > 0$ such that if $p_t(x,y)$ denotes the density of X_t given $X_0 = x$, then for $t \geq 1$,*

$$p_t(x,y) = f_{u,v}(y)\,[1+O(e^{-\alpha t})].$$

In particular, if F is a nonnegative function with

$$E_{u,v}(F) := \int_0^\infty F(y)\,f_{u,v}(y)\,dy < \infty$$

then

$$\mathbb{E}[X_t \mid X_0 = x] = E_{u,v}(F)\,\left[1+O(e^{-\alpha t})\right].$$

For later reference, we note that if $k > -1-u-v$, and

$$F(x) = \left[\frac{1-\cos x}{2}\right]^k,$$

then

$$\begin{aligned}
\mathbb{E}_{u,v}(F) &= \int_0^\pi c_{u,v}\,[\sin x]^{2u}\,[1-\cos x]^{2v}\,\left[\frac{1-\cos x}{2}\right]^k\,dx \\
&= 2^{-k}\,\frac{c_{u,v}}{c_{u,v+\frac{k}{2}}} \\
&= \frac{\Gamma(2u+2v+1)\,\Gamma(u+2v+k+\frac{1}{2})}{\Gamma(u+2v+\frac{1}{2})\,\Gamma(2u+2v+k+1)}. \qquad (2.58)
\end{aligned}$$

2.5 Identities for special functions

2.5.1 Asymptotics of h_a

Suppose $a > -1/2$ and

$$h_a(z) = \sum_{k=0}^{\infty} c_k\,z^{2k} \quad \text{where } c_k = c_{k,a} = \frac{1}{2^{a+2k-\frac{1}{2}}\,k!\,\Gamma(k+a+\frac{1}{2})}.$$

We note that the modified Bessel function of the first kind of order ν is given by $I_\nu(z) = z^\nu\,h_{\nu+\frac{1}{2}}(z)$. What we are discussing in this appendix are well known facts about I_ν, but we will state and prove them for the analytic function h_a. Since c_k decays like $[2^k\,k!]^{-2}$, is easy to see that the series has an infinite radius of convergence, and hence h_a is an entire function. Note that the c_k are given recursively by

$$c_0 = \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})}, \quad c_{k+1} = \frac{c_k}{(2k+2)\,(2k+2a+1)}. \qquad (2.59)$$

Proposition 2.5.1. *h_a is the unique solution to*

$$z\,h''(z) + 2a\,h'(z) - z\,h(z) = 0. \qquad (2.60)$$

with

$$h(0) = \frac{2^{\frac{1}{2}-a}}{\Gamma(a+\frac{1}{2})}, \quad h'(0) = 0.$$

Proof. Using term-by-term differentiation and (2.59), we see that h_a satisfies (2.60). A second, linearly independent solution of (2.60) can be given by

$$\tilde{h}_a(z) = \sum_{k=1}^{\infty} \tilde{c}_{k-1} z^{2k-1},$$

where $\tilde{c}_k$ are defined recursively by

$$\tilde{c}_0 = 1, \quad \tilde{c}_k = \frac{\tilde{c}_{k-1}}{(2k+1)\,(2k+2a)}.$$

Note that $\tilde{h}_a(0) = 0, \tilde{h}_a'(0) = 1$. By the uniqueness of second-order linear differential equations, every solution to (2.60) can be written as $h(z) = \lambda\, h_a(z) + \tilde{\lambda}\, \tilde{h}_a(z)$, and only $\lambda = 1, \tilde{\lambda} = 0$ satisfies the initial condition. □

Proposition 2.5.2. *Suppose h satisfies* (2.60)*, and*

$$\phi(x,y) = y^{2a} \exp\left\{-\frac{x^2+y^2}{2}\right\} h(xy).$$

Let

$$q_t(x,y;a) = \frac{1}{\sqrt{t}}\, \phi(x/\sqrt{t}, y/\sqrt{t}).$$

Then for every t,

$$\partial_t q_t(x,y:a) = L_x q_t(x,y;a) = L_y^* q_t(x,y;a),$$

where

$$Lf(x) = \frac{a}{x}\, f'(x) + \frac{1}{2}\, f''(x),$$

$$L^* f(x) = \frac{a}{x^2}\, f(x) - \frac{a}{x}\, f'(x) + \frac{1}{2}\, f''(x).$$

Proof. This is a straightforward computation. We first establish the equalities at $t = 1$. Note that

$$\partial_t q_t(x,y;a) \mid_{t=1} = -\frac{1}{2}\, \left[\phi(x,y) + x\, \phi_x(x,y) + y\, \phi_y(x,y)\right].$$

Hence we need to show that

$$\phi_{xx}(x,y) + \left[\frac{2a}{x} + x\right]\, \phi_x(x,y) + y\, \phi_y(x,y) + \phi(x,y) = 0,$$

$$\phi_{yy}(x,y) + x\, \phi_x(x,y) + \left[y - \frac{2a}{y}\right]\, \phi_y(x,y) + \left[\frac{2a}{y^2} + 1\right]\, \phi(x,y) = 0$$

Direct computation gives

$$\phi_x(x,y) = \left[-x + y\, \frac{h'(xy)}{h(xy)}\right] \phi(x,y),$$

$$\phi_{xx}(x,y) = \left[-1 + \frac{h''(xy)}{h(xy)}\, y^2 - 2xy\, \frac{h'(xy)}{h(xy)} + x^2\right]\, \phi(x,y),$$

$$\phi_y(x,y) = \left[\frac{2a}{y} - y + x\, \frac{h'(xy)}{h(xy)}\right]\, \phi(x,y),$$

$$\phi_{yy}(x,y) = \left[-1 - 4a + \frac{4a^2 - 2a}{y^2} + y^2 + x^2\, \frac{h''(xy)}{h(xy)} + \left(\frac{4ax}{y} - 2xy\right)\, \frac{h'(xy)}{h(xy)}\right]\, \phi(x,y).$$

If h satisfies (2.16), then

$$\frac{h''(xy)}{h(xy)} = 1 - \frac{2a}{xy}\, \frac{h'(xy)}{h(xy)},$$

so we can write

$$\phi_{xx}(x,y) = \left[-1 + x^2 + y^2 + \left(-2xy - \frac{2ay}{x}\right)\, \frac{h'(xy)}{h(xy)}\right]\, \phi(x,y),$$

$$\phi_{yy}(x,y) = \left[-1 - 4a + x^2 + y^2 + \frac{4a^2 - 2a}{y^2} + \left(\frac{4ax}{y} - 2xy \right) \frac{h'(xy)}{h(xy)} \right].$$

This gives the required relation.

For more general t, note that

$$\partial_t q_t(z,w;a) = \frac{1}{2t^{3/2}} \left[\phi(z/\sqrt{t}, w/\sqrt{t}; a) - \phi_x(z/\sqrt{t}, w/\sqrt{t}; a) - \phi_y(z/\sqrt{t}, w/\sqrt{t}; a) \right].$$

$$\partial_x q_t(z,w;a) = \frac{1}{t}\, \phi_x(z/\sqrt{t}, w/\sqrt{t}; a),$$

$$\partial_{xx} q_t(z,w;a) = \frac{1}{t^{3/2}}\, \phi_{xx}(z/\sqrt{t}, w/\sqrt{t}; a),$$

$$\partial_y q_t(z,w;a) = \frac{1}{t}\, \phi_y(z/\sqrt{t}, w/\sqrt{t}; a),$$

$$\partial_{yy} q_t(z,w;a) = \frac{1}{t^{3/2}}\, \phi_{yy}(z/\sqrt{t}, w/\sqrt{t}; a),$$

$$L_x q_t(z,w;a) = \frac{a}{(z/\sqrt{t})} \frac{1}{t^{3/2}}\, \phi_x(z/\sqrt{t}, w/\sqrt{t}; a) + \frac{1}{2} \frac{1}{t^{3/2}}\, \phi_{xx}(z/\sqrt{t}, w/\sqrt{t}; a),$$

$$L_y^* q_t(z,w;a) =$$

$$\frac{a}{(w/\sqrt{t})^2} \frac{1}{t^{3/2}}\, \phi(z/\sqrt{t}, w/\sqrt{t}; a) - \frac{a}{(w/\sqrt{t})} \frac{1}{t^{3/2}}\, \phi_y(z/\sqrt{t}, w/\sqrt{t}; a) + \frac{1}{2} \frac{1}{t^{3/2}}\, \phi_{yy}(z/\sqrt{t}, w/\sqrt{t}; a),$$

□

Proposition 2.5.3. *If h satisfies* (2.60)*, then exist an analytic function u with $u(0) \neq 0$ such that for all $x > 0$,*

$$h(x) = x^{-a}\, e^x\, u(1/x).$$

Proof. Let

$$v(x) = e^{-x}\, x^a\, h_a(x).$$

Then,

$$v'(x) = v(x) \left[-1 + \frac{a}{x} + \frac{h_a'(x)}{h_a(x)} \right],$$

$$\begin{aligned} v''(x) &= v(x) \left(\left[-1 + \frac{a}{x} + \frac{h_a'(x)}{h_a(x)} \right]^2 - \frac{a}{x^2} + \frac{h_a''(x)}{h_a(x)} - \frac{h_a'(x)^2}{h_a(x)^2} \right) \\ &= v(x) \left[1 + \frac{a^2 - a}{x^2} - \frac{2a}{x} + \left(\frac{2a}{x} - 2 \right) \frac{h_a'(x)}{h_a(x)} + \frac{h_a''(x)}{h_a(x)} \right] \\ &= v(x) \left[2 + \frac{a^2 - a}{x^2} - \frac{2a}{x} - 2\, \frac{h_a'(x)}{h_a(x)} \right] \\ &= -2v'(x) + \frac{a^2 - a}{x^2}\, v(x). \end{aligned}$$

The third equality uses the fact that h_a satisfies (2.60). If $u_a(x) = v(1/x)$, then

$$u'(x) = -\frac{1}{x^2}\, v'(1/x),$$

$$\begin{aligned} u_a''((x) &= \frac{2}{x^3}\, v'(1/x) + \frac{1}{x^4}\, v''(1/x) \\ &= \left[\frac{2}{x^3} - \frac{2}{x^4}\right] v'(1/x) + \frac{a^2 - a}{x^2}\, v(1/x) \\ &= \left[\frac{2}{x^2} - \frac{2}{x}\right] u_a'((x) + \frac{a^2 - a}{x^2}\, u_a((x). \end{aligned}$$

In other words, u satisfies the equation

$$x^2\, u''(x) + (2 - 2x)\, u'(x) + (a^2 - a)\, u(x) = 0.$$

We can find two linearly independent entire solutions to this equation of the form

$$u(z) = \sum_{k=0}^{\infty} b_k\, z^k$$

by choosing $b_0 = 1, b_1 = 0$ or $b_0 = 0, b_1 = 1$, and the recursively,

$$b_{k+2} = \frac{(2k + a - a^2)\, b_k - 2\,(k+1)\, b_{k+1}}{(k+1)(k+2)}.$$

Then,

$$u(z) = \sum_{k=0}^{\infty} b_k\, z^k,$$

$$u'(x) = \sum_{k=0}^{\infty} (k+1) b_{k+1}\, z^k,$$

$$zu'(z) = \sum_{k=1}^{\infty} k\, b_k z^k,$$

$$u''(z) = \sum_{k=0}^{\infty} (k+1)\,(k+2)\, b_{k+2} z^k,$$

then the differential equation induces the relation

$$b_{k+2} = \frac{(2k + a - a^2)\, b_k - 2\,(k+1)\, b_{k+1}}{(k+1)(k+2)}.$$

Note that

$$|b_{k+2}| \le \frac{2}{k+2}\, [|b_k| + |b_{k+1}|],$$

from which we can conclude that the power series converges absolutely for all z. By uniqueness $u_a(x)$ must be a linear combination of these solutions and hence must be the restriction of an entire function to the real line. □

2.5.2 Some integral identities

In this subsection we establish two "obvious" facts about the density by direct computation. We first prove that $\psi_t(x,\cdot;a)$ is a probability density.

Proposition 2.5.4. *For every $a > -1/2$ and $x > 0$,*

$$\int_0^\infty \psi_t(x,y;a)\,dy = 1.$$

We use a known relation about special functions, that we state here.

Lemma 2.5.5. *If $a > -1/2$ and $x > 0$,*

$$\int_0^\infty z^{2a} \exp\left\{-\frac{z^2}{2x^2}\right\} h_a(z)\,dz = x^{2a-1}\,e^{x^2/2}.$$

Proof. If we let $\nu = a - \frac{1}{2}$ and $r = x^2$, we see this is equivalent to

$$\int_0^\infty z^{\nu+1} \exp\left\{-\frac{z^2}{2r}\right\} I_\nu(z)\,dz = r^{\nu+1}\,e^{r/2}.$$

Equation 1.15.5 #4 of [3] gives the formula

$$\int_0^\infty x^{b-1}\,e^{-px^2}\,I_\nu(x)\,dx = 2^{-\nu-1}\,p^{-\frac{b+\nu}{2}}\,\frac{\Gamma(\frac{b+\nu}{2})}{\Gamma(\nu+1)}\,{}_1F_1\left(\frac{b+\nu}{2}, \nu+1, \frac{1}{4p}\right),$$

where ${}_1F_1$ is the confluent hypergeometric function. In our case, $b = \nu + 2$, $p = 1/(2r)$, so the right-hand side equals

$$2^{-\nu-1}\,(1/2r)^{-\nu-1}\,{}_1F_1\,(\nu+1, \nu+1, r/2) = r^{\nu+1}\,e^{r/2}.$$

where the last equality comes from the well-known identity ${}_1F_1(b,b,z) = e^z$. □

Proof of 2.5.4. Since

$$\int_0^\infty \psi_t(x,y;a)\,dy = \int_0^\infty t^{-1/2}\,\psi_1\left(x/\sqrt{t}, y/\sqrt{t}; a\right)\,dy = \int_0^\infty \psi_1(x/\sqrt{t}, z; a)\,dz,$$

it suffices to show that for all x,

$$\int_0^\infty \psi(x,y)\,dy = 1$$

where

$$\psi(x,y) = \psi_1(x,y;a) = y^{2a} \exp\left\{-\frac{x^2+y^2}{2}\right\} h_a(xy).$$

Using the substitution $xy = z$, we see that

$$\begin{aligned}\int_0^\infty \psi(x,y)\,dy &= e^{-x^2/2}\int_0^\infty y^{2a} \exp\left\{-\frac{y^2}{2}\right\} h_a(xy)\,dy \\ &= x^{-2a-1}\,e^{-x^2/2}\int_0^\infty z^{2a} \exp\left\{-\frac{z^2}{2x^2}\right\} h_a(z)\,dz = 1\end{aligned}$$

□

We now show that $\psi_t(x,y;a)$ satisfies the Chapman-Kolomogrov equations.

Proposition 2.5.6. *if $a > -1/2, 0 < t < 1$ and $x > 0$, then*

$$\int_0^\infty \psi_t(x,z;a)\,\psi_{1-t}(z,y;a)\,dz = \psi_1(x,y;a).$$

Proof. Using (2.36), we see that the proposition is equivalent to the identity

$$[t(1-t)]^{-a-\frac{1}{2}} \int_0^\infty z^{2a} \exp\left\{-\frac{x^2+z^2}{2t}\right\} \exp\left\{-\frac{z^2+y^2}{2(1-t)}\right\} h_a\left(\frac{xz}{t}\right) h_a\left(\frac{zy}{1-t}\right) dz = \exp\left\{-\frac{x^2+y^2}{2}\right\} h_a(xy).$$

We will use one integral identity which is equation 2.15.20 #8 in [3]: if $\nu > -1$, and $b, c > 0$,

$$\int_0^\infty x\, e^{-x^2/2}\, I_\nu(bx)\, I_\nu(cx)\, dx = \exp\left\{\frac{b^2+c^2}{2}\right\} I_\nu(bc). \tag{2.61}$$

If we set $\nu = a - \frac{1}{2}$, we have

$$\begin{aligned}
&\int_0^\infty z^{2a} \exp\left\{-\frac{z^2}{2t(1-t)}\right\} h_a\left(\frac{xz}{t}\right) h_a\left(\frac{zy}{1-t}\right) dz \\
&= \left(\frac{t}{x}\right)^{a-\frac{1}{2}} \left(\frac{1-t}{y}\right)^{a-\frac{1}{2}} \int_0^\infty z \exp\left\{-\frac{z^2}{2t(1-t)}\right\} I_\nu\left(\frac{xz}{t}\right) I_\nu\left(\frac{zy}{1-t}\right) dz \\
&= \left(\frac{t}{x}\right)^{a-\frac{1}{2}} \left(\frac{1-t}{y}\right)^{a-\frac{1}{2}} t(1-t) \int_0^\infty u\, e^{-u^2/2}\, I_\nu\left(\frac{ux\sqrt{1-t}}{\sqrt{t}}\right) I_\nu\left(\frac{uy\sqrt{t}}{\sqrt{1-t}}\right) du \\
&= [t(1-t)]^{a+\frac{1}{2}} \exp\left\{\frac{x^2(1-t)}{2t} + \frac{y^2 t}{2(1-t)}\right\} (xy)^{\frac{1}{2}-a}\, I_\nu(xy) \\
&= [t(1-t)]^{a+\frac{1}{2}} \exp\left\{\frac{x^2(1-t)}{2t} + \frac{y^2 t}{2(1-t)}\right\} h_a(xy)
\end{aligned}$$

□

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Chapter 3

Schramm-Loewner evolution

3.1 Definition

The Schramm-Loewner evolution (SLE) was defined by Oded Schramm as the only conformally invariant families of random curves that satisfy the domain Markov property. To make a precise definition we consider the implications of these assumptions on probability measures $\mathbb{P}$ on simple curves $\gamma : (0, \infty) \to \mathbb{H}$ with $\gamma(0+) = 0$. We will consider curves modulo (increasing) reparametrization, that is, we are only interested in the path that is traversed and not on how "quickly" one goes through the path. The assumptions on $\mathbb{P}$ are *scale invariance* and the *conformal Markov property*,

- **Scale invariance.** If $r > 0$ and $\mathbb{P}_r$ denotes the measure on curves obtained by considering $r\gamma$, then $\mathbb{P}_r = \mathbb{P}$. (Remember we are considering curves modulo reparametrization so, for example, the point mass on the straight line $\gamma(t) = it$ satisfies this property.)

- **Conformal Markov property**. Suppose the beginning segment $\gamma_t = \gamma[0, t]$ is observed and let $g : \mathbb{H} \setminus \gamma_t \to \mathbb{H}$ be a conformal transformation with $g(\gamma(t)) = 0, \hat{g}(\infty) = \infty$. Then the conditional distribution of $g[\gamma[t, \infty))$ given γ_t is $\mathbb{P}$.

The conformal transformation g is not unique, but any other such transformation is of the form rg for some $r > 0$; hence, scale invariance implies that the distribution of $g[\gamma[t, \infty))$ is independent of the choice of g.

Let us work towards the definition. Suppose $\mathbb{P}$ satisfies scale invariance and the conformal Markov property and is supported on simple curves γ with $\gamma(0, \infty) \subset \mathbb{H}$. Let $\gamma_t = \gamma[0, t]$ and let $g_t : \mathbb{H} \setminus \gamma_t \to \mathbb{H}$ with $g_t(z) = z + o(1)$ as $z \to \infty$. Theorem 1.8.18 implies that if we parametrize the curve so that $\text{hcap}[\gamma_t] = 2t$, then

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z,$$

where U_t is a random function with continuous paths. Conformal invariance and the domain Markov property imply that U_t is a continuous process with stationary, independent increments. This implies that U_t must be a (one-dimensional) Brownian motion with drift m and variance parameter κ. Scale invariance can be used to see that $m = 0$. This leaves one parameter, κ and SLE_κ was defined as the solution of this equation with driving function equal to a driftless Brownian motion with variance parameter κ.

For ease, we will choose a slightly different parametrization. If g_t is as above and $\tilde{g}_t = g_{t/\kappa}$ then

$$\partial_t \tilde{g}_t = \frac{2/\kappa}{\tilde{g}_t(z) - \tilde{U}_t}, \quad g_0(z) = z,$$

where $\tilde{U}_t = U_{t/\kappa}$ which is a standard Brownian motion. Here we have parametrized so that $\text{hcap}[\tilde{\gamma}_t] = at$ where $a = 2/\kappa$. At the moment, we can take all of this as motivation for the following definition.

Definition Suppose $a = 2/\kappa > 0$ and $U_t = -B_t$ is a standard one-dimensional Brownian motion. Let g_t denote the solution of the initial value problem

$$\dot{g}_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z. \tag{3.1}$$

Then g_t is called the *Schramm-Loewner evolution with parameter κ (SLE_κ) from 0 to ∞ in $\mathbb{H}$.*

We have started by defining SLE_κ as the random collection of maps $\{g_t\}$. We will also use the term for the curve induced by the maps, but it requires some work to show that the latter is well defined. We know from Section 1.8 that for all $z \in \mathbb{C} \setminus \{0\}$. the solution to (3.1) exists for all $t < T_z = \inf\{t : g_t(z) - U_t = 0\}$. If we write $Z_t(z) = g_t(z) - U_t$ we can write (3.1) as a stochastic differential equation

$$dZ_t(z) = \frac{a}{Z_t(z)}\, dt + dB_t, \quad Z_0(z) = z.$$

This is an example of a stochastic flow, that is, a family of process $Z_t(z)$ indexed by starting points $z \in \mathbb{C} \setminus \{0\}$ where the same Brownian motion B_t is used for all of the processes. One must take care in reading the equation. The quantity $Z_t(z)$ is complex-valued, but the Brownian motion B_t is real-valued.

If z is fixed and we write

$$Z_t = Z_t(z) = X_t + i\, Y_t, \tag{3.2}$$

then (3.2) becomes

$$dX_t = \frac{aX_t}{X_t^2 + Y_t^2}\, dt + dB_t = X_t \left[\frac{a}{X_t^2 + Y_t^2}\, dt + \frac{1}{X_t}\, dB_t\right], \quad X_0 = \text{Re}[z]; \tag{3.3}$$

$$\partial_t Y_t = -\, Y_t\, \frac{a}{X_t^2 + Y_t^2}, \quad Y_0 = \text{Im}[z]. \tag{3.4}$$

If $z = x \in \mathbb{R} \setminus \{0\}$, then $Y_t = 0$ for all t, and X_t satisfies the Bessel equation

$$dX_t = \frac{a}{X_t}\, dt + dB_t, \quad X_0 = x. \tag{3.5}$$

Let us review our logic. We started with trying to find probability measures on curves that satisfy scale invariance and the conformal Markov property. We conclude that the only possible candidates can be reparametrized so they are solutions to the Loewner equation whose driving function is a driftless Brownian motion. We then use the Loewner equation as the definition for SLE, but it remains to see if this is a measure on curves for all values of $\kappa > 0$. While this is true, we will see that it is not always a measure on *simple* curves.

Here and throughout, we will reserve partial derivative notation ∂_t for actual derivatives. Stochastic differentials will be denoted by d. As above, we will often drop the z dependence on Z_t, X_t, Y_t and similar quantities but it is important to remember that they depend on the initial point.

Our definition used a particular parametrization on the curves. This parametrization is very useful for analysis of the curve but is not always the most natural. We are really considering curves modulo reparametrization, so we should also think of SLE as a probability on random maps $\{g_t\}$ modulo reparametrization. We prove some easy facts about C^1 time changes.

Proposition 3.1.1. *Suppose g_t satisfies* (3.1)*, $\sigma : [0,\infty) \to [0,\infty)$ is a C^1 function with $\dot\sigma > 0$, and $\tilde g_t(z) = g_{\sigma(t)}(z)$. Then*

$$\partial_t \tilde g_t(z) = \frac{a\,\dot\sigma(t)}{\tilde g_t(z) - \tilde U_t}, \quad \tilde U_t = U_{\sigma(t)}.$$

Proof. Immediate from the chain rule. □

A particularly important application of this is the scaling rule for SLE which shows that SLE in the upper half plane (in the capacity parametrization) satisfies Brownian (heat equation) scaling.

Proposition 3.1.2 (Scaling)**.** *Suppose g_t satisfies* (3.1)*, $r > 0$, and $\tilde g_t(z) = r^{-1}\, g_{r^2 t}(rz)$. Then $\tilde g_t$ has the distribution of SLE_κ.*

Proof. Using the chain rule, we see that

$$\partial_t \tilde g_t(z) = r\,\dot g_{r^2 t}(rz) = \frac{ar}{g_{r^2 t}(rz) - U_{r^2 t}} = \frac{a}{\tilde g_t(z) - \tilde U_t},$$

where $\tilde U_t = r^{-1}\, U_{r^2 t}$. The scaling property of Brownian motion implies that $\tilde U_t$ is a standard Brownian motion.

□

We write $H_t = \{z \in \mathbb{H} : T_z > 0\}$ and let $K_t = \mathbb{H} \setminus H_t$ be the corresponding hull. Under our parametrization, $\operatorname{hcap}(K_t) = at$. Although we will not prove it at the moment, the following theorem holds.

Theorem 3.1.3. *There exists a curve $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ such that for each t, H_t is the unbounded component of $\mathbb{H} \setminus \gamma_t$ where $\gamma_t = \gamma[0,t]$.*

We call γ the *SLE_κ curve (parametrized so that* $\operatorname{hcap}[\gamma_t] = at$*).*

Proof. We will not prove this at the moment. We would like to define

$$\gamma(t) = g_t^{-1}(U_t) = \lim_{y \downarrow 0} g_t^{-1}(U_t + iy).$$

However, there are Loewner chains for which the limit does not exists for some t, and other chains for which the limit exists but does not give a continuous function of t. For $\kappa \neq 8$, we show in Section 3.8 that with probability one, the conditions of Proposition 1.8.24 hold. The case $\kappa = 8$ is much more delicate and will not be proved in this book. □

Many of the statements we make can be phrased without reference to the curve, but it will be easier to assume this theorem now. In order to do this, it will be useful to define the set γ_t in a way that does not require γ to be a curve. We say that z is a *pioneer point* for the Loewner chain $\{g_s\}$ at time t if $z \in H_s$ for all $s < t$ and $z \in \partial H_t$. We let γ_t be the union of all pioneer points with $s \leq t$. Then we have

- If the chain is generated by a curve γ, then $\gamma_t = \gamma[0, t]$.
- H_t is the unbounded connected component of $\mathbb{H} \setminus \gamma_t$.

Note that $\partial H_t \subset \mathbb{R} \cup \gamma_t$. If $T_z < \infty$, then there are two possibilities: either $\gamma(T_z) = z$, or there exists a bounded connected component of $\mathbb{H} \setminus \gamma_{T_z}$ containing z. In the latter case, this will also be the connected component of $\mathbb{H} \setminus \gamma_t$ containing z for all $t > T_z$.

In the definition of SLE, the curve is parametrized by half-plane capacity. This is what allows stochastic calculus to be used to analyze the curve. However, this is not necessarily the most intrinsic parametrization for curves. For this reason, it is often more useful to view SLE as a measure on curves *modulo reparameterization*.

Definition Suppose γ is a random curve from 0 to ∞ in $\mathbb{H}$ and let

$$a(t) = \inf\{s : \mathrm{hcap}(\gamma_s) = (2/\kappa)\, t\}.$$

- We say that γ is an SLE_κ *path* if $\tilde{\gamma}(t) := \gamma(a(t))$ is an SLE_κ curve as above.
- We say that γ has a *capacity parametrization* if $a(t)$ is a strictly increasing C^1 function of t.

Using this definition and Proposition 3.1.2, we can see that if γ is an SLE_κ then so is $r\gamma$ for all r; moreover, $r\gamma$ has a capacity parametrization if and only if γ does.

Definition Suppose D is a simply connected domain and z, w are distinct points on ∂D. Let $F : D \to \mathbb{H}$ be a conformal transformation with $F(z) = 0, F(w) = \infty$. We say that a random curve γ is a *chordal* SLE_κ from z to w in D if $\tilde{\gamma}(t) := F(\gamma(t))$ is an SLE_κ from 0 to ∞ in $\mathbb{H}$. We say that γ has a capacity parametrization if $\tilde{\gamma}$ has a capacity parametrization.

The conformal transformation F in the last definition is not unique. However, if $\hat{F}$ is another such transformation, then $\hat{F} = rF$ for some $r > 0$. Using the discussion before the last definition, we can see the definition is independent of which F we choose.

We are using parametrization by capacity to mean parametrizing so that $a(t)$ is a linear function of t but "a capacity paramterization" if $a(t)$ is C^1. The main other cases of capacity parametrizations will be the radial parameterization (with respect to a particular interior point) and the imaginary part parametrization in $\mathbb{H}$.

3.2 The curve near a point $z \in \overline{\mathbb{H}} \setminus \{0\}$

The Schramm-Loewner evolution is unusual in that the curve is defined indirectly in terms of the conformal maps g_t. If the curve grows with a capacity parametrization, the effect of the curve on points away from the curve is described by a relatively simple SDE. The curve itself is very irregular, and one cannot describe its motion as an SDE. Indeed, the dynamics of the curve are very nonMarkovian. The basic strategy is to use the behavior "away from the curve" to determine facts about the curve.

Consider the questions.

- How "thick" is the curve γ, that is, what is its fractal dimension?
- Could it possibly be plane filling and hit every point?

Addressing these questions may seem daunting, especially since we have not proved that the curve exists, but let us start with some heuristics. Suppose the curve had fractal dimension d. If we consider the intersection of the curve with a closed disk of, say, radius 1, then the expected number of disks of radius ϵ needed to cover the curve would be of order ϵ^{-d}. Since the number of discs needed to cover the disc of radius 1 of order ϵ^{-2}, we see that for a fixed disc of radius ϵ, we would expect that the probability of hitting that disk is order ϵ^{2-d}.

Using this as motivation, we will studying the following questions in this section.

- For a fixed $z \in \overline{\mathbb{H}} \setminus \{0\}$, does the curve hit z?
- If not, what is the probability that it gets within distance ϵ of z?
- If it misses z does it go to the "left" or to the "right" of z?

The answers will depend on κ and the results will be slightly different for $z \in \mathbb{H}$ and for $z \in \mathbb{R} \setminus \{0\}$.

We can phrase these questions in terms of the domains H_t, which we know are well defined, rather than in terms of γ, so we do not need to prove existence of the curve before discussing them. For example, if $\operatorname{dist}(z, \gamma_t) < \operatorname{Im}(z)$, then $\operatorname{dist}(z, \gamma_t) = \operatorname{dist}(z, \partial H_t)$. The latter quantity is comparable to the conformal radius $\operatorname{crad}_{H_t}(z)$.

We now fix z and let $Z_t = Z_t(z) = X_t + iY_t$ as in (3.2)–(3.5). If $z = x \in \mathbb{R} \setminus \{0\}$, then $Y_t = 0$ for all x, and we often write X_t instead of Z_t. By differentiating (3.1) with respect to z, we see that

$$\partial_t g_t'(z) = -\frac{a\, g_t'(z)}{(g_t(z) - U_t)^2} = -\frac{a\, g_t'(z)}{Z_t^2},$$

$$\partial_t [\log g_t'(z)] = -\frac{a}{Z_t^2}.$$

Although $\log g_t'(z)$ is defined only up to an initial additive multiple of $2\pi i$, the derivative $\partial_t[\log g_t'(z)]$ is independent of the choice. The next proposition follows immediately.

Proposition 3.2.1. *If $z \in \mathbb{H}$ and u_t, v_t are defined by $\log g_t'(z) = u_t(z) + iv_t(z)$, then for $t < T_z$,*

$$\partial_t u_t(z) = \frac{a\,(Y_t^2 - X_t^2)}{(X_t^2 + Y_t^2)^2}, \quad \partial_t v_t(z) = \frac{2aX_tY_t}{(X_t^2 + Y_t^2)^2}.$$

In particular,

$$\partial_t |g_t'(z)| = |g_t'(z)| \, \frac{a\,(Y_t^2 - X_t^2)}{(X_t^2 + Y_t^2)^2}. \tag{3.6}$$

If $x \in \mathbb{R} \setminus \{0\}$,

$$\partial_t g_t'(x) = -a \, \frac{g_t'(x)}{X_t^2}. \tag{3.7}$$

As before, we write $\partial_t g_t$ or $\dot{g}_t$ for derivatives with respect to time and reserve the prime notation $'$ for spatial derivatives.

Since g_t is a conformal transformation of H_t onto $\mathbb{H}$ we can see that $g_t'(x) > 0$ for all $x \in \mathbb{R}$ with $t < T_x$. For this reason we do not need absolute values in (3.7).

For now, let us assume that $z \in \mathbb{H}$. It is standard to represent a point in $\mathbb{H}$ in either rectangular coordinates $x + iy$ or polar coordinates $re^{i\theta}$. We will choose a compromise by representing the point by (y, θ). Here θ is the argument and y is the imaginary part which can also be considered as one-half the conformal radius of $\mathbb{H}$ with respect to y. The representation $(\mathrm{crad}/2, \theta)$ works well with conformal transformations and we will use this. Let

$$\Theta_t = \Theta_t(z) = \arg Z_t(z), \quad \Upsilon_t = \Upsilon_t(z) = \frac{Y_t}{|g_t'(z)|} = \frac{1}{2}\,\mathrm{crad}_{H_t}(z).$$

The last equality is justified in the following proposition.

Proposition 3.2.2. *If* $t < T_z$, *then* Υ_t *is one-half times the conformal radius of* H_t *with respect to* z. *In particular,*

$$\frac{\mathrm{dist}(z, \partial H_t)}{2} \le \Upsilon_t \le 2\,\mathrm{dist}(z, \partial H_t). \tag{3.8}$$

Proof. Recall that by definition, $\mathrm{crad}_D(z) = |f'(z)|^{-1}$ where $f : D \to \mathbb{D}$ is a conformal transformation with $f(z) = 0$. By using a conformal transformation of $\mathbb{H}$ onto $\mathbb{D}$ we see that $\mathrm{crad}_{\mathbb{H}}(x+iy) = 2y$. More generally, $\mathrm{crad}_{H_t}(z) = |g_t'(z)|^{-1}\,\mathrm{crad}_{\mathbb{H}}(g_t(z)) = 2Y_t/|g_t'(z)|$. The inequalities in (3.8) follow from the Koebe 1/4-theorem, see (1.23). □

If $T_z < \infty$, we define $\Upsilon_t = \Upsilon_{T_z} := \lim_{s \uparrow T_z} \Upsilon_s$ for $t \ge T_z$. This is either zero (if $\mathrm{dist}(\gamma_{T_z}, z) = 0$) or equals $\mathrm{crad}_{H_z}(z)/2$, where H_z is the component of $\mathbb{H} \setminus \gamma_{T_z}$ containing z. Similarly, we define

$$\Upsilon_\infty(z) = \lim_{t \to \infty} \Upsilon_t(z),$$

noting that $\Upsilon_\infty(z) = \Upsilon_{T_z}(z)$ if $T_z < \infty$. Using the Koebe 1/4-theorem as above, we see that

$$\Upsilon_\infty(z) \asymp_2 \mathrm{dist}(z, \gamma \cup \mathbb{R}).$$

(Here we write $f(x) \asymp_r g(x)$ if $r^{-1}\, f(x) \le g(x) \le r\, f(x)$.)

Let us consider what happens to the curve from the perspective of the point z up to the time T_z. There are three possibilities for a continuous curve going to infinity:

- The curve never reaches z and goes to infinity never trapping z. In this case $\Upsilon_t \downarrow \Upsilon_\infty > 0$ and as $z \to \infty$, $\Theta_z \to 0$ or $\Theta_z \to \pi$ depending on whether the curve goes to the "left" of z or to the "right" of z, respectively.
- The curve never reaches z and at the finite time T_z it disconnects z from infinity. Then $\Upsilon_{T_z} > 0$ and Θ_{T_z} equals 0 or π depending on whether the loop formed around z goes to the "left" of z or to the "right" of z, respectively.
- The curve reaches z in finite time in which case $\Upsilon_{T_z} = 0$ and we would expect Θ_t to fluctuate up to time T_z.

We will now investigate the quantities Υ_t, Θ_t using the Loewner equation and standard methods in stochastic calculus.

Proposition 3.2.3. *For $t < T_z$,*

$$\partial_t \Upsilon_t = -\Upsilon_t \frac{2aY_t^2}{(X_t^2 + Y_t^2)^2}. \tag{3.9}$$

$$d\Theta_t = \frac{(1-2a)\, X_t\, Y_t}{(X_t^2 + Y_t^2)^2}\, dt - \frac{Y_t}{X_t^2 + Y_t^2}\, dB_t. \tag{3.10}$$

Proof. Both Y_t and $|g_t'(z)|$ are differentiable with respect to t with

$$\partial Y_t = -\frac{a\, Y_t}{X_t^2 + Y_t^2} = -Y_t \frac{a(X_t^2 + Y_t^2)}{(X_t^2 + Y_t^2)^2},$$

so (3.9) follows from (3.6) and the product rule.

Let $L_t = \log Z_t$. Then Itô's formula shows that

$$dL_t = \frac{1}{Z_t}\, dZ_t - \frac{1}{2Z_t^2}\, d\langle Z\rangle_t = \frac{a - \frac{1}{2}}{Z_t^2}\, dt + \frac{1}{Z_t}\, dB_t$$

Since $\Theta_t = \mathrm{Im} L_t$, we see that

$$d\Theta_t = \mathrm{Im}\left[\frac{a-\frac{1}{2}}{Z_t^2}\right] dt + \mathrm{Im}\left[\frac{1}{Z_t}\right] dB_t = \frac{(1-2a)\, X_t\, Y_t}{(X_t^2 + Y_t^2)^2}\, dt - \frac{Y_t}{X_t^2 + Y_t^2}\, dB_t.$$

□

In the proof we used Itô's formula with a complex function and then took the imaginary part. The reader may wish to check that this was legitimate!

The equation (3.10) will become nicer if we change the parametrization. Since the conformal radius $\mathrm{crad}_{H_t}(z)$ is strictly decreasing, $\phi(\mathrm{crad}_{H_t}(z))$ will be strictly increasing if ϕ is strictly decreasing and can be used to reparametrize the curve. As the curve approaches z, the argument of z will tend to vary (roughly speaking) in a way that when one halves the distance to z the argument

has a change of order one. With this intuition, we can see that will be useful to parametrize the curve so that the logarithm of the conformal radius decays linearly. This is an example of a *radial parametrization* (with respect to z). Let

$$\sigma(t) = \sigma(t,z) = \inf\{s : \log(\Upsilon_0/\Upsilon_s) = 2at\}, \tag{3.11}$$

and define

$$\hat{Z}_t = \hat{X}_t + i\hat{Y}_t = Z_{\sigma(t)}, \quad \hat{\Theta}_t = \Theta_{\sigma(t)}, \quad \hat{\Upsilon}_t = \Upsilon_{\sigma(t)} = e^{-2at}\,\Upsilon_0.$$

(The choice of $2a$ in the exponent was made so that the coefficient of the Brownian motion in (3.12) below is 1.) Since $\partial_t\hat{\Upsilon}_t = -2a\hat{\Upsilon}_t$ and the chain rule gives $\partial_t\hat{\Upsilon}_t = \dot{\Upsilon}_{\sigma(t)}\,\dot{\sigma}(t)$, we see from (3.9) that

$$\dot{\sigma}(t) = \frac{(\hat{X}_t^2 + \hat{Y}_t^2)^2}{\hat{Y}_t^2}.$$

Using this, we see that (3.10) implies that

$$d\hat{\Theta}_t = (1-2a)\,\frac{\hat{X}_t}{\hat{Y}_t}\,dt + d\hat{B}_t = (1-2a)\,\cot\hat{\Theta}_t\,dt + d\hat{B}_t, \tag{3.12}$$

where $\hat{B}_t$ is a standard Brownian motion. This time change is very useful when viewing the flow from a particular starting point z. However, different starting points give different parametrizations so it is not as convenient when considering more than one initial point.

Here we have done a time change of an SDE. Since this will happen often, it is useful to discuss it in some detail. If

$$dK_t = R_t\,dt + A_t\,dB_t,$$

and σ is a C^1 time change, then $\hat{K}_t := K_{\sigma(t)}$ satisfies

$$d\hat{K}_t = R_{\sigma(t)}\,\dot{\sigma}(t)\,dt + \sqrt{\dot{\sigma}(t)}\,A_{\sigma(t)}\,d\hat{B}_t,$$

where $\hat{B}_t$ is a standard Brownian motion given by

$$\hat{B}_t = \int_0^{\sigma(t)} \sqrt{\dot{\eta}(s)}\,dB_s, \quad \eta = \sigma^{-1}. \tag{3.13}$$

Note that if $\hat{B}_t$ is defined as above, then

$$\langle\hat{B}\rangle_t = \int_0^{\sigma(t)} \dot{\eta}(s)\,ds = \eta(\sigma(t)) - \eta(\sigma(0)) = t,$$

from which we can see that $\hat{B}_t$ is a standard Brownian motion. In the example above,

$$K_t = \Theta_t, \quad R_t = \frac{(1-2a)\,X_t\,Y_t}{(X_t^2+Y_t^2)^2}, \quad A_t = -\frac{Y_t}{X_t^2+Y_t^2}, \quad \dot{\sigma}(t) = \frac{(X_t^2+Y_t^2)^2}{Y_t^2},$$

$$R_{\sigma(t)}\,\dot{\sigma}(t) = \frac{(1-2a)\,\hat{X}_t}{\hat{Y}_t} = (1-2a)\,\cot\hat{\Theta}_t, \quad \sqrt{\dot{\sigma}(t)}\,A_{\sigma(t)} = -1.$$

Therefore,

$$d\hat{\Theta}_t = (1-2a)\,\cot\hat{\Theta}_t\,dt - d\hat{B}_t.$$

Since $-\hat{B}_t$ is also a standard Brownian motion, we can also write (3.12) where the $\hat{B}_t$ there is the negative of the $\hat{B}_t$ defined in (3.13).

The next theorem shows the three "phases" of SLE_κ from the perspective of a point $z \in \mathbb{H}$.

