A note on the Brownian loop measure

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Abstract

The Brownian loop measure is a conformally invariant measure on loops that also satisfies the restriction property. In studying the Schramm-Loewner evolution (SLE), a quantity that arises is the measure of loops in a domain D that intersect both V_1 and V_2. If V_1, V_2 are nonpolar, and $D = \mathbb{C}$ this measure is infinite. We show the existence of a finite normalized quantity that can be used in its place. The motivation for studying this question comes from bulk SLE with boundary conditions, but this paper only discusses the loop measure.

1 Introduction

The Brownian loop measure [4] in $\mathbb{C}$ is a sigma-finite measure on unrooted loops in $\mathbb{C}$ that satisfies two important properties: conformal invariance and the restriction property. It arose in the study of the Schramm-Loewner evolution (SLE). An important quantity for SLE is

$$\Lambda(V_1, V_2; D),$$

which denotes the measure of the set of loops in a domain D that intersect both V_1 and V_2. This quantity comes up in comparison of SLE in two different domains. If $\mu_D(z_1, z_2)$ denotes the chordal SLE_κ ($\kappa \leq 4$) measure for z_1 to z_2 in a domain D and $\tilde{D} \subset D$ is a subdomain that agrees with D in neighborhoods of z_1 and z_2, then

$$\frac{d\mu_{\tilde{D}}(z_1, z_2)}{d\mu_D(z_1, z_2)}(\gamma) = \exp\left\{\frac{\mathbf{c}}{2}\Lambda(\gamma, D \setminus \tilde{D}; D)\right\},$$

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where

$$\mathbf{c} = \frac{(3\kappa - 8)(6-\kappa)}{2\kappa}$$

denotes the central charge.

If V_1, V_2 are nonpolar disjoint closed subsets of the Riemann sphere and D is a domain whose boundary is nonpolar, then this quantity is positive and finite (see Corollary 4.6). It is also conformally invariant in the sense that if $f : D \to f(D)$ is a conformal transformation, then

$$\Lambda(f(V_1), f(V_2); f(D)) = \Lambda(V_1, V_2; D).$$

In trying to generalize these ideas to SLE in the bulk, see [2], one is tempted to write similar quantities of the form

$$\Lambda(V_1, V_2; \mathbb{C}).$$

However this quantity as defined is infinite. The purpose of this note is to prove the existence of a normalized quantity $\Lambda^*(V_1, V_2)$ that has many of the properties one wants.

Theorem 1.1. *If V_1, V_2 are disjoint nonpolar closed subsets of the Riemann sphere, then the limit*

$$\Lambda^*(V_1, V_2) = \lim_{r \to 0+} [\Lambda(V_1, V_2; \mathcal{O}_r) - \log\log(1/r)], \tag{1}$$

exists where

$$\mathcal{O}_r = \{z \in \mathbb{C} : |z| > r\}.$$

Moreover, if f is a Möbius transformation of the Riemann sphere,

$$\Lambda^*(f(V_1), f(V_2)) = \Lambda^*(V_1, V_2).$$

We could write the assumption "disjoint nonpolar closed subsets of the Riemann sphere" as "disjoint closed subsets of $\mathbb{C}$, at least one of which is compact, such that Brownian motions hits both subsets at some positive time".

Invariance of Λ^* under Möbius transformations implies that the definition (1) does not change if we shrink down at a point on the Riemann sphere other than the origin. In other words if $\mathcal{O}_r(z) = z + \mathcal{O}_r$ and $\mathbb{D}_R$ denotes the open disk of radius R about the origin,

$$\Lambda^*(V_1, V_2) = \lim_{r \to 0+} [\Lambda(V_1, V_2; \mathcal{O}_r(z)) - \log\log(1/r)], \tag{2}$$

$$\Lambda^*(V_1, V_2) = \lim_{R\to\infty} [\Lambda(V_1, V_2; \mathbb{D}_R) - \log\log R]. \tag{3}$$

The goal of this paper is to prove Theorem 1.1. Theorem 4.7 establishes the existence of the limit in (1). Theorem 4.11 proves the alternate forms (2) and (3). If f is a Möbius transformation, then conformal invariance of the loop measure implies

$$\Lambda(V_1, V_2; \mathcal{O}_r) = \Lambda(f(V_1), f(V_2); f(\mathcal{O}_r)).$$

Invariance of Λ^* under dilations, translations, and inversions can be deduced from this and (1), (2), and (3), respectively.