- If $\kappa \le 4$, then $z \in H_t$ for all t.
- If $4 < \kappa < 8$, then $T_z < \infty$ and $z \notin H_t$ for $t \ge T_z$. However, $\operatorname{dist}(z, \gamma(T_z)) > 0$. At time T_z the path makes a loop that disconnect z from ∞.
- If $\kappa \ge 8$, then the curve actually reaches z.

Theorem 3.2.4. *If $z \in \mathbb{H}$,*

$$\mathbb{P}\{T_z = \infty, \operatorname{dist}(z,\gamma) > 0\} = 1, \qquad \text{if } 0 < \kappa \le 4,$$

$$\mathbb{P}\{T_z < \infty, \operatorname{dist}(z,\gamma) > 0\} = 1, \qquad \text{if } 4 < \kappa < 8,$$

$$\mathbb{P}\{T_z < \infty, \operatorname{dist}(z,\gamma) = 0\} = 1, \qquad \text{if } 8 \le \kappa < \infty.$$

Proof. Let $z = x + iy \in \mathbb{H}$. By scaling we may assume that $y = 1$. Note that $\operatorname{dist}(z,\gamma) > 0$ if and only if $\Upsilon_\infty(z) > 0$. To analyze $\Upsilon_\infty = \Upsilon_\infty(z)$ we parametrize as in (3.11) so that $\log \Upsilon_t$ decays linearly. Under this parametrization $\hat\Theta_t$ satisfies

$$d\hat\Theta_t = (1-2a)\,\cot\hat\Theta_t\,dt + d\hat B_t,$$

for a standard Brownian motion $\hat B_t$. This is the radial Bessel equation as discussed in Section 2.4. As $\theta \downarrow 0$, $\cot\theta = \theta^{-1}\,[1 + O(\theta^2)]$, and using this and comparison with the Bessel equation we see that $\hat\Theta_t$ reaches the origin in finite time if and only if $1 - 2a < 1/2$, that is, if $\kappa < 8$. Therefore,

$$\mathbb{P}\{\Upsilon_\infty > 0\} = \begin{cases} 1 & \text{if } \kappa < 8 \\ 0 & \text{if } \kappa \ge 8 \end{cases}.$$

We now address if $T_z = \infty$. There are several ways to do it, but we will use the *imaginary part parametrization*.

If $Z_t = X_t + iY_t = g_t(z) - U_t$, then the Loewner equations (3.3) and (3.4) can be written as

$$dX_t = \frac{aX_t}{X_t^2 + Y_t^2}\,dt + dB_t, \quad \partial_t[\log Y_t] = -\frac{a}{X_t^2 + Y_t^2}. \tag{3.14}$$

In the imaginary part parametrization $\log Y_t$ decreases linearly. Let

$$\sigma(t) = \inf\{s : \log Y_s = -at\}, \quad \tilde Y_t = Y_{\sigma(t)} = e^{-at}, \quad \tilde X_t = X_{\sigma(t)}, \quad K_t = e^{at}\,\tilde X_t.$$

Note that $\dot\sigma(t) = \tilde X_t^2 + \tilde Y_t^2 = e^{-2at}\,(K_t^2 + 1)$. By viewing the SDE for X_t in (3.14) in the new parametrization, we see that

$$\begin{aligned} d\tilde X_t &= a\,\tilde X_t\,dt + \sqrt{\tilde X_t^2 + \tilde Y_t^2}\,dW_t \\ &= a\,e^{-at}\,K_t\,dt + e^{-at}\,\sqrt{K_t^2 + 1}\,dW_t, \end{aligned}$$

where W_t is a standard Brownian motion. Since

$$dK_t = a\, e^{at}\, \tilde{X}_t\, dt + e^{at}\, d\tilde{X}_t,$$

we get

$$dK_t = 2a\, K_t\, dt + \sqrt{K_t^2 + 1}\, dW_t.$$

We will do a change of variables that will change the coefficient of the Brownian term to one. Let $J_t = \sinh^{-1}[K_t]$, that is, $K_t = \sinh J_t$. Itô's formula using the function $\sinh^{-1}(x)$ shows that

$$dJ_t = \left(2a - \frac{1}{2}\right) \tanh J_t\, dt + dW_t. \tag{3.15}$$

One can check (3.15) directly, or one could start with (3.15) where W_t is a standard Brownian motion and check that if $K_t = \sinh J_t$, then

$$\begin{aligned} dK_t &= \cosh J_t\, dJ_t + \frac{1}{2} \sinh J_t\, d\langle J\rangle_t \\ &= 2a \sinh J_t\, dt + \cosh J_t\, dW_t \\ &= 2a\, K_t\, dt + \sqrt{K_t^2 + 1}\, dW_t. \end{aligned}$$

In the imaginary part parametrization it takes an infinite amount of time for Y_t to reach 0. Indeed, Since $|\tanh x| < 1$, the solutions to the SDE (3.15) exist for all time. However, in the original capacity parametrization the total amount of time spent is

$$\sigma(\infty) = \int_0^\infty \dot{\sigma}(t)\, dt = \int_0^\infty e^{-2at}\, (K_t^2 + 1)\, dt = \int_0^\infty e^{-2at}\, \cosh^2 J_t\, dt.$$

Thus $T_z < \infty$ corresponds to $\sigma(\infty) < \infty$. There are three cases to be considered.

- $\kappa < 4$ ($a > 1/2$). Since $\tanh x = \pm 1 - O(e^{-2|x|})$ as $x \to \pm\infty$, we can see by comparison with a Brownian motion with drift, any solution to (3.15) must satisfy

  $$\lim_{t\to\infty} \frac{|J_t|}{t} = 2a - \frac{1}{2}.$$

  Indeed, the process has the same asymptotics as a Brownian motion with drift $\pm\left(2a - \frac{1}{2}\right)$. Choose r with $a < r < 2a - \frac{1}{2}$. We see that for all t sufficiently large, $\cosh^2 J_t \geq e^{2rt}/4$, and hence $\sigma(\infty) = \infty$.

- $\kappa = 4$ ($a = 1/2$). Then J_t satisfies

  $$dJ_t = \frac{1}{2} \tanh J_t\, dt + dW_t.$$

  If instead of this equation, we had $dJ_t = (1/2)\, dt + dW_t$, then we would have $J_t - (t/2)$ is a standard Brownian motion and hence there exists infinite number of n with $J_t \geq n/2$ (and hence $\cosh^2 J_t \geq e^n/4$) for $n \leq t \leq n+1$. This would imply that $\sigma(\infty) = \infty$. We need to do a little work, but the estimate $\tanh |J_t| = 1 - O(e^{-2|J_t|})$ can be used to establish this. We omit the details.

- $\kappa > 4$ ($a < 1/2$). As above, we see that

$$\lim_{t\to\infty} \frac{|J_t|}{t} = 2a - \frac{1}{2}.$$

Choose r with $2a - \frac{1}{2} < r < a$. We see that for all t sufficiently large $\cosh^2 J_t \le e^{2rt}$, and hence $\sigma(\infty) < \infty$.

□

We now prove the analogous theorem for $x \in \mathbb{R}$. We will state it for $x > 0$ but it obviously holds for $x < 0$ as well.

Theorem 3.2.5. *If $x > 0$,*

$$\mathbb{P}\{T_x = \infty, \operatorname{dist}(x,\gamma) > 0\} = 1, \qquad \text{if } 0 < \kappa \le 4,$$

$$\mathbb{P}\{T_x < \infty, \operatorname{dist}(x,\gamma) > 0\} = 1, \qquad \text{if } 4 < \kappa < 8,$$

$$\mathbb{P}\{T_x < \infty, \operatorname{dist}(x,\gamma) = 0\} = 1, \qquad \text{if } 8 \le \kappa < \infty.$$

If $z \in \mathbb{H}$ we considered Υ_t which is a constant multiple times $\operatorname{crad}_{H_t}(z)$ and is comparable to the distance. This will not be useful for $x > 0$ because $\operatorname{crad}_{H_t}(x) > 0$; however, we can use a different quantity based on a reflected domain. Let $H_t^+ = H_t \cup \{z : \overline{z} \in H_t\} \cup (x_t^+, \infty)$, where

$$x_t^+ = \max\{y \in \mathbb{R} : T_y \le t\}.$$

Note that $H_0^+ = \mathbb{C}\setminus(-\infty, 0]$. If the curve γ exists, then H_t^+ is the unbounded connected component of $\mathbb{C} \setminus [\gamma_t \cup \overline{\gamma}_t \cup (-\infty, 0])$; here $\overline{\gamma}_t = \{\overline{\gamma(s)} : 0 \le s \le t\}$. Let $X_t = g_t(x) - U_t, O_t = g_t(x_t^+) - U_t$, and

$$K_t = X_t - O_t, \quad J_t = \frac{K_t}{X_t}, \quad 1 - J_t = \frac{O_t}{X_t}, \quad \Psi_t = \frac{K_t}{g_t'(x)} = \frac{J_t\, X_t}{g_t'(x)}. \tag{3.16}$$

Proposition 3.2.6. *If $x > 0$ and $t < T_x$, then*

$$\Psi_t = \frac{1}{4} \operatorname{crad}_{H_t^+}(x).$$

In particular,

$$\Psi_t \le \operatorname{dist}(x,\gamma) \le 4\,\Psi_t.$$

Proof. Using the Koebe function, we can see that $\operatorname{crad}_{\mathbb{H}\setminus(-\infty,0]}(x) = 4x$. By the scaling rule for conformal radius

$$\operatorname{crad}_{H_t^+}(x) = \frac{\operatorname{crad}_{\mathbb{R}\setminus(-\infty, g_t(x_t^+)]}(g_t(x))}{g_t'(x)} = \frac{4\,(X_t - O_t)}{g_t'(x)} = 4\,\Psi_t.$$

The second inequality follows from the Schwarz lemma and the Koebe (1/4)-theorem.

□

Proof of Theorem 3.2.5. Without loss of generality assume that $x > 0$ and let $X_t = g_t(x) - U_t$. Then X_t satisfies

$$dX_t = \frac{a}{X_t}\,dt + dB_t, \quad X_0 = x.$$

This is a Bessel SDE for which it is known (see Proposition 2.2.1) that with probability one, the process reaches the origin if and only if $a < 1/2$, that is $\kappa > 4$. Hence

$$\mathbb{P}\{T_x < \infty\} = \begin{cases} 0, & \kappa \le 4 \\ 1, & \kappa > 4. \end{cases}$$

By monotonicity, we can conclude that if $y < x$, then $T_y \le T_x$. Hence, we can conclude the stronger fact

$$\mathbb{P}\{T_x = \infty \text{ for all } x > 0\} = 1, \quad 0 < \kappa \le 4,$$

$$\mathbb{P}\{T_x < \infty \text{ for all } x > 0\} = 1, \quad 4 < \kappa < \infty.$$

With more argument, see Proposition 2.2.11, we can see that if $0 < x < y$, then

$$\mathbb{P}\{T_x < T_y\} \begin{cases} < 1, & 4 < \kappa < 8 \\ = 1, & \kappa \ge 8 \end{cases},$$

and, in particular, for $\kappa \ge 8$,

$$\mathbb{P}\{T_x < T_y \text{ for all } 0 < x < y\} = 1. \tag{3.17}$$

For the remainder of this proof we consider $x \in \mathbb{R}$, and by scaling we can assume that $x = 1$. If $\kappa \ge 8$, we can use (3.17) to see that with probability one $\mathbb{R} \subset \gamma$. Let us consider the case $\kappa < 8$, and look at the quantities in (3.16). Using the Loewner equation and (3.7), we see that

$$\partial_t K_t = \frac{a}{O_t} - \frac{a}{X_t} = -a\,\frac{K_t}{O_t\,X_t} = -a\,\frac{K_t}{(1-J_t)\,X_t^2}, \quad \partial_t\, g_t'(x)^{-1} = g_t'(x)^{-1}\,\frac{a}{X_t^2}$$

Using the chain rule, (3.7), and Itô's formula, we can compute

$$\partial_t\,\Psi_t = -a\,K_t\,g_t'(x)^{-1}\,\left[\frac{1}{(1-J_t)\,X_t^2} - \frac{1}{X_t^2}\right] = -a\,\Psi_t\,\frac{J_t}{X_t^2\,(1-J_t)},$$

$$\begin{aligned} dJ_t &= K_t\,d[1/X_t] + [1/X_t]\,dK_t \\ &= K_t\,[-X_t^{-2}\,(a\,X_t^{-1}\,dt + dB_t) + X_t^3\,dt] - a\,\frac{K_t}{(1-J_t)\,X_t^3}\,dt \\ &= \frac{J_t}{X_t^2}\left(1 - a - \frac{a}{1-J_t}\right)\,dt - \frac{J_t}{X_t}\,dB_t. \end{aligned}$$

As in the case of the Υ_t, this equation becomes nicer if we parametrize so that $\log \Psi_t$ decays linearly. Let

$$\sigma(t) = \inf\{s : \Psi_s = e^{-at}\}, \tag{3.18}$$

and if $\sigma(t) < \infty$,

$$\hat{\Psi}_t = \Psi_{\sigma(t)} = e^{-at}, \quad \hat{X}_t = X_{\sigma(t)}, \quad \hat{K}_t = K_{\sigma(t)}, \quad \hat{J}_t = J_{\sigma(t)}.$$

By the chain rule, we see that

$$-a\,\hat{\Psi}_t = \partial_t \hat{\Psi}_t = -a\,\hat{\Psi}_t\,\frac{\hat{J}_t}{\hat{X}_t^2\,(1-\hat{J}_t)}\,\dot{\sigma}(t),$$

and hence

$$\dot{\sigma}(t) = \frac{\hat{X}_t^2\,(1-\hat{J}_t)}{\hat{J}_t}.$$

Using this, we see that

$$d\hat{J}_t = \left[1 - 2a - (1-a)\hat{J}_t\right]\,dt + \sqrt{\hat{J}_t\,(1-\hat{J}_t)}\,dW_t, \tag{3.19}$$

for a standard Brownian motion W. This is nicer if we do another change of variables. Define Q_t by

$$\hat{J}_t = \frac{1-\cos Q_t}{2},$$

and note that we can write (3.19) as

$$d\hat{J}_t = \left[\frac{1-3a}{2} + \frac{1-a}{2}\cos Q_t\right]\,dt + \frac{1}{2}\,\sin Q_t\,dW_t.$$

Using Itô's formula, we can see that this holds if Q_t satisfies the equation

$$d\,Q_t = \left[\left(\frac{1}{2} - a\right)\cot Q_t + \frac{1-3a}{\sin Q_t}\right]\,dt + dW_t.$$

We must actually define this with reflection at 1. The question is whether or not this process reaches 0 in finite (in the new parametrization) time. This is an example of a reflected "'asymptotically Bessel" process as discussed in Section 2.4. Near the origin

$$\left(\frac{1}{2} - a\right)\cot u + \frac{1-3a}{\sin u} = \frac{\frac{3}{2} - 4a}{u} + O(u).$$

By comparison with a Bessel process we see that Q_t reaches the origin in finite time if and only if $\frac{3}{2} - 4a < \frac{1}{2}$ or $a > \frac{1}{4}$. This corresponds to $\kappa < 8$.

□

The next proposition is a continuation of the description of the "phases" of SLE. In the proof we are assuming that SLE comes from a curve.

Proposition 3.2.7. *Let γ be an SLE_κ curve from 0 to ∞ in $\mathbb{H}$. Then the following holds with probability one.*

1. *If $\kappa \leq 4$, then γ is a simple curve with $\gamma(0,\infty) \cap \mathbb{R} = \emptyset$.*
2. *If $4 < \kappa < 8$, then γ has self-intersections and $\gamma(0,\infty) \cap \mathbb{R} \neq \emptyset$. However, for almost every $z \in \mathbb{C}$, $z \notin \gamma$.*

3. *If $8 \le \kappa < \infty$, then γ is plane-filling, that is,*

$$\gamma[0,\infty) = \overline{\mathbb{H}}.$$

Proof.

1. If $0 < x < y$, we know that $T_x < T_y$. We also know that with probability one $T_x < \infty$ for all rational x. Therefore, with probability one $T_x = \infty$ for all $x \in \mathbb{R}$ and hence $\gamma(0,\infty) \cap \mathbb{R} = \emptyset$. If $s > 0$, let
$$\gamma^s(t) = g_s[\gamma(s+t)] - U_s.$$
For each s, the distribution of γ^s is that of SLE_κ. Hence, with probability one, for every rational s, $\gamma^s(0,\infty) \cap \mathbb{R} = \emptyset$. If there existed $0 \le t_1 < t_2 < \infty$ with $\gamma(t_2) = \gamma(t_1)$, then $\gamma^s(0,\infty) \cap \mathbb{R} \neq \emptyset$ for all rational $t_1 < s < t_2$.

2. We have seen that for each $z \in \mathbb{C} \setminus \{0\}$, $\mathbb{P}\{\text{dist}(z,\gamma) > 0\} = 1$. Using Fubini's theorem, we see that with probabilty one the area of $\gamma[0,\infty)$ is zero.

3. Using Fubini's theorem, we know that with probability one for almost all z, $T_z < \infty$ and $z \in \gamma[0,T_z]$. Since γ is a continuous curve we can see that for such z $z = \gamma(T_z)$. This almost prove the result, and we delay the proof of the remainder.

□

The next proposition was one of the first calculations done for SLE reducing a geometric problem to the solution of a second-order ordinary differential equation.

Proposition 3.2.8. *Suppose γ is a chordal SLE_κ curve (from 0 to ∞) in $\mathbb{H}$ with $\kappa < 8$. Let $q(\theta)$ be the probability that the curve goes to the right of $re^{i\theta}$. Then*

$$q(\theta) = q_\kappa(\theta) = C \int_0^\theta \sin^{4a-2} u \, du, \qquad \text{where } C = \left[\int_0^\pi \sin^{4a-2} u \, du \right]^{-1}. \tag{3.20}$$

One may note that the condition $\kappa < 8$ is necessary for the integrals to be convergent. There are several interesting cases:

$$q_2(\theta) = \frac{\theta - \sin\theta\,\cos\theta}{\pi}, \quad q_{8/3}(\theta) = \frac{1-\cos\theta}{2}, \quad q_4(\theta) = \frac{\theta}{\pi}.$$

The $\kappa = 4$ case is just the gambler's ruin estimate for Brownian motion.

Proof. To say that the curve goes to the right is to say that $\Theta_T(z) := \Theta_{T-} = \pi$ where $T = T_z$. By scaling, the probability depends only on θ. If we parametrize so that the log conformal radius decays linearly as in (3.12), we see that $q(\theta) = \mathbb{P}^\theta\{\Theta_T = \pi\}$ where Θ_t satisfies

$$d\Theta_t = (1-2a)\,\cot\Theta_t\,dt + dB_t, \quad \Theta_t = \theta,$$

and $T = \inf\{t : \sin \Theta_t = 0\}$. Note that $q(\Theta_{t\wedge T})$ is a continuous martingale. If we also assume that q is C^2, then Itô's formula gives

$$\begin{aligned} dq(\Theta_t) &= q'(\Theta_t)\, d\Theta_t + \frac{1}{2}\, q''(\Theta_t)\, d\langle\Theta\rangle_t \\ &= \left[(1-2a)\, q'(\Theta_t)\, \cot \Theta_t + \frac{q''(\Theta_t)}{2}\right] dt + q'(\Theta_t)\, d\Theta_t. \end{aligned}$$

In order to make this a martingale, we solve the simple second order ODE,

$$(1-2a)\, q'(\theta)\, \cot \theta + \frac{q''(\theta)}{2} = 0,$$

which is a first order ODE in q'. Separation of variables shows that $q'(\theta) = c\,[\sin\theta]^{4a-2}$, and then using the boundary conditions $q(0) = 0$ and $q(\pi) = 1$ we get (3.20). Although we assumed that q was C^2 to start with, we could go back with q as in (3.20) and show that $q(\Theta_{t\wedge T})$ is a bounded martingale and use the optional sampling theorem.

□

The function $q(\theta)$ is the solution of a linear second-order ODE and could also be written as a hypergeometric function.

The arguments in the proof of Theorem 3.2.4 can be extended to establish two important "one-point" estimates for SLE_κ for $\kappa < 8$. The first discusses the probability of getting near an interior point z and the second deals with the probability of getting near a boundary point x.

Proposition 3.2.9. *If $\kappa < 8$, there exists $\alpha > 0$ such that if γ is a chordal SLE_κ curve from 0 to ∞ in $\mathbb{H}$, $z \in \mathbb{H}$, and $\Upsilon = \Upsilon_\infty(z)$, then for $0 < r \le 1/2$,*

$$\mathbb{P}\{\Upsilon \le r\, \Upsilon_0\} = c_*\, r^{2-d}\, [\sin(\arg z)]^{4a-1}[1 + O(r^\alpha)],$$

where

$$d = 1 + \frac{1}{4a} = 1 + \frac{\kappa}{8}, \quad c_* = 2\left[\int_0^\pi \sin^{4a}\theta\, d\theta\right]^{-1}.$$

The statement of the theorem is shorthand for the following. There exists c, α such that for all z and all $r < 1/2$,

$$\left|\mathbb{P}\{\Upsilon \le r\, \Upsilon_0\} - c_*\, r^\beta\right| \le c\, r^{\beta+\alpha}.$$

The same fact holds for all $r < 1 - \epsilon$ for any $\epsilon > 0$, but the constant c depends on ϵ.

Proof. By scaling, we may assume that $\Upsilon_0 = 1$, that is, $z = x + i$ for some $x \in \mathbb{R}$. We will use the radial parametrization as in (3.11) so that $\Upsilon_{\sigma(t)} = e^{-2at}$, and let $\hat{\Theta}_t = \Theta_{\sigma(t)}$, that satisfies

$$d\hat{\Theta}_t = (1-2a)\,\cot\hat{\Theta}_t\,dt + dB_t,$$

for a standard Brownian motion B_t. This equation is valid until $T = \inf\{t : \sin\hat{\Theta}_t = 0\}$ at which time $\hat{\Upsilon}_T = e^{-2aT}$. Hence, if we write $r = e^{-2as}$, we can write

$$\mathbb{P}\{\Upsilon < r\} = \mathbb{P}\{T > s\}.$$

Let $S_t = \sin\hat{\Theta}_t$ and let

$$M_t = e^{t(2a-\frac{1}{2})}\,S_t^{4a-1}. \tag{3.21}$$

We claim that $M_{t\wedge T}$ is a martingale satisfying

$$dM_t = (4a-1)\,[\cot\hat{\Theta}_t]\,M_t\,dB_t. \tag{3.22}$$

To check this, we use Itô's formula. Using $\cot^2\hat{\Theta}_t = S_t^{-2} - 1$, we see that for $t < T$,

$$\begin{aligned} dS_t &= \cos\hat{\Theta}_t\,d\hat{\Theta}_t - \frac{1}{2}\sin\hat{\Theta}_t\,dt \\ &= S_t\left[((1-2a)\,\cot^2\hat{\Theta}_t - \frac{1}{2})\,dt + \cot\hat{\Theta}_t\,dB_t\right] \\ &= S_t\left[(2a - \frac{3}{2} + \frac{1-2a}{S_t^2})\,dt + \cot\hat{\Theta}_t\,dB_t\right] \end{aligned}$$

$$\begin{aligned} dS_t^{4a-1} &= S_t^{4a-1}\left[\frac{4a-1}{S_t}\,dS_t + \frac{(2a-1)(4a-1)}{S_t^2}\,d\langle S\rangle_t\right] \\ &= (4a-1)\,S_t^{4a-1}\left[(2a - \frac{3}{2} + \frac{1-2a}{S_t^2})\,dt + (2a-1)\,[\frac{1}{S_t^2} - 1]\,dt + \cot\hat{\Theta}_t\,dB_t\right] \\ &= S_t^{4a-1}\left[(\frac{1}{2} - 2a)\,dt + (4a-1)\,\cot\hat{\Theta}_t\,dB_t\right] \end{aligned}$$

This verifies that (3.22) holds for $t < T$. It is easy to check that the martingale is continuous as $t \uparrow T$, and since the process is uniformly bounded on $[0, t_0]$ for all $t_0 < \infty$, we can see that $M_{t\wedge T}$ is, in fact, a continuous martingale.

Using the Girsanov theorem, we define a new probability measure $\mathbb{P}^*$ with expectations $\mathbb{E}^*$ by stating that if V is a nonnegative random variable measurable with respect to $B_s, 0 \le s \le t$, then

$$\mathbb{E}^*(V) = M_0^{-1}\,\mathbb{E}[V\,M_t] = S_0^{1-4a}\,\mathbb{E}[V\,M_t]\ .$$

By (3.22) and the Girsnaov theorem, we can see that

$$dB_t = (4a-1)\,\cot\hat{\Theta}_t\,dt + dW_t,$$

where W_t is a standard Brownian motion with respect to the probability measure $\mathbb{P}^*$. This equation only holds for $t < T$, but it is easy to see that $\mathbb{P}^*\{t < T\} = 1$. (Indeed, since $M_{t\wedge T} = 0$ on the event $T \le t$, this must be the case.) Therefore,

$$d\hat{\Theta}_t = 2a\,\cot\hat{\Theta}_t\,dt + dW_t. \tag{3.23}$$

We now write

$$\begin{aligned}
\mathbb{P}\{T > t\} &= \mathbb{E}\left[1\{T > t\}\right] \\
&= \mathbb{E}\left[M_t\, M_t^{-1}; T > t\right] \\
&= e^{t(\frac{1}{2}-2a)}\, \mathbb{E}\left[M_t\, S_t^{1-4a}; T > t\right] \\
&= e^{t(\frac{1}{2}-2a)}\, S_0^{4a-1}\, \mathbb{E}^*[S_t^{1-4a}; T > t] \\
&= e^{t(\frac{1}{2}-2a)}\, S_0^{4a-1}\, \mathbb{E}^*[S_t^{1-4a}].
\end{aligned}$$

The last equality holds because $\mathbb{P}^*\{T > t\} = 1$. We are left with estimating $\mathbb{E}^*[S_t^{1-4a}]$ which is a problem about the radial Bessel equation (3.23). As in Proposition 2.4.5, we see that the invariant density is

$$f_{4a}(\theta) = c_{4a}\, \sin^{4a}\theta, \qquad \text{where } c_r = \left[\int_0^\pi \sin^r\, y\, dy\right]^{-1},$$

and

$$\mathbb{E}^*[S_t^{1-4a}] = [1 + O(e^{-tu})] \int_0^\pi [\sin^{1-4a}\, \theta]\, f_{4a}(\theta)\, d\theta = 2\, c_{4a}\, [1 + O(e^{-tu})].$$

Therefore, if $r = e^{-2at}$,

$$\mathbb{P}\left\{\Upsilon < r\right\} = \mathbb{P}\{T > t\} = 2\, c_{4a}\, e^{t(\frac{1}{2}-2a)}\, S_0^{4a-1}\, [1 + O(e^{-tu})] = 2\, c_{4a}\, r^{1-\frac{1}{4a}}\, S_0^{4a-1}\, [1 + O(r^\alpha)],$$

where $\alpha = u/(2a)$.

□

It may seem that the martingale in (3.21) came out of the blue, but it arises very naturally. If we consider the radial Bessel equation

$$d\Theta_t = r\, \cot\Theta_t\, dt + dB_t,$$

with $r < 1/2$ and we want to compute $f(\theta, t) = \mathbb{P}\{T > t \mid \Theta_t = \theta\}$ we might hope that this would satisfy

$$f(\theta, t) \sim f(\theta)\, e^{-\lambda t} \quad t \to \infty.$$

We try to find f, λ so that

$$M_t = e^{\lambda t}\, f(\Theta_t)$$

is a martingale. Using Itô's formula, we see that a sufficient condition to make this a martingale is

$$\frac{1}{2}\, f''(x) + r\, [\cot x]\, f'(x) + \lambda\, f(x) = 0.$$

This is an eigenvalue/eigenfuction problem: try to find the (unique) positive solution to this second order differential equation with $f(0) = f(\pi) = 0$. If one tries functions of the form $f(x) = [\sin x]^\beta$, then one gets

$$f'(x) = \beta\, f(x)\, \cot x,$$

$$f''(x) = \beta\, f(x)\, \left[\beta\, \cot^2 x - \frac{1}{\sin^2 x}\right] = \beta\, f(x)\, \left[(\beta - 1)\, \cot^2 x - 1\right],$$

which leads to the equation

$$[\beta(\beta-1)+2r\beta]\,\cot^2 x + (2\lambda-\beta) = 0.$$

Equating coefficients gives

$$\beta = 1-2r, \qquad \lambda = r - \frac{1}{2}.$$

We therefore get the martingale

$$M_t = e^{t(r-\frac{1}{2})}\,[\sin\Theta_t]^{1-2r},$$

which satisfies

$$dM_t = (1-2r)\,M_t\,\cot\Theta_t\,dB_t.$$

When we tilt by the martingale to give a new measure $\mathbb{P}^*$, we see that

$$dB_t = (1-2r)\,\cot\Theta_t\,dt + dW_t,$$

where W_t is a standard Brownian motion with respect to $\mathbb{P}^*$. Therefore,

$$d\Theta_t = (1-r)\,\cot\Theta_t\,dt + dW_t.$$

Since $r < 1/2$, we see that $\mathbb{P}^*\{T<\infty\} = 0$. This diffusion has invariant density $g(\theta) = [\sin\theta]^{2-2r}$. The quasi-invariant density h is the limiting density (with respect to the original measure $\mathbb{P}$) as $t\to\infty$ of Θ_t given $T>t$. The density g, which is the density in the measure $\mathbb{P}^*$, can be written as $g(\theta) = c\,[\sin\theta]^{4a-1}\,h(\theta)$. In other words, $h(\theta) = c_1\,\sin\theta = \sin\theta/2$.

Corollary 3.2.10. *If $\kappa<8$, there exists $\alpha>0$ such that that the following holds. Suppose that $z,w\in\partial D$ are distinct, γ is a chordal SLE_κ path from z to w in D, and $\zeta\in D$. Let D' be the connected component of $D\setminus\gamma$ containing ζ. Then if $r\le 1/2$,*

$$\mathbb{P}\{\mathrm{crad}_{D'}(\zeta)\le r\,\mathrm{crad}_D(\zeta)\} = c_*\,r^{2-d}\,S_D(\zeta;z,w)^{4a-1}\,[1+O(r^\alpha)],$$

where d, c_* are as in Proposition 3.2.9.

Corollary 3.2.11. *If $\kappa<8$, there exists $\alpha>0$ such that that the following holds. Suppose that $z,w\in\partial D$ are distinct, γ is a chordal SLE_κ path from z to w in D, and $\zeta\in D$. Let D' be the connected component of $D\setminus\gamma$ containing ζ. Then if $r\le 1/2$,*

$$c_1\,S_D(\zeta;z,w)^{4a-1}\,r^{2-d} \le \mathbb{P}\left\{\mathrm{dist}(\zeta,\gamma)\le r\,\mathrm{dist}(\zeta,\partial D)\right\} \le c_2\,S_D(\zeta;z,w)^{4a-1}\,r^{2-d}.$$

Proof. Most of this follows from the previous corollary that $\mathrm{dist}_{D'}(\zeta)\asymp_2 2\,\mathrm{dist}(z,\partial D)$. For the upper bound for larger values of r, we also need the following fact that is easy to prove: there exists $\rho<1$, such that if $\zeta\in D_2\subset D_1$, then

$$\mathrm{crad}_{D_2}(\zeta)\le\rho\mathrm{crad}_{D_1}(\zeta).$$

□

We will now prove the analogue of Proposition 3.2.9 where conformal radius is replaced with distance to the boundary. We will get as strong a result although we will not get an explicit form for the constant.

Theorem 3.2.12. *If $\kappa < 8$, D is a simply connected domain, and $z, w \in \partial D$ are distinct, there exists a function $\mathbb{G}_D(\cdot; z, w)$ on D such that if γ is a chordal SLE path from z to w in D the following holds.*

1. *If $\zeta \in D$ and $r \le \operatorname{dist}(\zeta, D)/2$,*

$$\mathbb{P}\{\operatorname{dist}(\zeta, \gamma) \le r\} = \mathbb{G}_D(\zeta; z, w)\, r^{2-d}\, [1 + O(r^\alpha)].$$

2. *There exists $\hat{c} = \hat{c}(\kappa)$ such that*

$$\mathbb{G}_D(\zeta; z, w) = \hat{c}\, \operatorname{crad}_D(\zeta)^{d-2}\, S_D(\zeta; z, w)^{4a-1}.$$

We call $\mathbb{G}_D(\zeta; z, w)$ the *chordal SLE_κ Green's function (from z to w in D)*. The last assertion can be written as a combination of

$$\mathbb{G}_{\mathbb{D}}(0; 1, e^{2i\theta}) = \hat{c}\, (\sin\theta)^{4a-1},$$

and the conformal covariance rule: if $f : D \to f(D)$ is a conformal transformation, then

$$\mathbb{G}_D(\zeta; z, w) = |f'(\zeta)|^{2-d}\, \mathbb{G}_{f(D)}(f(\zeta); f(z), f(w)).$$

The proof will not give the value of the κ-dependent constant $\hat{c}$.

Proof. We first consider $D = \mathbb{D}, \zeta = 0, z = 1, w = e^{2\theta i}$. We will write $r = e^{-s}$.

We let $\mathbb{P}_\theta$ denote the probabilities for SLE_κ from 1 to $e^{2\theta i}$ in $\mathbb{D}$. We will show that there exists $\hat{c}$ such that for all $0 < \theta < \pi$,

$$\mathbb{P}_\theta\{\operatorname{dist}(0, \gamma) \le r\} = \hat{c}\, r^{2-d}\, (\sin\theta)^{4a-1}\, [1 + O(r^\alpha)]. \tag{3.24}$$

For each $r_0 > 0$ and $r_0 \le r \le 1/2$, small, we can use Corollary 3.2.11 to see that

$$\mathbb{P}_\theta\{\operatorname{dist}(0, \gamma) \le r\} \asymp [\sin\theta]^{4a-1}.$$

Hence it suffices to find r_0 so that the relation holds for $r \le r_0$.

We will assume that γ has the radial parametrization as in (3.11), so that if D_t is the connected component of $\mathbb{D} \setminus \gamma_t$ containing the origin, then

$$\operatorname{crad}_{D_t}(0) = e^{-2at}.$$

We let $\tilde{g}_t : D_t \to \mathbb{D}$ be the conformal transformation with $\tilde{g}_t(0) = 0, \tilde{g}_t(\gamma(t)) = 1$ and define Θ_t by $\tilde{g}_t(e^{2i\theta}) = e^{2i\Theta_t}$. Then Θ_t satisfies

$$d\Theta_t = (1 - 2a)\, \cot\Theta_t\, dt + dB_t,$$

where B_t is a standard Brownian motion. We write g_t for $\tilde{g}_t$ followed by a rotation such that $g_t'(0) > 0$. The curve has finite lifetime T in this parametrization. As before, using the Koebe 1/4-theorem,

$$2\operatorname{dist}(0, \partial D_t) \asymp_2 e^{-2at}.$$

Let $\tau_s = \inf\{t : |\gamma(t)| = e^{-s}\}$, so that (3.24) can be rephrased as

$$\mathbb{P}_\theta\{T > \tau_s\} = \hat{c}\, e^{s(d-2)}\, (\sin\theta)^{4a-1}\, [1 + O(e^{-\alpha s})]. \tag{3.25}$$

We will first give a sketch for why (3.25) holds. Note that $\tau_s \geq \frac{s-\log 4}{2a} > q$ where $q = q_s = \frac{s-2}{2a}$. The relation (3.25) can be established for large s by showing there exists u, C, C' such that

$$\mathbb{P}_\theta\, \{T > q\} = C\, e^{(d-2)s}\, (\sin\theta)^{4a-1}\, [1 + O(e^{-us})] \tag{3.26}$$

$$\mathbb{P}_\theta\, \{T > \tau_s \mid T > q\} = C' + O(e^{-us}). \tag{3.27}$$

Theorem 3.2.9 gives us (3.26). For the second equality note that conformal invariance implies that

$$\mathbb{P}_\theta\, \{T > \tau_s \mid T > q\} = \mathbb{P}_{\Theta_q}\{\gamma \cap \tilde{g}_q(e^{-s}\, \mathbb{D}) \neq \emptyset\}.$$

Using the Koebe 1/4-theorem, we can see that there exist $0 < \rho_1 < \rho_2 < 1$ such that

$$\rho_1\, \mathbb{D} \subset \tilde{g}_q(e^{-s}\, \mathbb{D}) \subset \rho_2\, \mathbb{D}.$$

If we can show that the conditional distribution of $(\Theta_q, \tilde{g}_q(e^{-s}\, \mathbb{D}))$ given $T > q$ converges exponentially fast to some distribution, then we would have (3.27).

For a proof, we consider the *quasi-invariant* density corresponding to the density of Θ_t given $T > t$. Let $\mathbb{P}^*_\theta, \mathbb{E}^*_\theta$ denote the probability measures obtained by tilting by the martingale

$$M_t = e^{t(a-\frac{1}{2})}\, [\sin\Theta_t]^{4a-1}$$

as in the proof of Theorem 3.2.9 , if $S_t = \sin\Theta_t$, Then, if F is a continuous function,

$$\begin{aligned}
\mathbb{E}_\theta[F(\Theta_t); T > t] &= e^{t(\frac{1}{2}-a)}\, \mathbb{E}_\theta\left[M_t\, F(\Theta_t)\, S_t^{1-4a}; T > t\right] \\
&= e^{t(\frac{1}{2}-a)}\, [\sin\theta]^{4a-1}\, \mathbb{E}^*_\theta\left[F(\Theta_t)\, S_t^{1-4a};\right] \\
&= c_{4a}\, e^{t(\frac{1}{2}-a)}\, [\sin\theta]^{4a-1}\, [1 + O(e^{-\alpha u})] \int_0^\pi F(x)\, [\sin x]^{4a}\, [\sin x]^{1-4a}\, dx \\
&= 2\, c_{4a}\, e^{t(\frac{1}{2}-a)}\, [\sin\theta]^{4a-1}\, [1 + O(e^{-\alpha u})] \int_0^\pi F(x)\, \frac{\sin x}{2}\, dx
\end{aligned}$$

In other words, $(\sin x)/2$ is the quasi-invariant density. Indeed, if we write $\mathbb{P}_\mu, \mathbb{E}_\mu$ for probabilities and expectations assuming that the initial θ has density $(\sin x)/2$, then

$$\mathbb{E}_\mu[F(\Theta_t); T > t] = e^{t(\frac{1}{2}-a)}\, \mathbb{E}_\mu[F(\Theta_0)] = \int_0^\pi F(x)\, \frac{\sin x}{2}\, dx.$$

Let

$$P(s) = e^{s(2-d)}\, \mathbb{P}_\mu\{T > s\}.$$

We will now show that there exists c_0, c such that

$$|P(s) - c_0| \leq c\, e^{-s}. \tag{3.28}$$

Suppose $0 \le \delta \le 1$, and note that $g'_\delta(0) = e^{2a\delta}$. Distortion estimates imply that there is a universal c (we could find an exact value but will not bother) such that for s sufficiently large and $|z| \le e^{-s}$,

$$|g_\delta(z) - e^{2a\delta}\, z| \le c\, e^{-2s}.$$

In particular, there exists universal c_1, c_2 such that

$$(e^{-s} - c_1 e^{-2s})\, \mathbb{D} \subset g_\delta(e^{-(s+2a\delta)}\mathbb{D}) \subset (e^{-s} + c_1 e^{-2s})\, \mathbb{D}$$

$$\mathbb{P}_\mu\{T > \tau_{s+c_2 e^{-s}}\} \le \mathbb{P}_\mu\{T > \tau_{s+\delta} \mid T > \delta\} \le \mathbb{P}_\mu\{T > \tau_{s-c_2 e^{-s}}\}.$$

Using the fact that μ is a quasi-invariant distribution, we see that this implies that for $0 \le \delta \le 1$ and s sufficiently large,

$$|\log P(s) - \log P(s+\delta)| \le c\, e^{-s}.$$

Similarly, if k is a positive integer

$$|\log P(s) - \log P(s+k)| \le \sum_{j=1}^{k} |\log P(s+j-1) - \log P(s+j)| = \sum_{j=1}^{k} O(e^{-s-j}) = O(e^{-s}).$$

We therefore get

$$\sup_{\delta>0} |\log P(s) - \log P(s+\delta)| \le c\, e^{-s},$$

which is another way to express (3.28).

If our initial condition is $\Theta_0 = \theta$, we write

$$\mathbb{P}_\theta\{T > \tau_{2s}\} = \mathbb{P}_\theta\left\{T > \frac{s}{2a}\right\} \mathbb{P}_\theta\left\{T > \tau_{2s} \mid T > \frac{s}{2a}\right\}.$$

Using distortion estimates and $g'_{s/2a}(0) = e^s$ as in the previous paragraph, we can see that there exists c_3 such that

$$\mathbb{P}_{\Theta_{s/2a}}\left\{T > \tau_{s+c_3\, e^{-s}}\right\} \le \mathbb{P}_\theta\{T > \tau_{2s} \mid T > \frac{s}{2a}\} \le \mathbb{P}_{\Theta_{s/2a}}\left\{T > \tau_{s-c_3\, e^{-s}}\right\},$$

Combining this, with (3.28), we see that

$$\begin{aligned} \mathbb{P}_\theta\{T > \tau_{2s} \mid T > \frac{s}{2a}\} &= [1 + O(e^{-us})]\, \mathbb{P}_\mu\{T > \tau_{s+O(e^{-s})}\} \\ &= c_0\, e^{-s(2-d)}\, [1 + O(e^{-us})]. \end{aligned}$$

$$\mathbb{P}_\theta\{T > s\} = c_*\, e^{-2as(2-d)}\, [\sin\theta]^{4a-1}\, [1 + O(e^{-us})].$$

Proposition 3.2.9 implies that

$$\mathbb{P}_\theta\left\{T > \frac{s}{2a}\right\} = c_*\, e^{-s(2-d)}\, [1 + O(e^{-us})].$$

This finishes the result for $D = \mathbb{D}, \zeta = 0, z = 1, w = e^{2i\theta}$. For more general D, z, w, let $f : D \to \mathbb{D}$ be the unique conformal transformation $f : D \to \mathbb{D}$ with $f(\zeta) = 0, f(z) = 1$ and define θ by $f(w) = e^{2i\theta}$. Note that $\sin\theta = S_D(\zeta; z, w)$. By scaling, we may assume that $|f'(\zeta)| = 1$. Note

that $\mathrm{crad}_D(\zeta)^{-1} = |f'(\zeta)| \asymp 1$. Let V_r denote the image under f of the open disk of radius r about ζ. By distortion estimates, there exists a universal c' such that if $r \le \mathrm{dist}(\zeta, \partial D)/2$, then

$$(r - c'\, r^2)\mathbb{D} \subset f(V_r) \subset (r + c'\, r^2)\, \mathbb{D}.$$

Hence, we can use the result for the disk.