The proof really only uses standard arguments about planar Brownian motion but we need to control the error terms. In order to make the paper easier to understand, we have split it into three sections. The first section considers estimates for planar Brownian motion. Readers who are well acquainted with planar Brownian motion may wish to skip this section and refer back as necessary. This section does assume knowledge of planar Brownian motion as in [1, Chapter 2]. The next section discusses the Brownian (boundary) bubble measure and gives estimates for it. The Brownian loop measure is a measure on unrooted loops, but for computational purposes it is often easier to associate to each unrooted loop a particular rooted loop yielding an expression in terms of Brownian bubbles. The last section proves the main theorem by giving estimates for the loop measure.

We will use the following notation:

$$\mathbb{D}_r = \{z : |z| < r\}, \quad \mathbb{D} = \mathbb{D}_1,$$

$$\mathcal{O}_r = \{z : |z| > r\}, \quad \mathcal{O} = \mathcal{O}_1, \quad \mathcal{O}_r(w) = w + \mathcal{O}_r,$$

$$A_{r,R} = \mathbb{D}_R \cap \mathcal{O}_r = \{z : r < |z| < R\}, \quad A_R = A_{1,R},$$

$$C_r = \partial\mathbb{D}_r = \partial\mathcal{O}_r = \{|z| = r\}, \quad C_r(w) = \partial\mathcal{O}_r(w) = \{z : |w - z| = r\}.$$

We say that a subset V of $\mathbb{C}$ is *nonpolar* if it is hit by Brownian motion. More precisely, V is nonpolar if for every $z \in \mathbb{C}$, the probability that a Brownian motion starting at z hits V is positive. Since Brownian motion is recurrent we can replace "is positive" with "equals one". For convenience, we will call a domain (connected open subset) D of $\mathbb{C}$ nonpolar if ∂D is nonpolar.

2 Lemmas about Brownian motion

If B_t is a complex Brownian motion and D is a domain, let

$$\tau_D = \inf\{t : B_t \not\in D\}.$$

A domain D is nonpolar if and only if $\mathbf{P}^z\{\tau_D < \infty\} = 1$ for every z. In this case we define harmonic measure of D at $z \in D$ by

$$h_D(z, V) = \mathbf{P}^z\{B_{\tau_D} \in V\}.$$

If V is smooth then we can write

$$h_D(z, V) = \int_V h_D(z, w)\, |dw|,$$

where $h_D(z, w)$ is the *Poisson kernel*. If $z \in \partial D \setminus V$ and ∂D is smooth near z, we define the *excursion measure* of V in D from z by

$$\mathcal{E}_D(z, V) = \mathcal{E}(z, V; D) = \partial_{\mathbf{n}} h_D(z, V),$$

where $\mathbf{n} = \mathbf{n}_{z,D}$ denotes the unit inward normal at z. If V is smooth, we can write

$$\mathcal{E}_D(z, V) = \int_V h_{\partial D}(z, w)\, |dw|,$$

where $h_{\partial D}(z, w) := \partial_{\mathbf{n}} h_D(z, w)$ is the *excursion or boundary Poisson kernel.* (Here the derivative $\partial_{\mathbf{n}}$ is applied to the first variable.) One can also obtain the excursion Poisson kernel as the normal derivative in both variables of the Green's function; this establishes symmetry, $h_{\partial D}(z, w) = h_{\partial D}(w, z)$. If $f : D \to f(D)$ is a conformal transformation, then (assuming smoothness of f at boundary points at which f' is taken)

$$h_D(z, V) = h_{f(D)}(f(z), f(V)),$$

$$h_D(z, w) = |f'(w)|\, h_{f(D)}(f(z), f(w)),$$
$$\mathcal{E}_D(z, V) = |f'(w)|\, \mathcal{E}_{f(D)}(f(z), f(V)),$$
$$h_{\partial D}(z, w) = |f'(z)|\, |f'(w)|\, h_{\partial f(D)}(f(z), f(w)).$$