□

The estimates above can be considered as "interior" or "bulk" estimates for the probability that the SLE curve gets close to an interior point. We will now give the estimates for getting near a boundary point. By scaling, we can consider the probability that an SLE_κ path from 0 to ∞ gets near 1. The interior estimate gives us a (correct) guess for the correct exponent of decay. Suppose $\zeta = 1 + i\epsilon$. Then one might expect that the probability of getting within distance ϵ of 1 is comparable to the probability of getting within distance $\epsilon/2$ of ζ. Since $S_{\mathbb{H}}(1+\epsilon; 0, \infty) = \epsilon$, we see that the latter probability is comparable to ϵ^{4a-1}. Indeed, this is correct, and the next proposition gives a much finer estimate. We will use the alternative conformal radius Ψ_t that was introduced in the proof of Theorem 3.2.4. Recall that Ψ_t is one-fourth times the conformal radius of 1 in the unbounded component of

$$\mathbb{C} \setminus \left(\gamma_t \cup \{\overline{\gamma(s)} : 0 \le s \le t\} \cup (-\infty, 0] \right).$$

Proposition 3.2.13. *There exists $\alpha > 0$ such that if $x > 0$,*

$$\mathbb{P}\{\Psi_\infty(x) \le rx\} = \frac{\Gamma(6a)}{\Gamma(4a)\,\Gamma(2a+1)}\, r^{4a-1}\, [1 + O(r^\alpha)]\,.$$

Proof. Without loss of generality we assume that $x = 1$. Let $X_t = X_t(1) = g_t(1) - U_t$ and

$$M_t = X_t^{1-4a}\, g_t'(1)^{4a-1}.$$

Since

$$\begin{aligned} dX_t^{1-4a} &= X_t^{1-4a} \left[\frac{1-4a}{X_t}\, dX_t + \frac{(1-4a)(-2a)}{X_t^2}\, d\langle X \rangle_t \right] \\ &= X_t^{1-4a} \left[\frac{-a(1-4a)}{X_t^2}\, dt + \frac{1-4a}{X_t}\, dB_t \right]. \end{aligned}$$

$$\partial_t g_t'(1)^{4a-1} = -g_t'(1)^{4a-1}\, \frac{a(4a-1)}{X_t^2},$$

we see that $M_t = g_t'(1)^{4a-1}\, X_t^{1-4a}$ is a local martingale satisfying

$$dM_t = \frac{1-4a}{X_t}\, M_t\, dB_t.$$

Let $\mathbb{P}^*, \mathbb{E}^*$ denote probabilities and expectations obtained by tilting by the local martingale M_t. Then the Girsanov theorem implies that

$$dB_t = \frac{1-4a}{X_t}\, dt + dB_t^*,$$

where B_t^* is a standard Brownian motion with respect to $\mathbb{P}^*$. Using the notation as in (3.16), we get

$$dJ_t = \frac{J_t}{X_t^2}\left(3a - \frac{a}{1-J_t}\right)dt - \frac{J_t}{X_t}\,dB_t^*.$$

If we reparametrize as in (3.18) so that $\Psi_{\sigma(t)} = e^{-at}$, we see that $\hat{J}_t := J_{\sigma(t)}$ satisfies

$$dJ_t = \left(2a - 3a\hat{J}_t\right)dt + \sqrt{\hat{J}_t\,(1-\hat{J}_t)}\,W_t, \tag{3.29}$$

where W_t is also a standard Brownian motion with respect to $\mathbb{P}^*$. In the new parametrization,

$$\hat{M}_t := \hat{\Psi}_t^{1-4a}\,\hat{J}_t^{4a-1} = M_{\sigma(t)} = e^{(4a-1)at}\,\hat{J}_t^{4a-1}.$$

As before, we define Q_t by

$$\hat{J}_t = \frac{1-\cos\hat{Q}_t}{2},$$

so that (3.29) can be written as

$$dJ_t = \left(\frac{a}{2} + \frac{3a}{2}\cos Q_t\right)dt + \frac{\sin Q_t}{2}\,dW_t.$$

Since

$$d\hat{Q}_t = \frac{1}{2}\,\sin Q_t\,dQ_t + \frac{1}{4}\,\cos Q_t\,d\langle Q\rangle_t,$$

we can see that

$$dQ_t = \left[\frac{a}{\sin Q_t} + \left(3a - \frac{1}{2}\right)\cot Q_t\right]dt + dW_t.$$

This is the SDE studied in Section 2.4.5 with

$$v = a, \quad u = 2a - \frac{1}{2}.$$

As in the Proof of Proposition 3.2.9,

$$\begin{aligned}
\mathbb{P}\{\Psi_\infty < e^{-at}\} &= \mathbb{E}\left[1\{\Psi_\infty < e^{-at}\}\right] \\
&= e^{(1-4a)at}\,\mathbb{E}\left[\hat{M}_t\,\hat{J}_t^{1-4a}\;;\;\Psi_\infty < e^{-at}\right] \\
&= e^{(1-4a)at}\,\mathbb{E}^*\left[\hat{J}_t^{1-4a}\right].
\end{aligned}$$

So it suffices to show there exists α such that

$$\mathbb{E}^*\left[\hat{J}_t^{1-4a}\right] = \frac{\Gamma(6a)}{\Gamma(4a)\,\Gamma(2a+1)} + O(e^{-t\alpha}).$$

This is obtained by plugging $v = a, u = 2a - \frac{1}{2}, k = 1 - 4a$ into (2.58).

□

It follows from the last proposition and the Koebe-1/4 theorem that for $0 < r \le 1$, and $x \in \mathbb{R}$,

$$\mathbb{P}\{\text{dist}(x,\gamma) \le xr\} \asymp r^{4a-1}.$$

This can be improved using essentially the same proof as Theorem (3.2.12) so we omit it. As in that theorem, we are unable to determine the value of the constant c'.

Theorem 3.2.14. *There exist $c' = c'_\kappa$ and $\alpha > 0$ such that for $r \le 1$,*

$$\mathbb{P}\{\text{dist}(x,\gamma) \le r|x|\} = c\, r^{4a-1}\, [1 + O(r^\alpha)]\,.$$

For practical purposes, the following simple corollary of Proposition 3.2.13 usually suffices. It is useful to write it in terms of a conformally invariant quantity, excursion measure.

Proposition 3.2.15. *If $\kappa < 8$, there exists $c < \infty$ such that the following holds. Suppose D is a simply connected domain and z, w are distinct boundary points. Let ∂_- denote one of the two arcs of ∂D with endpoints z, w and let η be a crosscut of D whose endpoints are on $\partial D \setminus \partial_-$. Let D' be the connected component of $D \setminus \eta$ whose boundary contains z and w. If γ is an SLE_κ curve from z to w in D, then*

$$\mathbb{P}\{\gamma \cap \eta \ne \emptyset\} \le c\, \mathcal{E}_{D'}(\eta, \partial_-)^{4a-1}.$$

Proof. By conformal invariance, we may assume $D = \mathbb{H}, z = 0, w = \infty$ and $\partial_-$is the negative real axis. We can also assume that one of the endpoints of η is 1. In this case (see Lemma 1.8.4),

$$\text{cap}_{\mathbb{H}}(\eta) \ge c\, \text{diam}(\eta),$$

and from this we see that $\mathcal{E}_{D'}(\eta, \partial_-) \ge c\, r$ where $r = 1 \wedge \text{diam}(\eta)$.

$$\mathbb{P}\{\gamma \cap \eta \ne \emptyset\} \le \mathbb{P}\{\text{dist}(\gamma, 1) \le r\} \le c\, r^{4a-1} \le c\, \mathcal{E}_{D'}(\eta, \partial_-)^{4a-1}.$$

□

Here we give a similar result which is a little easier to prove because we only need to consider a real SDE.

Proposition 3.2.16. *Suppose $4 < \kappa < 8$ and γ is an SLE_κ curve from 0 to ∞. Let*

$$T = \inf\{t : \gamma(t) \in [1, \infty)\}.$$

Then

$$\mathbb{P}\{\gamma(T) < 1 + x\} = \frac{\Gamma(2a)}{\Gamma(4a-1)\,\Gamma(1-2a)} \int_0^{\frac{x}{1+x}} \frac{du}{u^{2-4a}\,(1-u)^{2a}}.$$

Note in particular

$$\mathbb{P}\{\gamma(T) < 1 + x) \sim \frac{\Gamma(2a)}{\Gamma(4a-1)\,\Gamma(1-2a)}\, x^{4a-1}, \quad x \downarrow 0.$$

Proof. Let $X_t = g_t(x) - U_t, Y_t = g_t(1+x) - U_t$. Then $T = \inf\{t : X_t = 0\}$. Let $T_x = \inf\{t : Y_t = 0\}$. Then the event $\{\gamma(T) < 1 + x\}$ is the same as the event $\{T < T_x\}$. Let $R_t = [Y_t - X_t]/X_t$. Then this event can also be described as $\{R_{T-} = \infty\}$. Since

$$dX_t = \frac{a}{X_t}\, dt + dB_t, \quad dY_t = \frac{a}{Y_t}\, dt + dB_t,$$

an Itô's formula calculation shows that

$$dR_t = \left[\frac{(1-a)\, R_t}{X_t^2} - \frac{a\, R_t}{(R_t+1)\, X_t^2}\right] dt - \frac{R_t}{X_t}\, dB_t.$$

Under a suitable time change we get that $\hat{R}_t := R_{\sigma(t)}$ satisfies

$$\hat{R}_t = \left[\frac{1-2a}{\hat{R}_t} + \frac{a}{\hat{R}_t + 1}\right] dt + dW_t,$$

for a Brownian motion W_t.

If $\psi(x) = \mathbb{P}\{T_x > T\}$, then scaling shows that $\psi(\hat{R}_t)$ is a martingale and hence (assuming ψ is C^2),

$$\frac{1}{2}\, \psi''(x) + \left[\frac{1-2a}{x} + \frac{a}{x+1}\right] \psi'(x) = 0.$$

This is a first-order differential equation for ψ' that can be solved easily. Using the boundary conditions $\psi(0) = 0, \psi(\infty) = 1$, we get the formula.

□

3.3 Dimension and natural parametrization

The definition of *SLE* was given modulo reparametrization. When the curve is parametrized by capacity, the conformal transformation removing the curve evolves so that it is C^1 in time. However, the capacity parametrization has some major disadvantages. For example, if one takes a particular realization of the curve and changes the domain that the curve lies in, then the parametrization changes. Also, *SLE* curves are models for scaling limits of discrete lattice curves that are parametrized so that edge is traversed at the same rate. For this reason, we would like to parametrize the *SLE* curve by length. If the curve γ were a C^1 curve, then we could use the *natural parametrization*, that is, parametrization by arclength. However, Theorem 3.2.12 implies that $SLE_\kappa, \kappa < 8$ paths are d-dimensional where

$$d = d_\kappa = 1 + \frac{\kappa}{8}.$$

In this section, we will assume $\kappa < 8$ and define the *natural parametrization* for *SLE* curves. We need a concept of "d-dimensional length" to replace arclength. One possibility is to consider Hausdorff d-measure of the curve, but it turns out that this is zero. The correct analogue is (*d-dimensional*) *Minkowski content.*We will state some theorems in this section but will delay the proof until ???. The main technical tool needed to prove these theorems is a "two-point" analogue of Theorem 3.2.12.

Definition If $V \subset \mathbb{C}$ is bounded and $1 < d < 2$, the *d-dimensional Minkowski content* of V is defined by

$$\mathrm{Cont}_d(V) = \lim_{\epsilon \downarrow 0} \epsilon^{d-2} \,\mathrm{Area}\{z : \mathrm{dist}(z, V) \le \epsilon\}, \tag{3.30}$$

provided that the limit exists.

One can also define the upper and lower Minkowski contents by taking limsups and liminfs instead of limits. It is easy to see that if the box dimension of V is strictly less than d, then $\mathrm{Cont}_d(V) = 0$. The limit on the right-hand side of (3.30) often does not exist, but for SLE_κ paths it exists and is nontrivial.

Theorem 3.3.1. *If $\kappa < 8$, and γ is an SLE_κ path from 0 to ∞ in $\mathbb{H}$, then with probability one, for all t, the Minkowski content $\mathrm{Cont}_d(\gamma_t)$ exists. Moreover, the function $t \mapsto \gamma(t)$ is continuous, strictly increasing, and satisfies the additivity rule: if $s < t$,*

$$\mathrm{Cont}_d(\gamma_t) = \mathrm{Cont}_d(\gamma_s) + \mathrm{Cont}_d(\gamma[s,t]).$$

The additivity rule is immediate (given the rest of the theorem) if $\kappa \le 4$ since the curve is simple. For $4 < \kappa < 8$, it requires a little more work but essentially follows from the fact that the double points of the paths have fractal dimension strictly less than d. Since the function $t \mapsto \gamma_t$ is continuous and strictly increasing, one can reparametrize the curve so that the content grows linearly.

Although we delay the full proof to later, let us discuss the construction. We will, in fact, construct the *occupation measure* ν^{occ} for SLE which is a nontrivial, positive measure supported on γ such that for each $s < t$, $0 < \nu^{\mathrm{occ}}(\gamma_t \setminus \gamma_s) < \infty$. We say that S is a (closed) dyadic square in $\mathbb{H}$ if it is of the form

$$S = \left\{ x + iy : \frac{j}{2^n} \le x \le \frac{j+1}{2^n},\ \frac{k}{2^n} \le y \le \frac{k+1}{2^n} \right\}$$

where j, n are integers and k is a positive integer. Let

$$\nu^{\mathrm{occ}}_\epsilon(S) = \epsilon^{d-2} \int_S 1\{\mathrm{dist}(z,\gamma) \le \epsilon\}\, dA(z),$$

and note that Theorem 3.2.12 implies that

$$\lim_{\epsilon \downarrow 0} \mathbb{E}\left[\nu^{\mathrm{occ}}_\epsilon(S)\right] = \int_S \mathbb{G}(z)\, dA(z).$$

We will show that for each dyadic S, the limit $\nu^{\mathrm{occ}}(S) = \nu^{\mathrm{occ}}_{0+}(S)$ exists with probability one and in L^2. With a little more, we see that any subsequential limit ν of the measures $\nu^{\mathrm{occ}}_\epsilon$ must satisfy $\nu(\partial S) = 0$, from this see that any two limit measures agree on all open sets from which we conclude that the limit is unique and we can define

$$\nu^{\mathrm{occ}} = \lim_{\epsilon \downarrow 0} \nu^{\mathrm{occ}}_\epsilon.$$

Definition We say that an SLE_κ curve γ from 0 to ∞ in $\mathbb{H}$ has the *natural parametrization* if for each t,

$$\mathrm{Cont}_d[\gamma_t] = t, \quad d = d_\kappa = 1 + \frac{\kappa}{8}.$$

If $\kappa \geq 8$, we could say that SLE_κ has the natural parametrization if for each t, Area$[\gamma_t] = t$. In this book we will only talk about natural parametrizaton for $\kappa < 8$.

We want to extend this definition to talk about SLE_κ from z to w in D in the natural parametrization. We will take a little care to avoid pathological domains.

Definition Let $\mathcal{Q}_{\rm sc}$ denote the collection of triples (D, z, w) where D is a simply connected domain; z, w are distinct points in ∂D; $z \neq \infty$; and satisfying

- If $w \neq \infty$, then
$$\mathbb{G}_D(D; z, w) := \int_D \mathbb{G}_D(\zeta; z, w)\, dA(\zeta) < \infty. \tag{3.31}$$
- If $w = \infty$, then for every bounded set V,
$$\mathbb{G}_D(V; z, \infty) := \int_V \mathbb{G}_D(\zeta; z, \infty)\, dA(\zeta) < \infty. \tag{3.32}$$

Recall that SLE_κ from z to w in D is defined as the image of SLE_κ from 0 to ∞ under a conformal map $f : D \to f(D)$. Let us define the curve $f \circ \gamma = f \circ_d \gamma$ as follows:

$$f \circ \gamma(t) = f[\gamma(\sigma(t))]$$

where $\sigma(t)$ is defined by

$$\int_0^{\sigma(t)} |f'(\gamma(s))|^d\, ds = t.$$

In other words, the time for $f \circ \gamma$ to traverse $f[\gamma_t]$ is

$$\int_0^t |f'(\gamma(s))|^d\, ds.$$

This definition assumes that D, z are sufficiently nice so that

$$\int_0^t |f'(\gamma(s))|^d\, ds < \infty,$$

for all $t < \infty$. This will be guaranteed, for example, if for all $t < \infty$,

$$\max_{0 \leq s \leq t} |f'(\gamma(s))| < \infty.$$

In order to keep the notation short, we have decided to write $f \circ \gamma$ rather than $f \circ_d \gamma$. However, it is important to remember that the notation implicitly assumes the dimension d. When studying SLE_κ paths, we fix one value of κ, and then d is implied by $d = \min\{1 + \frac{\kappa}{8}, 2\}$. If one is only interested in the curves modulo reparametrization, then one does need to worry about d.

Here we are doing an analogue of what is done in establishing the conformal invariance of two-dimensional Brownian motion. If B_t is a complex Brownian motion and f is a conformal map, then $f \circ B = f \circ_2 B$ is also a complex Brownian motion. The 2 represents the dimension of Brownian paths, and this gives a shorthand for the change in time parametrization needed to state conformal invariance.

Proposition 3.3.2. *Suppose γ is an $SLE_\kappa(\kappa < 8)$ path from 0 to ∞ in $\mathbb{H}$ with the natural parametrization. Suppose $f : \mathbb{H} \to D$ is a conformal transformation with $f(0) = z, f(\infty) = w$ satisfying* (3.31)*. Then with probability one, for every t,*

$$\mathrm{Cont}_d[f(\gamma_t)] = \int_0^t |f'(\gamma(s))|^d \, ds.$$

In other words, the curve $\tilde{\gamma} := f \circ \gamma$ has the natural parametrization in the sense

$$\mathrm{Cont}_d[\tilde{\gamma}_t] = t.$$

Proof. We will not give the details of the proof. If f' is constant, this is immediate from the definition of the Minkowski content. Otherwise, we can use the additivity of the content to partition time $t_0 = 0 < t_1 < t_2 < \cdots < t_n = t$ so that the derivative does not change much in each interval. The condition (3.31) is used to see that the contribution of the content for the curve near the boundary (for which the derivative can be very large) is negligible. □

The total "natural time duration" T of a path is $\mathrm{Cont}_d[\gamma]$. We can see that if (3.31) holds, then

$$\mathbb{E}[T] = \mathbb{G}_D(D; z, w) < \infty.$$

In the case $z = 0, w = \infty, D = \mathbb{H}$, $T = \infty$ with probability one. but if D is a bounded domain and z, w are analytic boundary points, then $\mathbb{E}[T] < \infty$.

One major advantage of the natural parametrization over the capacity parametrization is that the time to traverse a curve γ depends only on the curve γ and not on the domain it sits.

Definition Suppose $\kappa < 8$ and $(D, z, w) \in \mathcal{Q}_{\mathrm{sc}}$.

- We define $\mu_D^{\#}(z, w)$ to be the probability measure induced by SLE_κ from z to w with the curves curves parametrized naturally.
- If $f : D \to f(D)$ is a conformal transformation and μ is a measure on naturally parametrized d-dimensional curves from z to w in D, we define the measure $f \circ \mu$ by

$$f \circ \mu[V] = f \circ_d \mu[V] = \mu\{\gamma : f \circ \gamma \in V\}.$$

  Note that there is an implicit d in the definition.

Under this definition, the conformal invariance of SLE_κ can be stated as follows.

Proposition 3.3.3. *Suppose $(D, z, w), (D_1, z_1, w_1) \in \mathcal{Q}_{\rm sc}$, and $f : D \to D_1$ is a conformal transformation with $f(z) = z_1, f(w) = w_1$. Then*

$$f \circ \mu_D^{\#}(z, w) = \mu_{D_1}^{\#}(z_1, w_1).$$

Throughout this book we use superscripts of # to indicate probability measures. As we will see, it is very natural to consider measures on paths that are not probability measures.

The measure $\mu_D^{\#}(z, w)$ is just another way to define SLE_κ in the same way that Brownian motion can be defined equivalently as the Wiener measure on continuous paths. The Minkowski content gives the convenient way to parametrize the paths for the path measure formulation.

We have used d-dimensional Minkowski content to characterize the fractal dimension of the curve. One can also use Hausdorff dimension, and this agrees. This proof also needs the two-point estimates, and so we delay it.

Theorem 3.3.4. *If $\kappa < 8$ and γ is an SLE_κ curve from 0 to ∞ in $\mathbb{H}$, then with probability one the Hausdorff dimension of the curve is $d = 1 + \frac{\kappa}{8}$.*

Proof. Consider γ restricted to a compact subset K in $\mathbb{H}$. There exist $c = c_K$ such that for $0 < \epsilon \le 1$, $z \in K$,

$$c_K^{-1}\, \epsilon^{2-d} \le \mathbb{P}\{\text{dist}(z, \gamma_\infty) \le \epsilon\} \le c_K\, \epsilon^{2-d}.$$

In Section 3.9, we prove a two-point estimate,

$$\mathbb{P}\{\text{dist}(z, \gamma_\infty) \le \epsilon,\ \text{dist}(w, \gamma_\infty) \le \epsilon\} \le c_K\, \epsilon^{2-d}\, |z - w|^{d-2}.$$

This is enough to show that the dimension is at most d with probability one, and that with positive probability it is at least d; see Section 3.10. To show that it is equal to d with probability one, we can use the fact that the dimension of $\gamma[0, 1]$ is the maximum of the dimensions of $\gamma[\frac{k-1}{n}, \frac{k}{n}], k = 1, \ldots, n$; we leave the details as the next exercise.

□

Exercise 2. *Complete the proof by showing that with probability one, for all $s < t$, $\dim_h(\gamma[s, t]) = d$.*

It is typical for random curves with Hausdorff dimension d to have zero d-Hausdorff dimension. Indeed, this true for SLE_κ curves for $\kappa < 8$ but we will not prove it here. This is different than the case of deterministic fractals such as the Koch snowflake curve. It is possible that there is a Hausdorff gauge function such that the measure with this gauge function is strictly between zero and infinity, but this is unknown.

3.4 Rate of escape for SLE

Let γ be an SLE_κ path from 0 to ∞ in $\mathbb{H}$. Let

$$\xi_s = \inf\{t : |\gamma(t)| = e^s\}.$$

Since $\mathrm{hcap}[e^s\,\overline{\mathbb{D}}_+] = e^{2s}$, we can see that $\xi_s \le e^{2s}/a$.

In this section we prove the following theorem.

Proposition 3.4.1. *For every $\kappa < 8$, there exists $c < \infty$ such that the following holds. Suppose $\eta : [0,1] \to \overline{\mathbb{H}}$ is a curve with $\eta(0+) = 1, \eta(1) = w \in \partial\mathbb{D}$ and $0 \le |\eta(t)| < 1$ for $0 < t < 1$. Let D denote the unbounded component of $\mathbb{H} \setminus \eta$. Let γ be an SLE_κ curve from w to ∞ in D. Then*

$$\mathbb{P}\{\gamma \cap \{|z| \le r\} \ne \emptyset\} \le c\, r^{4a-1}.$$

We will only give a complete proof for $\kappa \le 4$. For $4 < \kappa < 8$, we will sketch a proof (this second proof also works for $\kappa \le 4$) but we omit some details.

Corollary 3.4.2. *For every $\kappa < 8$ and $\beta > (4a-1)^{-1} = \kappa/(8-\kappa)$, with probability one, for all integers n sufficiently large,*

$$\gamma[\xi_n, \infty) \cap (n^{-\beta}\, e^n\, \overline{\mathbb{D}}_+) \ne \emptyset, \tag{3.33}$$

and for all t sufficiently large

$$|\gamma(t)| \ge t^{1/2} (\log t)^{-\beta}.$$

Proof. If E_n denotes the event in (3.33), then Proposition 3.4.1 applied to $D = H_{\xi_n}$ and scaling imply that $\mathbb{P}(E_n) = O(n^{-u})$ for some $u > 1$ and hence the first assertion follows from the Borel-Cantelli lemma. The second follows from $\xi_n \le e^{2n}/a$. □

Proof of Proposition 3.4.1 for $\kappa \le 4$. Let C denote the half-circle of radius r about the origin in $\mathbb{H}$. Let $g : D \to \mathbb{H}$ be a conformal transformation with $g(w) = 0, g(\infty) = \infty$. Note that $D \cap \partial\mathbb{D}$ consists of two circular crosscuts L_+, L_- where $g \circ L_+$ (resp., L_-) is a crosscut of $\mathbb{H}$ from 0 to a positive (resp., negative) number.

Note that $C \cap D$ consists of a finite or countably infinite number of circular crosscuts $\mathcal{L} = \{\ell_j\}$. In order for γ to hit $\{|z| \le r\}$ it must hit one of the crosscuts, and so

$$\mathbb{P}\{\gamma \cap \{|z| \le r\} \ne \emptyset\} \le \sum_{\ell \in \mathcal{L}} \mathbb{P}\{\gamma \cap \ell \ne \emptyset\}.$$

A curve from ℓ_j to infinity in D must first hit the unit circle at L_+ or L_-. We will restrict ourselves to the positive crosscuts $\mathcal{L}_+$, that is, those that go through L_+. (A similar argument will handle the negative crosscuts). Using Proposition 3.2.15, we have

$$\sum_{\ell \in \mathcal{L}_+} \mathbb{P}\{\gamma \cap \ell \ne \emptyset\} = \sum_{\ell \in \mathcal{L}_+} \mathbb{P}\{\tilde{\gamma} \cap (g \circ \ell) \ne \emptyset\} \le c \sum_{\ell \in \mathcal{L}_+} \mathcal{E}_{\mathbb{H}}(\mathbb{R}_-, g \circ \ell)^{4a-1} \le c \left[\sum_{\ell \in \mathcal{L}_+} \mathcal{E}_{\mathbb{H}}(\mathbb{R}_-, g \circ \ell) \right]^{4a-1}.$$

The last inequality uses $4a - 1 \ge 1$; this is where the assumption $\kappa \le 4$ is being used. Hence we have reduced to showing the estimate

$$\sum_{\ell \in \mathcal{L}_+} \mathcal{E}_{\mathbb{H}}(\mathbb{R}_-, g \circ \ell) \le c\, r.$$

Note that by continuity and monotonicity of excursion measure, $\mathcal{E}_{\mathbb{H}}(\mathbb{R}_-, g \circ \ell) \le \mathcal{E}_D(L_+, \ell)$. Note that

$$\mathcal{E}_D\left(L_+, \bigcup_{\ell \in \mathcal{L}_+} \ell\right) \le \mathcal{E}_{\mathbb{H}}(\partial\mathbb{D}, C) = O(r).$$

We claim (and this will finish the proof) that

$$\sum_{\ell \in \mathcal{L}_+} \mathcal{E}_{\mathbb{H}}(L_+, \ell) \le 2\,\mathcal{E}_D\left(L_+, \bigcup_{\ell \in \mathcal{L}_+} \ell\right).$$

If $z \in D$, let $f(z)$ be the probability that a Brownian motion starting at z hits $\cup_{\ell \in \mathcal{L}_+} \ell$ before leaving D or hitting L_+. Let $f_\ell(z)$ be the probability that a Brownian motion starting at z hits ℓ before leaving D or hitting L_+. Then,

$$\sum_{\ell \in \mathcal{L}_+} \mathcal{E}_{\mathbb{H}}(L_+, \ell) = \int_{L_+} \sum_{\ell \in \mathcal{L}_+} \partial_n\, f_\ell(z)\, d|z|,$$

$$\mathcal{E}_D\left(L_+, \bigcup_{\ell \in \mathcal{L}_+} \ell\right) = \int_{L_+} \partial_n\, f(z)\, d|z|.$$

where ∂_n denotes inward normal derivative. It suffices to show for each $z \in D$,

$$\sum_{\ell \in \mathcal{L}_+} f_\ell(z) \le 2 f(z).$$

Note that the left-hand side gives the *expected number* of crosscuts hit by the Brownian motion. So it suffices to show that for every z, the expected number of crosscuts hit given that at least one is hit is at most two. In turn, to show this it suffices to show that if we start a Brownain motion on a crosscut ℓ, then the probability that it will reach any other crosscut before leaving D is at most $1/2$. This last fact can be seen by noting that if we have any curve starting on the crosscut and ending on a different crosscut, there is a dual path obtained by reflecting across the circle C. These are equally likely for the Brownian motion, but simple connectedness implies that at most one of these paths is in D. □

We will need another proof to handle $4 < \kappa < 8$. We start with the notation of the proof and let $h(z) = \log g(z)$ which is a conformal transformation of D onto the doubly infinite strip $S = \{x + iy : 0 < y < \pi\}$ sending 0 to $-\infty$ and ∞ to $+\infty$. Here $-\infty, \infty$ refer to the left and right boundary points of S, respectively. For $\ell \in \mathcal{L}_+$, $h \circ \ell$ is a crosscut of S with both endpoints on the real line. Let

$$K = \bigcup_{\ell \in \mathcal{L}_+} (h \circ \ell).$$

Proposition 3.2.15 and conformal invariance show that the probability that an SLE_κ path from $-\infty$ to ∞ in S hits one of the crosscuts is bounded above by a constant times

$$\sum_{\ell \in \mathcal{L}_+} [\mathrm{diam}(h \circ \ell)]^{4a-1}.$$

This estimate is not sufficient. However, we not that if $U_1, U_2, \ldots$ is any cover of K with balls centereed on the real line, then the probability of hitting K is bounded above by the probability of hitting $\bigcup U_n$ which in turn is bounded by a constant times

$$\sum_{n=1}^{\infty} \operatorname{diam}(U_n)^{4a-1}. \tag{3.34}$$

If we ease the restriction that the balls be centered on the real line, one can only reduce this quantity by a multiplicative constant and so we get that the probability of hitting K is bounded above by a constant times the infimum of the quantity in (3.34) where the infimum is over all covers of K. This infimum is called the $(4a-1)$-Hausdorff content.

The problem then reduces to an estimate for the content. We omit it here because we will not need the proposition for $4 < \kappa < 8$.

3.5 SLE in a smaller domain

3.5.1 Introduction

We will study chordal SLE_κ from 0 to ∞ in a subdomain of $\mathbb{H}$. Suppose D is a simply connected subdomain of $\mathbb{H}$ with $K := \mathbb{H} \setminus D$ bounded and $\operatorname{dist}(0, K) > 0$. Let $\Phi = \Phi_D : D \to \mathbb{H}$ denote the unique conformal transformation with $\Phi(z) = z + o(1)$ as $z \to \infty$. Let $F = \Phi^{-1}$ which takes $\mathbb{H}$ onto D. Then SLE_κ in D from 0 to infinity is defined as follows.

- Let γ be an SLE_κ from 0 to $\mathbb{H}$ and consider $F \circ \gamma$.

We can consider this as a probabiity measure on curves modulo reparametrization or for curves with the natural parametrization.

If $\kappa \leq 4$, we will see that SLE_κ in D, is absolutely continuous with respect to SLE_κ in $\mathbb{H}$. Let us see what this will imply. Since the measure is a probability measure on simple curves γ, we can parametrize the curves so that $\operatorname{hcap}[\gamma_t] = at$ and the conformal maps g_t satisfy the Loewner equation

$$\dot{g}_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{3.35}$$

where $U_t = g_t(\gamma(t))$ is the driving function. To say that this probability measure is absolutely continuous with respect to usual SLE_κ is to say that the probability distribution on the driving function is absolutely continuous with respect to a standard Brownian motion. The way to get such measures is to give a drift to the Brownian motion, that is if B_t is a standard Brownian motion, we can let

$$dU_t = A_t\, dt - dB_t, \tag{3.36}$$

where A_t is a continuous, adapted process. One way to find A_t is starting with an SLE_κ $\gamma^*(t)$ in $\mathbb{H}$ with driving function $U_t^* = -B_t^*$ and carefully using a version of Itô's formula to compute $dF(U_t^*)$. We will do essentially the equivalent, by finding which A_t to choose so that $\Phi(U_t)$ is a (time change of a) Brownian motion.

One way to put drifts onto Brownian motion is to find an appropriate local martingale and use Girsanov theorem. We will start with SLE_κ in $\mathbb{H}$ using driving function $-B_t$ and then tilt by a local martingale to get (3.36). Recall that if $x \in \mathbb{R} \setminus \overline{K}$, then $\Phi'(x)$ denotes the probability that a

Brownian excursion from x to ∞ in $\mathbb{H}$ stays in D; in particular, $0 < \Phi'(x) \le 1$. We will "tilt" by $\Phi_t'(U_t)^b$ where $b = b_\kappa$ is an exponent that we will call the boundary scaling exponent. It will be chosen so the calculations work out nicely but the choice will help describe the SLE path. More precisely, we will consider a local martingale $M_t = C_t\, \Phi_t'(U_t)^b$ where C_t is a C^1 compensator. The calculations will be valid for all κ for t sufficiently small but for $\kappa \le 4$ it will be valid for all t and we will be able to write $\mathbb{E}[C_\infty = \mathbb{E}[M_\infty] = \mathbb{E}[M_0] = \Phi'(0)^b$.

The term C_∞ will be written in terms of integrals over time of spatial derivatives, but it can also be interpreted in terms of Brownian loops. Indeed, if $\kappa \le 4$,

$$M_\infty = \exp\left\{\frac{\mathbf{c}}{2}\,\Lambda(\gamma, K; \mathbb{H})\right\}\, 1\{\gamma(0,\infty) \subset D\}, \tag{3.37}$$

where $\Lambda(\gamma, K; \mathbb{H})$ denote the Brownian loop measure of loops in $\mathbb{H}$ that intersect both γ and K. Here $\mathbf{c} = \mathbf{c}_\kappa$ is an exponent that is also computed along the way.

The strongest results hold only for $\kappa \le 4$, but some things still hold for $\kappa > 4$. If $\kappa > 4$, then there is a positive probability that the SLE_κ path hits K; if $\kappa \ge 8$, this probability equals one. If the curve hits K, then SLE_κ in D will act differently than SLE_κ in H since the curve bounces off of K.

- If $\kappa \ge 8$, SLE_κ in D is singular with respect to SLE_κ in $\mathbb{H}$.
- If $4 < \kappa < 8$, SLE_κ in D has a singular part and an absolutely continuous part with respect to SLE_κ in $\mathbb{H}$. The absolutely continuous part corresponds to curves in $\mathbb{H}$ that do not hit K,

Since they will appear in the computations, we define two parameters now.

Definition

The *boundary scaling exponent* is

$$b = b_\kappa = \frac{6-\kappa}{2\kappa} = \frac{3a-1}{2}.$$

The *central charge* $\mathbf{c} = \mathbf{c}_\kappa$ is defined by

$$\mathbf{c} = \frac{2b(3-4a)}{a} = \frac{(3a-1)\,(3-4a)}{a} = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}.$$

We make several remarks.

- The value $b = 0$ for $\kappa = 6$ gives a property we call the *locality property.*
- We have $\mathbf{c} = 0$ for $\kappa = 8/3, 4$. The value $\kappa = 8/3$ gives the measure on simple curves such that the Brownian loop term in (3.37) is trivial. This is called the *restriction property.*
- We can invert the formula for central charge

$$a = \frac{(13-\mathbf{c}) \pm \sqrt{(13-\mathbf{c})^2 - 144}}{24}, \quad \kappa = \frac{(13-\mathbf{c}) \pm \sqrt{(13-\mathbf{c})^2 - 144}}{3}.$$

Note that $\kappa \mapsto \mathbf{c}_\kappa$ is a two-to-one mapping from $(0,\infty)$ to $(-\infty, 1]$; more precisely it is two-to-one except that $\mathbf{c} = 1$ has a single preimage $\kappa = 4$.

- For any $\mathbf{c} < 1$, there are two corresponding values of κ, κ' with $\mathbf{c}_\kappa = \mathbf{c}_{\kappa'} = \mathbf{c}$. They satisfy $\kappa\,\kappa' = 16$ (as does the double root $\kappa = \kappa' = 4$ for $\mathbf{c} = 1$).
- The term central charge comes from conformal field theory. There are algebraic meanings of the term which we will not discuss here. The standard notation for central charge is c, but it would be too confusing to use this when c so often means an arbitrary constant. As a compromise, we use $\mathbf{c}$.

3.5.2 Computations and locality

Let γ be an SLE_κ curve from 0 to infinity in $\mathbb{H}$ satisfying (3.35) where $U_t = -B_t$ is a standard Brownian motion, and, as before, let H_t denote the unbounded component of $\mathbb{H} \setminus \gamma_t$. Let $T_\epsilon = \inf\{t : \mathrm{dist}(\gamma_t, \mathbb{H} \setminus D) \le \epsilon\}$ and $T = T_{0+} = \inf\{t : \gamma(t) \in \overline{\mathbb{H} \setminus D}\}$. For fixed $\epsilon > 0$, we can consider SLE_κ in $\mathbb{H}$ (from 0 to ∞) and chordal SLE_κ in D (from 0 to ∞). Note that

$$\mathbb{P}\{T = \infty\} > 0, \quad \kappa < 8,$$

$$\mathbb{P}\{T = \infty\} = 0, \quad \kappa \ge 8.$$

We will show that the two distributions are absolutely continuous if viewed as measures on curves up to time T_ϵ. Also, on the event $\{T = \infty\}$, the measures on the full curves are absolutely continuous.

We will start by doing some computations that are valid for all κ for $t < T$. We will use some deterministic results from Section 1.8.7 which we now recall. Let $D_t = g_t(D) = \mathbb{H} \setminus K_t$ where $K_t = g_t(K)$. Let $\Phi_t = \Phi_{D_t}$ which is the unique conformal transformation of D_t onto $\mathbb{H}$ with $\Phi_t(z) = z + o(1), z \to \infty$. Let $\gamma^*(t) = \Phi(\gamma(t))$ and $H_t^* = \Phi(H_t \setminus K)$. Let $g_t^* : H_t^* \to \mathbb{H}$ be the conformal transformation with $g_t^*(z) = z + o(1), z \to \infty$. Then (1.45) tells us that

$$\mathrm{hcap}[\gamma_t^*] = a \int_0^t \Phi_s'(U_s)^2\, ds, \quad \partial_t g_t^*(z) = \frac{a\,\Phi_t'(U_t)^2}{g_t^*(z) - U_t^*}, \tag{3.38}$$

where $U_t^* = \Phi_t(U_t)$. (Here we have replaced the factor 2 with the factor a because we are growing our curve so that the hcap grows at rate a.) We will use the following deterministic results which are restatements of (1.46) and (1.47), where they are stated for $a = 2$,

$$\dot\Phi_t(U_t) = \lim_{z \to U_t} \dot\Phi_t(z) = -\frac{3a}{2}\,\Phi_t''(U_t), \tag{3.39}$$

$$\dot\Phi_t'(U_t) = \frac{a}{4}\,\frac{\Phi_t''(U_t)^2}{\Phi_t'(U_t)} - \frac{2a}{3}\,\Phi_t'''(U_t). \tag{3.40}$$

We have mentioned before that the capacity parametrization of a curve depends on the domain. We have parametrized γ so that the half plane capacity of γ_t in $\mathbb{H}$ (from infinity) grows at rate a. We can interpret (3.38) as saying that the capactiy of γ_t viewed as a curve in D grows at rate $a\,\Phi_t'(U_t)^2$.

Theorem 3.5.1. *Suppose $\gamma(t), 0 \le t < T$ is a solution to the Loewner equation (3.35) where the driving function U_t satisfies*

$$dU_t = b\,\frac{\Phi_t''(U_t)}{\Phi_t'(U_t)}\,dt + dW_t, \quad t < T,$$

where W_t is a standard Brownian motion. Then γ has the distribution of SLE_κ in D up to time T. Here $b = (3a-1)/2$ is the boundary scaling exponent.

Proof. Itô's formula and (3.39) give

$$\begin{aligned} dU_t^* = d[\Phi_t(U_t)] &= \left[\dot\Phi_t(U_t) + \frac{1}{2}\,\Phi_t''(U_t)\right]\,dt + \Phi_t'(U_t)\,dU_t \\ &= -b\,\Phi_t''(U_t)\,dt + \Phi_t'(U_t)\,dU_t \\ &= \Phi_t'(U_t)\,dW_t. \end{aligned} \tag{3.41}$$

Let $\gamma^*(t) = \Phi[\gamma(t)]$ with corresponding conformal maps g_t^* as in (3.38). Let $\hat\gamma(t) = \gamma^*(\sigma(t))$ where the time change $\sigma(t)$ is chosen so that $\mathrm{hcap}[\hat\gamma_t] = at$, that is

$$\int_0^{\sigma(t)} \Phi_s'(U_s)^2\,ds = t.$$

In other words, U^* is a continuous local martingale with quadratic variation

$$\langle U^*\rangle_t = \int_0^t \Phi_s'(U_s)^2\,ds = \mathrm{hcap}[\gamma_t^*]$$

Using scaling (Proposition 3.1.1), we can see that γ^* is (a time change of) SLE_κ. □

Here we are using Itô's formula on a function $\Phi_t(\cdot)$ that is random but adapated to the Brownian motion. The results (1.46) and (1.47) show that these are C^1 in t and hence the usual proof of Itô's formula is valid. We will often write quantitites such as $\dot\Phi_t(U_t)$; this denotes the function $\dot\Phi_t(z)$ evaluated at the point $z = U_t$. The time derivative is not taken on the (in fact, not even differentiable) function U_t.