The exact form of the Poisson kernel in the unit disk shows that there is a c such that for all $|z| \le 1/2$, $|w| = 1$

$$|2\pi\, h_{\mathbb{D}}(z, w) - 1| \le c\, |z|.$$

By taking an inversion, we get that if $|z| \geq 2$,

$$|2\pi\, h_{\mathcal{O}}(z,w) - 1| \leq \frac{c}{|z|}. \tag{4}$$

It is standard that

$$h_{A_R}(z, C_R) = \frac{\log |z|}{\log R}, \quad 1 < |z| < R. \tag{5}$$

In particular,

$$\mathcal{E}_{A_R}(1, C_R) = \frac{1}{\log R}, \quad \mathcal{E}_{A_R}(R, C_1) = \frac{1}{R \log R}, \tag{6}$$

If $V \subset \partial D$ is smooth, let $\overline{h}_D(z,w;V) = h_D(z,w)/h_D(z,V)$ for $w \in V$. In other words, $\overline{h}_D(z,w;V)$ is the density of the exit distribution of a Brownian motion *conditioned* so that it exits at V. We similarly define $\overline{h}_{\partial D}(z,w;V)$.

Lemma 2.1. *There exists $c < \infty$ such that the following holds. Suppose $R > 0$ and D is a domain with $A_R \subset D \subset \mathcal{O}$. Then*

$$|2\pi\, \overline{h}_D(z,w) - 1| \leq c\, \frac{\log R}{R}, \quad |w| = 1, z \in D \cap \overline{\mathcal{O}}_{R/2}. \tag{7}$$

Remark The conclusion of this lemma is very reasonable. If a Brownian motion starting at a point z far from the origin exits D at C_1, then the hitting distribution is almost uniform. This uses the fact that $D \cap \mathbb{D}_R$ is the same as A_R. The important result is the estimate of the error term.

Proof. Assume $|w| = 1$. Let $\tau = \tau_D$ and let $\partial^* = \partial D \cap \mathcal{O}$. It suffices to prove the estimate for $|z| = R/2$. For every $|\zeta| \geq R/2$, (4) gives

$$|2\pi h_{\mathcal{O}}(\zeta, w) - 1| \leq \frac{c}{R}. \tag{8}$$

Note that

$$h_{\mathcal{O}}(z,w) = h_D(z,w) + \mathbf{E}^z[h_{\mathcal{O}}(B_\tau, w); B_\tau \in \partial^*].$$

Using (8), we get

$$2\pi \mathbf{E}^z[h_{\mathcal{O}}(B_\tau, w); B_\tau \in \partial^*] = h_D(z, \partial^*)\,[1 + O(R^{-1})].$$

Therefore,

$$2\pi\, h_D(z,w) = h_D(z, C_1) + O(R^{-1}). \tag{9}$$

Since $h_D(z, C_1)$ is bounded below by the probability of reaching C_1 before C_R, (5) implies

$$h_D(z, C_1) \geq \frac{\log 2}{\log R},$$

and hence (9) implies

$$2\pi \, h_D(z, w) = h_D(z, C_1) \left[1 + O\left(\frac{\log R}{R}\right)\right]. \tag{10}$$

□

Corollary 2.2. *There exists $c < \infty$ such that if $R \geq 2, |z| = 1, |w| = R$, Then*

$$\left| h_{\partial A_R}(z, w) - \frac{1}{2\pi R \, \log R} \right| \leq \frac{c}{R^2}. \tag{11}$$

Proof. Recall that $h_{\partial A_r}(z, w) = h_{\partial A_r}(w, z)$. We know from (6) that

$$\int_{C_1} h_{\partial A_R}(w, \zeta) \, |d\zeta| = \frac{1}{R \log R}.$$

Also, by definition,

$$h_{\partial A_R}(w, z) = \frac{\overline{h}_{\partial A_R}(w, z)}{R \, \log R}.$$

Note that $\overline{h}_{\partial A_R}(w, z)$ is bounded by the minimum and maximum values of $\overline{h}_{A_R}(\hat{w}, z)$ over $|\hat{w}| = R/2$, and hence (7) gives

$$\overline{h}_{A_R}(\hat{w}, z) = \frac{1}{2\pi} + O\left(\frac{\log R}{R}\right).$$

□

Corollary 2.3. *Suppose $R > 1, D$ is a domain containing $\mathbb{D}_R$, $\hat{D}$ is a nonpolar domain containing $\overline{\mathcal{O}}$ with smooth boundary and $\tilde{D} = D \cap \hat{D}$. For $w \in \partial\hat{D}$, let*

$$q_w = \frac{1}{2\pi} \int_0^{2\pi} h_{\tilde{D}}(e^{i\theta}, w) \, d\theta,$$

and let

$$q = \int_{\partial\hat{D}} q_w \, |dw| = \frac{1}{2\pi} \int_0^{2\pi} h_{\tilde{D}}(e^{i\theta}, \partial\hat{D}) \, d\theta,$$

Then if $z \in D$ with $|z| \geq R/2$ and $w \in \partial\tilde{D}$,

$$h_{\tilde{D}}(z, w) = \frac{q_w}{q} \, h_{D \cap \mathcal{O}}(z, C_1) \, \left[1 + O\left(\frac{\log R}{R}\right)\right].$$

Remark The implicit constants in the $O(\cdot)$ term are uniform and do not depend on $z, w, D, \tilde{D}$. The key fact is that (up to the error term) $h_{\tilde{D}}(z,w)$ factors into two terms q_w/q and $h_{D\cap\mathcal{O}}(z, C_1)$.

Proof. Let $\tau = \tau_{\tilde{D}}$ and Let $\sigma = \tau_{D\cap\mathcal{O}}$. Then (10) implies that if $z \in D$, $|z| \geq R/2$,

$$h_{\tilde{D}}(z, \partial\hat{D}) = q\, h_{D\cap\mathcal{O}}(z, C_1) \left[1 + O\left(\frac{\log R}{R}\right)\right].$$

Similarly,

$$h_{\tilde{D}}(z, w) = q_w\, h_{D\cap\mathcal{O}}(z, C_1) \left[1 + O\left(\frac{\log R}{R}\right)\right].$$

□

Lemma 2.4. *Suppose D is a nonpolar domain containing $\mathbb{D}$. If $0 < s < 1$, let $D_s = D \cap \mathcal{O}_s$. Then if $s < r \leq 1/2$ and $|z| = r$,*

$$\frac{\log r}{\log s} \leq h_{D_s}(z, C_s) \leq \frac{\log r}{\log s} \left[1 - \frac{p \log 2}{(1-p)\log(1/r)}\right]^{-1},$$

where

$$p = p_D = \sup_{|\tilde{w}|=1} h_{D_{1/2}}(\tilde{w}, C_{1/2}) < 1.$$

Remark The inequality $p_D < 1$ follows immediately from the fact that D is nonpolar and contains $\mathbb{D}$.

Proof. Let $T = T_s = \inf\{t : B_t \in C_s \cup C_1\}$ and $\sigma = \sigma_{s,r} = \inf\{t \geq T : B_t \in C_r\}$. Then if $|z| \leq 1/2$,

$$\mathbf{P}^z\{B_{\tau_{D_s}} \in C_s\} = \mathbf{P}^z\{B_T \in C_s\} + \mathbf{P}^z\{B_T \in C_1, B_{\tau_{D_s}} \in C_s\}.$$

By (5),

$$\mathbf{P}^z\{B_T \in C_s\} = \frac{\log r}{\log s},$$

which gives the lower bound. Let

$$q = q(r, s, D) = \sup_{|\tilde{z}|=r} \mathbf{P}^{\tilde{z}}\left\{B_{\tau_{D_s}} \in C_s\right\},$$

$$u = u(r, D) = \sup_{|\tilde{w}|=1} \mathbf{P}^{\tilde{w}}\left\{B_{\tau_{D_r}} \in C_r\right\}.$$

Then,

$$\mathbf{P}^z\{B_T \in C_1, B_{\tau_{D_s}} \in C_s\} \leq$$

$$\mathbf{P}^z\{\sigma < \tau_{D_s} \mid B_T \in C_1\}\, \mathbf{P}^z\{B_{\tau_{D_s}} \in C_s \mid B_T \in C_1, \sigma < \tau_{D_s}\} \leq u\, q.$$

Applying this to the maximizing $\tilde{z}$, gives

$$q \leq \frac{\log r}{\log s} + u\, q, \qquad q \leq \frac{\log r}{(1-u)\, \log s}.$$