The following is an immediate corollary since $b = 0$ if $\kappa = 6$.

Theorem 3.5.2 (Locality for SLE_6)**.** *Suppose γ is an SLE_6 in $\mathbb{H}$ and T is defined as above. Then the distribution of γ_T is the same as the distribution of SLE_6 from 0 to ∞ in D stopped at the first visit to K.*

The two distributions are clearly different after time T since SLE_6 in D "bounces off" K which SLE_6 in $\mathbb{H}$ does not see K.

This is somewhat analogous to a reflected Brownian motion in a domain D. Such a Brownian motion does not "see" the boundary ∂D until it hits it, that is, the distribution of the path before the hitting time is the same whether or not the boundary is there. For other values of κ, the SLE_κ does feel the boundary before reaching it.

The discrete model that satisfies the locality property is the percolation exploration process and this is why $\kappa = 6$ is the natural choice for the scaling limit.

3.5.3 The fundamental local martingale

In the section we will do one of the early fundamental calculations about SLE. It was first done when the restriction property was conjectured for $SLE_{8/3}$; more precisely, it was conjectured that the scaling limit of self-avoiding walks would be a probability measure on simple curves and that the restriction propertry, which held trivially on the discrete level, would also hold for this process. So it was asked which valuse of κ give a process that satisfies restriction. If it did it could be shown that there must be a β such that the probability that SLE_κ does not hit K is $\Phi'(0)^\beta$. Using the conformal Markov property for SLE_κ that would imply that $M_t = \Phi'(U_t)^\beta$ was a martingale.

Using this as motivation, we will consider for all κ what happens when one tilts by $\Phi'(U_t)^\beta$. This is not always a local martingale but by choosing an approriate value of β we can also find the compensator to make this a martingale. Recall the exponents $b, \mathbf{c}$ defined above. The next proposition is an exercise in Itô's formula using the deterministic identity (3.40).

Proposition 3.5.3. *Suppose D is as above and g_t satisfies* (3.35) *where U_t is a standard Brownian motion. Let*

$$M_t = \Phi_t'(U_t)^b \, \exp\left\{ -\frac{a\mathbf{c}}{12} \int_0^t S\Phi_s(U_s)\, ds \right\}, \quad t < T,$$

where b is the boundary scaling exponent, $\mathbf{c}$ is the central charge, and S denotes the Schwarzian derivative

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3f''(z)^2}{2f'(z)^2}.$$

Then M_t is a local martingale for $t < T$ satisfying

$$dM_t = b\, \frac{\Phi_t''(U_t)}{\Phi_t'(U_t)}\, M_t\, dU_t. \tag{3.42}$$

Proof. This is a straightforward calculation. Let $\psi_t(z) = \Phi_t'(z)^\beta$. Then, using (3.40) we see that

$$\dot\psi_t(U_t) = \beta\, \psi_t(U_t)\, \frac{\dot\Phi_t'(U_t)}{\Phi_t(U_t)} = \beta\ \psi_t(U_t)\ \left[\frac{a\, \Phi_t''(U_t)^2}{4\, \Phi_t'(U_t)^2} - \frac{2a\, \Phi_t'''(U_t)}{3\Phi_t'(U_t)} \right],$$

$$\psi_t'(z) = \beta\, \psi_t(z)\, \frac{\Phi_t''(z)}{\Phi_t'(z)},$$

$$\psi_t''(z) = \beta\, \psi_t(z)\ \left[(\beta - 1)\, \frac{\Phi_t''(z)^2}{\Phi_t'(z)^2} + \frac{\Phi_t'''(z)}{\Phi_t'(z)} \right],$$

Therefore, Itô's formula gives

$$d\Phi_t'(U_t)^\beta = \beta\, \Phi_t'(U_t)^\beta\ \left(\left[\frac{a + 2\beta - 2}{4}\, \frac{\Phi_t''(U_t)^2}{\Phi_t'(U_t)^2} + \frac{3 - 4a}{6}\, \frac{\Phi_t'''(U_t)}{\Phi_t'(U_t)} \right] dt + \frac{\Phi_t''(U_t)}{\Phi_t'(U_t)}\, dU_t \right).$$

The term inside the square brackets can be written as

$$\frac{3 - 4a}{6}\, S\Phi_t(U_t) + \frac{2\beta + 1 - 3a}{4}\, \frac{\Phi_t''(U_t)^2}{\Phi_t'(U_t)^2}.$$

If we choose $\beta = b = (3a-1)/2$, then the second term vanishes and we have

$$\begin{aligned} d\Phi_t'(U_t)^b &= \Phi_t'(U_t)^b \left[b\,\frac{3-4a}{6}\, S\Phi_t(U_t)\,dt + b\,\frac{\Phi_t''(U_t)}{\Phi_t'(U_t)}\,dU_t \right] \\ &= \Phi_t'(U_t)^b \left[\frac{a\mathbf{c}}{12}\, S\Phi_t(U_t)\,dt + b\,\frac{\Phi_t''(U_t)}{\Phi_t'(U_t)}\,dU_t \right]. \end{aligned}$$

Given this, the computation for dM_t follows from the product rule. □

One surprise that comes from the calculation is that choosing $\beta = b$ not only makes the compensator term nicer, it also gives the equation (3.42) which tells us what the drift will be if we tilt by M_t and use the Girsanov theorem.

Proposition 3.5.4. *Under the assumptions of the previous theorem, if γ is an SLE_κ with driving function $U_t = -B_t$, and if we tilt by the local martingale $M_t, t < T$, then*

$$dU_t = b\,\frac{\Phi_t''(U_t)}{\Phi_t'(U_t)}\,dt + dW_t,$$

where W_t is a standard Brownian motion in the new measure. In particular, in the new measure, γ is SLE_κ in D stopped at time T.

Proof. The equation for dU_t follows immediately from the Girsanov theorem and the last comment follows by comparison with (3.5.1). □

The proof of Proposition 3.5.3 was a straightforward, if a bit tedious, calculation. We will give a more geometric and probabilistic interpretation to the compenstor term in the martingale. We start by recalling theh relationship among $S\Phi$, half-plane capacity and "Brownian bubbles". If $D = \mathbb{H} \setminus K$ is a subdomain of $\mathbb{H}$ (not necessarily simply connected) with $\operatorname{dist}(0, \mathbb{H} \setminus D) > 0$, we define

$$\Gamma(0, D) = \operatorname{hcap}[-1/K] = \lim_{y \downarrow 0} y^{-1}\, \mathbb{E}^{iy} \left[\operatorname{Im}(-1/B_\tau) \right],$$

where B_t is a standard complex Brownian motion, τ_D is the first exit time from D. Recall that $\operatorname{Im}(-1/z) = H_{\mathbb{H}}(z, 0)$ where H denotes π times the Poisson kernel. This can be interpreted as the measure of Brownian bubbles at 0 in $\mathbb{H}$ that do not stay in D. We recall (4.11).

- Suppose D is a simply connected subdomain of $\mathbb{H}$ with $\operatorname{dist}(0, \partial D) > 0$. Let $\Phi : D \to \mathbb{H}$ be a conformal transformation with $\Phi(0) = 0$. Let $D' = \{-1/z : z \in D\}$ and $K' = \mathbb{H} \setminus D'$. Then

$$\Gamma(0, D) = \operatorname{hcap}(K') = -\frac{1}{6}\, S\Phi(0).$$

We can write the martingale in terms of the Brownian loop measure discussed in Section 4.2. By (4.17), we can see that

$$a \int_0^t \Gamma(U_t, D_t)\, ds = \Lambda(\mathbb{H}; \mathbb{H} \setminus D, \gamma_t),$$

where the right-hand side is the Brownian loop measure of loops in $\mathbb{H}$ that intersect both $\mathbb{H} \setminus D$ and γ_t. This representation focuses on the first point of the curve γ hit by a loop. (The factor a

is there because we have parametrized our curve so that the capacity grows at rate a.) For this reason, we will write our local martingale as

$$M_t = \Phi_t'(U_t)^b \exp\left\{\frac{\mathbf{c}}{2}\,\Lambda(\mathbb{H};\mathbb{H}\setminus D,\gamma_t)\right\}, \quad t<T,$$

where T is the first time that γ hits K. One advantage of this representation is that looks the same for all parametrizations of the SLE curve. This local martingale is valid for all $\kappa>0$, but for $\kappa\le 4$ it is a martingale.

Theorem 3.5.5. *Suppose $0<\kappa\le 4$ and $D=\mathbb{H}\setminus K$ is a simply connected domain with K bounded and $\operatorname{dist}(0,K)>0$. Let $\Phi_D: D\to\mathbb{H}$ be the conformal transformation with $\Phi_D(z)=z+o(1), z\to\infty$. Let γ denote an SLE_κ curve from 0 to ∞ in $\mathbb{H}$. Let*

$$M_\infty = \exp\left\{\frac{\mathbf{c}}{2}\,\Lambda(\mathbb{H};K,\gamma)\right\}\,1\{\gamma\cap K=\emptyset\}.$$

Then,

$$\mathbb{E}[M_\infty] = \Phi_D'(0)^b.$$

Moreover, if Q denotes the probability measure

$$\frac{dQ}{d\mathbb{P}} = \frac{M_\infty}{M_0} = \frac{M_\infty}{\Phi_D'(0)^b},$$

then with respect to Q, γ has the distribution of SLE_κ from 0 to ∞ in D.

Proof. We first make a remark about SLE_κ in D. Let $\tilde\eta$ denote an SLE_κ from 0 to ∞ in $\mathbb{H}$ and $\eta=\Phi_D^{-1}\circ\eta$. Then η has the distribution of chordal SLE_κ from 0 to infinity in D. It follows from Corollary 3.4.2 that with probability one, $\operatorname{dist}(\eta,K)>0$ and $\eta_t\to\infty$ as $t\to\infty$. In particular, $\Gamma(\mathbb{H};K,\eta)<\infty$.

We now prove the theorem. Recall that $\Phi_t'(U_t)$ denotes the probability that a Brownian excursion from U_t to infinity in $\mathbb{H}$ stays in the domain D_t. In particular, $0<\Phi_t'(U_t)\le 1$. If $\mathbf{c}\le 0$ and $b>0$ (that is, if $0<\kappa\le 8/3$), then M_t is a bounded local martingale. However, for $8/3<\kappa\le 4$, we have $\mathbf{c}>0$ and hence M_t is not uniformly bounded. However, even if this case, we can find a sequence of stopping times T_n with $T_n\uparrow T$ such that $M_{t\wedge T_n}$ is bounded. For example, we could choose

$$T_n = n\wedge\inf\{t: M_t\ge n \text{ or } \operatorname{dist}(\gamma_t,\mathbb{H}\setminus D)\le 1/n\}.$$

For each n, there is a probability measure Q_n on paths $\gamma_t, 0\le t\le T_n$ given by

$$dQ_n = \frac{M_{T_n}}{M_0}\,d\mathbb{P}.$$

Using the Girsanov theorem and (3.42), we see that if

$$W_t = U_t - \int_0^t b\,\frac{\Phi_s''(U_s)}{\Phi_s'(U_s)}\,ds,$$

Then $W_t, 0\le t\le T_n$ is a standard Brownian motion with respect to the measure Q_n. In other words,

$$dU_t = b\,[\log\Phi_t(U_t)]'\,dt + dW_t.$$

Note that T_n does not appear in this equation so we can consider Q as a measure on paths $\gamma(s), 0 \le s < T$ satisfying the above SDE. This is exactly the equation one gets in Proposition 3.5.1. This shows that up to time T_n, the Q-distribution of γ is that of SLE_κ in D. But, as we remarked in the first paragraph of this proof, with probability one in this new measure, the path stays a positivie distance away from K and goes to infinity. When this happens, we see that $M_n \to M_\infty$ where M_∞ is as above.

To be a bit more precise, the martingale convergence theorem tells us that with $\mathbb{P}$-probability one, the limit M_{T-} exists. Also, we see that M_t is tight with respect to the measure Q; indeed we have $M_t \le 1$ for $\kappa \le 8/3$. For $8/3 < \kappa \le 4$ with $\mathbf{c} > 0$, we have with Q-probability one $T = \infty$ and

$$M_t \le \exp\left\{\frac{\mathbf{c}}{2}\Lambda(\mathbb{H}; K, \gamma)\right\}.$$

Therefore with Q-probability one the limit $\tilde{M}_\infty = \lim_{t\to\infty} M_t$ exists. We will have $\tilde{M}_\infty = M_\infty$ as above provided that $\Phi'(U_t) \to 1$. Since we know we have convergence and $\exp\left\{\frac{\mathbf{c}}{2}\Lambda(\mathbb{H}; K, \gamma_t)\right\}$ is increasing in t, it suffices to show that $\limsup \Phi'(U_t) = 1$ and we can do a deterministic estimate taking a conveninent increasing sequence of stopping times, e.g., the first time that curve reaches the circle of radius R.

□

Theorem 3.5.6. *(Restriction property) Under the assumptions of the previous theorem, the only value of $\kappa \le 4$ such that SLE_κ from 0 to ∞ in D is the same as SLE_κ from 0 to ∞ conditioned to avoid K is $\kappa = 8/3$. If γ is an $SLE_{8/3}$ path, then*

$$\mathbb{P}\{\gamma \cap K \neq \emptyset\} = \Phi'_D(0)^{5/8}.$$

For $\kappa \le 4$ with $\kappa \neq 8/3$, there is simple formula for $\mathbb{P}\{\gamma \cap K \neq \emptyset\}$. This is the primary reason that $SLE_{8/3}$ is the conjectured scaling limit for self-avoiding walks.

Here are the ideas of stochastic calculus used in the last proof.

Suppose X_t satisfies

$$dX_t = m_t\, dt + dB_t,$$

where B_t is a standard Brownian motion and m_t is adapted. Suppose Ψ_t is a positive function and T is a stopping time such that for $t < T$,

$$d\Psi_t = \Psi_t\, [A_t\, dB_t + R_t\, dt]\,.$$

Then the measure Q obtained by "tilting by Ψ" will mean the measure obtained by tilting by the local martingale

$$M_t = \Psi_t\, \exp\left\{-\int_0^t R_s\, ds\right\}$$

which satisfies

$$dM_t = A_t\, M_t\, dB_t.$$

If T is not given explicitly, we let $T = \lim_{\epsilon\downarrow 0} T_\epsilon$ where

$$T_\epsilon = \min\{t : M_t \le \epsilon \text{ or } M_t \ge 1/\epsilon\}.$$

By the Girsanov theorem,

$$dB_t = A_t\,dt + dW_t, \quad t < T,$$

where W_t is a standard Brownian motion with respect to Q. In particular,

$$dX_t = [m_t + A_t]\,dt + dW_t, \quad t < T$$

The martingale convergence theorem implies that with $\mathbb{P}$-probability 1, the limit

$$M_T = \lim_{t \uparrow T} M_t,$$

exists. If we also have tightness in the measure Q, that is, if $M^* = \sup_t M_t$,

$$\lim_{\epsilon \downarrow 0} Q\{T_\epsilon < \infty\} = 0,$$

then we also have that convergence to M_∞ with probability one with respect to Q and

$$M_\infty = \frac{dQ}{d\mathbb{P}}.$$

3.5.4 The chordal partition function (in simply connected domains)

The term partition function in statistical mechanics is ofteen used for the total mass of the measure for some weight on discrete models. It often depeends on a parameter such as inverse time and perhaps the lattice spacing. We will use the term here for the total mass of a nonprobabilty measure; in some cases it is conjected to be the normalized limit of a lattice partition function. We will define it for all κ but the meaning will be clearest for $\kappa \le 4$ as we will see in the next subsection. For now we will define it only for simply connected domains, and some smoothness assumptions are needed on the boundary. We start with the assumptions.

Definition Suppose $(D, z, w) \in \mathcal{Q}_{\rm sc}$.

- If $w \neq \infty$, and ∂D is locally analytic in neighborhoods of z, w, then $(D, z, w) \in \mathcal{Q}_{\rm a}$.
- If $D \subset \mathbb{H}$, $\mathbb{H} \setminus D$ is bounded, and ∂D is locally analytic in a neighborhood of z, then $(D, z, \infty) \in \mathcal{Q}_{\rm a}$.

Definition Suppose $0 < \kappa \le 4$ and $(D, z, w) \in \mathcal{Q}_{\rm a}$. We define the *$SLE_\kappa$ partition function* by

- If $w \neq \infty$, then
$$\Psi(D; z, w) = H_D(z, w)^b, \tag{3.43}$$
where $H_D(z, w)$ denotes the Poisson kernel normalized so that $H_{\mathbb{H}}(0, x) = x^{-2}$
- If $w = \infty$ and $\Phi_D : D \to \mathbb{H}$ is the conformal transformation with $\Phi_D(\zeta) = \zeta + o(1)$ as $\zeta \to \infty$, then
$$\Psi(D; z, \infty) = \Psi_\kappa(D; z, \infty) = |\Phi_D'(z)|^b.$$

We make some remarks.

- Although $\Psi(D;z,\infty)$, depends on κ, we will usually write them without the κ.

- It may seem to be a waste of notation to use a letter Ψ to denote the right-hand side of (3.43). However, when we extend the definition of $\Psi(D;z,w)$ to multiply connected domains we will see that (3.43) does not hold.

- If $D_1 \subset D$ and D_1, D agree in neighborhoods of z, w, then

$$H_{D_1|D}(z,w) := \frac{H_{D_1}(z,w)}{H_D(z,w)}$$

gives the probability that a Brownian excursion from z to w in D does not hit $D \setminus D_1$. Hence it is possible to define

$$\Psi(D_1 \mid D;z,w) := \frac{\Psi(D_1;z,w)}{\Psi(D;z,w)} = \left[\frac{H_{D_1}(z,w)}{H_D(z,w)}\right]^b \tag{3.44}$$

even if $D \in \mathcal{Q}_{\rm sc} \setminus \mathcal{Q}_{\rm a}$ (again, assuming that D, D_1 agree in neighborhoods of z, w).

- If $(D,z,w), (D_1.z_1,w_1) \in \mathcal{Q}_{\rm a}$ and $f : D \to D_1$ is a conformal transformation with $f(z) = z_1, f(w) = w_1$, then

$$\Psi(D;z,w) = |f'(z)|^b \, |f'(w)|^b \, \Psi(D_1;z_1,w_1). \tag{3.45}$$

If $w, w_1 \neq \infty$, this follows from the corresponding scaling property of the boundary Poisson kernel. Although the map f is not unique, the product $|f'(z)|\,|f'(w)|$ does not depend on the choice. One can choose a unique f by requiring that $|f'(w)| = 1$; for this particular, choice we get $\Psi(D;z,w) = |f'(z)|^b \, \Psi(D_1;z_1,w_1)$. This formula also holds, if either or both of w, w_1 equal infinity provided we choose the correct analogue of the derivative. It is easiest to choose the unique f such that "$|f'(w)| = 1$". If $w = w_1 = \infty$, then this becomes the familiar condition

$$f(\zeta) = \zeta + O(1), \quad \zeta \to \infty.$$

If $w \neq \infty, w_1 = \infty$ and $\mathbf{n}$ denotes the normal derivative at w, the condition becomes

$$f(w + \epsilon\, \mathbf{n}) = \frac{i}{\epsilon} + O(1),$$

and if $w = \infty, w_1 \neq \infty$, we get a similar condition on f^{-1}. We will write (3.45) with this understanding about derivatives at infinity.

- For example if $f(z) = z/(1-z)$, which is the Möbius transformation of $\mathbb{H}$ with $f(0) = 0$ and

$$f(1 + \epsilon i) = \frac{i}{\epsilon},$$

then $f'(0) = 1$ and

$$1 = H_{\mathbb{H}}(0,1)^b = \Psi_{\mathbb{H}}(0,1) = |f'(0)|^b \, \Psi_{\mathbb{H}}(0,\infty) = \Psi_{\mathbb{H}}(0,\infty).$$

We will see an interpretation of the partition function as a total mass of a measure in the next subsection. Even when this interpretation is not available, one can use the partition function as a way to write the (local) martingale used to tilt paths to go from one type of SLE_κ to another SLE_κ. The case we have seen is comparing SLE_κ from 0 to ∞ in D to that in $\mathbb{H}$. The local martingale M_t can be written as

$$M_t = C_t \frac{\Psi(D_t; \gamma(t), \infty)}{\Psi(\mathbb{H}; \gamma(t), \infty)}, \tag{3.46}$$

where C_t is C^1 in t. In other word,

- **Different versions of SLE_κ (with the same κ) tend to be "locally absolutely continuous". One type of SLE_κ is obtained from another type by tilting locally by the ratio of the partition functions.**

Sometimes a compensator C_t will be needed but not always.

3.5.5 The SLE_κ measure for $\kappa \leq 4$

While the partition function is defined for all κ, we will see that for $\kappa \leq 4$ it can be used to redefine SLE_κ as a nonprobablity measure whose total mass is given by the partiition function. For this section we will assume that $0 < \kappa \leq 4$, and as before we will write $\mu_D^\#(z, w)$ for the probability measure associated to chordal SLE_κ in D from z to w with the curves parametrized by the Minkowski content (naturally). This assumes that $(D, z, w) \in \mathcal{Q}_{\rm sc}$. If z, w are nice boundary points, then it is convenient to consider a nonprobability measure on paths. We first define nice.

Definition Suppose $0 < \kappa \leq 4$ and $(D, z, w) \in \mathcal{Q}_{\rm a}$. The SLE_κ measure $\mu_D(z, w)$ is defined by

$$\mu_D(z, w) = \mu_{D,\kappa}(z, w) = \Psi(D, z, w)\, \mu_D^\#(z, w).$$

We make some remarks.

- As in the previous subsection, it is possible to define the ratio

$$\frac{\mu_{D_1}(z, w)[V]}{\mu_D(z, w)[V]},$$

  even if the two measures $\mu_{D_1}(z, w), \mu_D(z, w)$ are not well defined. Indeed,

$$\mu_{D_1|D}(z, w) := \frac{\mu_{D_1}(z, w)[V]}{\mu_D(z, w)[V]} = \frac{\mu_{D_1}^\#(z, w)[V]}{\mu_D^\#(z, w)[V]} \frac{\Psi(D_1; z, w)}{\Psi(D; z, w)}, \tag{3.47}$$

  where the right-hand side makes sense using (3.44).

- If $(D, z, w), (D_1.z_1, w_1) \in \mathcal{Q}_{\rm a}$ and $f : D \to D_1$ is a conformal transformation with $f(z) = z_1, f(w) = w_1$,

$$f \circ \mu_D(z, w) = |f'(z)|^b\, |f'(w)|^b\, \mu_{f(D)}(f(z), f(w)),$$

  where the derivatives are interpreted as above.

We now restate the results of the last section in terms of the measures $\mu_D(z,w)$. As a slight abuse of notation, we will write $\gamma \subset D$ to mean that the curve lies in D except perhaps the endpoints. For example, since $\kappa \leq 4$, the measure $\mu_D(z,w)$ is carried on curves γ with $\{\gamma \subset D\}$.

Theorem 3.5.7. *Suppose $0 < \kappa \leq 4$, $(D,z,w) \in \mathcal{Q}_{\rm a}$ and $D' \subset D$ is a simply connected domain that agrees with D in neighborhoods of z,w. Then $\mu_{D'}(z,w) \ll \mu_D(z,w)$ with*

$$\frac{d\mu_{D'}(z,w)}{d\mu_D(z,w)}\,(\gamma) = 1\{\gamma \subset D'\}\,\exp\left\{\frac{\mathbf{c}}{2}\,\Lambda(D;\gamma,D\setminus D')\right\}.$$

In particular, if $\kappa = 8/3$,

$$\frac{d\mu_{D'}(z,w)}{d\mu_D(z,w)}\,(\gamma) = 1\{\gamma \subset D'\}.$$

The last part of the theorem is the most succinct way of stating the *restriction property* for $SLE_{8/3}$: if $D' \subset D$, then $\mu_{D'}(z,w)$ is the restriction of $\mu_D(z,w)$ to curves that lie in D'.

We now return to the martingale in the previous section. Suppose that $D = \mathbb{H}\setminus K$ is a simply connected domain with K bounded and $\operatorname{dist}(0,K) > 0$. We can write the martingale as

$$M_t = \Phi_t'(U_t)^b\,\exp\left\{\frac{\mathbf{c}}{2}\,\Lambda(\mathbb{H};\gamma_t,K)\right\}, \quad t < T.$$

As before, we write $\mathbb{P}$ for the original probability measure and Q for the measure obtained by tilting by M_t. Since $\kappa \leq 4$, we know that with Q-probability one, $\operatorname{dist}(\gamma_\infty,K)0.$ and $\Lambda(\mathbb{H};\gamma_\infty,K) < \infty$. In particular, with $\mathbb{P}$-probability one

$$\lim_{t\uparrow T} M_t = 0,$$

on the event $\{T < \infty\}$. For this reason we can write

$$M_t = \Phi_t'(U_t)^b\,\exp\left\{\frac{\mathbf{c}}{2}\,\Lambda(\mathbb{H};\gamma_t,K)\right\}\,1\{\gamma_t \subset D\},$$

and stop the martingale at time T knowing that $M_{t\wedge T}$ is a continuous martingale. Since time changes of martingale are martingales (under suitable conditions), it does not matter how we parametrize γ_t. Let us assume that γ has the natural parametrization so that, in particular, the parametrization is the same whether we consider γ as an SLE path in $\mathbb{H}$ or an SLE in D. Note that if $t < T$, then

$$\Phi_t, (U_t)^b = \Psi(D_t;U_t,\infty) = \frac{\Psi(D_t;U_t,\infty)}{\Psi(\mathbb{H},U_t,\infty)} = \frac{\Psi(D\setminus\gamma_t;\gamma(t),\infty)}{\Psi(\mathbb{H}\setminus\gamma_t;\gamma(t),\infty)}.$$

The numerator and the denominator of the expression on the right do not technically make sense, but the ratio is well defined by (3.44). Formally, we can write

$$\frac{\Psi(D\setminus\gamma_t;\gamma(t),\infty)}{\Psi(\mathbb{H}\setminus\gamma_t;\gamma(t),\infty)} = \frac{|g_t'(\gamma(t))|^b\,|g_t'(\infty)|^b\,\Psi(D_t;U_t;\infty)}{|g_t'(\gamma(t))|^b\,|g_t'(\infty)|^b\,\Psi_(\mathbb{H};U_t.\infty)} = \Psi(D_t;U_t,\infty),$$

if we are willing to cancel the $|g_t'(\gamma(t))|^b$ even though this derivative does not exist. We will reiterate this as a proposition, replacing $(\mathbb{H},D)$ with a more general (D,D_1).

Proposition 3.5.8. *Suppose $(D,z,w) \in \mathcal{Q}_{\rm a}$ and D_1 is a simply connected subdomain of D that agrees with D in neighborhoods of z,w. Suppose S is a stopping time and $\mu_{D_1}(z,w), \mu_D(z,w)$ are viewed as measures on the stopped paths $\gamma(t), 0 \le t \le S$. Then*

$$\frac{d\mu_{D_1|D}(z,w)}{d\mu_D(z,w)}\,(\gamma_S) = \Psi[D_1 \setminus \gamma_S \mid D \setminus \gamma_S; \gamma(S), w]\, \exp\left\{\frac{\mathbf{c}}{2}\,\Lambda(D;\gamma_S, D\setminus D_1)\right\}\, 1\{\gamma_S \subset D_1\}.$$

This proposition does not care how the curves were originally parametrized since it refers to the stopping time S. The assumption $(D,z,w) \in \mathcal{Q}_{\rm a}$ can be replaced with $(D,z,w) \in \mathcal{Q}_{\rm sc}$ provided that we interpret the left-hand side as in (3.47).

Proposition 3.5.9. *Suppose $(D,z,w), (D_1,z,w_1) \in \mathcal{Q}_{\rm a}$ with $D_1 \subset D$ and such that D and D_1 agree in neighborhoods of z,w. Suppose S is a stopping time and $\mu_{D_1}(z,w_1), \mu_D(z,w)$ are viewed as measures on the stopped paths $\gamma(t), 0 \le t \le S$. Then*

$$\frac{d\mu_{D_1}(z,w_1)}{d\mu_D(z,w)}\,(\gamma_S) = \frac{\Psi_{D_1\setminus\gamma_S}(\gamma(S),w_1)}{\Psi_{D\setminus\gamma_S}(\gamma(S),w)}\, \exp\left\{\frac{\mathbf{c}}{2}\,\Lambda(D;\gamma_S, D\setminus D_1)\right\}\, 1\{\gamma_S \subset D_1\}.$$

3.5.6 SLE in $\mathbb{H}$ connecting two real points

In this subsection we will view SLE_κ in $\mathbb{H}$ from 0 to $x>0$ as SLE_κ from 0 to ∞ weighted by the partition function. Here we will take any $\kappa > 0$ and let $T = \inf\{t > 0 : \gamma(t) \in [x,\infty)\}$. For $t < T$, we let H_t be the unbounded component of $\mathbb{H} \setminus \gamma_t$ and we consider the local martingale

$$M_t = \frac{\Psi_{H_t}(\gamma(t),x)}{\Psi_{H_t}(\gamma(t),\infty)}, \quad t < T$$

which is in the form similar to (3.46). It may not be obvious that this is a local martingale and no compensator is needed, or even that it is well defined. To justify this, we first do the formal calculation

$$\frac{\Psi_{H_t}(\gamma(t),x)}{\Psi_{H_t}(\gamma(t),\infty)} = \frac{|g_t'(\gamma(t))|^b\, g_t'(x)^b\, \Psi_{H_t}(U_t, g_t(x))}{|g_t'(\gamma(t))|^b\, g_t'(\infty)\, \Psi_{\mathbb{H}}(U_t,\infty)} = g_t'(x)^b\, X_t^{-2b} = g_t'(x)^b\, X_t^{1-3a},$$

where $X_t = g_t(x) - U_t$. This shows that M_t is well defined and this allows us to use Itô's formula to show it is a local martingale.

Proposition 3.5.10. *If $M_t = g_t'(x)^b\, X_t^{1-3a}$, then M_t is a local martingale for $t < T$ satisfying*

$$dM_t = \frac{1-3a}{X_t}\, M_t\, dB_t, \quad t < T.$$

Proof. Itô's formula and (3.7) give

$$\begin{aligned} dX_t^{1-3a} &= X_t^{1-3a}\left[\frac{1-3a}{X_t}\, dX_t + \frac{3a(3a-1)}{2\,X_t^2}\, dt\right] \\ &= X_t^{1-3a}\left[\frac{ab}{X_t^2}\, dt + \frac{1-3a}{X_t}\, dB_t\right] \end{aligned}$$

$$\partial_t g_t'(x)^b = -\frac{ab}{X_t^2}\, dt,$$

and hence the proposition follows from the product rule. □

If we use the Girsanov theorem to tilt by M_t, we see that

$$dB_t = \frac{1-3a}{X_t}\,dt + dW_t, \quad dX_t = \frac{1-2a}{X_t}\,dt + dW_t, \tag{3.48}$$

where W_t is a standard Brownian motion in the new measure. We emphasize that this description is valid only up to time T. Although we will not prove it now, we explain what this means in the three regimes. When we talk about "time" we mean in the usual capacity parametrization such that $\mathrm{hcap}[\gamma_t] = at$.

- $0 < \kappa \le 4, a \ge 1/2$. This process stops in finite time and $\gamma(T) = x$. The amount of time to reach x is finite.
- $4 < \kappa < 8, 1/4 < a < 1/2$. The process stops in finite time but γ "overshoots" x, $\gamma(T) > x$.
- $\kappa \ge 8, a \le 1/4$. Here the process survives for infinite time. This can be understood in the plane filling nature of the path. By the time SLE_κ going to x reaches x it has to visit all other points in the plane and hence has infinite capacity.

We summarize as follows.

Theorem 3.5.11. *Suppose $0 < \kappa < 8$ and $x > 0$. There are three different ways to construct SLE_κ from 0 to x in $\mathbb{H}$ up to time $T = T_x$.*

1. *The conformal image of SLE_κ in $\mathbb{H}$ from 0 to x under the transformation Φ^{-1} where*

$$\Phi(z) = \frac{xz}{z-x}.$$

2. *The solution of the Loewner equation (3.1) where $X_t := g_t(x) - U_t$ satisfies*

$$dX_t = -\frac{2b}{X_t}\,dt + dB_t = \frac{1-3a}{X_t}\,dt + dB_t,$$

and B_t is a Brownian motion. Equivalently, U_t satisfies

$$dU_t = \frac{2a-1}{X_t}\,dt - dB_t.$$

3. *Take SLE_κ from 0 to ∞ in $\mathbb{H}$ and tilt by $H_{\mathbb{H}}(0, X_t)^b$.*

Proof. Note that Φ is the conformal transformation of $\mathbb{H}$ onto $\mathbb{H}$ with $\Phi(0) = 0, \Phi(\infty) = -x, \Phi(x) = \infty$. Let $F = \Phi^{-1}$. Let γ be an SLE_κ path from 0 to ∞ and as in Section 1.8.7, let $\gamma^*(t) = F(\gamma(t))$. As before, for $t < T$, let g_t^* be the conformal transformation of the unbounded component of $\mathbb{H} \setminus \gamma_t^*$ satisfying $g_t^*(z) = z + o(1), z \to \infty$, and let $F_t = g_t^* \circ F \circ g_t^{-1}$, $U_t^* = F_t(U_t)$. Then, Itô's formula as in (3.41) shows that

$$dU_t^* = -b\,F_t''(U_t)\,dt + F_t'(U_t)\,dU_t.$$

If we reparametrize, setting $\hat\gamma(t) = \gamma^*(\sigma(t))$ so that $\mathrm{hcap}[\hat\gamma_t] = at$, then we get

$$d\hat U_t = -b\,\frac{F''_{\sigma(t)}(\hat U_{\sigma(t)})}{F'_{\sigma(t)}(\hat U_{\sigma(t)})^2}\,dt + dW_t = b\,\frac{\hat\Phi_t''(\hat U_t)}{\hat\Phi_t'(\hat U_t)}\,dt + dW_t,$$

where $\hat{\Phi}_t = F^{-1}_{\sigma(t)}$ and W_t is a standard Brownian motion. Note that $\hat{\Phi}_t$ is a conformal transformation of $\mathbb{H}$ with $\Phi_t(\hat{g}_t(x)) = \infty$ where $\hat{g}_t = g^*_{\sigma(t)}$. There is a two-parameter family of such maps, but for each one,

$$\frac{\hat{\Phi}_t''(\hat{U}_t)}{\hat{\Phi}_t'(\hat{U}_t)} = \frac{2}{\hat{g}_t(x) - \hat{U}_t}.$$

If we let $X_t = \hat{g}_t(x) - \hat{U}_t$, then we get

$$dX_t = -\frac{2b}{X_t}\,dt + dB_t,$$

where $B_t = -W_t$. The upshot of this is that if we let g_t be the solution of the Loewner equation (3.1) with U_t satisfying

$$dU_t = \frac{2b}{X_t}\,dt - dB_t = \frac{3a-1}{X_t}\,dt - dB_t,$$

for a standard Brownian motion, then the distribution is the same as the image of SLE_κ in $\mathbb{H}$ from 0 to ∞ under F. In other words, this gives the distribution of SLE_κ from 0 to x up to time $T = \inf\{t : X_t = 0\}$. To see that this is the same as tilting by M_t, see (3.48).

This shows the equivalence of the first two definitions and the third follows from $H_{\mathbb{H}}(0,x) = x^{-2}$ and Itô's formula which shows that

$$dX_t^{-2b} = X_t^{-2b}\left[-\frac{2b}{X_t}\,dB_t + \frac{2b(2b+1)}{2X_t^2}\,dt\right].$$

□

3.6 Radial SLE

Chordal SLE_κ is a measure on curves connecting two boundary points of a domain. Radial SLE_κ is a similar construction, but in this case connecting a boundary point to an interior point. There are several ways to construct the measure. We will start with the original definition in terms of a process from the boundary of the unit disk to the origin and then we will give another construction that shows that radial and chordal SLE are locally absolutely continuous. Recall the discussion of the radial Loewner equation in Section 1.8.8.

Definition If $\kappa = 2/a > 0$, then *radial* SLE_κ is the solution to the Loewner equation

$$\dot{g}_t(z) = 2a\, g_t(z)\,\frac{e^{2iU_t} + g_t(z)}{e^{2iU_t} - g_t(z)}, \tag{3.49}$$

where $U_t = -B_t$ is a standard Brownian motion.

There exists a curve $\gamma : (0,\infty) \to \mathbb{D}$ with $\gamma(0+) = e^{2iU_t}$ such if D_t is the connected component of $\gamma \setminus \gamma_t$ containing the origin, then g_t is the unique conformal transformation of D_t on $\mathbb{D}$ with $g_t'(0) = 0, g_t'(0) > 0$. The curve has been parametrized so the $g_t'(0) = e^{2at}$. An equivalent (up to time change) definition, and the original one due to Schramm, is parametrize the curve so that $g_t'(0) = e^t$ and then to let g_t be the solution of

$$\dot{g}_t(z) = g_t(z)\,\frac{w_t + g_t(z)}{w_t - g_t(z)},$$

where w_t is a Brownian motion on the unit circle with variance parameter κ. We will use (3.49).

Recall that if h_t is defined by

$$g_t(e^{2iz}) = \exp\{2ih_t(z)\},$$

then h satisfies the equation

$$\dot{h}_t(z) = a\,\cot(h_t(z) - U_t),$$

that is, if $X_t = X_t(z) = h_t(z) - U_t$, then X_t satisfies the radial Bessel equation

$$dX_t = a\,\cot X_t\,dt + dB_t. \tag{3.50}$$

Note that if $\theta \in \mathbb{R}$, then

$$\dot{h}'_t(\theta) = -\frac{a\,h'_t(\theta)}{\sin^2(h_t(\theta) - U_t)}, \qquad h'_t(\theta) = \exp\left\{-a\int_0^t \frac{ds}{\sin^2(h_t(\theta) - U_t)}\right\},$$

and since $|g'_t(e^{2i\theta})| = h'_t(\theta)$,

$$|g'_t(e^{2i\theta})| = \exp\left\{-a\int_0^t \frac{ds}{\sin^2(h_s(\theta) - U_s)}\right\}. \tag{3.51}$$

In Section 3.5.6, two SLE_κ paths in $\mathbb{H}$ with different boundary "target points" were compared. The measures on paths were locally absolutely continuous and could be described using a local martingale obtained by the ratio of the partition functions. We will do the same thing here with one of the target points being an interior point. Along the way we will have to determine the radial partition function.

Suppose that $U_0 = 0$ and let $w_0 = e^{2i\theta_0}$ with $0 < \theta_0 < \pi$. We will compare radial SLE_κ from 0 to ∞ with chordal SLE_κ from 0 to w_0. Suppose γ is chordal SLE_κ from 0 to w_0 and we parametrized the curve using a radial parametrization with respect to 0. In other words we parametrize so that the conformal radius at time t of $\mathbb{D}\setminus\gamma_t$ with respect to the origin is e^{-2at}. Let D_t be the connected component of $\mathbb{D}\setminus\gamma_t$ containing the origin and $\hat{g}_t$ the unique conformal transformation of D_t onto $\mathbb{D}$ with $\hat{g}_t(0) = 0$ and define Θ_t by $\hat{g}_t(w_0) = e^{2i\Theta_t}$. Let T be the first time t that γ_t disconnects w_0 from the oirign We have seen in (3.12) that Θ_t satisfies

$$d\Theta_t = (1-2a)\,\cot\Theta_t\,dt + dW_t, \quad t < T$$

where W_t is a standard Brownian motion. However, if it were radial SLE, we see from (3.50) that is should satisfy

$$d\Theta_t = a\,\cot\Theta_t\,dt + dB_t. \tag{3.52}$$

To get from one to the other, we use the Girsanov theorem. We want to get a drift of $(1-3a)\,\cot\Theta_t$ and this leads us to tilt by $\sin^{3a-1}\Theta_t$. We omit the proof of the next proposition which is an exercise in Itô's formula. It uses another scaling exponent so we will define it first.

Definition The *interior scaling exponent* $\tilde{b}$ is defined by

$$\tilde{b} = \frac{1-a}{2a}\,b = \frac{\kappa-2}{4}\,b.$$

Proposition 3.6.1. *If*

$$M_t = e^{2a\hat{b}t} \exp\left\{ ab \int_0^t \frac{ds}{\sin^2 \Theta_s} \right\} [\sin \Theta_t]^{3a-1} = e^{2a\hat{b}t} \, |g_t'(w_0)|^{-b} \, [\sin \Theta_t]^{3a-1}, \tag{3.53}$$

where $\hat{b} = b(\kappa - 4)/2$ is the interior scaling exponent, then M_t is a local martingale satisfying

$$dM_t = [\cot \Theta_t]^{3a-1} \, M_t \, dB_t.$$

Proof. The first equality is an exercise in Itô's formula which we omit. The second follows from (3.51). □

We can think of this proposition as the computation of the interior exponent $\tilde{b}$; it is the unique number such that M_t as defined in (3.53) is a local martingale.

Girsanov's theorem now shows that radial SLE_κ can be obtained from chordal SLE_κ up to a stopping time by tilting by M_t.