By (5), the probability that a Brownian motion starting on $C_{1/2}$ reaches C_r before reaching C_1 is $\log 2/\log(1/r)$. Using a similar argument as in the previous paragraph, we see that

$$u \leq p\, \frac{\log 2}{\log(1/r)} + p\, u, \qquad u \leq \frac{p}{1-p}\, \frac{\log 2}{\log(1/r)}. \tag{12}$$

□

Proposition 2.5. *Suppose D is a nonpolar domain containing the origin. Then there exists $c = c_D < \infty$ such that if $0 < r \leq 1/2$, $D_r = D \cap \mathcal{O}_r$, and $z \in D, |z| \geq 1$,*

$$h_{D_r}(z, C_r) \leq \frac{c}{\log(1/r)}. \tag{13}$$

Also, if $|w| = r$,

$$h_{D_r}(z, w) \leq \frac{c}{r\, \log(1/r)}. \tag{14}$$

Proof. Find $0 < \beta < 1/2$ such that $\partial D \cap \mathcal{O}_{2\beta}$ is nonpolar. It suffices to prove (13) for $r < \beta$. Since $\partial D \cap \mathcal{O}_{2\beta}$ is nonpolar, there exists $q = q_{D,\beta} > 0$ such that for every $|z| \geq 2\beta$, the probability that a Brownian motion starting at z leaves D before reaching C_β is at least q. If $r < \beta$, the probability that a Brownian motion starting at C_β reaches C_r before reaching $C_{2\beta}$ is

$$p(r) = \log 2/\log(2\beta/r) \leq \frac{c_1}{\log(1/r)}.$$

Let $Q(r) = \sup_{|z| \geq 2\beta} h_{D_r}(z, C_r)$. Then arguing similarly to the previous proof, we have

$$Q(r) \leq (1-q)\; [p(r) + [1 - p(r)]\, Q(r)] \leq p(r) + (1-q)\, Q(r),$$

which yields $Q(r) \leq p(r)/q$. This gives (13) and (14) follows from

$$h_{D_r}(z, w) \leq h_{D_{2r}}(z, C_{2r}) \sup_{|\zeta| = 2r} h_{\mathcal{O}_r}(\zeta, w).$$

□

Proposition 2.6. *There exists $c < \infty$ such that the following holds. Suppose $|z| = 1/2$ and $0 < s < r < 1/8$. Let $D_{s,r} = \mathcal{O}_s \cap \mathcal{O}_r(z)$. Then for $|w| \geq 1$,*

$$\left| \frac{\log(rs)}{\log r} h_{D_{s,r}}(w, C_s) - 1 \right| \leq \frac{c}{\log(1/r)}. \tag{15}$$

Proof. Without loss of generality, we assume $z = 1/2$. Let L denote the line $\{x + iy : x = 1/4\}$. Let $\tau = \tau_{D_{s,r}}$, T the first time a Brownian motion reaches $C_r \cup C_r(z)$, and σ the first time after T that the Brownian motion returns to L. By symmetry, for every $w \in L$,

$$\mathbf{P}^w\{B_T \in C_r\} = \frac{1}{2}.$$

Using Lemma 2.4, we get

$$\mathbf{P}^w\{\tau < \sigma \mid B_T \in C_r\} = \frac{\log r}{\log s} \left[1 + O\left(\frac{1}{\log(1/r)}\right)\right].$$

Therefore, for every $w \in L$,

$$\mathbf{P}^w\{\tau < \sigma; B_\tau \in \mathcal{O}_r(z)\} = \frac{1}{2},$$

$$\mathbf{P}^w\{\tau < \sigma; B_\tau \in \mathcal{O}_s\} = \frac{\log r}{2 \log s} \left[1 + O\left(\frac{1}{\log(1/r)}\right)\right].$$

This establishes (15) for $w \in L$. If $|w| \geq 1$, then the probability of reaching $\mathcal{O}_r(z)$ before reaching L is $O(1/\log(1/r))$ and the probability of reaching $\mathcal{O}_s$ before reaching L is $O(1/\log(1/s))$. Using this we get (15) for $|w| \geq 1$. $\square$

Remark The end of the proof uses a well known fact. Suppose one performs independent trials with three possible outcomes with probabilities $p, q, 1 - p - q$, respectively. Then the probability that an outcome of the first time occurs before one of the second type is $p/(p + q)$.