Definition Suppose D is a simply connected domain, z a locally analytic point on ∂D; and $\zeta \in D$. The *radial partition function* $\Psi(D; z, \zeta)$ is defined by $\Psi(\mathbb{D}; 1, 0) = 1$ and the scaling rule: if $f : D \to f(D)$ is a conformal transformation,

$$\Psi_D(z, \zeta) = \Psi(D; z, \zeta) = |f'(z)|^b \, |f'(\zeta)|^{\tilde{b}} \, \Psi(f(D); f(z), f(\zeta)).$$

We will give another interpretation which helps understand how to view this local martingale. Recall that the partition function for chordal SLE_κ from 1 to w_0 is $H_{\mathbb{D}}(1, w_0)^b$ which (see Section 1.11.1) is a constant times $[\sin \theta]^{-2b} = \sin^{1-3a} \theta$. Hence we can write

$$\frac{M_t}{M_0} = \frac{|g_t'(0)|^{\tilde{b}} \, \Psi_{\mathbb{D}}(1, w_0)}{|g_t'(w_0)|^b \, \Psi_{\mathbb{D}}(1, w_t)} = \frac{\Psi_{D_t}(\gamma(t), 0)}{\Psi_{D_t}(\gamma(t), w_0)}.$$

Proposition 3.6.2. *Suppose γ is a chordal SLE_κ curve with distribution $\mu_{\mathbb{D}}^{\#}(1, w)$ where $w = e^{2i\Theta_0}, 0 < \Theta_0 < \pi$. Let $\epsilon > 0$ and let T be a stopping time such that $\operatorname{dist}(0, \gamma_t) \geq \epsilon$ and w is on the boundary of the connected component of $\mathbb{D} \setminus \gamma_T$ containing the origin. Then radial SLE_κ from 1 to 0 is $\mathbb{D}$, viewed as a measure on paths $\gamma(t), 0 \leq t \leq T$, is absolutely continuous with respect to $\mu_{\mathbb{D}}^{\#}(1, w)$ with Radon-Nikodym derivative*

$$\frac{M_T}{M_0} = |g_t'(0)|^{\tilde{b}} \, |g_t'(w)|^{-b} \, \frac{\Psi_{\mathbb{D}}(1, w)}{\Psi_{\mathbb{D}}(1, e^{2i\Theta_T})} = e^{2a\hat{b}T} \exp\left\{ ab \int_0^T \frac{ds}{\sin^2 \Theta_s} \right\} \left[\frac{\sin \Theta_T}{\sin \Theta_0} \right]^{3a-1}.$$

Recalling that $a = 1/3, b = \hat{b} = 0$ for $\kappa = 6$, we get the following corollary.

Corollary 3.6.3. *Let T be the first time t that w is not on the boundary of the connected component of $\mathbb{D} \setminus D_t$ containing the origin. Then the distribution of chordal SLE_6 and radial SLE_6 up to time T is the same.*

After time T the distributions are different because chordal SLE_6 goes toward w while radial SLE_6 goes toward the origin.

For $\kappa > 4$, we can compare radial SLE to chordal SLE only up to the point that the curve separates the boundary target point from the origin. However, at this time, we can choose a new boundary point in the domain D_T and compare to that, stopping when the curve close the new point. Alternatively, we can go until we almost separate the boundary point from the interior point and at that time change to a new target point. The key point is that the path is absolutely continuous with respect to some chordal SLE_κ path and almost sure facts about chordal paths also hold for radial paths. For example, radial SLE_κ paths have fractal dimension $d = 1 + \frac{\kappa}{8}$ and can be parametrized naturally (by d-dimensional Minkowski content). The following can be proved but we omit the proofs.

Proposition 3.6.4.

1. *The radial Green's function*

$$\mathbb{G}_{\mathbb{D}}(\zeta; 1, 0) = \lim_{r \downarrow 0} r^{d-2}\, \mathbb{P}\{\text{dist}(\zeta, \gamma) \le r\}$$

   exists for $\zeta \in \mathbb{D} \setminus \{0\}$. Moreover,

$$\int_{\mathbb{D}} \mathbb{G}_{\mathbb{D}}(\zeta; 1, 0)\, dA(\zeta) < \infty.$$

2. *With probability one, the path is continuous at the origin.*

Definition For $\kappa \le 4$, we define the radial SLE_κ measure $\mu_D(z, w)$ where $z \in \partial D, w \in D$, by

$$\mu_D(z, w) = \Psi_D(z, w)\, \mu_D^{\#}(z, w).$$

Here $\Psi_D(z, w)$ is the radial partition function as above and $\mu_D^{\#}(z, w)$ is the corresponding probability measure. Then we can write

$$f \circ \mu_D^{\#}(z, w) = |f'(z)|^b\, |f'(w)|^{\hat{b}}\, \mu_{f(D)}(f(z), f(w)),$$

$$f \circ \mu_D(z, w) = \mu_{f(D)}^{\#}(f(z), f(w)),$$

where the first relation requires that the boundaries be nice at $z, f(z)$.

3.6.1 Fundamental local martingale and radial restriction

We are going to consider the fundamental local martingale and the restriction property for radial SLE_κ. While this could be done directly using stochastic calculus we will take a short cut that allows us to use the local martingales that we have already defined.

Suppose $D = \mathbb{D} \setminus K$ is a simply connected subdomain of the unit disk including the origin, with $\text{dist}(1, K) > 0$. Let $\Phi = \Phi_D : D \to \mathbb{D}$ be the conformal transformation with $\Phi(0) = 0, \Phi'(0) > 0$. Let $\gamma(t)$ be a radial SLE_κ path from 1 to 0 in $\mathbb{D}$ and let $T = \inf\{t : \text{dist}(\gamma(t), K) = 0\}$. It will not be important for use the exact parametrization; it can be by capacity or by Minkowski content.

Let $D_t = g_t(D)$. Let $\gamma^*(t) = \Phi(\gamma(t))$ and $\Phi_t = g_t^* \circ \Phi \circ g_t^{-1}$. Note that $\Phi_t : D_t \to \mathbb{D}$ is the unique conformal transformation with $\Phi_t(0) = 0, \Phi_t'(0) > 0$.

As in the chordal case, we will consider the ratio of partition functions,

$$\frac{\Psi_{D\setminus\gamma_t}(\gamma(t),0)}{\Psi_{\mathbb{D}\setminus\gamma_t}(\gamma(t),0)} = \frac{|g_t'(\gamma_t)|^b\,|g_t'(0)|^{\tilde{b}}\,\Psi_{D_t}(z_t,0)}{|g_t'(\gamma_t)|^b\,|g_t'(0)|^{\tilde{b}}\Psi_{\mathbb{D}}(z_t,0)} = \Psi_{D_t}(z_t,0) = |\Phi_t'(z_t)|^b\,\Phi_t'(0)^{\tilde{b}}.$$

where $z_t = g_t(\gamma(t))$. We can also, at least for small t, take another boundary point w_0, let $w_t = g_t(w_0)$ and write

$$\frac{\Psi_{D\setminus\gamma_t}(\gamma(t),0)}{\Psi_{\mathbb{D}\setminus\gamma_t}(\gamma(t),0)} = N_t\,R_t\,\tilde{N}_t,$$

where

$$N_t = \frac{\Psi_{D\setminus\gamma_t}(\gamma(t),0)}{\Psi_{D\setminus\gamma_t}(\gamma(t),w_t)}, \quad R_t = \frac{\Psi_{D\setminus\gamma_t}(\gamma(t),w_t)}{\Psi_{\mathbb{D}\setminus\gamma_t}(\gamma(t),w_t)}, \quad \tilde{N}_t = \frac{\Psi_{\mathbb{D}\setminus\gamma_t}(\gamma(t),w_t)}{\Psi_{\mathbb{D}\setminus\gamma_t}(\gamma(t),0)}.$$

We then do a series of tilting starting with the probability measure for radial SLE_κ from 1 to 0 in $\mathbb{D}$.

- In this measure, $\tilde{N}_t$ is a local martingale, and if we tilt by this we get chordal SLE_κ from 1 to w_0.
- If we are in the new measure, and

  $$\tilde{M}_t = 1\{\gamma_t \subset D\}\,\exp\left\{\frac{\mathbf{c}}{2}\,\lambda_{\mathbb{D}}(\gamma_t,K)\right\},$$

  , then $\tilde{M}_t$ is a local martingale in the new measure and tilting by $\tilde{M}_t$ gives a new measure that is chordal SLE_κ from 1 to w in $\mathbb{D}$.
- Finally in this newer measure, N_t is a local martingale, and tilting by N_t yields SLE_κ from 1 to 0 in D.

Combining this we see that if

$$M_t = N_t\,\tilde{M}_t\,\tilde{N}_t = \{\gamma_t \subset D\}\,\exp\left\{\frac{\mathbf{c}}{2}\,\lambda_{\mathbb{D}}(\gamma_t,K)\right\}\,|\Phi_t'(z_t)|^b\,\Phi_t'(0^{\tilde{b}},$$

then M_t is a local martingale and tilting by M_t gives SLE_κ in D from 1 to 0. We summarize here. We state the theorem for radial SLE_κ from 1 to 0 in $\mathbb{D}$ but it clearly applies to other domains by conformal invariance. We have stated the proposition so that the parametrization of the curve is not important.

Proposition 3.6.5. . *Suppose D is a simply connected subdomain of $\mathbb{D}$ with $0 \in D$ and $\operatorname{dist}(1, \mathbb{D}\setminus D) > 0$. Let $\gamma(t)$ be a radial SLE_κ path from 1 to 0 and let T be the first time that γ hits $\overline{\mathbb{D}\setminus D}$. Let $D_t = g_t(D), z_t = g_t(\gamma(t))$, and let Φ_t denote the conformal transformation of D_t onto $\mathbb{D}$ with $\Phi_t(0) = 0, \Phi_t'(0) > 0$. Then,*

$$M_t = \{\gamma_t \subset D\}\,\exp\left\{\frac{\mathbf{c}}{2}\,\lambda_{\mathbb{D}}(\gamma_t,K)\right\}\,|\Phi_t'(z_t)|^b\,\Phi_t'(0)^{\tilde{b}},$$

is a local martingale for $t < T$. If one tilts by M_t, then in the new measure $\gamma(t), 0 \le t < T$ is an SLE_κ from 1 to 0 in D.

If $\kappa \le 4$, then $T = \infty$ in the new measure and

$$\mathbb{E}[M_\infty] = \mathbb{E}\left[\{\gamma_t \subset D\}\,\exp\left\{\frac{\mathbf{c}}{2}\,\lambda_{\mathbb{D}}(\gamma,K)\right\}\right] = \Phi_0'(0)^{\tilde{b}}\,\Phi_0'(1)^b.$$

3.7 Computing an expectation

Here we use stochastic calculus to compute an expectation for chordal SLE with $\kappa < 8$.

Theorem 3.7.1. *If $\kappa < 8$, $0 < x < y$, $\lambda \geq -(a-\frac{1}{2})^2/2a$, and γ denotes an SLE_κ path from 0 to ∞ in $\mathbb{H}$, then*

$$\mathbb{E}\left[\left(\frac{H_{\mathbb{H}\setminus\gamma}(x,y)}{H_{\mathbb{H}}(x,y)}\right)^{\lambda}; H_{\mathbb{H}\setminus\gamma(x,y)} > 0\right] = \frac{\Gamma(2a)\,\Gamma(4a+2r-1)}{\Gamma(2r+2a)\,\Gamma(4a-1)}\,(x/y)^r\,F(2r,1-2a,2r+2a;x/y),$$

where

$$r = \frac{1}{2} - a + \sqrt{\left(a-\frac{1}{2}\right)^2 + 2\lambda a},$$

is the larger root of the equation $r(r-1)+2ar-2a\lambda = 0$ and $F = {}_2F_1$ denotes the hypergeometric function.

An important case is $\lambda = b = (3a-1)/2$ for which $r = a$ and

$$\mathbb{E}\left[\left(\frac{H_{\mathbb{H}\setminus\gamma}(x,y)}{H_{\mathbb{H}}(x,y)}\right)^{b}; H_{\mathbb{H}\setminus\gamma(x,y)} > 0\right] = \frac{\Gamma(2a)\,\Gamma(6a-1)}{\Gamma(4a)\,\Gamma(4a-1)}\,(x/y)^a\,F(2a,1-2a,4a;x/y).$$

It is instructive to see what happens as $\kappa \to 0$, that is, as $a \to \infty$. In this case, if $u = a - \frac{1}{2}$,

$$r = -u + u\sqrt{1 + \frac{2\lambda(u+\frac{1}{2})}{u^2}} = \lambda + o(1).$$

As $\kappa \to 0$, we expect that $\mathbb{H}\setminus\gamma$ has a "limit" of D, the northeast quadrant, since γ has a limit of the imaginary axis. Using conformal invariance, one can see that

$$H_D(x,1) \sim x\,H_{\mathbb{H}}(x,1), \quad x \downarrow 0.$$

Proof. Note that $H_{\mathbb{H}\setminus\gamma}(x,y)/H_{\mathbb{H}}(x,y)$ denotes the probability that a Brownian excursion from x to y in $\mathbb{H}$ stays in $\mathbb{H}\setminus\gamma$. Scaling shows that its distribution depends only on the ratio x/y so without loss of generality, we will assume that $y = 1$. We will use

$$r + a \geq \frac{1}{2}.$$

Let $X_t = g_t(x) - U_t, Y_t = g_t(1) - U_t$. Let $K_t = K_t(x,1)$ be the probability that a Brownian excursion from x to 1 in $\mathbb{H}$ stays in $\mathbb{H}\setminus\gamma_t$, that is,

$$K_t(x,y) = \frac{H_{\mathbb{H}\setminus\gamma_t}(x,y)}{H_{\mathbb{H}}(x,y)} = (y-x)^2\,H_{\mathbb{H}\setminus\gamma_t}(x,y) = K_0(x,y)\,\frac{g_t'(x)\,g_t'(y)}{(Y_t-X_t)^2}.$$

We need to compute $\phi(x) = \phi_{\lambda,\kappa}(x) = \mathbb{E}\left[K_\infty^\lambda; K_\infty > 0\right]$.

The Loewner equations give

$$d[\log X_t] = \frac{a - \frac{1}{2}}{X_t^2} + \frac{1}{X_t}\, dB_t, \quad \partial_t g_t'(x)^\lambda = -\frac{a\lambda g_t'(x)}{X_t^2},$$

and similarly for Y_t. If $Q_t = X_t/Y_t, L_t = \log Q_t$, we have

$$\begin{aligned} dL_t &= \left[\frac{a-\frac{1}{2}}{X_t^2} - \frac{a-\frac{1}{2}}{Y_t^2}\right] dt + \frac{Y_t - X_t}{X_t Y_t}\, dB_t. \\ &= \frac{a-\frac{1}{2}}{Y_t^2}\,\frac{1-Q_t^2}{Q_t^2}\, dt + \frac{1}{Y_t}\,\frac{1-Q_t}{Q_t}\, dB_t \\ &= \frac{a-\frac{1}{2}}{Y_t^2}\left[\frac{1-Q_t}{Q_t}\right]^2 \frac{1+Q_t}{1-Q_t}\, dt + \frac{1}{Y_t}\,\frac{1-Q_t}{Q_t}\, dB_t, \end{aligned}$$

and,

$$\begin{aligned} \partial_t K_t^\lambda &= a\lambda K_t \left[\frac{2}{X_t Y_t} - \frac{1}{X_t^2} - \frac{1}{Y_t^2}\right] \\ &= -\frac{a\lambda(1-Q_t)^2}{Y_t^2\, Q_t^2}. \end{aligned}$$

This suggests a time change. Let $\hat{X}_t = X_{s(t)}$, and $\hat{Y}_t, \hat{Q}_t, \hat{L}_t$ similarly where

$$\dot{s}(t) = \left[\frac{\hat{Q}_t\, \hat{Y}_t}{1-\hat{Q}_t}\right]^2.$$

Then $\hat{K}_t^\lambda = e^{-a\lambda t}$ and

$$d\hat{L}_t = (a - \frac{1}{2})\,\frac{1+\hat{Q}_t}{1-\hat{Q}_t}\, dt + dW_t,$$

for a standard Brownian motion W_t. Since $\hat{Q}_t = e^{\hat{L}_t}$, Itô's formula gives

$$\begin{aligned} d\hat{Q}_t = \hat{Q}_t\left[d\hat{L}_t + \frac{1}{2}dt\right] &= \hat{Q}_t\left[[(a-\frac{1}{2})\frac{1+\hat{Q}_t}{1-\hat{Q}_t} + \frac{1}{2}]\, dt + dW_t\right] \\ &= \hat{Q}_t\left[\left(a + \frac{(2a-1)\hat{Q}_t}{1-\hat{Q}_t}\right) dt + dW_t\right] \end{aligned}$$

We now use standard methods. Note that

$$\hat{M}_t := \mathbb{E}\left[\hat{K}_\infty^\lambda\, 1\{\hat{K}_\infty > 0\} \mid \gamma_{\sigma(t)}\right] = \hat{K}_t^\lambda\, 1\{\hat{K}_t > \infty\}\, \phi(\hat{Q}_t).$$

The left-hand side is a martingale and hence the dt term on the right-hand side must vanish giving the differential equation

$$x^2\, \phi''(x) + 2x\left(a + \frac{(2a-1)x}{1-x}\right) \phi'(x) - 2a\lambda\, \phi(x) = 0. \tag{3.54}$$

To solve this equation, we first consider only the largest order terms near the origin in (3.54) to get the equation

$$x^2 f'' + 2axf' - 2a\lambda f = 0. \tag{3.55}$$

Since $r(r-1) + 2ar - 2a\lambda = 0$, we see that x^r is a solution to (3.55). This observation leads us to try solutions of the form $\phi(x) = x^r\, h(x)$ for (3.54). Since

$$\phi'(x) = x^r\, h'(x) + rx^{r-1}h(x),$$

$$\phi''(x) = x^r\, h''(x) + 2rx^{r-1}h'(x) + r(r-1)x^{r-2}\, h(x),$$

we can plug in to get

$$h''(x) + 2\left[\frac{r+a}{x} + \frac{2a-1}{1-x}\right] h'(x) - \frac{r(2-4a)}{x(1-x)}\, h(x) = 0,$$

$$x(1-x)\, h''(x) + [2(r+a) - (2r-2a+2)x]\, h'(x) - 2r(1-2a)\, h(x) = 0.$$

This is a hypergeometric equation with $\alpha = 2r, \beta = 1-2a, \gamma = 2r+2a$. Note that $\gamma - \alpha - \beta = 4a - 1 > 0$. One solution to this equation is $F(x) := F(2r, 1-2a, 2r+2a; x)$. Two important fact about this function (which use $\gamma > \alpha + \beta$) are the following.

- As $x \uparrow 1$,
$$F(1) = \lim_{x\uparrow 1} F(x) = \frac{\Gamma(\gamma)\,\Gamma(\gamma - (\alpha+\beta))}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} = \frac{\Gamma(2r+2a), \Gamma(4a-1)}{\Gamma(2a)\,\Gamma(2r+4a-1)}.$$
See [Lebedev, (9.3.1)].

- There exists $u < 1$ such that as $x \uparrow 1$,
$$F'(x) = O((1-x)^{-u}).$$
If $\gamma > \beta > 0$, this follows from the integral representation [Lebedev, (9.14)]
$$F(z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}\,(1-t)^{\gamma-\beta-1}\,(1-tx)^{-\alpha}\, dx.$$
For other values, of (α, β, γ) with $\gamma \notin \{0, -1, -2, \ldots\}$ and $\gamma > \alpha + \beta$ it can be deduced using the recursion formula [Lebedev, (9.1.6)]
$$\gamma(\gamma+1)\, F(\alpha, \beta, \gamma; x) = \gamma(\gamma - \alpha + 1)F(\alpha, \beta+1; \gamma_2); x) \\ +\alpha[\gamma - (\gamma-\beta)x]F(\alpha+1, \beta+1, \gamma+2; x).$$

At this point, we step back and define $\phi(x) = x^r\, F(x)/F(1)$ and our goal is to show that $\phi(x) = \mathbb{E}[K_\infty^\lambda; K_\infty > 0]$. We know that ϕ is C^2 on $(0,1)$; is continuous on $[0,1]$; Also, as $x \to 0$,

$$[\log \phi(x)]' = \frac{r}{x} + O(1), \quad [\log \phi(1-x)]' = O((1-x)^{-u}),$$

for some $u < 1$. It is also true that $\phi(x) > 0$ for all $0 < x \le 1$; we will use this fact now but we will show later how this could actually be deduced separately. Let $\tau = \inf\{t : K_t = 0\}$. Itô's formula shows that $\phi(Q_t)\, K_t^\lambda$ is a local martingale for $0 \le t < \tau$.

Let us do another time change so that $R_t = \hat{Q}_{\sigma(t)}$ satisfies

$$dR_t = \left(\frac{a}{R_t} + \frac{2a-1}{1-R_t} \right) dt + dB_t,$$

for a standard Brownian motion B_t. The usual chain rule implies that

$$K_{\sigma(t)} = \exp \left\{ -a \int_0^t \frac{ds}{R_s^2} \right\}.$$

Then, $M_t := K_{\sigma(t)}^{\lambda} \, \phi(R_t)$ is a local martingale for $0 < t < T$ where $T = \inf\{t : R_t \in \{0,1\}\}$. It satisfies

$$dM_t = \frac{\phi'(R_t)}{\phi(R_t)} \, M_t \, dW_t, \quad 0 \le t < T.$$

If we use Girsnaov and tilt by the local martingale M_t, then

$$dB_t = \frac{\phi'(R_t)}{\phi(R_t)} \, dt + dB_t^*$$

where B_t^* is a Brownian motion in the new measure. In other words,

$$dR_t = \left(\frac{a+r}{R_t} + \frac{2a-1}{1-R_t} + \delta(R_t) \right) dt + dB_t^*,$$

where

$$\delta(x) = O(1), \quad \delta(1-x) = O((1-x)^{-u}) \quad x \downarrow 0.$$

Since $a + r \ge \frac{1}{2}$, then solutions to this equation never reach the origin and since $a > 1/4$ they reach 1 in finite time so that $K_\infty = K_{\sigma(T)} > 0$. Hence M_t is actually a martingale and

$$\phi(x) = \mathbb{E}^x[M_0] = \mathbb{E}^x[M_\infty] = \mathbb{E}^x[\phi(1) \, K_\infty^{\lambda}; K_\infty > 0].$$

This assumed that $\phi > 0$ on $(0,1]$ which could be derived from facts about hypergeometric functions. If we do not want to use this fact, we can first note that it is immediate that $\phi(x) > 0$ for $0 < x \le \delta$. Suppose y were the smallest positive number with $\phi(y) = 0$. Then we could do the same argument with $T = \inf\{t : R_t \in \{0, y\}\}$. Then we would have

$$\phi(x) = \mathbb{E}^x \left[\phi(R_T) \, K_{\sigma(T)}^{\lambda} \right] = 0,$$

which would be a contradiction since in the tilted measure $K_{\sigma(T)} > 0$.

□

Suppose X_t satisfies

$$dX_t = \left(\frac{1}{2X_t} - r \right) dt + dB_t,$$

where r is a constant. Then X_t never reaches the origin. One way to see this is to write

$$dX_t = \frac{1}{2X_t} \, dt + dW_t,$$

where W_t is a Brownian motion with drift. But on every compact time interval B and W are absolutely continuous.

3.7.1 Multiple SLE paths

We will define the measure on n-tuples of $SLE_\kappa, \kappa \leq 4$ paths connecting different boundary points in a domain D. The measure will lie of paths that do not intersect, but the measure is more complicated then just restricting to paths that are noninterecting.

It will be useful to set up notation. We will write

$$\mathbf{z} = (z_1, \ldots, z_n), \quad \mathbf{w} = (w_1, \ldots, w_n)$$

for $2n$ distinct points on ∂D, and we write

$$\boldsymbol{\gamma} = (\gamma^1, \ldots, \gamma^n)$$

for an n-tuple of curves with γ^j connecting z_j to w_j in D. We let $I(\boldsymbol{\gamma})$ be the indicator function that the curves are nonintersecting, that is, $\gamma^j \cap \gamma^k = \emptyset$ for $j \neq k$. We define the measure

$$\mu_D(\mathbf{z}, \mathbf{w}) = \Psi_D(\mathbf{z}, \mathbf{w})\, \mu_D^{\#}(\mathbf{z}, \mathbf{w})$$

to be the measure absolutely continuous with respect to the product measure $d[\mu_D(z_1, w_1) \times \cdots \times \mu_D(z_n, w_n)]$ with Radon-Nikodym derivative

$$\frac{d\mu_D(\mathbf{z}, \mathbf{w})}{d[\mu_D(z_1, w_1) \times \cdots \times \mu_D(z_n, w_n)]}(\boldsymbol{\gamma}) = I(\boldsymbol{\gamma}) \exp\left\{ \frac{\mathbf{c}}{2} \sum_{j=2}^{n} \Lambda(D; \gamma^j, \gamma^1 \cup \cdots \cup \gamma^{j-1}) \right\}.$$

This formulation assume that D is locally analytic at the points z_j, w_j; however, the right-hand side does not require any smoothness. In particular, we have

$$\frac{\Psi_D(\mathbf{z}, \mathbf{w})}{\prod_{j=1}^n \Psi_D(z_j, w_j)} = \mathbb{E}\left[I(\boldsymbol{\gamma}) \exp\left\{ \frac{\mathbf{c}}{2} \sum_{j=2}^{n} \Lambda(D; \gamma^j, \gamma^1 \cup \cdots \cup \gamma^{j-1}) \right\} \right],$$

where the expectation on the right is with respect to the product measure $\mu_D^{\#}(z_1, w_1) \times \cdots \times \mu_D^{\#}(z_n, w_n)$. The left-hand side is a conformal invariant and the measures satisfy the scaling rules

$$f \circ \mu_D(\mathbf{z}, \mathbf{w}) = |f(\mathbf{z})|^b\, |f'(\mathbf{w})|^b\, \mu_{f(D)}(f(\mathbf{z}), f(\mathbf{w})), \quad f \circ \mu_D^{\#}(\mathbf{z}, \mathbf{w}) = \mu_{f(D)}^{\#}(f(\mathbf{z}), f(\mathbf{w})),$$

where $f'(\mathbf{z})$ is shorthand for $f'(z_1) \cdots f'(z_n)$.

Proposition 3.7.2. *Let $\hat{\mu}$ denote the marginal distribution on γ^1 in $\mu_D(\mathbf{z}, \mathbf{w})$. Then*

$$\frac{d\hat{\mu}}{\mu_D(z_1, w_1)}(\gamma^1) = \Psi_{D \setminus \gamma^1}(\hat{\mathbf{z}}, \hat{\mathbf{w}}).$$

Moreover, given γ^1, the conditional distribution on $\hat{\boldsymbol{\gamma}}$ is that of $\mu_{D \setminus \gamma^1}^{\#}(\hat{\mathbf{z}}, \hat{\mathbf{w}})$. Here

$$\hat{\mathbf{z}} = (z_2, \ldots, z_n), \quad \hat{\mathbf{w}} = (w_2, \ldots, w_n), \quad \hat{\boldsymbol{\gamma}} = (\gamma^2, \ldots, \gamma^n).$$

The set $D \setminus \gamma^1$ has two components, but we can define $\mu_{D\setminus\gamma^1}(\hat{\mathbf{z}}, \hat{\mathbf{w}})$ in the obvious way. We will compute a crossing exponent. Let $\mathcal{R}_L$ denote the rectangle

$$\mathcal{R}_L = \{x + iy : 0 < x < L, 0 < y < \pi\}.$$

Using conformal invariance, we can see that

$$H_{\mathcal{R}_L}(iy, L + iy') \asymp e^{-L} (\sin y) (\sin y).$$

We define the crossing exponents $\xi_n = \xi_{n,\kappa}$ by $\xi_1 = b$ and the relation

$$\xi_{n+1} = b + \xi_n + r_{\xi_n} = b + \xi_n + \frac{1}{2} - a + \sqrt{\left(a - \frac{1}{2}\right)^2 + 2a\xi_n}.$$

A simple induction argument shows that

$$\xi_n = \frac{an^2 + (2a-1)n}{2} = bn + \frac{an(n-1)}{2} = bn + \frac{n(n-1)}{\kappa}.$$

The following proposition can be derived from Theorem 3.7.1.

Proposition 3.7.3. *For every n, there exists $C = C_n < \infty$ such that if*

$$\mathbf{z} = (iy_1, iy_2, \ldots, iy_n), \quad \mathbf{w} = (L + i\tilde{y}_1, \ldots, L + i\tilde{y}_n),$$

with $y_0 = 0 < y_1 < \cdots < y_n < y_{n+1} = \pi, \tilde{y}_0 = 0 < \tilde{y}_1 < \cdots < \tilde{h}_n < \tilde{y}_{n+1} = \pi$, then for all $L \geq 1$,

$$\Psi_{\mathcal{R}_L}(\mathbf{z}, \mathbf{w}) \leq C\, e^{-\xi_n L}.$$

Moreover, for every $\epsilon > 0$, there exists $c_\epsilon > 0$ such that if $y_j - y_{j-1} \geq \epsilon$ and $\tilde{y}_j - \tilde{y}_{j-1} \geq \epsilon$ for $j = 1, 2, \ldots, n-1$, then fpr $L \geq 1$,

$$\Psi_{\mathcal{R}_L}(\mathbf{z}, \mathbf{w}) \geq c_\epsilon\, e^{-\xi_n L}.$$

The last inequality can be written as

$$\Psi_{\mathcal{R}_L}(\mathbf{z}, \mathbf{w}) \asymp e^{-\zeta_n L} \prod_{j=1}^{n} \Psi_{\mathcal{R}_L}(z_j, w_j), \qquad \zeta_n = \frac{an(n-1)}{2} = \frac{n(n-1)}{\kappa}.$$

3.8 Reverse SLE_κ chains and existence of the curve

Suppose g_t is defined as in (3.1) where U_t is a standard Brownian motion. In this section we will show that for $\kappa \neq 8$, that g_t is generated by a curve. This result is also true for $\kappa = 8$ but it is harder to show and the method in this section is insufficient. Recall from Section 1.8.6 that if the curve exists, then it can be given by

$$\gamma(t) = \lim_{y \downarrow 0} g_t^{-1}(U_t + iy).$$

In order to show the existence of the lemma, we will study the derivative of g_t^{-1} near U_t. Our strategy will be to study a *reverse Loewner chain* which has the same marginal density as g_t^{-1}.

The reverse SLE_κ chain is the solution to the reverse Loewner flow

$$\partial_t h_t(z) = -\frac{a}{h_t(z) - U_t}, \quad h_0(z) = z \tag{3.56}$$

where $U_t = -B_t$ is a standard Brownian motion. For each t, $h_t : \mathbb{H} \to h_t(\mathbb{H})$ is a conformal transformation with

$$h_t(z) = z - \frac{at}{z} + O(|z|^{-2}), \quad z \to \infty.$$

It is related to the usual (forward) SLE_κ with maps g_t and driving function V_t as the next proposition shows.

Proposition 3.8.1. *For fixed $t > 0$, the distribution of h_t is the same as the distribution of the function $\hat{f}_t$ defined by*

$$\hat{f}_t(z) = g_t^{-1}(z + V_t) - V_t.$$

In particular, $\hat{f}_t'$ and $\hat{h}'$ have the same distribution.

Proof. For fixed t, the maps $h_s, 0 \le s \le t$, are exactly the reverse flow for the forward maps $\tilde{g}_s, 0 \le s \le t$ where $\tilde{g}$ has driving function $\tilde{V}_s = U_{t-s}$. We let g_t be the corresponding maps with driving function $V_s = \tilde{V}_s - \tilde{V}_0 = U_{t-s} - U_t$. □

The proposition shows that for a fixed t, h_t' and $\hat{f}_t'$ has the same distribution. However, it is not true that that the joint distribution (h_s', h_t') is the same as $(\hat{f}_s', \hat{f}_t')$.

The properties of reverse SLE_κ chains are similar but not identical to those of the forward flow. We start with the scaling relation whose proof is identical.

Proposition 3.8.2. *Suppose h_t is a reverse SLE_κ chain and $r > 0$.*

- *If $\tilde{h}_t(z) = r^{-1} h_{r^2 t}(rz)$, then $\tilde{h}_t$ is a reverse SLE_κ chain.*
- *The distribution of $h'_{r^2 t}(rz)$ is the same as that of $h_t'(z)$.*

Proof.

$$\partial_t \tilde{h}_t(z) = \frac{ar}{h_{t^2}(rz) - U_{t^2}} = \frac{a}{\tilde{h}_t(z) - \tilde{U}_t},$$

where $\tilde{U}_t = r^{-1} U_{r^2 t}$ is a standard Brownian motion. This gives the first assertion and the second follows from $\tilde{h}_t'(z) = h'_{r^2 t}(rz)$.

□

As in the forward case, if $z \in \mathbb{H}$, we let $Z_t = Z_t(z) = h_t(z) - U_t$ which in this case satisfies

$$dZ_t = -\frac{a}{Z_t}\, dt + dB_t.$$

If $Z_t = X_t + iY_t$, then

$$dX_t = -\frac{a\, X_t}{X_t^2 + Y_t^2}\, dt + dB_t, \quad \partial_t Y_t = \frac{a\, Y_t}{X_t^2 + Y_t^2}.$$

Note that the imaginary part increases, so this is valid for all times. Since

$$\partial_t Y_t^2 = \frac{2aY_t^2}{X_t^2 + Y_t^2} \le 2a,$$

we get

$$Y_t^2 \le 2at + \operatorname{Im}(z).$$

By differentiating (3.56) with respect to z, and then taking real parts, we see that

$$\partial_t \log |h_t'(z)| = \operatorname{Re}\left[\frac{a}{Z_t^2}\right] = \frac{a\,(X_t^2 - Y_t^2)}{(X_t^2 + Y_t^2)^2}.$$

We will use a particular martingale which will depend on some parameters. We define the parameters now. These values will be used in this section only.

- If $r \in \mathbb{R}$ we define

$$\lambda = \lambda(r) = r\left(1 + \frac{1}{2a}\right) - \frac{r^2}{4a}, \quad \zeta = \zeta(r) = \lambda - \frac{r}{2a} = r - \frac{r^2}{4a}. \tag{3.57}$$

Note that there is an implicit dependence on a. For future reference, we note that

$$\lambda(1) + \zeta(1) = 2, \qquad \lambda(4a) = 2, \quad \zeta(4a) = 0.$$

Proposition 3.8.3. *If $r \in \mathbb{R}$, λ, ζ are defined as in* (3.57) *and*

$$N_t = |h_t'(z)|^\lambda\, Y_t^\zeta \left[\frac{Y_t}{\sqrt{X_t^2 + Y_t^2}}\right]^{-r},$$

then N_t is a martingale. In particular, if $r \ge 0$,

$$\mathbb{E}\left[|h_t'(i)|^\lambda\, Y_t^\zeta\right] \le 1. \tag{3.58}$$

We could prove this directly but the calculation will be easier if we do a change of time in which $\log Y_t$ grows linearly. Let $\sigma(t) = \inf\{s : Y_s = e^{at}\}$, and let $\hat{X}_t = X_{\sigma(t)}$, etc. Then

$$a\, Y_{\sigma(t)} = \partial_t\, Y_{\sigma(t)} = \frac{a\, Y_{\sigma(t)}}{X_{\sigma(t)}^2 + Y_{\sigma(t)}^2}\, \dot{\sigma}(t).$$

which implies

$$\dot\sigma(t) = \hat X_t^2 + e^{2at}.$$

$$d\hat X_t = -a\hat X_t\,dt + \sqrt{\hat X_t^2 + e^{2at}}\,dW_t.$$

Define K_t by $\hat X_t = e^{at}\,K_t$ and let $L_t = \sqrt{K_t^2+1}$. Then

$$\dot\sigma(t) = e^{2at}\,L_t^2,$$

$$dK_t = e^{-at}\,d\hat X_t - a\,K_t\,dt = -2a\,K_t\,dt + L_t\,dW_t.$$

$$\partial_t \log|\hat h'(z)| = \frac{a\,(\hat X_t^2 - \hat Y_t^2)}{\hat X_t^2 + \hat Y_t^2} = a\,\left(1 - 2\,L_t^{-2}\right).$$

We do a change of variables using the following easy Itô's formula calculation.

Lemma 3.8.4. *If J_t satisfies*

$$dJ_t = -\left(\frac12 + 2a\right)\,\tanh J_t\,dt + dW_t,$$

where W_t is a standard Brownian motion, and $K_t = \sinh J_t, L_t = \cosh J_t$, then

$$dK_t = -2a\,K_t\,dt + L_t\,dW_t,$$

$$dL_t = \left[\frac{L_t}{2} - \left(\frac12 + 2a\right)\frac{K_t^2}{L_t}\right]\,dt + K_t\,dW_t.$$

This gives us another way to construct the Loewner flow starting at z. For ease, let us assume $\mathrm{Im}(z) = 1$, so that $\sigma(0) = 0$. Let W_t be a standard Brownian motion, and define J_t, K_t, L_t as above. We define

$$\sigma(t) = \int_0^t \dot\sigma(s)\,ds = \int_0^t e^{as}\,L_s^2\,ds \ge \frac1a\,\left[e^{at}-1\right].$$

$$\tau = \sigma^{-1}(s) \le \frac1a\,\log(as+1), \quad Y_s = e^{a\tau(s)}, \quad X_s = K_{\tau(s)}\,Y_s.$$

As the analogue of $\log|\hat h'(z)|$ we can just define

$$\Delta_t = \exp\left\{a\int_0^t \left(1 - 2\,L_s^{-2}\right)\,ds\right\} = \exp\left\{a\int_0^t (2\tanh^2 J_s - 1)\,ds\right\}.$$

This gives a one parameter family of martingales.

Proposition 3.8.5. *If $r \in \mathbb{R}$ and λ, ζ are as defined in* (3.57) *then*

$$M_t := \Delta_t^\lambda\,[\cosh J_t]^r\,e^{a\zeta t},$$

is a martingale satisfying

$$dM_t = r\,[\tanh J_t]\,M_t\,dW_t. \tag{3.59}$$

We note that Proposition 3.8.3 will follow immediately since $N_t = M_{\tau(t)}$ and $\tau(t)$ is uniformly bounded on every compact time interval $[0, t_0]$.

Proof. Showing that M_t is a local martingale satisfying (3.59) is a straightforward exercise using

$$\partial_t \Delta_t^\lambda = a\lambda \left[2\tanh^2 J_t - 1\right] \Delta_t^\lambda, \qquad \partial_t e^{a\zeta t} = a\zeta\, e^{a\zeta t},$$

$$\begin{aligned} dL_t^r &= dL_t^r \left[\frac{r}{L_t}\, dL_t + \frac{r(r-1)}{2L_t^2}\, d\langle L\rangle_t \right] \\ &= dL_t^r \left[\left([\frac{r(r-1)}{2} - \frac{r}{2} - 2ar]\, \tanh^2 J_t + \frac{r}{2} \right) dt + r\, \tanh J_t\, dW_t \right] \end{aligned}$$

Setting the dt term equal to zero gives the equations

$$2\lambda a + \frac{r^2}{2} - (1+2a)r = 0, \qquad -a\lambda + a\zeta + \frac{r}{2} = 0.$$

If we choose λ, ζ satisfying (3.57), then the dt term vanishes and we have a local martingale. To see that it is actually a martingale, we consider the measure obtained by tilting by M_t. Note that

$$dJ_t = -q\, \tanh J_t\, dt + dB_t, \qquad q = \frac{1}{2} + 2a - r,$$

where B_t is a Brownian motion in the new measure. This is well defined if we use stopping times, but since $|\tanh x| \le 1$, we can see from this equation that there is no blow-up or other bad behavior in finite time with respect to the new measure, and hence it is actually a martingale. □

To prove that the curve exists for $\kappa \neq 8$, we need to find an r such that the combination (r, λ, ζ) suffices. This lemma proves what we need.

Lemma 3.8.6. *If $a \neq 1/4$ ($\kappa \neq 8$), then we can find r satisfying*

$$0 < r < \frac{1}{2} + 2a, \qquad \lambda + \zeta > 2,$$

and

$$\zeta > 0 \ \text{ if } \ a > 1/4, \qquad \text{and} \qquad \zeta < 0 \ \text{ if } a < 1/4.$$

Proof. Note that

$$\lambda'(r) = 1 + \frac{1}{2a} - \frac{r}{2a}, \quad \zeta'(r) = 1 - \frac{r}{2a}.$$

- If $a > 1/4$, we use

  $$\lambda(1) + \zeta(1) = 2, \quad \zeta(1) > 0, \quad \lambda'(1) + \zeta'(1) = 2 - \frac{1}{2a} > 0.$$

  to see that we can find r slightly larger than 1 with $\lambda(r) + \zeta(r) > 2$ and $\zeta(r) > 0$.

- If $a < 1/4$, we use

  $$\lambda(4a) = 2, \quad \zeta(4a) = 0, \quad \lambda'(4a) + \zeta'(4a) = \frac{1}{2a} - 2 > 0, \quad \zeta'(4a) = -1,$$

  to see that we find r slightly larger than $4a$ with $\lambda(r) + \zeta(r) > 2$ and $\zeta(r) < 0$.

□

Proposition 3.8.7. *If $\kappa \neq 8$, then there exists $\delta > 0$ such that that the following holds with probability one for every $t_0 < \infty$ for SLE_κ. If n is sufficiently large and $t = j2^{-2n}$ is a dyadic time with $t \leq t_0$, then*

$$|f_t'(U_t + i2^{-n})| \leq 2^{(1-\delta)n}.$$

Proof. By scaling it suffices to prove the result for $t_0 = 1$. By the Borel-Cantelli lemma it suffices to find $\delta, \epsilon > 0, C < \infty$ such that for all n and all $t \leq 1$,

$$\mathbb{P}\left\{|f_t'(U_t + i2^{-n})| \geq 2^{(1-\delta)n}\right\} \leq C\, 2^{-(2+\epsilon)n}, \tag{3.60}$$

Indeed, (3.60) implies that

$$\sum_{n=1}^{\infty} \sum_{j=1}^{2^{2n}} \mathbb{P}\left\{|f_{j2^{-2n}}'(U_{j2^{-2n}} + i2^{-n})| \geq 2^{(1-\delta)n}\right\} < \infty,$$

Proposition 3.8.1 shows that (3.60) is equivalent to

$$\mathbb{P}\left\{|h_t'(i2^{-n})| \geq 2^{(1-\delta)n}\right\} \leq C\, 2^{-(2+\epsilon)n},$$

and by scaling (see Proposition 3.8.2), this will follow if we show for all t sufficiently large,

$$\mathbb{P}\left\{|h_{t^2}'(i)| \geq t^{1-\delta}\right\} \leq t^{-(2+\epsilon)}.$$

We choose r, λ, ζ as in Lemma 3.8.6 and use (3.58). Recall that $Y_{t^2} \leq \sqrt{2at^2 + 1} \leq ct$ (for $t \geq 1$ say). The Koebe 1/4-theorem implies that the image under h_{t^2} of the disk of radius 1 about i has radius at least $|h_{t^2}'(i)|/4$; hence $Y_{t^2} \geq |h_{t^2}'(i)|/4$.

- If $a \geq 1/4$, then $\zeta > 0$ and hence

$$\mathbb{E}\left[|h_{t^2}'(i)|^{\lambda+\zeta}\right] \leq 4^{\zeta}\, \mathbb{E}\left[|h_{t^2}'(i)|^{\lambda}\, Y_{t^2}^{\zeta}\right] \leq 4^{\zeta}.$$

- If $a < 1/4$, then $\zeta < 0$ and $Y_{t^2} \leq t$ for t large enough, hence for t large enough

$$\mathbb{E}\left[|h_{t^2}'(i)|^{\lambda+\zeta}\, ;\, |h_{t^2}'(i)| \geq t^{1-\delta}\right] \leq t^{-\delta\zeta}\, \mathbb{E}\left[|h_{t^2}'(i)|^{\lambda}\, Y_{t^2}^{\zeta}\right] \leq t^{-\delta\zeta}.$$

Combining these estimates, we see there exists a $C < \infty$ such that

$$\mathbb{P}\left\{|h_{t^2}'(i)| \geq t^{1-\delta}\right\} \leq C\, t^{-v}, \quad v = (1-\delta)(\lambda+\zeta) + \delta(0 \wedge \zeta).$$

We now choose δ sufficiently small so that $(1-\delta)(\lambda+\zeta) + \delta\,(0 \wedge \zeta) = 2 + 2\epsilon$ with $\epsilon > 0$. Then for all t sufficiently large

$$\mathbb{P}\left\{|h_{t^2}'(i)| \geq t^{1-\delta}\right\} \leq t^{-(2+\epsilon)}.$$

□

Proposition 3.8.8. *If $\kappa \neq 8$, then there exists $\delta > 0$ such that that the following holds with probability one for SLE_κ and $t_0 < \infty$: for all y sufficiently small,*

$$\max_{0 \le t \le t_0} |f_t'(U_t + iy)| \le y^{\delta - 1}.$$

Proof. By scaling, it suffices to prove this for $t_0 = 1$. Using the distortion estimate, it suffices to establish this for dyadic $y = 2^{-n}$. Let

$$I_n = \max\left\{|f_t'(U_t + i2^{-2n})| : t = k2^{-2n}, k = 0, 1, \ldots, 2^{2n}\right\}.$$

$$I_n^* = \max\left\{|f_t'(U_t + i2^{-2n})| : 0 \le t \le 1\right\}.$$

If $t = k2^{-2n} \le s \le t + 2^{-2n}$, then distortion estimates imply that

$$|f_s'(U_s + i2^{-n})| \le c\,[2^n|U_t - U_s| + 1]^4\ |f_s'(U_t + i2^{-n})|.$$

By the inverse Loewner equation (see Lemma 1.8.22), we have

$$|f_s'(U_t + i2^{-n})| \le c\,|f_t'(U_t + i2^{-n})|.$$

Using the modulus of continuity for Brownian motion we know that for n sufficiently large

$$|U_t - U_s| \le C\,\sqrt{n}\,2^{-n}.$$

Combining this with the distortion theorem we can see that for all n sufficiently large

$$I_n^* \le C\,n^2\,I_n.$$

Hence the result follows from the previous proposition. □

Theorem 3.8.9. *If $\kappa \neq 8$, SLE_κ paths are generated by a curve. Moreover, the curves are weakly Hölder continous of order δ for any $\delta > 0$ satisfying Proposition 3.8.8.*

Proof. We have shown that with probability one the conditions of Proposition 1.8.24 hold. Here we also use the fact that the paths of Brownian motion are weakly Hölder continuous of order $1/2$. □

3.9 Two-point estimate

An important, but technically a bit tricky, estimate used to establish fractal properties of an SLE curve is the two-point estimate for the Green's function and related probabilities. We state a version here. Throughout this section we assume that $\kappa < 8$ and allow all constants to depend on κ. Let γ denote an SLE_κ path from 0 to infinity in $\mathbb{H}$.

If $z \in \mathbb{H}$ let

$$\xi_r^z = \min\{t : |\gamma(t) - z| = e^{-r}\,\mathrm{Im}(z)\}.$$

Theorem 3.9.1. *There exists c such that the following is true. If $w, z \in \mathbb{H}$, then for all $r, s \ge 0$,*

$$\mathbb{P}\{\xi_r^z < \infty, \xi_s^w < \infty\} \le c\,e^{(d-2)(r+s)}\ \left(\frac{|w - z|}{\mathrm{Im}(z) + \mathrm{Im}(w)} \wedge 1\right)^{d-2}. \tag{3.61}$$

By scaling, it suffices to prove this when $\mathrm{Im}(w) \le \mathrm{Im}(z) = 1$ and in this case the right-hand side is comparable to

$$\mathbb{P}\{\xi_r^z < \infty\}\, \mathbb{P}\{\xi_s^w < \infty\}\, |w-z|^{d-2}.$$

In particular, if $|w-z| \asymp 1$, then the events $\{\xi_r^z < \infty\}$ and $\{\xi_s^w < \infty\}$ are "independent up to a multiplicative constant". This will allow second moments to be bounded. The hardest work is needed to establish this when $|w-z| \asymp 1$; we state this as a proposition.

Proposition 3.9.2. *For every $\delta > 0$, there exist $c < \infty$, such that the following is true. If $w, z \in \mathbb{H}$ with $\mathrm{Im}(w) \le \mathrm{Im}(z) = 1$ and $|w-z| \ge \delta$. then for all $r, s \ge 0$,*

$$\mathbb{P}\{\xi_r^z < \infty, \xi_s^w < \infty\} \le c\, e^{(d-2)(r+s)}.$$

Proof of Theorem 3.9.1 given Proposition 3.9.2. By scaling, we may assume that $\mathrm{Im}(w) < \mathrm{Im}(z) \le 1$ and define u by $|w-z| = e^{-u-4}$. If $u \le 0$, then the result follows immediately from Proposition 3.9.2. If $s \le u+4$ then the right-hand side of (3.61) is greater than $O(e^{(d-2)r})$ and hence (3.61) follows from

$$\mathbb{P}\{\xi_r^z < \infty, \xi_s^w < \infty\} \le \mathbb{P}\{\xi_r^z < \infty\} \le c\, e^{(d-2)r}.$$

Similarly, if $r \le u+4$, the one-point estimate applied to ξ_s^w gives the result.

Hence without loss of generality we assume $r, s \ge u+4 \ge 4$. We then write

$$\begin{aligned} \mathbb{P}\{\xi_r^z < \infty, \xi_s^w < \infty\} &= \mathbb{P}\{\xi_u^z < \infty\}\, \mathbb{P}\{\xi_r^z < \infty, \xi_s^w < \infty \mid \xi_u^z < \infty\} \\ &\le c\, e^{(d-2)u}\, \mathbb{P}\{\xi_r^z < \infty, \xi_s^w < \infty \mid \xi_u^z < \infty\}. \end{aligned}$$

On the event $\{\xi_u^z < \infty\}$, we consider the image under the map $g_{\xi_u^z}$. Using distortion estimates (details omitted) and Proposition 3.9.2, we get

$$\mathbb{P}\{\xi_r^z < \infty, \xi_s^w < \infty \mid \xi_u^z < \infty\} \le c\, e^{(d-2)(r-u)}\, e^{(d-2)(s-u)},$$

and hence

$$\mathbb{P}\{\xi_r^z < \infty, \xi_s^w < \infty\} \le c\, e^{(d-2)r}\, e^{(d-2)s}\, e^{(2-d)u}.$$

□

We will now discuss the proof of Proposition 3.9.2. This next lemma is important. We start with some notation. Let $w, z \in \mathbb{H}$. Let H_t denote the unbounded component of $\mathbb{H} \setminus \gamma_t$ and assume that t is a time with $z, w \in H_t$ and $|z-w| \max\{\mathrm{dist}(w, \partial H_t), \mathrm{dist}(z, \partial H_t)\}$. Let $\mathcal{B}_t^w$ denote the open disk of radius $\mathrm{dist}(w, \gamma_t)$ about w with boundary circle C_t^w. There is a unique crosscut ℓ_t^w of H_t that lies in C_t^w and such that z and w are in different components of $H_t \setminus \ell_t^w$. Let V_t^w denote the connected component of $H_t \setminus \ell_t^w$ containing w, and note that $z \notin V_t^w$. We define $\mathcal{B}_t^z, \ell_t^z, V_t^z$ analogously. A key observation is that if $\gamma(t) \in \partial V_t^w$, then the distance from γ to z cannot decrease without hitting the crosscut ℓ_t^z.

We will write

$$\xi_r^w = \inf\{t : \mathrm{dist}(w, \gamma_t) = e^{-r}\}$$

(this is slightly different than defined earlier). Note that if $\xi_r^w > 0$, then $\ell_{\xi_r^w}$ is the entire circle of radius e^{-r} about w (with a single endpoint $\gamma(t)$).

Lemma 3.9.3. *If $0 < \kappa < 8$, there exist $c < \infty$ such that the following holds. Suppose γ is an SLE_κ path from 0 to ∞ in $\mathbb{H}$, $|w - z| \geq 1$. Suppose ρ is a stopping time with $w, z \in H_\rho$ satisfying the following:*

- $|\gamma(\rho) - w| = e^{-s} = \operatorname{dist}(\gamma_\rho, w)$,
- $\operatorname{dist}(\gamma_\rho, z) = e^{-r}$,

and let V denote the connected component of $H_\rho \cap \mathcal{B}_r^z$ containing z. Let $\tau_r = \inf\{t \geq \rho : \gamma(t) \in \ell_\rho^z$ and for positive integer j, let $\tau_{r+j} = \inf\{t \geq \rho : |\gamma(t) - z| = e^{-(r+j)}\}$. Let $\tilde{\rho}_j = \inf\{t \geq \tau_{r+j} : \gamma(t) \in \ell^w_{\tau_{r+j}}\}$ Then,

$$\mathbb{P}\{\tilde{\rho}_j < \infty \mid \gamma_\rho\} \leq c\, e^{-\beta(r+s)}\, e^{-(2-d)j}, \quad \beta = \frac{4a-1}{4}.$$

With more work one can prove this result with $\beta = (4a-1)/2$ but we will not need it so we will just prove this proposition.

The basic idea of the proof is the following. If the SLE curve is near w and it has also gotten near z then at one of the following is true: it will be unlikely starting near w to get closer to z, or, if we succeed, then it will be unlikely to return close to w after we have gotten close to z. Which of these is the case depends on the location of w and z. For example, if w is near the origin and z is far away, then it is not difficult to get near z but returning to w is difficult. The opposite is true if z is near the origin and w is far away.

Proof. Let $g = g_\rho - U_\rho$ and let $\hat{\ell}^w = g(\ell_\rho^w), \hat{\ell}^z = g(\ell_\rho^z)$. Note that $\hat{\ell}^w, \hat{\ell}^z$ are disjoint crosscuts of $\mathbb{H}$ with one of the endpoints of $\hat{\ell}^w$ being the origin. For ease we will assume that ℓ^z has at least one endpoint on the positive real axis $\mathbb{R}_+$ (there is an identical argument if both endpoints are on the negative axis). By conformal invariance and the Beurling estimate,

$$\mathcal{E}_{\mathbb{H}}(\hat{\ell}^w, \hat{\ell}^z) = \mathcal{E}_{H_\rho}(\ell_\rho^w, \ell_\rho^z) \leq c\, e^{-(r+s)/2}.$$

The second inequality uses only the facts that H_ρ is simply connected, its boundary intersects C_r^w and C_r^z, and $|w - z| \geq 1$.

We split into four cases. In two of the cases we will show that

$$\mathbb{P}\{\tau_r < \infty \mid \gamma_\rho\} \leq c\, e^{-\beta(r+s)}, \tag{3.62}$$

It then follows from the interior estimate that

$$\mathbb{P}\{\tilde{\rho}_j < \infty \mid \gamma_\rho\} \leq \mathbb{P}\{\tau_r < \infty \mid \gamma_\rho\}\, \mathbb{P}\{\tau_{r+j} < \infty \mid \tau_r < \infty\} \leq c\, e^{-\beta(r+s)}\, e^{-(2-d)j}.$$

In the other two cases, we will show that

$$\mathbb{P}\{\tilde{\rho}_j < \infty \mid \tau_{r+j} < \infty\} \leq c\, e^{-\beta(r+s)}. \tag{3.63}$$

It then follows from the interior estimate that

$$\mathbb{P}\{\tilde{\rho}_j < \infty \mid \gamma_\rho\} \leq \mathbb{P}\{\tau_{r+j} < \infty \mid \gamma_\rho\}\, \mathbb{P}\{\tilde{\rho}_j < \infty \mid \tau_{r+j} < \infty\} \leq c\, e^{-(2-d)j}\, e^{-\beta(r+s)}.$$

- **Case I:** $\hat{\ell}^z$ is in the bounded component of $\mathbb{H} \setminus \hat{\ell}^w$. In this case $\hat{\ell}^w$ separates $\hat{\ell}^z$ from $\mathbb{R}_-$ and

$$\mathcal{E}_{\mathbb{H}}(\hat{\ell}^z, \mathbb{R}_-) \le \mathcal{E}_{\mathbb{H}}(\hat{\ell}^w, \hat{\ell}^z) \le c\, e^{-(r+s)/2}.$$

  Hence from the boundary estimate, we get (3.62).

- **Case II:** $\hat{\ell}^w$ is in the bounded component of $\mathbb{H} \setminus \hat{\ell}^z$; let us denote this component by U. Assume $\tau_{r+j} < \infty$, let $\tilde{H} = H_{\tau_{r+j}}$. We assume $w \in \tilde{H}$; otherwise, the result is trivial. Let V be the component of $U \cap \tilde{H}$ containing w; V is clearly bounded. Let $\tilde{\ell}$ denote the unique crosscut of $\tilde{H}$ contained in $\hat{\ell}^z$ that is also contained in ∂V. If $j = 0$, then $\gamma(\tau_r)$ is an endpoint of $\tilde{\ell}$. If $j > 0$, then $\gamma(\tau_r)$ is not on the boundary of V. In either case, we see that $\tilde{\ell}$ separates $\ell^w_{\tau_{r+j}}$ and the probability that the SLE hits $\ell^w_{\tau_{r+j}}$ after time τ_{r+j} is bounded above by

$$\mathcal{E}_{\tilde{H}}(\ell^w_{\tau_{r+j}}, \tilde{\ell}) \le \mathcal{E}_{H_\rho}(\ell^w_\rho, \ell^z_\rho) \le e^{-(r+s)/2}.$$

  Hence, using the boundary exponent, we get (3.63).

For the remaining cases we assume implicitly that neither I nor II holds. Let $\phi^z = \operatorname{diam}(\hat{\ell}^z)/\operatorname{diam}(\hat{\ell}^w)$, $\delta = \operatorname{dist}(\hat{\ell}^w, \hat{\ell}^z)/\operatorname{diam}(\hat{\ell}^w)$. If $\delta \le 1 \wedge \phi_z$, then $\mathcal{E}_{\mathbb{H}}(\hat{\ell}^w, \hat{\ell}^z) \ge c$ which is impossible for $r+s$ sufficiently large. Hence we may assume that $\delta \ge 1 \wedge \phi_z$.

- **Case III:** Neither I nor II and $\phi_z \le 1$. If $\delta \le 2$, then

$$\mathcal{E}_{\mathbb{H}}(\hat{\ell}^w, \hat{\ell}^z) \ge c\, \mathcal{E}_{\mathbb{H}}(R_-, \hat{\ell}^z).$$

  If $\delta \ge 2$, then

$$\mathcal{E}_{\mathbb{H}}(R_-, \hat{\ell}^z) \asymp \phi_z/\delta, \quad \mathcal{E}_{\mathbb{H}}(\hat{\ell}^w, \hat{\ell}^z) \asymp \frac{\phi_z}{\delta^2}.$$

  Then

$$\mathcal{E}_{\mathbb{H}}(\hat{\ell}^w, \hat{\ell}^z) \ge c\, (\phi_z/\delta)^2 \asymp \mathcal{E}_{\mathbb{H}}(R_-, \hat{\ell}^z)^2.$$

  Therefore, by the boundary estimate, (3.62) holds.

- **Case IV:** Neither I nor II and $\phi_z > 1$. Note that $\mathcal{E}_{\mathbb{H}}(\hat{\ell}^w, \hat{\ell}^z) \ge c/\delta^2$ and hence there exists c_1 such that

$$\delta \ge 2c_1\, e^{(r+s)/4}.$$

  Let C be the half-circle of radius $c_1 e^{(r+s)/4}$ about the origin and let $\eta = g^{-1}(C)$. Then η is a crosscut of H_ρ with

$$\mathcal{E}_{H_\rho}(\ell^w_\rho, \eta) = \mathcal{E}_{\mathbb{H}}(\hat{\ell}^w, C) \asymp e^{(r+s)/4}.$$

  Let $\tau = \tau_{r+j}$ and let $\tilde{H} = H_\tau$. Let U be the component of $\tilde{H} \setminus \eta$ that contains w (if $w \notin \tilde{H}$, the result is trivial). There is a unique crosscut l of $\tilde{H}$ that is a subset of η and is contained in ∂U. Note that l disconnects w from ℓ^z_ρ and that

$$\mathcal{E}_{\tilde{H}}(\ell^w_\tau, l) \le \mathcal{E}_{H_\rho}(\ell^w_\rho, \eta) \le e^{(r+s)/4}.$$

  Using the boundary exponent, we see that the probability that $\gamma[\tau, \infty)$ hits ℓ^w_τ is $O(e^{(4a-1)(r+s)/4})$ and hence (3.63) holds.

□

Proof of Theorem 3.9.1. We will consider paths that go to w first and then z; the same argument works for paths that get to z first and then w. We will consider "excursions" between points near w and near z.

Let $\rho_{s,r} = \xi^w_s \vee \xi^z_r$. We will be considering the case where s, r are positive integers (this clearly suffices for our result). We think of $\rho_{s,r}$ as first stopping at time ξ^w_s, and then, if necessary, continuing until time ξ^z_r. Using this viewpoint, on the event $\{\rho_{s,r} < \infty\}$ define the finite sequence $\mathbf{j} = (j_1, \ldots, j_m), \mathbf{k} = (k_1, \ldots, k_m)$ by the following,

- Each integer is strictly positive except perhaps k_m which can equal zero.
- $k_m = 0$ if and only if $\xi^z_r < \xi^w_s$.
- $j_1 + \cdots + j_m = s,\ k_1 + \cdots + k_m = r$
- For $q = 1, \ldots, m-1$,

$$\xi^w_{j_1+\cdots+j_q} < \xi^z_{(k_1+\cdots+k_q)\wedge r} < \xi^w_{j_1+\cdots+j_q+1},$$
$$\xi^z_{k_1+\cdots+k_{q-1}} < \xi^w_{j_1+\cdots+j_q} < \xi^z_{k_1+\cdots+k_{q-1}+1},$$

We can now write

$$\mathbb{P}\{\xi^w_s \vee \xi^z_r < \infty; \xi^w_1 < \xi^z_1\} = \sum_{m=1}^{\infty} \sum_{\mathbf{j}=(j_1,\ldots,j_m),\mathbf{k}=(k_1,\ldots,k_m)} \mathbb{P}\{\xi^w_s \vee \xi^z_r < \infty; \mathbf{j}, \mathbf{k}\}$$

where the sum is over all finite sequences $\mathbf{j} = (j_1, \ldots, j_m), \mathbf{k} = (k_1, \ldots, k_m)$ of nonnegative integers all of which are positive except perhaps k_m.

Using the proposition, we can see that there exist c, β such that for a given $m, \mathbf{j}, \mathbf{k}$,

$$e^{(2-d)(r+s)}\mathbb{P}\{\xi^w_s \vee \xi^z_r < \infty; \mathbf{j}, \mathbf{k}\} \le c^m \prod_{i=1}^{m-1} \exp\left\{-\beta(j_1 + \cdots + j_i + k_1 + \cdots + k_{i-1})\right\}.$$

Let

$$S_m = \sum c^m \prod_{i=1}^{m-1} \exp\left\{-\beta(j_1 + \cdots + j_i + k_1 + \cdots + k_{i-1})\right\}$$

where the sum is over all finite sequences $(j_1, \ldots, j_m)$, $(k_1, \ldots, k_{m-1})$ of finite integers. By focusing on the possible values of $(j_1, k_1) = (j, k)$ we get the inequality

$$S_{m+1} \le c \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} e^{-\beta(j+k)(m-1)} S_m.$$

Using this, we see that for m sufficiently large $S_{m+1} \le S_m/2$ and hence

$$\sum_{m=1}^{\infty} S_m < \infty.$$

□

3.10 Notes about fractal dimension

In this section we will prove some general facts about how to compute dimensions of random subsets. Since we do not need to use planarity, we will state our results for $\mathbb{R}^d$. Throughout this section we will make the following assumption.

- There is a fixed compact $K \subset \mathbb{R}^d$ and V is a random compact subset of K such that with probability one $\operatorname{dist}(V, \partial K) \geq 1$.

This may be more restrictive than one wants. If V is not bounded, we can use these results for the intersection of V with compact sets. We fix $0 < \alpha < d$, set $\zeta = d - \alpha$ and allow all constants to depend implicitly on α, ζ, d, K. We will consider ways to rigorously state that "V has fractal dimension α."

Let $V^\epsilon = \{x : \operatorname{dist}(x, V) \leq \epsilon\}$ denote the closed ϵ-"sausage" around V, and $\ell(V^\epsilon)$ the Lebesgue measure of V^ϵ. The sets V^ϵ increase with ϵ and $V^1 \subset K$. If V is α-dimensional, then we expect that $\ell(V^\epsilon) \approx \epsilon^{d-\alpha}$ as $\epsilon \to 0$. The upper and lower *box or Minkowski dimensions* of V are defined by

$$\underline{\dim}_b(V) = \liminf_{\epsilon \downarrow 0} \frac{\log[\epsilon^{-d}\ell(V_\epsilon)]}{\log(1/\epsilon)} \leq \limsup_{\epsilon \downarrow 0} \frac{\log[\epsilon^{-d}\ell(V_\epsilon)]}{\log(1/\epsilon)} = \overline{\dim}_b(V)$$

If the liminf and limsup agree, we call the common value $\dim_b(V)$ the *box or Minkowski dimension*. Note that we would get the same definition if we took the limit just along a geometric sequence, $\{\rho^n\}$ where $0 < \rho < 1$.

The box dimension α is often defined by saying that if N_ϵ is the minimum number of balls of radius ϵ need to cover V, then $N_\epsilon \approx \epsilon^{-\alpha}$ (with corresponding liminf and limsup versions). It is a straightfoward exercise to check that this agrees with our definition.

To define the Hausdorff dimension, we start by defining

$$\mathcal{H}^\alpha_\epsilon(V) = \inf \sum_{j=1}^{\infty} [\operatorname{diam}(U_j)]^\alpha,$$

where the infimum is over all covers of V by sets $U_1, U_2, \ldots,$ with $\operatorname{diam}(U_j) \leq \epsilon$. Since $\mathcal{H}^\alpha_\epsilon(V)$ is decreasing in ϵ, we can define the Hausdorff measure $\mathcal{H}^\alpha(V) = \mathcal{H}^\alpha_{0+}(V)$. The *Hausdorff dimension* $\dim_h(V)$ is the unique α such that

$$\mathcal{H}^\beta(V) = \begin{cases} 0, & \beta > \alpha \\ \infty, & \beta < \alpha. \end{cases}$$

This is well defined for all V.

The major difference between box and Hausdorff dimension is that the former uses cover by sets of size ϵ while the latter uses covers by sets of size *at most* ϵ. The former is more natural for numerical approximations and for comparison to lattice models, but the latter has the advantage of being defined for all sets and producing a mathematically nicer object.

It is not hard to see that $\dim_h(V) \le \underline{\dim}_b(V)$ but the inequality can be strict. We can give upper bounds for Hausdorff dimension by estimating the box dimension. Lower bounds are trickier. We will use the following fact that states that if a set supports an "α-dimensional measure", then it must be at least α-dimensional. If μ is a positive measure on $\mathbb{R}^d$ we define the β-energy by

$$\mathcal{E}_\beta(\mu) = \int\int \frac{\mu(dx)\,\mu(dy)}{|x-y|^\beta}.$$

Proposition 3.10.1. *Suppose V is a compact set and there exists a probability measure μ supported on V with $\mathcal{E}_\beta(\mu) < \infty$, then $\dim_h(V) \ge \beta$.*

For each $0 < \epsilon \le 1$, define

$$I_\epsilon(x) = \epsilon^{-\zeta}\, 1_{V^\epsilon}(x) = \epsilon^{-\zeta}\, 1\{\mathrm{dist}(x,V) \le \epsilon\},$$

and let μ_ϵ be the (random) measure whose Radon-Nikodym derivative with respect to Lebesgue measure is I_ϵ. Let

$$J_\epsilon = \mu_\epsilon(\mathbb{R}^d) = \int_K I_\epsilon(x)\,dx = \epsilon^{-\zeta}\,\ell(V^\epsilon).$$

Proposition 3.10.2. *Suppose V is a random subset of a compact set $K \subset \mathbb{R}^d$ and $0 < \zeta < d, \alpha = d - \zeta$. Suppose that there exist $0 < c_1 < c_2 < \infty$ and $\epsilon_0 > 0$ such that for all x, y and all $0 < \epsilon < \epsilon_0$,*

$$\mathbb{E}[I_\epsilon(x)] \le c_2, \quad \mathbb{E}[J_\epsilon] \ge c_1, \quad \mathbb{E}[I_\epsilon(x)\, I_\epsilon(y)] \le c_2\, |x-y|^{-\zeta}. \tag{3.64}$$

Then, there exists $\delta = \delta(c_1, c_2, K, d, \zeta) > 0$ such that

$$\delta \le \mathbb{P}\{\dim_h(V) = \alpha\} \le \mathbb{P}\{\overline{\dim}_b(V) \le \alpha\} = 1.$$

Proof. Note that the first inequality in (3.64) implies that $\mathbb{E}[J_\epsilon] \le c_2\,\ell(K)$. Also, using the third inequality,

$$\begin{aligned}
\mathbb{E}[J_\epsilon^2] &\le \mathbb{E}\left[\left(\int_K I_\epsilon(x)\,dx\right)\left(\int_K I_\epsilon(y)\,dy\right)\right] \\
&= \int_K \int_K \mathbb{E}[I_\epsilon(x)\, I_\epsilon(y)]\,dy\,dx \\
&\le c_2 \int_K \int_K |x-y|^{-\zeta}\,dy\,dz := c_3 < \infty.
\end{aligned}$$

For any nonnegative random variable X, we have

$$\mathbb{P}\left\{X \ge \frac{\mathbb{E}(X)}{2}\right\} \ge \frac{(\mathbb{E}[X])^2}{4\mathbb{E}[X^2]}.$$

Therefore, using the second inequality,

$$\mathbb{P}\left\{J_\epsilon \ge \frac{c_1}{2}\right\} \ge \frac{c_1^2}{4c_3} > 0. \tag{3.65}$$

If $\beta < \alpha$, let

$$\mathcal{E}_\beta(\mu_\epsilon) = \int_K \int_K \frac{\mu_\epsilon(dx)\,\mu_\epsilon(dy)}{|x-y|^\beta},$$

and note that the third inequality gives

$$\mathbb{E}\left[\mathcal{E}_\beta(\mu_\epsilon)\right] \le c_2 \int_K \int_K \frac{dx\,dy}{|x-y|^\zeta\,|x-y|^\beta} := C_\beta < \infty.$$

Therefore,

$$\mathbb{P}\left\{\mathcal{E}_\beta(\mu_\epsilon) \ge \frac{8c_3\,C_\beta}{c_1^2}\right\} \le \frac{c_1^2}{8c_3}$$

Combining this with (3.65), we see that

$$\mathbb{P}\left\{\mu_\epsilon(K) \ge \frac{c_1}{2}, \mathcal{E}_\beta(\mu_\epsilon) \le \frac{8c_3\,C_\beta}{c_1^2}\right\} \ge \delta := \frac{c_1^2}{8c_3} > 0. \tag{3.66}$$

- **Upper bound**. Since $\mathbb{E}[J_{2^{-n}}]$, is uniformly bounded, we can see from the Markov inequality that

  $$\sum_{n=1}^{\infty} \mathbb{P}\{J_{2^{-n}} \ge n^2\} < \infty,$$

  and hence from the Borel-Cantelli lemma that with probability one for all n sufficiently large

  $$2^{nd}\,\ell(V^{2^{-n}}) = 2^{n\alpha}\,J_{2^{-n}} \le n^2\,2^{n\alpha}.$$

- **Lower bound.** It suffices to show for every $\beta < \alpha$, $\mathbb{P}\{\dim_h(V) \ge \beta\} \ge \delta$. Let $E = E_\beta$ be the event that for infinitely many n,

  $$\mu_\epsilon(K) \ge \frac{c_1}{2}, \quad \mathcal{E}_\beta(\mu_\epsilon) \le \frac{8c_3\,C_\beta}{c_1^2}. \tag{3.67}$$

  Using (3.66), we see that $\mathbb{P}(E) \ge \delta$. On the event E, find a subsequence that satisfies (3.67), and then take a further subsequence that converges to a measure μ. Using compactness of K and Fatou's lemma we see that

  $$\mu(K) \ge \frac{c_1}{2}, \quad \mathcal{E}_\beta(\mu) \le \frac{8c_3\,C_\beta}{c_1^2},$$

  and since μ_ϵ is supported on V^ϵ, we can see that μ is supported on V. Using Proposition 3.10.1, on the event E we have $\dim_h(V) \ge \beta$.

□

We let $\mathcal{S}_n$ denote the set of dyadic cubes of side length 2^{-n}, that is, the set of closed cubes of the form

$$S = [(k_1-1)2^{-n}, k_1 2^n] \times \cdots \times [(k_d-1)2^{-n}, k_d 2^n],$$

where $k_1, \ldots, k_d$ are integers. Let $\mathcal{S} = \cup_n \mathcal{S}_n$ and

$$J_\epsilon(S) = \mu_\epsilon(S) = \int_S I_\epsilon(x)\,dx.$$

Definition We say that the Minkowski content exists if (3.64) hold, and with probability one, for every $S \in \mathcal{S}$, the limit

$$J(S) = \lim_{\epsilon \downarrow 0} J_\epsilon(S) \tag{3.68}$$

exists.

Lemma 3.10.3. *Suppose the Minkowski content exists. If $S \in \mathcal{S}_m$ and $n \geq m$, let S^n denote the union of all squares in $\mathcal{S}_n$ that intersect ∂S. Then, with probability one, for all n sufficiently large,*

$$J(S^n) \leq 2^{-dm}\, 2^{(m-n)/2}. \tag{3.69}$$

In particular, $\lim_{n\to\infty} J(S^n) = 0$.

Proof. Since $\mathbb{E}[J(S^n)] \leq c\,\ell[J(S^n)] \approx 2^{-dm}\, 2^{m-n}$ we have $\mathbb{P}\{J(S^n) \geq 2^{-dm}\, 2^{(m-n)/2}\} \leq c\, 2^{(m-n)/2}$, and the result follows by Borel-Cantelli. □

Proposition 3.10.4. *Suppose that a random subset V satisfies the hypothesis of Proposition 3.10.2, A sufficient condition for the existence of the limit* (3.68) *for $S \in \mathcal{S}$ is to show that for every $\rho < 1$ there exists $u > 0$ such that for all n sufficiently large,*

$$\mathbb{E}[(Y_{n+1} - Y_n)^2] \leq n^{-(3+2u)} \qquad \text{where } Y_n = J_{\rho^n}(S). \tag{3.70}$$

The hard work in establishing the existence of Minkowski content is to show (3.70). In practice, one often shows a stronger estimate, $\mathbb{E}[(Y_{n+1} - Y_n)^2] = O(e^{-\beta n})$ for some $\beta = \beta_\rho > 0$.

Proof. We fix S and write $J_\epsilon = J_\epsilon(S)$. For a fixed ρ, (3.70) immediately shows that $\{Y_n\}$ is a Cauchy sequence in L^2, and hence there exists $Y_\infty = Y_{\infty,\rho}$ such that $Y_n \to Y_\infty$ in L^2. Using Markov's inequality,

$$\sum_{n=1}^{\infty} \mathbb{P}\{(Y_{n+1} - Y_n)^2 \geq n^{-(2+u)}\} < \infty,$$

and hence with probability one $|Y_{n+1} - Y_n| \leq n^{-(1+\frac{u}{2})}$ for all n sufficiently large. On this event $Y_n \to Y_\infty$.

We will choose $\rho_{k+1} = \sqrt{\rho_k}$, that is $\rho_k = 2^{-2^{-k}}$. For fixed k, the condition (3.70) implies that there exists a random variable Y_∞ such that $Y_n \to Y_\infty$ in L^2 and with probability one. Since $\{\rho_k^n : n \geq 1\} \subset \{\rho_{k+1}^n : n \geq 1\}$, we see that the limit Y_∞ must be the same for all k. If $\rho_k^{n+1} \leq \epsilon \leq \rho_k^n$, then

$$J_{\rho_k^{n+1}} \leq \rho_k^\zeta\, J_\epsilon \leq \rho_k^{2\zeta}\, J_{\rho_k^n}.$$

Using this and the uniform bound on $\mathbb{E}[J_\epsilon^2]$, we see that $J_\epsilon \to Y_\infty$ in L^2 and with probability one. □

If U is an open set, we can write U uniquely as

$$U = \bigcup_{S \in \mathcal{S}_U} S, \tag{3.71}$$

where $\mathcal{S}_U$ is the set of $S \in \mathcal{S}$ with $S \subset U$ but whose "parent" cube is not contained in U. Note that $\{\text{int(S)} : S \in \mathcal{S}_U\}$ are disjoint

Proposition 3.10.5. *If the Minkowski content exists, then with probability one the measure* $\mu = \lim_{\epsilon \downarrow 0} \mu_\epsilon$ *exists. Moreover, if* U *is an open set written as in* (3.71)*,*

$$\mu(U) = \sum_{S \in \mathcal{S}_U} J(S). \tag{3.72}$$

Proof. We will consider the event of probability one on which (3.68) and (3.69) holds for all $S \in \mathcal{S}$. By choosing $S \in \mathcal{S}$ containing K in its interior, we can see that with probability one, $\{\mu_\epsilon\}$ is a collection of measures whose support is contained in S with $\mu_\epsilon(S) \to J(S)$. Hence the collection is precompact and every subsequence contains a convergent subsequence. To prove that the limit exists, it suffices to show that every subsequential limit is the same. Since these are finite Borel measures, it suffices to show that all subsequential limits agree on every open set U. Hence, it suffices to show that any subsequential limit satisfies (3.72).

Let μ be a subsequential limit. We claim that $\mu(\partial S) = 0$ for all $S \in \mathcal{S}$. To see this we use Lemma 3.10.3. Suppose $S \in \mathcal{S}_m$ and let S^n be defined as in that lemma. Then for all n sufficiently large, $J(S^n) \le 2^{-dm} 2^{(m-n)/2}$. If we now fix n (so that there are only a finite number of cubes from $\mathcal{S}_n$ in S^n), we can see that there exists ϵ' such that for $\epsilon < \epsilon'$, $\mu_\epsilon(S') = J_\epsilon(S') \le 2 \cdot 2^{-dm} 2^{(m-n)/2}$. From this we conclude that $\mu(\partial S) \le 2 \cdot 2^{-dm} 2^{(m-n)/2}$. If we now let $n \to \infty$, we get $\mu(\partial S) = 0$. With the claim, we now see that

$$\mu(U) = \mu \left[\bigcup_{S \in \mathcal{S}_U} \operatorname{int}(S) \right] = \sum_{S \in \mathcal{S}_U} \mu(\operatorname{int}(S)) = \sum_{S \in \mathcal{S}_U} \mu(S) = \sum_{S \in \mathcal{S}_U} J(S).$$

□

Chapter 4

Brownian measures

4.1 Brownian measures

4.1.1 Introduction

Probabilists generally like to view Brownian motion as a random process. Indeed, many of the powerful tools from probability require this viewpoint. However, there are other times, e.g., when considering models from statistical physics, to consider Brownian motion and related processes as a measure on paths, not necessarily a probability measure. We will look at Brownian motion this way although we will take advantage of the fact that the reader already has a good feeling for the probabilistic construct. We will consider a number of path measures but the most important will be the *Brownian loop measure* and the corresponding *Brownian bubble measure*.

It will be necessary to set up some notation. We will need a metric on the space of measures in order to topologize the space. This will allow us to consider continuous measure-valued functions and to integrate these functions. This can be skipped at a first reading.

- Let $\mathcal{K}$ denote the set of curves, that is, the set of continuous functions $\gamma : [0, t_\gamma] \to \mathbb{C}$. We do not make any further assumptions of γ of either smoothness or number of self-intersections.
- We call t_γ the *time duration* and and $\gamma(0), \gamma(t_\gamma)$ the *initial and terminal points*, respectively, of γ.
- If D is a domain, we will abuse notation slightly and write $\gamma \subset D$ if $\gamma(0, t_\gamma) \subset D$. The initial and terminal points may be in D or on ∂D. We write $\mathcal{K}_D = \{\gamma \in \mathcal{K} : \gamma \subset D\}$.
- If $\gamma \in \mathcal{K}_D$ and $f : D \to f(D)$ is holomorphic such that

$$t_{f\circ\gamma} := \int_0^{t_\gamma} |f'(\gamma(s))|^2 \, ds < \infty, \tag{4.1}$$

and the limits

$$\lim_{t\downarrow 0} f(\gamma(0)), \quad \lim_{t\uparrow 0} f(\gamma(t_\gamma)) \tag{4.2}$$

exist, then we define $f \circ \gamma \in \mathcal{K}_{f(D)}$ by

$$f \circ \gamma(t) = f[\gamma(\sigma(t))],$$

where σ is defined by

$$\int_0^{\sigma(t)} |f'(\gamma(s))|^2 \, ds = t.$$

The conditions in (4.1) and (4.2) obviously hold if the endpoints of γ are in D but need to be checked if one of the endpoints is a boundary point.

- For any $\gamma \in \mathcal{K}$, we define the oscillation by

$$\operatorname{osc}(\gamma, \delta) = \max\{|\gamma(s) - \gamma(t)| : 0 \le s, t \le t_\gamma, |s - t| \le \delta\}.$$

- We can associate to each curve γ an order pair $(t_\gamma, \tilde{\gamma})$ where $t_\gamma \in [0, \infty)$ and $\tilde{\gamma} \in \mathbb{C}[0, 1]$ given by

$$\tilde{\gamma}(t) = \gamma(t/t_\gamma).$$

We define the metric on curves

$$\rho(\gamma, \gamma_1) = |t_\gamma - t_{\gamma_1}| + \|\gamma - \gamma_1\|,$$

where $\|\cdot\|$ denotes the supremum norm.

There is another natural metric to put on curves. Let

$$\rho_1(\gamma, \tilde{\gamma}) = \inf_\phi \left[\|\mathrm{id} - \phi\| + \|\gamma - \tilde{\gamma} \circ \phi\|\right],$$

where id denotes the identity function, $\|\cdot\|$ the supremum norm, and the infimum is over all increasing homeomorphism $\phi : [0, t_\gamma] \to [0, t_{\tilde{\gamma}}]$. If we choose ϕ to be the linear homeomorphism $\phi(s) = (st_{\tilde{\gamma}}/t_\gamma)$ then

$$\|\mathrm{id} - \phi\| + \|\gamma - \tilde{\gamma} \circ \phi\| = \rho(\gamma, \tilde{\gamma}).$$

Hence $\rho_1 \le \rho$. We claim that the two metrics give the same topology.

Lemma 4.1.1. *For every $\gamma \in \mathcal{K}$ and $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\{\eta \in \mathcal{K} : \rho_1(\gamma, \eta) < \delta\} \subset \{\eta \in \mathcal{K} : \rho(\gamma, \eta) < \epsilon\},$$

Moreover, we can choose

$$\delta = \min\left\{\frac{u}{2 \vee t_\gamma}, \frac{\epsilon}{4}\right\},$$

where u satisfies $\operatorname{osc}(\gamma, u) \le \epsilon/4$.

Proof. Let u, δ be as above. We will show that if $\rho(\gamma, \eta) \ge \epsilon$ then $\rho_1(\gamma, \eta) \ge \delta$. Note that for every ϕ,

$$\|\mathrm{id} - \phi\| \ge |t_\gamma - \phi(t_\gamma)| = |t_\gamma - t_\eta|.$$

Hence if $|t_\gamma - t_\eta| \ge \delta$, then $\rho_1(\gamma, \delta) \ge |t_\gamma - t_\eta| \ge \delta$. Hence, we may assume that $|t_\gamma - t_\eta| \le \delta \le \epsilon/4$.

Since $\rho(\gamma, \eta) \ge \epsilon$ and $|t_\gamma - t_\eta| < \epsilon/2$, there exists $s \in [0, 1]$ such that

$$|\gamma(st_\gamma) - \eta(st_\eta)| \ge \epsilon/2.$$

Note that if $|s' - st_\gamma| \le u$, then

$$|\gamma(s') - \eta(st_\eta)| \ge |\gamma(st_\gamma) - \eta(st_\eta)| - |\gamma(s') - \gamma(st_\gamma)| \ge \epsilon/4.$$

Since $|t_\gamma - t_\eta| \le u/2t_\gamma$,

$$|st_\gamma - st_\eta| \le t_\gamma \, |t_\gamma - t_\eta| \le u/2,$$

and hence

$$|\gamma(s') - \eta(st_\eta)| \ge \epsilon/4, \quad |s' - st_\eta| \le u/2.$$

Suppose $\phi : [0, t_\gamma] \to [0, t_\eta]$ is an increasing homeomorphism and suppose that $\phi(t) = st_\eta$. Then either $|t - \phi(t)| \ge u/2$ or $|\gamma(t) - \eta(\phi(t))| \ge \epsilon/4$. Therefore, $\rho_1(\gamma, \eta) \ge \delta$.

□

The metric ρ induces a metric on probability measures $\mathcal{M}^\#$ on $\mathcal{K}$ called the Prokhorov metric which we also denote by ρ. There are several equivalent definitions, If μ_1, μ_2 are probability measures we say that $\rho(\mu_1, \mu_2)$ is the infimum of all ϵ such that we can define (X, Y) on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ so that X has marginal distribution μ_1, Y has marginal distribution μ_2 and

$$\mathbb{P}\{\rho(X, Y) \ge \epsilon\} \le \epsilon.$$

There is also a corresponding Prokhorov metric ρ_1 and the two metrics give the same topology on $\mathcal{M}^\#$.

We will want to consider measures on $\mathcal{K}$ that are not probability measures. Let $\mathcal{M}$ denote the set of measures on $\mathcal{K}$ of total mass strictly between 0 and ∞. We write such a measure μ as $\mu = (\mu^\#, q)$ where $q = \|\mu\| = \mu(\mathcal{K})$ is the total mass and $\mu^\# = \mu/q$ is the probability measure obtained by normalization. The metric on $\mathcal{M}$ is the induced measure on $\mathcal{M}^\# \times (0, \infty)$. In another words, $\mu_n \to \mu$ if and only if

$$\mu_n(\mathcal{K}) \to \mu(\mathcal{K}), \quad \text{and} \quad \mu_n^\# \to \mu^\#.$$

The zero measure is not in $\mathcal{M}$, but we will say that μ_n converges to the zero measure if $\mu_n(\mathcal{K}) \to 0$. In this case we do not require that $\mu_n^\#$ converge.

If $E \subset \mathcal{K}$ and μ is a measure we write $\mu|_E$ for the measure restricted to the subset E. If $D \subset \mathbb{C}$, we write μ_D for the measure restricted to the event $\{\gamma \subset D\}$.

4.1.2 Some easy lemmas

Here we collect some easy facts about the metrics ρ, ρ_1.

- If $\gamma \in \mathcal{K}$, then the reverse path γ^R is defined by

  $$\gamma^R(t) = \gamma(t_\gamma - t), \quad 0 \le t \le t_\gamma.$$

  Note that $\rho(\gamma, \eta) = \rho(\gamma^R, \eta^R), \operatorname{osc}(\gamma) = \operatorname{osc}(\gamma^R)$.

- If $\gamma \in \mathcal{K}$ and $z \in \mathbb{C}$, we define γ_z by $\gamma_z(t) = \gamma(t) + z$, $0 \le t \le t_\gamma$.

- If $\gamma, \eta \in \mathcal{K}$ with $\eta(0) = \gamma(t_\gamma)$ we define the *concatenation* $\gamma \oplus \eta$ by $t_{\gamma\oplus\eta} = t_\gamma + t_\eta$ and

$$\gamma \oplus \eta(t) = \begin{cases} \gamma(t) & 0 \le t \le t_\gamma \\ \gamma(t_\gamma) + \eta(s - t_\gamma) & t_\gamma \le s \le t_\gamma + t_\eta \end{cases}$$

- If $\gamma, \eta \in \mathcal{K}$, we define another concatenation $\gamma \otimes \eta$ by

$$\gamma \otimes \eta = \gamma \oplus \eta_{\gamma(t_\gamma)-\eta(0)}.$$

  This is defined for all curves. It is a concatentation where we translate the initial point of η so that if matches up with the terminal point of γ.

- If $\gamma \in \mathcal{K}$ define

$$\|\gamma\| = t_\gamma + \max_{0\le t\le t_\gamma} |\gamma(t) - \gamma(0)|.$$

  We have defined this so that $\|\gamma_z\| = \|\gamma\|$.

- If $\gamma, \eta \in \mathcal{K}$, then

$$\rho_1(\gamma, \gamma \otimes \eta) \le \|\eta\|, \quad \rho_1(\gamma \otimes \eta, \gamma) \le \|\eta\|.$$

  Indeed, for the first inequality we can consider the parametrizations

$$\phi_\delta(t) = \begin{cases} (1-\delta)\, t, & t \le t_\gamma, \\ (1-\delta)\, t_\gamma + (\delta/t_\eta)\,(t - t_\gamma), & t_\gamma \le t \le t_\gamma + t_\eta \end{cases},$$

  as $\delta \to 0$.

- If $\gamma \in \mathcal{K}$, and $s < t_\gamma$, we define $\Phi_s\gamma$ to be the curve with time duration s with $\gamma(t) = \Phi_s\gamma(t), 0 \le t \le s$. Then using concatenation, we see that

$$\lim_{s\uparrow t_\gamma} \rho_1(\gamma, \Phi_s\gamma) = 0,$$

  and hence this also holds for ρ.

4.1.3 Brownian measures

In this section we will introduce many measures in $\mathcal{M}$ that arise from complex Brownian motions. We will use ν for all of these measures with extra parameters included. Let us start easy by defining $\nu(z, \cdot; t)$ to be the probability measure which is the distribution of the curve $B_s, 0 \le s \le t$ where B_s is a standard Brownian motion starting at z. This is a probability measure. The $\cdot$ indicates which variable is being integrated over. We will write

$$\nu(z, \cdot; t) = \int_C \nu(z, w; t)\, dA(w),$$

for appropriate measures $\nu(z, w; t)$ that we now describe. Here and throughout dA represents integration with respect to area. The integral can be considered a Riemann integral. In order to justify the expression we will show that $w \mapsto \nu(z, w; t)$ is a continuous function.

Our reason to discuss topologies on measures on curves was to allow us to integrate measure-valued functions. Measurability of the functions is with respect to the Borel σ-algebra generated by the topology. We will consider continuous or piecewise continuous functions and so the integrals can be considered as Riemann integrals.

There are several ways to handle the technical details here and this should not be considered the main issue.

Let us describe the measure $\nu(z, w; t)$; for ease we will choose $z = 0$. Let

$$p_s(z, w) = (2\pi s)^{-1} \exp\left\{-|z - w|^2/2s\right\}$$

be the transition probabilities for complex Brownian motion. For each $\epsilon > 0$, let $\mu_\epsilon = \mu_\epsilon(0, w; t)$ be $\nu(0, \cdot; t)$ restricted to the event $\{|\gamma(t) - w| \le \epsilon\}$. We define

$$\nu(z, w; t) = \lim_{\epsilon \downarrow 0} (\pi\epsilon^2)^{-1} \mu_\epsilon.$$

Of course, we need to show that the limit exists. Since

$$\mu_\epsilon(\mathcal{K}) = \int_{|w-\zeta| \le \epsilon} p_t(0, \zeta)\, dA(\zeta),$$

it is easy to see that

$$\lim_{\epsilon \downarrow 0} (\pi\epsilon^2)^{-1} \mu_\epsilon(\mathcal{K}) = p_t(0, w).$$

We will show that the limit

$$\lim_{\epsilon \downarrow 0} \mu_\epsilon^\# = \nu^\#(0, w)$$

exists and in the process we will describe the limit.

Fix $\epsilon > 0$ and let B_t be a Brownian motion starting at 0. For each $0 \le s \le t$, let

$$F_\epsilon(s, z) = F_\epsilon(s, z; w) = \int_{|w-\zeta| \le \epsilon} p_{t-s}(z, \zeta)\, dA(\zeta).$$

$$J_s = F_\epsilon(s, B_s).$$

Note that $F_\epsilon(z, s)$ is C^2 in z and C^1 in s for $s < t$. Hence we can write

$$\begin{aligned} dJ_s &= \left[\dot{F}_\epsilon(s, B_s) + \frac{\Delta F_\epsilon(s, B_s)}{2}\right] ds + \nabla F_\epsilon(s, B_s) \cdot dB_s \\ &= J_s \left[\frac{\dot{F}_\epsilon(s, B_s)}{F_\epsilon(s, B_s)} + \frac{\Delta F_\epsilon(s, B_s)}{2F_\epsilon(s, B_s)}\, ds + \frac{\nabla F_\epsilon(s, B_s)}{F_\epsilon(s, B_s)} \cdot dB_s\right] \end{aligned}$$

Let $t_1 < t$. Then

$$M_s = J_s \exp\left\{-\int_0^s \left[\frac{\dot{F}_\epsilon(r, B_r)}{F_\epsilon(r, B_r)} + \frac{\Delta F_\epsilon(r, B_r)}{2F_\epsilon(r, B_r)}\right] dr\right\}$$

is a positive local martingale satisfying

$$dM_s = \nabla[\log F_\epsilon(s, B_s)] \cdot dB_s.$$

If we use the Girsanov theorem to tilt by the local martingale M_s, we see that

$$dB_t = \nabla[\log F_\epsilon(s, B_s)]\, dt + dW_t,$$

where W_t is a complex Brownian motion with respect to the new measure. This drift always points toward w, and from this we can see that there is no blowup of the process or the local martingale by time t_1.

In our new notation, Let $\mu^{\#}_{\epsilon,t_1}$ denote the measure on paths of time duration t_1 that is obtained from $\mu^{\#}_\epsilon$ by truncation. The Girsanov theorem tells us that $\mu^{\#}_{\epsilon,t_1}$ is absolutely continuous with respect to $\nu(0;\cdot;t_1)$ with Radon-Nikodym derivative

$$\frac{d\mu^{\#}_{\epsilon,t_1}}{d\nu(0,\cdot;t_1)}(\gamma) = F_\epsilon(t_1,\gamma(t_1)) \exp\left\{ -\int_0^{t_1} \left[\frac{\dot F_\epsilon(s,\gamma(s))}{F_\epsilon(s,\gamma(s))} + \frac{\Delta F_\epsilon(s,\gamma(s))}{2F_\epsilon(s,\gamma(s))} \right] ds \right\}.$$

With this expression, we can take the limit and see that

$$\frac{d\mu^{\#}_{0,t_1}}{d\nu(0,\cdot;t_1)}(\gamma) = F(t_1,\gamma(t_1)) \exp\left\{ -\int_0^{t_1} \left[\frac{\dot F(s,\gamma(s))}{F(s,\gamma(s))} + \frac{\Delta F(s,\gamma(s))}{2F(s,\gamma(s))} \right] ds \right\},$$

where $F(s,z) = \lim_{\epsilon\downarrow 0} F_\epsilon(s,z) = p_{t-s}(z,w)$.

As $\epsilon \downarrow 0$, the process approaches Brownian motion weighted by $p_{t-s}(z,w)$. Note that

$$\nabla \log p_{t-s}(z,w) = \left(\frac{w - X_s}{t-s}, \frac{w-Y_s}{t-s} \right). \tag{4.3}$$

Having seen how to take the limit, it is useful to describe the process $\nu(0,w;t)$ directly. There are a number of equivalent ways of writing this. Let us write this as a process (W^1_s, W^2_s) in $\mathbb{R}^2$ beginning at the origin and ending at $w = (w^1, w^2)$ at time t.

- Let B_s be a standard Brownian motion in $\mathbb{R}^2$ and let

$$W_s = B_s + (s/t)\,(w - B_t).$$

- Take a Brownian motion B_s and "tilt locally" by $p_{t-s}(B_s, w)$.

- Equivalently, consider a process satisfying the diffusion equation

$$dX_s = \frac{(w^1 - X^1_t) + (w^2 - X^2_t)}{t-s}\, dt + dB_t,$$

  where B_t is a standard Brownian motion.

The process is often called the *Brownian bridge* (for Brownian motion conditioned to be at w at time t). The bridge construction shows that $w \mapsto \nu(z,w;t)$ is continuous in w and using this we can see that

$$\int_D \nu(z,w;t)\, dA(w)$$

is the same as the measure $\nu(z,\cdot;t)$ restricted to the event that $\gamma(t) \in D$. We also can see that $t \mapsto \nu(z,w;t)$ is continuous for $t > 0$, and we can define

$$\nu(z,w;[0,T])) = \int_0^T \nu(z,w;t)\,dt.$$

We can also define

$$\nu(z,w) = \int_0^\infty \nu(z,w;t)\,dt,$$

although this is now an infinite measure.

Suppose that D is a regular domain. then $\nu_D(z,w;t)$ and $\nu_D(z,w)$ are defined to be the measures anove restricted to curves $\gamma \subset D$. Let us denote the total mass of $\nu_D(z,w;t)$ by $p_t^D(z,w)$, and the total mass of $\nu_D(z,w)$ is

$$\|\nu_D(z,w)\| = \int_0^\infty p_t^D(z,w)\,dt = \frac{1}{\pi}\,G_D(z,w).$$

Here G_D is the Green's function normalized so that $G_{\mathbb{D}}(0,z) = -\log|z|$.

Here is one way to check the constant for $\|\nu_D(z,w)\|$. Assume that we know that

$$\|\nu_D(z,w)\| = c_0\,G(z,w)$$

for some constant c_0. Consider the unit disk $\mathbb{D}$ with $\tau = \tau_D$. If B_t is a Brownian motion starting at the origin, then $M_t = |B_t|^2 - 2t$ is a martingale. Therefore,

$$0 = M_0 = \mathbb{E}[M_\tau] = \mathbb{E}[1 - 2\tau],$$

that is, $\mathbb{E}[\tau] = 1/2$. We should also have

$$\begin{aligned}
\mathbb{E}[\tau] &= \int_0^\infty \mathbb{P}\{\tau > t\}\,dt \\
&= \int_0^\infty \int_{\mathbb{D}} p_t(0,z)\,dA(z)\,dt \\
&= \int_{\mathbb{D}} c_0\,G(0,z)\,dA(z) \\
&= -2\pi c_0 \int_0^1 r\,\log r\,dr = \frac{c_0\,\pi}{2}.
\end{aligned}$$

Therefore, $c_0 = 1/\pi$.

If μ is a measure on $\mathcal{K}$, we let μ^R denote the reversal,

$$\mu^R(K) = \mu\{\gamma : \gamma^R \in K\}.$$

Proposition 4.1.2 (Reversibility)**.**

$$[\nu_D(z,w;t)]^R = \nu_D(w,z;t),$$
$$[\nu_D(z,w;t)]^R = \nu_D(w,z;t),$$
$$[\nu_D(z,w)]^R = \nu_D(w,z).$$

Proof. It is immedate that $p_t(z,w) = p_t(w,z)$. To show that $[\mu^{\#}(z,w;t)]^R = \mu^{\#}(w,z;t)$, we can use the Brownian bridge representation of this measure. For ease assume that $z = 0$. We know that $\mu^{\#}(0,w;t)$ is induced by

$$W_s = B_s + (s/t)\,(w - B_t), \quad 0 \le s \le t.$$

If $Y_s = W_{t-s} - w$, $Z_s = B_{t-s} - B_t$, then

$$Y_s = B_{t-s} - \frac{s}{t}\,w - \frac{t-s}{t}\,B_t = [B_{t-s} - B_t] + \frac{s}{t}\,[-w + B_t] = Z_s + \frac{s}{t}\,[-w - Z_t].$$

Since Z_t is a standard Brownian motion, we see that the distribution of the reversal of the bridge is the same as a bridge. This shows that $[\nu(z,w;t)]^R = \nu(w,z;t)$. Since $\gamma \subset D$ if and only if $\gamma^R \subset D$, we see that $[\nu_D(z,w;t)]^R = \nu_D(w,z;t)$, and by integrating over t, we get the second result. Although $\nu(z,w)$ is not a finite measure, we can write is as

$$\nu(z,w) = \lim_{s\to\infty} \nu_{\mathbb{D}_{-s}}(z,w).$$

□

Suppose $f : D \to f(D)$ is a conformal transformation. The measures $\mu_D(z,w;t)$ do not transform very well because one needs to change time in the parametrization. However, the integrated versions do well.

Proposition 4.1.3. *If D is a regular domain and $f : D \to f(D)$ is a conformal transformation, then $f \circ \mu_D(z,w) = \mu_{f(D)}(f(z), f(w))$.*

Proof. We can consider this proposition as a combination of two results:

$$\|f \circ \mu_D(z,w)\| = \|\mu_{f(D)}(f(z), f(w))\|,$$

$$f \circ \mu_D^{\#}(z,w) = \mu_{f(D)}^{\#}(f(z), f(w)).$$

The first follows from $G_D(z,w) = G_{f(D)}(f(z), f(w))$, so let us concentrate on the second. It is useful to consider the measure $\mu_D^{\#}(z,w)$. Indeed, this measure can be obtained from Brownian motion in D tilted by the Green's function.

Let us be more explicit. For ease, assume that $w = 0$ and write $G(z) = G_D(z,w)$. For each $\delta, \epsilon > 0$, let $\tau_\epsilon = \inf\{t : B_t \notin D \text{ or } |B_t| = \epsilon\}$. We can consider a probability measure $\mu_\epsilon = \mu_{\epsilon,\delta}$ given by the paths $B_t, 0 \le t \le \tau_\delta$ *conditioned* on the event $\{|B_{\tau_\epsilon}| = \epsilon\}$. If we fix δ and let $\epsilon \downarrow 0$, then by properties of the Green's function we see that the limit measure μ_0 is the measure on paths $B_t, 0 \le t \le \tau_\delta$ weighted by the Green's function G. We then can take $\delta \downarrow 0$ and get our measure $\nu_D^{\#}(z,0)$

To phrase this in terms of Girsanov's theorem, we let $M_t = G(B_t)$ which satisfies

$$dM_t = \nabla L_D(B_t)\, M_t\, dB_t.$$

If we use tilt by this local martingale, then

$$dB_t = \nabla L_D(B_t)\, dt + dW_t, \tag{4.4}$$

where W_t is a standard Brownian motion in the new measure. Note that

$$df(B_t) = f'(B_t) \cdot \frac{\nabla G_D(B_t)}{G_D(B_t)}\, dt + f'(B_t) \cdot dW_t,$$

Conformal invariance of Brownian motion and the Green's function shows that the measures $f \circ \nu^{\#}(z,w) = \nu^{\#}_{f(D)}(f(z), f(z))$. □

Rather than going through the limiting process, we could use Itô's formula to show that solutions to (4.4) with appropriate change of parametrization are conformally invariant.

Example Suppose $D = \mathbb{D}$ and $w = 0$. Then $G(\zeta) = -\log|\zeta|$ and $\nabla \log G(\zeta)$ is a vector pointing radially towards the origin with norm $[r \log(1/r)]^{-1}$ where $r = |\zeta|$. We want to show that with probability one the path reaches the origin in finite time. Consider the visits to the circles $C_n = \{|z| = e^{-n}\}$. Suppose that $|B_0| = e^{-n}$ and let $\tau = \tau_m = \inf\{t : B_t \in C_{n-1} \text{ or } B_t \in C_m\}$. In the tilted measure, we have

$$\mathbb{P}^*\{B_\tau \in C_{n-1}\} = \frac{\mathbb{P}\{B_\tau \in C_{n-1}\ \log e^{n-1}}{\mathbb{P}\{B_\tau \in C_{n-1}\ \log e^{n-1} + \mathbb{P}\{B_\tau \in C_m\}\ \log e^m} =$$

Starting at C_r, for usual Brownian motion the probability is $1/2$ that the path reaches C_{n-1} before C_{n+1}. Therefore, in the weighted measure $\mathbb{P}^*$

$$\begin{aligned}
\lim_{m\to\infty} \mathbb{P}^*\{\tau_{n-1} < \tau_m \mid B_0 = e^{-n}\} &= \lim_{m\to\infty} \frac{\mathbb{P}\{B_\tau \in C_{n-1}\}\ \log e^{n-1}}{\mathbb{P}\{B_\tau \in C_{n-1}\ \log e^{n-1} + \mathbb{P}\{B_\tau \in C_m\}\ \log e^m} \\
&= \frac{\frac{m-1-n}{m-n}\, n - 1}{\frac{m-1-n}{m-n}\, n - 1 + \frac{1}{m-n}\, m} = \frac{n-1}{n}.
\end{aligned}$$

In particular, if we let J_n be the total number of excursions between C_n and C_{n-1}, then

$$\mathbb{P}\{J_n \geq n\} = (n - 1/n)^n \leq 1/e,$$

and hence there exists $c < \infty$ such that $\mathbb{P}\{J_n \geq c\, n \log n\} \leq n^{-2}$. By the Borel-Cantelli lemma, we see that with $\mathbb{P}^*$-probability one, except for a finite number of integers n, $J_n \leq c\, n \log n$. Using this (we omit the argument), we can see that with $\mathbb{P}^*$-probability one, the process reaches 0 in finite time and is continuous at the terminal time.

4.1.4 Radial and angular part

When studying Brownian motion in $\mathbb{C}$ it is often to consider the radial and angular parts to be independent. To make this precise, let $W_t = X_t + iY_t$ be a complex Brownian motion, and let

$$\tilde{B}_t = \exp\{-W_t\}.$$

Then $\tilde{B}_t$ is a time change of a standard Brownian motion. To be more precise, let $f(z) = e^{-z}$ and define $\sigma(t)$ by

$$t = \int_0^{\sigma(t)} |f'(W_s)|^2 ds = \int_0^{\sigma(t)} e^{-2X_s}\, ds.$$

Then $B_t = \tilde{B}_{\sigma(t)}$ is a complex Brownian motion. Note that

$$\dot{\sigma}(t) = |f'(W_{\sigma(t)})|^{-2} = e^{-2X_{\sigma(t)}} = |B_t|^{-2}.$$

Itô's formula shows that

$$d|\tilde{B}_t| = de^{-X_t} = -e^{-X_t}\, dX_t + \frac{e^{-X_t}}{2}\, dt = \frac{|\tilde{B}_t|}{2}\, dt - |\tilde{B}_t|\, dX_t.$$

Apply the time change rule, we see that if we write $B_t = R_t \exp\{i\Theta_t\}$,

$$dR_t = \frac{1}{2\, R_t}\, dt + d\hat{X}_t, \quad d\Theta_t = \frac{1}{R_t}\, d\hat{Y}_t, \tag{4.5}$$

for a standard independent Brownian motions $\hat{X}_t, \hat{Y}_t$

Let W_t be a complex Brownian motion and let

$$\mu_s = \inf\{t : |W_t| = e^{-s}\}.$$

Let ν_s denote the measure induced by $B_{t+\sigma_s}, 0 \le t \le \sigma - \sigma_s$. Then

$$\nu_{\mathbb{D}}(0, \partial D) = \lim_{s\to\infty} \mu_s.$$

The probability measure μ_s can be constructed as follows, Let $\hat{X}_t, \hat{Y}_t$ be independent Brownian motions and let Θ be an independent random variable uniform on $[0, 2\pi]$. Consider the solution to (4.5) with initial condition $R_0 = e^{-s}, \Theta_0 = \Theta$ stopped at $T = \inf\{t : R_t = 1\}$. Then μ_s is the measure generated by $\exp\{R_t + i\Theta_t\}, 0 \le t \le T$. Since the initial condition Θ is indpendent of the Brownian motions, we can see that Θ_T and T are independent random variables. Letting $s \to 0$, we get the following. We state this for the unit disk but a similar result holds for other disks.

Proposition 4.1.4. *Let B_t be a Brownian motion starting at the origin and let $\tau = \tau_{\mathbb{D}}$. Then the random variables τ and $\arg(B_\tau)$ are independent. The latter is uniform on $[0, 2\pi]$ while the form has the distribution of*

$$\inf\{t : R_t = 1\},$$

where R_t is a Bessel process satisfying

$$dR_t = \frac{dt}{2\, R_t} + dW_t, \quad R_0 = 0.$$

Let us emphasize that this proposition strongly uses the fact that the domain is a disk and that the Brownian motion starts at the center of the disk. If the Brownian motion started very close to the boundary, then the random variables σ_s and $\arg(B_{\sigma_s})$ would not be independent. Indeed, if we know that τ is very small, then the argument would more likely to be very close to the starting argument, while if τ is large, then the argument would be almost uniformly distributed. Not so obvious, but true, is that if D is a domain, $z \in D$, and τ_D and B_{τ_D} are independent random variables, then D is a disk and z is the center point.

4.1.5 Interior to boundary

Suppose that D is a regular point, B_t is a Brownian motion starting at $z \in D$, and, as usual, $\tau = \tau_D = \inf\{t : B_t \notin D\}$. Suppose $B_0 = z \in D$. Then there is a probability measure $\nu_D(z, \partial D)$ which is the distribution of the random curve $B_t, 0 \le t \le \tau$. If $V \subset \partial D$, then we write $\nu_D(z, V)$ to be the restriction of $\nu_D(z, \partial D)$ to curves that lie in V. We can write

$$\nu_D(z, V) = \|\nu_D(z, V)\|\, \nu_D^{\#}(z, V),$$

where

$$\|\nu_D(z, V)\| = \mathrm{hm}_D(z, V),$$

and if $\mathrm{hm}_D(z, V) > 0$, $\nu_D^{\#}(z, V)$ is the measure obtained by tilting by the harmonic function

$$h(\zeta) = \mathrm{hm}_D(\zeta, V).$$

In this case the conditioning is elementary, but we can use the Girsanov formula to write down the SDE satisfied by the conditioned process. If w is a locally analytic point of ∂D, then we can define $\nu_D(z, w)$ so that if V is analytic curve

$$\nu_D(z, V) = \int_V \nu_D(z, w)\, |dw|.$$

There are two equivalent definitions (both of which need to be verified but this is not so difficult).

$$\nu_D(z, w) = \frac{1}{2} \lim_{\epsilon \downarrow 0} \nu_D(z, w + \epsilon\, \mathbf{n}),$$

where $\mathbf{n} = \mathbf{n}_w$ denotes the inward normal at w. Equivalently,

$$\|\nu_D(z, w)\| = \frac{1}{\pi}\, H_D(z, w) = \frac{1}{2\pi} \lim_{\epsilon \downarrow 0} G_D(z, w + \epsilon\, \mathbf{n}),$$

and $\nu_D^{\#}(z, w)$ is the h-process obtained by tilting by the harmonic function $h(\zeta) = H_D(\zeta, w)$.

To help keep our constants straight, consider $D = \mathbb{D}$ and $w = 1$. Under our normalization, $H_{\mathbb{D}}(0, 1) = 1/2$, $\|\nu_{\mathbb{D}}(0, 1)\| = 1/2\pi$. $G_{\mathbb{D}}(0, 1 - \epsilon) = -\log[1 - \epsilon] \sim \epsilon$.

4.1.6 h-process

The h-process for Brownian motion conditioned to leave a domain D can be done easiest for Brownian motion in the $\mathbb{D}$ starting at the origin. Indeed, in this case if we start with a Brownian motion and let $\tau = \tau_D$, then the process $W_t = \overline{B_\tau}\, B_t$ can be shown to be an h-process conditioned to leave the disk at 1. To get the h-process in other simply connected domains (or from other starting points in $\mathbb{D}$), we can use conformal invariance. This does not require smoothness at the boundary, but we will assume that the conformal map extends continuously to the boundary point.

Let h be any positive harmonic function on a domain D. We can define the Green's function for the h-process

$$G_{D,h}(w,z) = \lim_{r\to\infty} r\,\mathbb{P}_h^w\left\{\text{dist}\,(z, B[0,\tau]) \le e^{-r}\right\},$$

where $\mathbb{P}_h$ denotes the tilted measure. Note that as $r \to \infty$,

$$\mathbb{P}_h^w\left\{\text{dist}\,(z, B[0,\tau]) \le e^{-r}\right\} \sim \mathbb{P}_h^w\left\{\text{dist}\,(z, B[0,\tau]) \le e^{-r}\right\}\,\frac{h(w)}{h(z)},$$

from which we derive

$$G_{D,h}(w,z) = G_D(w,z)\,\frac{h(z)}{h(w)}.$$

If $f : D \to f(D)$ is a conformal transformation and h is a positive harmonic function on D, let $h_f(z) = h(f^{-1}(z))$ he the corresponding harmonic function on $f(D)$. Then, connformal invariance of the Green's function implies that

$$G_{f(D),h_f}(f(w), f(z)) = G_{D,h}(w,z).$$

Lemma 4.1.5. *There exists $c < \infty$ such that for every simply connected regular domain D, every positive harmonic function h on D, and all $z, w \in D$,*

$$G_{D,h}(z,w) \le c\,[1 + G_D(z,w)]\,.$$

Proof. Without loss of generality assume that $z = 0$ and $h(0) = 1$. Let ζ be a point on ∂D closest to the origin. If $|w| \le |\zeta|/2$, then the Harnack inequality implies that $h(w) \asymp 1$ and hence $G_{D,h}(0,w) \asymp G_D(0,w)$.

For any w, let V denote the closed disk of radius $\text{dist}(w, \partial D)/2$ about w. Note that for $w' \in V$, $h(w) \asymp h(0,w'), G(w) \asymp G(0,w')$. Moreover, $G(0,w) \asymp \mathbb{P}^0\{B[0,\tau] \cap V \neq \emptyset\}$ and $\mathbb{P}_h^0\{B[0,\tau] \cap V \neq \emptyset\} \asymp \mathbb{P}^0\{B[0,\tau] \cap V \neq \emptyset\}\,h(w) \asymp G_D(0,w)\,h(w) = G_{D,h}(0,w)$. In particular, $G_{D,h}(0,w) \le c$. □

Corollary 4.1.6. *There exists $c < \infty$ such that if D is a bounded simply connected domain, and h is a positive harmonic function on D, then for every $w \in D$,*

$$\int_D G_{D,h}(w,z)\,dw < \infty.$$

In particular, with $\mathbb{P}_h$-probability one the time duration of an h-process is finite.

We are only assuming that D is bounded. This may be a bit surprising. Consider a saw tooth domain of the form $D = ((0,1) \times i(0,1)) \setminus V$ where $V = V_1 \cup V_2$ with

$$V_1 = \bigcup_{n=1}^{\infty} (2^{-2n}, 2^{-2n} + i(3/4)], \quad V_2 = \bigcup_{n=1}^{\infty} [2^{-2n+1} + i(1/4), 2^{-2n+1} + i).$$

We can find a positive harmonic function h whose boundary value is zero on $\partial D \setminus [0,i]$. Although the boundary behavior of the function f near the boundary is bad, we can still consider the h-process starting at, say, $(3/4) + i(3/4)$. Although the path must go "up and down" infinitely often to avoid the teeth of ∂D, the total time duration of the path is finite. However, in this case the path can not be extended to be continuous at its terminal time.

Corollary 4.1.7. *Suppose $f : \mathbb{D} \to D$ is a conformal transformation and suppose that f has a continuous extension to the boundary in a neighborhood of 1. Then we can write*

$$\nu_D^{\#}(f(0), f(1)) = f \circ \nu_{\mathbb{D}}^{\#}(0, 1).$$

If f is analytic in a neighborhood of 1, then we can write

$$\nu_D(f(0), f(1)) = |f'(1)|^{-1} f \circ \nu_{\mathbb{D}}^{\#}(0, 1).$$

The following will be used later in this chapter.

Proposition 4.1.8. *Suppose $D' \subset D$ are domains, $\zeta \in D$, and z is an analytic boundary point of ∂D. Let B_t be a standard Brownian motion starting at ζ under the measure $\mathbb{P}$ and let $\mathbb{P}_*$ be the measure under which B_t is an h-process in D from ζ to z. Let $T = \inf\{t : B_t \notin D'\}$ and $\tau = \tau_z = \inf\{t : B_t = z\}$. Then*

$$\mathbb{P}_*\{T < \tau\} = \frac{\mathbb{E}^{\zeta}\left[H_D(B_T, z); T < \tau\right]}{H(\zeta, z)}.$$

Proof. Let $M_t = H(\zeta, B_{t\wedge\tau})$ which is a local martingale for $t < \tau$. Then $\mathbb{P}_*$ denote the probability measure obtained by tilting by M_t. The result follows from the Girsanov theorem.

□

4.1.7 Upper half plane

The h-process is perhaps easiest seen in the upper half plane $\mathbb{H}$ with boundary point infinity. Here the path has infinite length and hence is not in $\mathcal{K}$. However, the SDE is rather nice. Indeed the Poisson kernel, that is, the positive harmonic function that vanishes on the boundary except for the target point, is $h(x + iy) = y$. When we tilt by this measure, in the new measure the real and imaginary components are independent. Indeed, the real part is a regular Brownian motion and the imaginary part satisfies the Bessel equation

$$dY_t = \frac{dt}{Y_t} + dB_t.$$

If we choose the boundary point on the real line, the formulas are not as nice but there are still not bad. Let us choose the origin for the boundary point in which case

$$h(x + iy) = \frac{y}{x^2 + y^2},$$

with

$$\frac{\nabla h(x + iy)}{h}(x + iy) = \left(-\frac{2x}{y(x^2 + y^2)}, \frac{x^2 - y^2}{y(x^2 + y^2)}\right).$$

Hence in the tilted measure we get

$$dX_t = \frac{-2X_t}{Y_t(X_t^2 + Y_t^2)}\, dt + d\hat{X}_t,$$

$$dY_t = \frac{X_t^2 - Y_t^2}{Y_t(X_t^2 + Y_t^2)}\, dt + d\hat{Y}_t,$$

where $\hat{X}_t + i\hat{Y}_t$ is a Brownian motion in the tilted measure.

In the upper half plane, we can take advantage of the independence of the real and imaginary parts, to give an explicit form for the measures $\nu_{\mathbb{H}}(z, x; t)$ so that

$$\nu_{\mathbb{H}}(z, x) = \int_0^\infty \nu_{\mathbb{H}}(z, x; t)\, dt.$$

Lemma 4.1.9. *Let B_t be a one-dimensional Brownian motion starting at $y > 0$ and let $T = \inf\{t : B_t = 0\}$. Then T has density*

$$\frac{y}{\sqrt{2\pi}\, t^{3/2}} \exp\left\{-y^2/2t\right\}. \tag{4.6}$$

Proof. Using symmetry and the reflection principle for Brownian motion,

$$\begin{aligned}
\mathbb{P}\{T \le t\} &= \mathbb{P}^y \left\{ \min_{0\le s\le t} B_s \le 0 \right\} \\
&= \mathbb{P}^0 \left\{ \max_{0\le s\le t} B_s \ge y \right\} \\
&= 2\mathbb{P}^0 \{B_t \ge y\} \\
&= 1 - 2\,\mathbb{P}^0\{0 \le B_t \le y\} \\
&= 1 - 2\int_0^{y/\sqrt{t}} \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx
\end{aligned}$$

By differentiating, we get (4.6). □

Suppose we start a two-dimensional Brownian motion at yi and let $\tau = \tau_{\mathbb{H}}$. Then τ has the same distribution as T of the preceding lemma. We define

$$\|\nu_{\mathbb{H}}(yi, x; t)\| = \frac{y}{\sqrt{2\pi}\, t^{3/2}}\, e^{-y^2/2t}\, \frac{y}{\sqrt{2\pi t}}\, e^{-x^2/2t} = \frac{y}{2\pi t^2}\, e^{-(x^2+y^2)/2t}.$$

Note that

$$\int_0^\infty \|\nu_{\mathbb{H}}(yi, x; t)\|\, dt = \int_0^\infty \frac{y}{2\pi t^2}\, e^{-(x^2+y^2)/2t}\, dt = \frac{y}{\pi(x^2+y^2)} = \frac{1}{\pi}\, H_{\mathbb{H}}(yi, x).$$

The probability measure $\nu_{\mathbb{H}}^{\#}(yi, x; t)$ can be described by saying that the real and imaginary components are independent; the real component is a bridge of time duration t from 0 to x; and the imaginary part is a Brownian motion starting at y conditioned to hit 0 first at time t.

We now define

$$\nu_{\mathbb{H}}(0, x; t) = \lim_{y\downarrow 0} y^{-1}\, \nu_{\mathbb{H}}(yi, x; t),$$

and define $\nu_{\mathbb{H}}(x', x; t)$ similarly by translation. We can describe $\nu_{\mathbb{H}}(0, x; t)$ as the measure of total mass $(2\pi t^2)^{-1}\, e^{-x^2/2t}$ whose probability measure $\nu_{\mathbb{H}}^{\#}(0, x; t)$ is described as follows:

- The real and imagainary parts are independent.

- The real part is a Brownian bridge from 0 to x of time duration t. We will write this probability measure as $\lambda_1^{\#}(x;t)$
- The imaginary part is the probability measure associated to a positive excursion of a Brownian motion conditioned to return to the origin at the first time at time t. We write this probability measure as $\lambda_2^{\#}(t)$.

Using the decomposition above, we also write

$$\lambda_1(x;t) = \frac{1}{\sqrt{2\pi t}}\, e^{-x^2/2}\, \lambda_1^{\#}(x;t),$$

$$\lambda_2(t) = \frac{1}{\sqrt{2\pi}\, t^{3/2}}\, e^{-y^2/2t}\, \lambda_2^{\#}(t).$$

Definition The *(Brownian) excursion measure* in the upper half-plane is defined by

$$\nu_{\mathbb{H}}(x,x') = \lim_{y\downarrow 0} y^{-1}\, \nu_{\mathbb{H}}(x+iy,x') = \int_0^{\infty} \nu_{\mathbb{H}}(x,x';t)\, dt.$$

If $x \neq x'$, then $\nu_{\mathbb{H}}(x,x')$ is a finite measure with total mass

$$\|\nu_{\mathbb{H}}(x,x')\| = \frac{1}{\pi} H_{\partial\mathbb{H}}(x,x') = \frac{1}{\pi\,(x'-x)^2}.$$

If $x = x'$, then $\nu_{\mathbb{H}}(x,x')$ and the limit in the definition must be taken in an appropriate sense. For example $\nu_{\mathbb{H}}(x,x)$ restricted to loops of time duration at least t, is given by

$$\lim_{y\downarrow 0} y^{-1} \int_t^{\infty} \nu_{\mathbb{H}}(x+iy,x).$$

Definition Suppose D is a domain and $z, w \in \partial D$ such that ∂D is analytic in neighborhoods of z and w, Then the *(Brownian) excursion measure* $\nu_D(z,w)$ is defined by

$$\nu_D(z,w) = \lim_{\epsilon\downarrow 0} \epsilon^{-1}\, \nu_D(z+\epsilon \mathbf{n}_z, w),$$

where $\mathbf{n}_z$ denotes the unit interior normal at z. If V, V' are closed analytic arcs of ∂D, then we define

$$\nu_D(z,V) = \int_V \nu_D(z,w)\, |dw|,$$

$$\nu_D(V',V) = \int_{V'} \nu_D(z,V)\, |dz| = \int_{V'} \int_V \nu_D(z,w)\, |dw|\, |dz|.$$

If $z = w$, then the excursion measure $\nu_D(z,z)$ is an infinite measure while if $z \neq w$ it is a finite measure. Similarly, if $\operatorname{dist}(z,V) > 0$, then $\nu_D(z,V)$ is a finite measure and if V, V' are compact disjoint arcs, $\nu_D(V',V)$ is a finite measure.

Proposition 4.1.10. *Suppose $f : D \to f(D)$ is a conformal transformation that can be extended locally in neighborhoods of $z, w \in \partial D$. Let V, V' be analytic arcs and that f is locally analytic at points of V, V'. Then*

$$f \circ \nu_D(z, w) = |f'(z)|\,|f'(w)|\,\nu_{f(D)}(f(z), f(w)),$$
$$f \circ \nu_D(z, V) = |f'(z)|\,\nu_{f(D)}(f(z), f(V)),$$
$$f \circ \nu_D(V', V) = \nu_{f(D)}(f(V'), f(V)).$$

In fact, from this we see that we do not need the assumption that the arcs V, V" be analytic. We write

$$\nu_{\partial D} = \nu_D(\partial D, \partial D),$$

and note that this is a conformally invariant measure.

Definition If z is a locally analytic boundary point of a domain D, the *(boundary) bubble measure* ${\nu^{\rm bub}}_D(z)$ is defined by

$${\nu^{\rm bub}}_D(z) = \pi\,\nu)D(z, z) == \pi \lim_{\epsilon \downarrow 0} \epsilon^{-1}\,\nu_D(z + \epsilon \mathbf{n}_z, z),$$

We include the factor π in order to agree with definitions in other places. Note that the bubble measure satisifes the conformal covariance rule

$$f \circ {\nu^{\rm bub}}_D(z) = |f'(z)|^2\,{\nu^{\rm bub}}_{f(D)}(f(z)).$$

This assumes analtyicity of f in a neighborhood of z.

4.1.8 Concatenation formulas

When analyzing Brownian measures, it is useful to have formulas that break up a path into smaller pieces. Such path decompositions are standard for studying discrete processes. In the continuous situation, similar arguments are generally given in terms of the strong Markov property using stopping times for the process. When dealing with finite measures that are not probability measures it is useful to write down such expressions in terms of path decompositions. We will state some results here. The proofs are not difficult and will be omitted.

For ease, we will assume that the boundaries at which we split the path are locally analytic. One can give expressions without these assumptions, but they are not quite as nice. For practial purposes, one generally tries to split a path at a nice boundary.

Proposition 4.1.11. *Let D be a domain, let $D' \subset D$ be a subdomain, and let η denote the closure of $D \cap \partial D'$. Assume that η is a finite union of analytic curves, and let z, w be distinct analytic points in $\overline{D} \setminus \eta$.*

- *If $z \in \overline{D'}$,*
$$\nu_D(z, w) - \nu_{D'}(z, w) = \int_{D \cap \partial D_1} [\nu_{D'}(z, \zeta) \oplus \nu_D(\zeta, w)]\;|d\zeta|, \tag{4.7}$$
- *If $w \in \overline{D'}$,*
$$\nu_D(z, w) - \nu_{D'}(z, w) = \int_{D \cap \partial D'} [\nu_D(z, \zeta) \oplus \nu_{D'}(\zeta, w)]\;|d\zeta|, \tag{4.8}$$

- *If* $z, w \in \overline{D'}$,

$$\nu_D(z,w) - \nu_{D'}(z,w) = \int_{D\cap\partial D_1}\int_{D\cap\partial D'} \left[\nu_{D'}(z,\zeta_1)\oplus\nu_D(\zeta_1,\zeta_2)\oplus\nu_{D'}(\zeta_2,w)\right]\,|d\zeta_1|\,|d\zeta_2|. \tag{4.9}$$

We note that in (4.7), if $w \notin \overline{D_1}$, then $\nu_{D'}(z,w)$ is the zero measure, and hence the right-hand side equals $\nu_D(z,w)$. Similarly if either $z \notin \overline{D_1}$ or $w \notin \overline{D_1}$, the right-hand side of (4.9) equals $\nu_D(z,w)$.

Proof. These are all applications of the strong Markov property and reversibility. We will do (4.7) in the case $z \in D', w \in D$. Let $\tau = \tau_D, \tau' = \tau_{D'}$ and let $\sigma_s = \sigma_{\epsilon,w} = \inf\{t > \tau' : |B_t - w| \le \epsilon\}$. Let μ_ϵ denote the measure on paths $B_t, 0 \le t \le \sigma_\epsilon$ given by the Brownian measure restricted to the event $\{\sigma_\epsilon < \tau\}$. Then,

$$\nu_D(z,w) - \nu_{D'}(z,w) = \lim_{\epsilon\downarrow 0}(\pi\epsilon^2)^{-1}\,\mu_\epsilon.$$

We write $B[0,\sigma_\epsilon] = B[0,\tau'] \oplus B[\tau',\sigma_\epsilon]$. By the strong Markov property, the distribution of $B[\tau',\sigma_\epsilon]$ given $\mathcal{F}_{\tau'}$ is that of a Brownian motion starting at $B_{\tau'}$ stopped when it gets within distance ϵ of w. If we restrict to the event that this occurs before the path leaves D, multiply by $(\pi\epsilon^2)^{-1}$, and then take the limit, we get the measure $\nu_D(B_{\tau'}, w)$. The expression (4.7) is obtained by integrating over all possible paths $B[0,\tau']$. If $z \in \partial D$ or $w \in \partial D$, we can do a similar argument or we can take limits as the points are approached from the inside.

We get (4.8) by path reversal,

$$\nu_D(z,w) - \nu_{D'}(z,w) = \left[\nu_D(w,z) - \nu_{D'}(w,z)\right]^R,$$

and we get (4.9) by combining (4.7) and (4.8).

□

4.2 Brownian loop measure

Let $\mathcal{K}_L$ denotes the set of curves $\gamma \in \mathcal{K}$ with $\gamma(0) = \gamma(t_\gamma)$. We call elements of $\mathcal{K}_L$ *(rooted) loops.*

Let us define the measure ν on $\mathcal{K}_L$ by

$$\nu = \int_{\mathbb{C}} \nu(z)\,dA(z) = \int_C \int_0^\infty \nu(z;t)\,dt\,dA(z).$$

This is an infinite measure. Indeed, there are three places that the integral blows up. For fixed z, the integral over t blows up both near the origin and at infinity since $||\nu(z,t)|| = (2\pi t)^{-1}$. Also, there is a blowup in z because we are integrating over the unbounded set $\mathbb{C}$. If $D \subset \mathbb{C}$ is a domain, then the measure ν_D is defined by restriction,

$$\nu_D = \int_D \nu_D(z)\,dA(z) = \int_D \int_0^\infty \nu_D(z;t)\,dt\,dA(z).$$

If D is a bounded domain, then this measure is infinite, but there is only one source of blowup: the dt integral as $t \to 0$. In particular, if D is bounded, $\epsilon > 0$, and $V = V_{\epsilon,D}$, $V' = V'_{\epsilon,D}$ denote the set

of loops $\gamma \subset D$ with $t_\gamma \geq \epsilon$ and $\mathrm{diam}(\gamma) \geq \epsilon$, respectively, then $\nu_D(V) < \infty$, $\nu_D(V') < \infty$. One way to view $\nu(z)$ in terms of finite measures is as a consistent collection of measures $\{\nu_D(z) - \nu_{D'}(z)\}$ where D, D' are bounded domains with $z \in D' \subset D$.

The measure $\nu_D(z)$ satisfies: if $w \to z$, then $\nu_D(w, z) \to \nu_D(z)$. More precisely, every subdomain $z \in D' \subset D$,

$$\lim_{w \to z} [\nu_D(w, z) - \nu_{D'}(w, z)] = \nu_D(z) - \nu_{D'}(z).$$

Using conformal invariance of the measures $\nu_D(w, z)$, we can see that the measures $\nu_D(z)$ are conformally invariant:

- If $f : D \to f(D)$ is a conformal transformation, then

$$f \circ \nu_D(z) = \nu_{f(D)}(f(z)).$$

However, the integrated measure ν_D is not conformally invariant.

Proposition 4.2.1. *If $f : D \to f(D)$ is a conformal transformation, and $g = f^{-1}$, then*

$$f \circ \nu_D = \int_{f(D)} |g'(w)|^2 \, \nu_{f(D)}(w) \, dA(w).$$

In other words,

$$\frac{d[f \circ \nu_D]}{d\nu_{f(D)}}(\gamma) = |g'(\gamma(0))|^2.$$

Proof.

$$\begin{aligned} f \circ \nu_D &= \int_D f \circ \nu_D(z) \, dA(z) \\ &= \int_D \nu_D(f(z)) \, dA(z) \\ &= \int_D |f'(z)|^{-2} \, \nu_{f(D)}(f(z)) \, |f'(z)|^2 \, dA(z) \\ &= \int_{f(D)} |g'(w)|^2 \, \nu_{f(D)}(w) \, dA(w). \end{aligned}$$

□

We would like to define a conformally invariant measure. As a step towards this we define a new measure here.

Definition The *rooted Brownian loop measure* is the measure $\tilde{\mu}$ defined by

$$\frac{d\tilde{\mu}}{d\nu}(\gamma) = \frac{1}{t_\gamma}.$$

The measure $\tilde{\mu}_D$ is obtained by restriction.

This measure is not quite conformally invariant. However, if we restrict the measure to certain sets of curves, then we will get a conformally invariant measure. If γ is a loop, which we can view as a periodic function $\gamma : (-\infty, \infty) \to \mathbb{C}$, we define $\theta_s\gamma$ by $t_{\theta_s\gamma} = t_\gamma$ and

$$\theta_s\gamma(t) = \gamma(s+t), \quad -\infty < t < \infty.$$

Suppose $\mathcal{F}$ is the Borel σ-algebra of curves under the topology induced by the metric ρ. Let $\mathcal{F}^u$ be the sub σ-algebra of sets $E \in \sigma$ that satisfy

- If $\gamma \in E$, then $\theta_s\gamma \in E$ for all $0 \le s \le t_\gamma$.

It is easy to check that $\mathcal{F}^u$ is a σ-algebra. Examples of sets in $\mathcal{F}^u$ are:

$$\{\gamma : s \le t_\gamma \le t\},$$

$$\{\gamma : \gamma \subset D\},$$

$$\{\gamma : \gamma \cap V \neq \emptyset\}.$$

Also, if $E \in \mathcal{F}^u$ and $f : D \to f(D)$ is a conformal transformation, then $\{\gamma : f \circ \gamma \in E\} \subset \mathcal{F}^u$. Indeed, we can define an equivalence relation $\sim$ on curves by $\gamma \sim \gamma_1$ if $\gamma_1 = \theta_s\gamma$ for some s. Then, it is easy to see that if $\gamma \sim \gamma_1$, then $f \circ \gamma \sim f \circ \gamma_1$. The equivalence classes of loops γ under the equivalence relation $\sim$ are called *(oriented) unrooted loops*.

If ϕ is a continuous, nonnegative function on $\mathcal{K}$, let μ^ϕ denote the measure on $\mathcal{F}^u$ defined by

$$\frac{d\mu^\phi}{d\nu}(\gamma) = \phi(\gamma).$$

Note that if ϕ_1, ϕ_2 are two such functions such that for every $\gamma \in \mathcal{K}_L$,

$$\int_0^{t_\gamma} \phi_1(\theta_s\gamma)\, ds = \int_0^{t_\gamma} \phi_2(\theta_s\gamma)\, ds,$$

then $\mu^{\phi_1} = \mu^{\phi_2}$. We write μ^ϕ_D for the measures restricted to D.

Definition The *(unrooted) Brownian loop measure* μ is the measure $\tilde{\mu}$ restricted to the σ-algebra $\mathcal{F}^u$.

Equivalently, we can consider μ as a measure on unrooted loops, and perhaps this is more natural. Define an equivalence relation by $\gamma \sim \gamma'$ if $\gamma' = \theta_s\gamma$ for some s. The set of equivalence classes are the *unrooted loops*, and measures on unrooted loops can be defined by taking the quotient topology and then the corresponding Borel σ-algebra.

For the remainder, when we refer to the Brownian loop measure we are referring to μ. If we wish to refer to $\tilde{\mu}$, we will explicitly say the rooted loop measure. Note that $\mu = \mu^\phi$ where $\phi(\gamma) = 1/t_\gamma$ and hence

$$\int_0^{t_\gamma} \phi(\theta_s\gamma)\, ds = 1. \tag{4.10}$$

Using this we could define:

- The Brownian loop measure is the measure μ^ϕ where ϕ is any continuous, nonnegative function on $\mathcal{K}_L$ satisfying (4.10) for very loop $\gamma \in \mathcal{K}_L$.

Theorem 4.2.2. *Suppose $f : D \to f(D)$ is a conformal transformation. Then*

$$f \circ \mu_D = \mu_{f(D)}.$$

Proof. Let $g = f^{-1}$. By Proposition 4.2.1,

$$\frac{d(f \circ \nu_D)}{d\nu_{f(D)}}(\gamma) = |g'(\gamma(0))|^2,$$

and hence,

$$\frac{d(f \circ \tilde{\mu}_D)}{d\nu_{f(D)}}(\gamma) = \phi(\gamma),$$

where

$$\phi(\eta) = \frac{|g'(\eta(0))|^2}{t_{g\circ\eta}} = |g'(\eta(0))|^2 \left[\int_0^{t_\eta} |g'(\eta(s))|^2 \, ds\right]^{-1}.$$

Note that

$$\phi(\theta_s\eta) = |g'(\eta(s))|^2 \left[\int_0^{t_\eta} |g'(\eta(s))|^2 \, ds\right]^{-1},$$

and hence

$$\int_0^{t_\eta} \phi(\theta_s\eta) \, ds = 1.$$

Therefore, $\mu^\phi_{f(D)} = \mu_{f(D)}$.

□

It is worth noting that no topological constraints were put on the domain D. The domain can be multiply connected.

It will be useful to give an equivalent version of the measures $\nu(z)$ and $\mu(z)$. For ease assume that $z = 0$. To each loop $\gamma : [0, t_\gamma] \to \mathbb{C}$ with $\gamma(0) = \gamma(t_\gamma) = 0$, we associate the ordered pair $(t_\gamma.\tilde{\gamma})$ where $\tilde{\gamma} : [0, 1] \to \mathbb{C}$ is the scaled loop:

$$\tilde{\gamma}(s) = t_\gamma^{-1/2} \, \gamma(s \, t_\gamma), \quad 0 \le s \le 1.$$

We can write

$$\nu(0) = \left(\frac{1}{2\pi t} \, dt\right) \times \nu^\#(0, 0; 1),$$

$$\mu(0) = \left(\frac{1}{2\pi t^2} \, dt\right) \times \nu^\#(0, 0; 1),$$

Here we recall that $\nu^\#(0, 0; 1)$ is the probability measure associated to a Brownian bridge conditioned to return to the origin at time 1. Similarly we can consider the measures ν and μ as measures

on triples $(z, t_\gamma, \tilde{\gamma})$, where z is the root of the loop. To be explicit, the triple $(z, t_\gamma, \tilde{\gamma})$ corresponds to the rooted loop

$$\gamma(t) = z + t_\gamma^{1/2}\, \tilde{\gamma}(t/t_\gamma), \quad 0 \le t \le t_\gamma.$$

and the measures ν and $\tilde{\mu}$ are given by

$$\nu = \text{area} \times \left(\frac{1}{2\pi t}\, dt\right) \times \nu^{\#}(0,0;1),$$

$$\tilde{\mu} = \text{area} \times \left(\frac{1}{2\pi t^2}\, dt\right) \times \nu^{\#}(0,0;1).$$

The Brownian loop measure by nature is defined up to an arbitrary multiplicative constant. It will be important for us to keep track of the constant we have chosen. Roughly speaking, in the measure μ, the density of loops of time duration t in a domain D looks like $(2\pi t^2)^{-1}\,\text{area}(D)$. This can be made precise if D goes to infinity for fixed t or t goes to zero for fixed D.

4.2.1 Brownian bubbles in $\mathbb{H}$

We have defined the measure Brownian loop measure as the measure obtained from a particular measure on rooted loops. This definition is best to prove certain properties such as conformal invariance and convergence of random walk loop measure to the Brownian loop measure, but it is not the best definition for computations. It is more convienient to choose roots of unrooted loops in a different way. This will lead us to *Brownian (boundary) bubbles*.

We will start by considering what happens when one chooses a point of smallest imaginary part as the root. Let $\mathcal{K}^{\text{bub}}$ be the set of $\gamma \in \mathcal{K}_L$ such that

$$\text{Im}(\gamma(s)) > \text{Im}(\gamma(0)), \quad 0 < s < t_\gamma.$$

Let $\mathcal{K}^{\text{bub}}(z)$ denote the set of $\gamma \in \mathcal{K}^{\text{bub}}$ with $\gamma(0) = z$.

We define the measure $\nu^{\text{bub}} = \nu^{\text{bub}}{}_{\mathbb{H}}(0)$ on $\mathcal{K}^{\text{bub}}(0)$ roughly as

$$\nu^{\text{bub}} = \pi\, \nu_{\mathbb{H}}(0,0) = \lim_{\epsilon \downarrow 0} \frac{\pi}{\epsilon}\, \nu_{\mathbb{H}}(\epsilon i, 0).$$

The factor π is just a conventional choice. This definition is imprecise, because we are taking a limit of measures whose total mass is going to infinity. To make it precise we will do the following. If $D \subset D'$, and D, D' agree in a neighborhood of z, we let $\Gamma_{D'}(D; z)$ be the $\nu^{\text{bub}}{}_{D'}(z)$ measure of loops that do not stay in D, that is, $\Gamma_{D'}(D; z)$ is the total mass of the measure $\nu^{\text{bub}}{}_{D'}(z) - \nu^{\text{bub}}{}_{D}(z)$. If $D \subset \mathbb{H}$ with $\text{dist}(0, \mathbb{H} \cap \partial D) > 0$, we write just $\Gamma(D)$ for $\Gamma_{\mathbb{H}}(D; 0)$. More generally, if D is a simply connected domain with analytic boundary point z, and z is an analytic boundary point and $f : D \to \mathbb{H}$ is a conformal transformation with $f(z) = 0$, then

$$\Gamma_D(D'; z) = |f'(z)|^2\, \Gamma[f(D')].$$

If $D \subset \mathbb{H}$ is a domain with $\operatorname{dist}(0, \mathbb{H} \cap \partial D) > 0$, let $\Gamma(D)$ denote the ν^{bub} measure of loops that do not stay in D. Note that

$$\Gamma(D) = \int_{\mathbb{H}\setminus\partial D} H(w,0)\, d\mathcal{E}_D(0,dw).$$

For example, if $D = \mathbb{D}_+$, then

$$\Gamma(\mathbb{D}_+) = \int_0^\theta [\sin\theta] \left[\frac{2}{\pi}\,\sin\theta\right] d\theta = 1.$$

Proposition 4.2.3. *Let $F(z) = -1/z$. Then*

$$\Gamma(D) = \operatorname{hcap}\left[F(\mathbb{H}\cap\partial D)\right].$$

Proof. Let B_t be a Brownian motion in $\mathbb{C}$ and let $\tau = \tau_D = \inf\{t : B_t \notin D\}$. Then we can write

$$\Gamma(D) = \lim_{y\downarrow 0} \frac{1}{y}\mathbb{E}^{iy}\left[\operatorname{Im}[H(B_\tau,0)]; B_\tau \notin \mathbb{R}\right] = \lim_{y\downarrow 0} \frac{1}{y}\mathbb{E}^{iy}\left[\operatorname{Im}[H(B_\tau,0)]\right].$$

If $z = re^{i\theta}$, then $F(z) = r^{-1}\, e^{i\theta}$, and $H(z,0) = \operatorname{Im}\left[F(z)\right]$. Hence

$$\Gamma(D) = \lim_{y\to 0} \frac{1}{y}\,\mathbb{E}^{iy}[\operatorname{Im} f(B_\tau)].$$

If $W = F\circ B$ and $\sigma = \inf\{t : W_t \notin F(D)\}$, then conformal invariance shows that

$$\Gamma(D) = \lim_{y\to\infty} y\,\mathbb{E}^{iy}[\operatorname{Im} F(W_\sigma)].$$

□

Proposition 4.2.4. *Suppose D is a simply connected domain with $\operatorname{dist}(0, \mathbb{H}\cap\partial D) > 0$ and let $f : D \to \mathbb{H}$ be a conformal transformation with $f(0) = 0$. Then*

$$\Gamma(D) = -\frac{1}{6}\, Sf(0), \tag{4.11}$$

where S denotes the Schwarzian derivative

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3f''(z)^2}{2f'(z)^2}.$$

Proof. Without loss of generality assume that $f'(0) = 1$. This determines f up to one real constant. Let $F(z) = -1/z$ and write $f = F\circ g\circ F$ where g is a conformal transformation of $F(D)$ onto $\mathbb{H}$ satisfying $F(z) \sim z$ as $z\to\infty$. By the previous proposition we know that

$$g(z) = z + x + \frac{\Gamma(D)}{z} + O(|z|^{-2})$$

for some $x \in \mathbb{R}$ (this is the one degree of freedom in the choice of f). Therefore,

$$f(x) = -\left[-\frac{1}{z} + x - \Gamma(D)z\right]^{-1} = z\,\left[1 - xz + \Gamma(D)\, z^2\right]^{-1} = z + xz^2 - \left[x^2 - \Gamma(D)\right] z^3,$$

and,

$$f''(0) = 2x, \quad f'''(x) = 6\,x^2 - 6\,\Gamma(D), \quad Sf'(0) = -6\Gamma(D).$$

□

Note that the left hand side of (4.11) is independent of the choice of conformal transformation f. Any other transformation would be of the form $T \circ f$ where T is a Möbius transformation of $\mathbb{H}$ fixing the origin. One could check directly that $S(T \circ f)(0) = Sf(0)$.

The definition of the bubbles is essentially the same as the excursion measure except for the factor of π and the fact that it is an infinite measure. Recall that if $f : \mathbb{H} \to D$ is a conformal transformation that is locally analytic at 0 and $x > 0$, then

$$f \circ \nu_{\mathbb{H}}(0, x) = |f'(0)|\,|f'(x)|\,\nu_D(f(0), f(x)).$$

We can do similarly for the Brownian bubbles and define ${\nu^{\text{bub}}}_D(z)$ at analytic boundary points z by

$$f \circ \nu^{\text{bub}}(0) = |f'(0)|^2\,{\nu^{\text{bub}}}_D(z).$$

Here $f : \mathbb{H} \to D$ is a conformal transformation with $f(0) = z$. It is not hard to check that this is the same as

$$\lim_{\epsilon \downarrow 0} \frac{\pi}{\epsilon} \nu_D(z + \epsilon \mathbf{n}_z, z),$$

where $\mathbf{n}_z$ is the inward unit normal. For this reason we will also use the excursion measure notation

$$\nu_D(z, z) = \frac{1}{\pi}\,{\nu^{\text{bub}}}_D(z) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \nu_D(z + \epsilon \mathbf{n}_z, z),$$

where we must remember this is not exactly a limit in $\mathcal{K}$ since the limit is an infinite measure. If $D' \subset D$, and z is an analytic boundary point, we define $\Gamma_D(D'; z)$ to be the ${\nu^{\text{bub}}}_D(z)$ measure of bubbles that intersect $D \cap \partial D'$. If f is as above and $D_1 \subset \mathbb{H}$, then we get the scaling rule

$$\Gamma(D_1) = |f'(0)|^2\,\Gamma_D(f(D'), f(0)).$$

There is a relationship between Brownian bubbles and the Brownian loop measure.

Theorem 4.2.5. *Considered as a measure on the unrooted σ-algebra $\mathcal{F}^u$,*

$$\mu = \frac{1}{\pi} \int_{\mathbb{C}} {\nu^{\text{bub}}}_{\mathbb{H}+iy}(x + iy)\,dx\,dy = \int_{\mathbb{C}} \nu_{\mathbb{H}+z}(z, z)\,dA(z).$$

The factor of $1/\pi$ is to compensate for the π in the definition of the bubble. The basic idea is to attach to each unrooted loop the rooted loop obtained by rooting at the (unique up to a set of loops of measure zero) point of minimal imaginary part. Technically, it is easier to go the other way — we start with the measure

$$\int_{\mathbb{C}} \nu_{\mathbb{H}+z}(z, z)\,dA(z), \tag{4.12}$$

then assign the root of each loop uniformly, and see that we get the rooted Brownian loop measure. As before, let us write a rooted loop as a triple $(z, t_\gamma, \tilde{\gamma})$.

We give another argument here that splits the path into the real and imaginary parts. We start by giving an alternative construction of the measure

$$\nu_{\mathbb{H}}(0,0) = \lim_{y \downarrow 0} y^{-1} \, \nu_{\mathbb{H}}(iy, 0).$$

We recall that the limit on the right-hand side must be taken in some sense, that is, considered as a finite measure restricted to some nice set of curves. Another way to take the limit is as follows. Let use write

$$\nu_{\mathbb{H}}(iy, 0) = \int_0^\infty \nu_{\mathbb{H}}(iy, 0; t) \, dt,$$

where $\nu_{\mathbb{H}}(iy, 0; t)$ denotes the measure restricted to curves of time duration t. Then we define

$$\nu_{\mathbb{H}}(0,0) = \int_0^\infty \nu_{\mathbb{H}}(0, 0; t) \, dt.$$

where the measures $\nu_{\mathbb{H}}(0, 0; t)$ can be defined by

$$\nu_{\mathbb{H}}(0, 0; t) = \lim_{y \downarrow 0} y^{-1} \, \nu_{\mathbb{H}}(iy, 0; t).$$

Since the real and imaginary parts of a Brownian motion are independent, we can write

$$\nu_{\mathbb{H}}(iy, 0; t) = \lambda_1(t) \times \lambda_2(t; y)$$

where $\lambda_1(t)$ is the one-dimensional Brownian loop measure at the origin for time t, and $\lambda_2(t; y)$ is the one-dimensional Brownian measure corresponding to paths starting at y and exiting the positive real axis at time t. Note that $\|\lambda_1(t)\| = 1/\sqrt{2\pi t}$ and $\lambda_1^{\#}(t)$ is the one-dimensional Brownian bridge measure.

The total mass $\|\lambda_2(t; y)\|$ is the density of the hitting time T of the origin for a one-dimensional Brownian motion starting at y, which is given by

$$\|\lambda_2(t; y)\| = \frac{y}{\sqrt{2\pi}\, t^{3/2}} \, \exp\left\{-y^2/2t\right\}. \tag{4.13}$$

This formula is derived using symmetry and the reflection principle for Brownian motion,

$$\begin{aligned}
\mathbb{P}\{T \le t\} &= \mathbb{P}^y \left\{ \min_{0 \le s \le t} B_s \le 0 \right\} \\
&= \mathbb{P}^0 \left\{ \max_{0 \le s \le t} B_s \ge y \right\} \\
&= 2\mathbb{P}^0 \left\{ B_t \ge y \right\} \\
&= 1 - 2\,\mathbb{P}^0\{0 \le B_t \le y\} \\
&= 1 - 2 \int_0^{y/\sqrt{t}} \frac{1}{\sqrt{2\pi}} \, e^{-x^2/2} \, dx
\end{aligned}$$

By differentiating, we get (4.13). In particular, for $t > 0$,

$$\lim_{y \downarrow 0} y^{-1} \, \|\mu_{\mathbb{H}}(iy, 0; t)\| = \lim_{y \downarrow 0} y^{-1} \, \|\lambda_1(t) \times \lambda_2(t; y)\| = \frac{1}{2\pi t^2}.$$

The probability measure $\lambda_2^{\#}(t; iy)$ can be described as Brownian motion B_t starting at y tilted by the function

$$J_s = F(t-s, B_s),$$

where

$$F(s,x) = \frac{x}{\sqrt{2\pi}\, s^{3/2}} \exp\left\{-x^2/2s\right\}.$$

Note that

$$F'(s,x) = F(s,x)\left[\frac{1}{x} - \frac{x}{t-s}\right],$$

and hence we have

$$dB_s = \left[\frac{1}{B_s} - \frac{B_s}{t-s}\right] ds + dW_s,$$

where W_s is a Brownian motion in the tilted measure. By comparison with a Bessel process, we can see that this process does not hit the origin before time t. However, as $s \uparrow t$, the drift towards the origin is so strong to force $B_t = 0$ in the tilted measure. Using this equation, we can construct the limit

$$\lambda_2^{\#}(t;0) = \lim_{y \downarrow 0} \lambda_2^{\#}(t; iy).$$

This measure on one-dimensional paths is essentially the same as what is known as Itô excursion measure for one-dimensional Brownian motion. We have described the measure on one-dimensional excursions in the positive half-line "conditioned" to return to the origin at time t. We put "conditioned" in quotes because we really multiply this measure by $[\sqrt{2\pi}\, t^{3/2}]^{-1}$

Putting this all together, we see that $\nu_{\mathbb{H}}(0,0;t)$ is a measure of total mass $(2\pi t^2)^{-1}$ whose probability measure $\nu_{\mathbb{H}}^{\#}(0,0;t)$ is a one-dimensional Brownian bridge in the first component and an independent one-dimensional excursion of in the second component, both of time duration t. Then

$$\nu_{\mathbb{H}}(0,0) = \int_0^\infty \nu_{\mathbb{H}}(0,0;t)\, dt.$$

In particular, if we consider the integrated measure

$$\int_{\mathbb{C}} \nu_{\mathbb{H}+z}(z,z)\, dA(z),$$

then the density of loops of time duration t is a domain D is asymptotic to $\text{area}(D)\,(2\pi t^2)^{-1}$. This asymptotics can be taken as $D \uparrow \mathbb{C}$ for fixed t or $t \downarrow 0$ for fixed D. In particular, the constant is the same as for the Brownian loop measure μ.

It is not hard to show that Brownian scaling holds for the excursion measures $\nu_{\mathbb{H}}^{\#}(0,0;t)$, i.e., if γ is distributed according to $\nu_{\mathbb{H}}^{\#}(0,0;t)$ and $\tilde{\gamma}(s) = t^{-1/2}\,\gamma(st), 0 \le s \le t$, then $\tilde{\gamma}$ has the distribution $\nu^{\#}(0,0;1)$. Therefore, we can write the measure (4.12) as a measure on triples $(z, t_\gamma, \tilde{\gamma})$ as

$$\text{area} \times \left(\frac{1}{2\pi t^2}\, dt\right) \times \nu_{\mathbb{H}}^{\#}(0,0,1).$$

We now use an important fact about one-dimensional bridges and one-dimensional excursions. If $\gamma(t), 0 \le t \le 1$ is a path with $\gamma(0) = \gamma(1) = 0$, considered as function of period one, we write $\Theta_s\gamma(t) = \gamma(s+t) - \gamma(s)$.

- If γ has the distribution $\lambda_1^{\#}(1)$ and U is an independent uniform random variable on $[0,1]$, then $\Theta_U\gamma$ has distribution $\lambda_1^{\#}(1)$.
- If γ has the distribution $\lambda_2^{\#}(0;1)$ and U is an independent uniform random variable on $[0,1]$, then $\Theta_U\gamma$ has distribution $\lambda_1^{\#}(1)$.

This can be proved in a number of ways. One way is to approximate by simple random walk for which the analogous facts are easy. Applying this to the real and imaginary parts of a two-dimensional process we get the following.

- If γ has the distribution $\nu^{\#}(0,0;1)$ and U is an independent uniform random variable on $[0,1]$, then $\Theta_U\gamma$ has distribution $\nu^{\#}(0,0;1)$.
- If γ has the distribution $\nu_{\mathbb{H}}^{\#}(0,0;1)$ and U is an independent uniform random variable on $[0,1]$, then $\Theta_U\gamma$ has distribution $\nu^{\#}(0,0;1)$.

It may be useful to consider a random walk analogue. Consider simple random walk on the integer lattice $\mathbb{Z}$. It is known that

$$\mathbb{P}\{S_{2n} = 0\} = \frac{1}{\sqrt{\pi n}} + O(n^{-3/2}).$$

Note that $1/\sqrt{\pi n}$ is $2p_{2n}(0,0)$ where p_t is the transition probability for Brownian motion. The factor of 2 comes from the bipartite nature of the random walk, that is, the fact that $\mathbb{P}\{S_{2n+1} = 0\}$. Let us fix an integer n and let $\omega = [\omega_0, \ldots, \omega_{2n}]$ be a random walk loop with $\omega_0 = \omega_{2n} = 0$. Let $X = X_{2n}$ be the largest integer x such that $-x$ is visited by ω and let $\sigma = \sigma_{2n}$ be the the smallest index s such that $\omega_s = -X$. Note that s and x (and, hence, also $2n - s$) must have the same parity. We have split $\omega = \omega^1 \oplus \omega^2$ in a unique fashion such that the the terminal point, say $-x$, of ω^1 is smaller than the other vertices of ω^1, the initial of ω^2 is the same as the terminal point of ω^1, and no other vertex of ω^2 is smaller than its initial point. (However, $-x$ may appear more than once in ω^2.)

If S is a simple random walk starting at the origin and n is an integer, then

$$\sum_{x=0}^{\infty} \sum_{s=0}^{2n-1} \mathbb{P}\{S_{2n} = 0, X = x, \sigma - s\} = \mathbb{P}\{S_{2n} = 0\} \sim \frac{1}{\sqrt{\pi n}}.$$

Let us consider n large and let $x_n = x/\sqrt{2n}$. Recall that $W_t^{(n)} := S_{2nt}/\sqrt{2n}$ converges to a Brownian motion.

By comparison with Brownian motion, we expect that if $s + x$ is even,

$$\mathbb{P}\{S_s = -x; S_j > -x, j < x\} \sim 2\, F(s,x),$$

where $F(s,x)$ is the Brownian quantity,

$$F(s,x) = \frac{x}{\sqrt{2\pi}\, s^{3/2}} \exp\left\{-x^2/2s\right\}.$$

The factor of 2 comes from the bipartite nature of the walk. However, we expect

$$\mathbb{P}\{S_s = -x; S_j \geq -x, j < x\} \sim 4\, F(s,x).$$

The extra factor of two represents the expected number of visits to $-x$ before reaching $-x-1$. Therefore, if x is even,

$$\begin{aligned}
\mathbb{P}\{S_{2n} = 0, \sigma = 2s\} &= \sum_{k=0}^{\infty} \mathbb{P}\{S_{2n} = 0, \sigma = 2s, X = 2k\} \\
&\sim \sum_{k=0}^{\infty} 2F(2r,2k)\, 4\, F(2n-2r,2k) \\
&= \frac{8}{(2n)^2} \sum_{k=0}^{\infty} F\left(\frac{r}{n}, \frac{2k}{\sqrt{2n}}\right) F\left(\frac{n-r}{n}, \frac{2k}{\sqrt{2n}}\right) \\
&\asymp \frac{8}{(2n)^{3/2}} \int_0^{\infty} F(t,z)\, F(1-t,z)\, dz \\
&= \frac{8}{(2n)^{3/2}}\, \frac{1}{2\sqrt{2\pi}} = \frac{1}{2n\sqrt{\pi n}}.
\end{aligned}$$

Similarly, we get

$$\mathbb{P}\{S_{2n} = 0, \sigma = 2s+1\} \sim \frac{1}{2n\sqrt{\pi n}},$$

which is consistent wth

$$\sum_{s=0}^{2n-1} \mathbb{P}\{S_{2n} = 0, \sigma = 2s+1\} = \mathbb{P}\{S_{2n} = 0\} \sim \frac{1}{\sqrt{\pi n}}.$$

Proposition 4.2.6. *Considered as a measure on the unrooted σ-algebra $\mathcal{F}^u$,*

$$\mu_{\mathbb{H}} = \frac{1}{\pi} \int_0^{\pi} \int_0^{\infty} \nu_{U_r}(re^{i\theta}, re^{i\theta})\, r\, dr\, d\theta, \tag{4.14}$$

where $U_r = \{z \in \mathbb{H} : |z| > r\}$.

Proof. Let $f(z) = e^z$ which is a conformal transformation of $D := \{x + iy \in \mathbb{H} : 0 < y < \pi\}$ onto $\mathbb{H}$. Therefore, if $D_y = D \cap (\mathbb{H} + y)$,

$$\begin{aligned}
\mu_{\mathbb{H}} = f \circ \mu_D &= \int_0^{\pi} \int_0^{\infty} f \circ \nu_{D_y}(x+iy, x+iy)\, dx\, dy \\
&= \int_0^{\pi} \int_0^{\infty} e^{2x}\, \nu_{U_{e^x}}(e^{x+iy}, e^{x+iy})\, dx\, dy \\
&= \int_0^{\pi} \int_0^{\infty} r\, \nu_{U_r}(re^{iy}, re^{iy})\, dr\, dy
\end{aligned}$$

□

Proposition 4.2.7. *There exists $c < \infty$ such that if $K \subset \mathbb{H}$ such that $D = \mathbb{H} \setminus K$ is a domain with $\operatorname{dist}(0, K) \geq 1$, and $q(\epsilon, D)$ denotes the $\mu_{\mathbb{H}}$ measure of loops that intersect both the circle of radius ϵ and K, then*

$$q(\epsilon, D) = \epsilon^2 \, \Gamma(D) \, [1 + O(\epsilon)], \quad \epsilon \downarrow 0.$$

Proof. The ${\nu^{\rm bub}}_{U_r}(re^{i\theta})$ measure of loops that intersect K is given by $\Gamma_{U_r}(D \cap U_r; re^{i\theta})$. Then (4.14) implies that

$$q(\epsilon, D) = \frac{1}{\pi} \int_0^{\pi} \int_0^{\epsilon} \Gamma_{U_r}(D \cap U_r; re^{i\theta}) \, r \, dr \, d\theta.$$

Let

$$f_r(z) = z + \frac{r^2}{z},$$

which is a conformal transformation of U_r onto $\mathbb{H}$. Then

$$\Gamma_{U_r}(D \cap U_r; re^{i\theta}) = |f_r'(re^{i\theta})|^2 \, \Gamma_{\mathbb{H}}(f_r(D), f_r(re^{i\theta})).$$

Since $|f_r(re^{i\theta})| = O(r)$, we can use derivative estimates to see that

$$\Gamma_{\mathbb{H}}(f_r(D), f_r(re^{i\theta})) = \Gamma(f_r(D)) \, [1 + O(r)].$$

Using $f_r(z) = f_r(1) + O(r^2)$ for $|z| = 1$ and
we see that

$$\Gamma(f(D)) = \Gamma(D)[1 + O(r)].$$

Also,

$$|f'(re^{i\theta})|^2 = \left|1 - e^{-2i\theta}\right|^2 = 4 \, \sin^2 \theta.$$

Therefore,

$$\begin{aligned} q(\epsilon, D) &= \frac{\Gamma(D)}{\pi} \int_0^{\pi} \int_0^{\epsilon} 4[\sin^2 \theta] \, r \, [1 + O(r)] dr \, d\theta \\ &= \epsilon^2 \, \Gamma(D) \, [1 + O(\epsilon)]. \end{aligned}$$

□

Let $\Lambda_D(V_1, V_2)$ denote the μ_D measure of loops that intersect both V_1 and V_2.

Proposition 4.2.8. *There exists $c < \infty$ such that the following holds. Suppose $D \subset \mathbb{H}$ is a domain with $\operatorname{dist}(0, \partial D) \geq 1$. Let $0 < r \leq 1/2$ and let V be a closed subset of $\mathbb{H}$ contained in $\{|z| \leq r\}$. Then,*

$$|\Lambda_{\mathbb{H}}(\mathbb{H} \setminus D, V) - \operatorname{hcap}(V) \, \Gamma(D)| \leq c \, r \operatorname{hcap}(V) \, \Gamma(D).$$

Proof. (Sketch) If V is the half disk of radius r, then this is the previous lemma. More generally, for any loop that hits both K and $r\mathbb{D}_+$, we consider the excursions from the unit circle to $r\mathbb{D}_+$. For each of these we can ask the probability that the excursion hits V.

We claim that for an excursion in $\mathbb{D}_+$ connecting two boundary points in $\mathbb{D}_+ \setminus \mathbb{H}$, the probability to hit V given that the excursion hits $r\mathbb{D}_+$ is $r^{-2}\,\mathrm{hcap}(V)\,[1+O(r)]$. To see this we start by recalling that the probability that the excursion from $z = e^{\beta i}$ to $w = e^{\psi i}$ hits $r\mathbb{D}_+$ is $H_{\partial\mathbb{D}_+}(z,w)^{-1}$ times

$$\begin{aligned}\int_{|\zeta|=r} H_{\mathbb{D}_+}(\zeta,w)\,d\mathcal{E}_{\mathbb{D}_+\setminus r\mathbb{D}_+}(z,d\zeta) &= [1+O(r)]\,\frac{2r\sin\beta}{\pi}\int_0^\pi H_{\mathbb{D}_+}(re^{i\theta},w)\,\sin\theta\,d\theta\\ &= [1+O(r)]\,\frac{2r^2\sin\beta\,\sin\psi}{\pi}\int_0^\pi \sin^2\theta\,d\theta\\ &= r^2\,\sin\beta\,\sin\psi\,[1+O(r)]. \qquad (4.15)\end{aligned}$$

Since K is contained in $r\mathbb{D}_+$, we can use the strong Markov property to see that

$$\int_{|\zeta|=r} H_{\mathbb{D}_+}(\zeta,w)\,d\mathcal{E}_{\mathbb{D}_+\setminus V}(z,d\zeta) = [1+O(r)]\,\frac{2r\sin\beta}{\pi}\int_0^\pi \mathbb{E}^{re^{i\theta}}[H_{\mathbb{D}_+}(B_\sigma,w)]\,\sin\theta\,d\theta,$$

where σ is the first time that a Brownian motion leaves $\mathbb{D}_+ \setminus V$. We can see that

$$\begin{aligned}\mathbb{E}^{re^{i\theta}}[H_{\mathbb{D}_+}(B_\sigma,w)] &= \sin\psi\,\mathbb{E}^{re^{i\theta}}[\mathrm{Im}(B_\sigma); B_\sigma\in V]\,[1+O(r)]\\ &= \sin\psi\,\mathbb{E}^{re^{i\theta}}[\mathrm{Im}(B_{\sigma'})]\,[1+O(r)]\end{aligned}$$

where σ' is the first time that the Brownian motion leaves $\mathbb{H}\setminus V$. Recall that

$$\mathrm{hcap}(V) = \frac{2r}{\pi}\int_0^\infty \mathbb{E}^{re^{i\theta}}[\mathrm{Im}(B_{\sigma'})]\,\sin\theta\,d\theta.$$

Therefore,

$$\int_{|\zeta|=r} H_{\mathbb{D}_+}(\zeta,w)\,d\mathcal{E}_{\mathbb{D}_+\setminus V}(z,d\zeta) = \sin\beta\,\sin\psi\,\mathrm{hcap}(V)\,[1+O(r)]. \qquad (4.16)$$

Comparing (4.15) and (4.16) gives the claim. □

By considering derivatives we get the following corollary.

Corollary 4.2.9. *Suppose that $D\subset\mathbb{H}$ is a subdomain, γ is a continuous curve in $\mathbb{H}$ with $\gamma(0+) = 0$, and let $\gamma_t = \gamma(0,t]$. Suppose that $\gamma_t \subset D$. Let $a(t) = \mathrm{hcap}[\gamma_t]$ and assume that a is C^1. Let g_t denote the unique conformal transformation of the unbounded component of $\mathbb{H}\setminus\gamma_t$ onto $\mathbb{H}$ with $g_t(z) - z \to 0$ and let $\beta_s = g_s(\gamma(s))$. Then*

$$\Lambda_{\mathbb{H}}(\gamma_t, \mathbb{H}\setminus D) = \int_0^t \dot a(s)\,\Gamma(g_s(D);\beta_s)\,ds. \qquad (4.17)$$

4.2.2 Brownian loop measure on cylinder

Let $\mathcal{C}yl$ denote the cylinder $[0,2\pi]\times\mathbb{C}$ or equivalently the complex plane with the identification $z\sim z+2\pi k$ for all integers k. The Brownian loop measure on $\mathcal{C}yl$ can be defined as before

$$\text{area}\times\left(\frac{1}{2\pi t^2}\,\nu^{\#}_{\mathcal{C}yl,t}dt\right),$$

where now $\nu_t^{\mathcal{C}yl}$ denotes the appropriate Brownian bridge of time duration on the cylinder. This latter term can be considered in two independent parts: the vertical part is a one-dimensional Brownian bridge on $\mathbb{R}$ and the horizontal part is a Brownian bridge on the circle, that is, on the interval $[0, 2\pi]$ with periodic boundary conditions. Similarly, we can consider a Brownian bubble decomposition where we choose the y part from a Brownian excursion and again choose the x part from a Brownian motion on the circle.

Conformal invariance can be proved as before and this is most important when consider the exponential map $z \mapsto e^{iz}$ that conformally maps $\mathcal{C}yl$ onto $\mathbb{C} \setminus \{0\}$. The image of the Brownian loop measure is the Brownian loop measure.

Proposition 4.2.10. *Considered as a measure on the unrooted σ-algebra $\mathcal{F}^u$,*

$$\mu_{\mathbb{C}} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \nu_{O_r}(re^{i\theta}, re^{i\theta})\, r\, dr\, d\theta = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \nu_{r\mathbb{D}}(re^{i\theta}, re^{i\theta})\, r\, dr\, d\theta, \tag{4.18}$$

where $O_r = \{z \in \mathbb{C} : |z| > r\}$.