3 Brownian bubble measure

If D is a nonpolar domain and $z \in \partial D$ is an analytic boundary point (i.e., ∂D is analytic in a neighborhood of D), the Brownian bubble measure $m_D(z)$ of D at z is a sigma-finite measure on loops $\gamma : [0, t_\gamma] \to \mathbb{C}$ with $\gamma(0) = \gamma(t_\gamma) = z$ and $\gamma(0, t_\gamma) \subset D$. It can be defined as the limit as $\epsilon \to 0+$ of $\pi \, \epsilon^{-1} \, h_D(z + \epsilon \mathbf{n}, z)$ times the probability measure on paths obtained from starting a Brownian motion at $z + \epsilon \mathbf{n}$ and conditioning so that the path leaves D at z. Here $\mathbf{n} = \mathbf{n}_{z,D}$ is the inward unit normal. If $\tilde{D} \subset D$ agrees with D in a neighborhood of z, then the bubble measure at $\tilde{D}$, $m_{\tilde{D}}(z)$ is obtained from $m_D(z)$ by restriction. This is also an infinite measure but the difference $m_D(z) - m_{\tilde{D}}(z)$ is a finite measure. We will denote its total mass by

$$m(z; D, \tilde{D}) = \|m_D(z) - m_{\tilde{D}}(z)\|.$$

The normalization of m is chosen so that

$$m(0; \mathbb{H}, \mathbb{H} \cap \mathbb{D}) = 1. \tag{16}$$

Remark The factor of π in the bubble measure was put in so that (16) holds. However, the loop measure in the next section does not have this factor so we will have to divide it out again. For this paper, it would have been easier to have defined the bubble measure without the π but we will keep it in order to match definitions elsewhere.

From the definition, we see that if $\partial \tilde{D} \cap D$ is smooth

$$m(z; D, \tilde{D}) = \pi \int_{\partial \tilde{D} \cap D} \partial_{\mathbf{n}} h_{\tilde{D}}(z, w)\, h_D(w, z)\, |dw|.$$

This is also equal to $\pi\, \partial_{\mathbf{n}} f(z)$ for the function $f(\zeta) = h_D(\zeta, z) - h_{\tilde{D}}(\zeta, z)$. Let $h_{D,-}(V, z), h_{D,+}(V, z)$ denote the infimum and supremum, respectively, of $h_D(w, z)$ over $w \in V$. Then a simple estimate is

$$h_{D,-}(\partial \tilde{D} \cap D, z) \leq \frac{m(z; D, \tilde{D})}{\pi\, \mathcal{E}_{\tilde{D}}(z, \partial \tilde{D} \cap D)} \leq h_{D,+}(\partial \tilde{D} \cap D, z). \tag{17}$$

Lemma 3.1. *If $R > 1$, let*

$$\rho(R) = m(1; \mathcal{O}, A_R).$$

There exists $c < \infty$ such that for all $R \geq 2$,

$$\left| \rho(R) - \frac{1}{2 \log R} \right| \leq \frac{c}{R \log R}.$$

Remark Rotational invariance implies that $m(z; \mathcal{O}, A_R) = \rho(R)$ for all $|z| = 1$.

Proof. By (6),

$$\mathcal{E}(1;\mathcal{O},A_R) = \frac{1}{\log R}.$$

and by (4),

$$2\pi\, h_{\mathcal{O}}(w,1) = 1 + O(R^{-1}).$$

We now use (17). □

The next lemma generalizes this to domains D with $A_R \subset D \subset \mathcal{O}$. The result is similar but the error term is a little larger. Note that the q in the next lemma equals 1 if $D = \mathcal{O}$.

Lemma 3.2. *Suppose $R \geq 2$ and D is a domain satisfying $A_R \subset D \subset \mathcal{O}$. Let q be the probability that a Brownian motion started uniformly on C_R exits D at C_1, i.e.,*

$$q = q(R,D) = \frac{1}{2\pi R}\int_{C_R} h_D(z,C_1)\,|dz|.$$

Then if $|w| = 1$,

$$m(w;D,A_R) = \frac{q}{2\,\log R}\left[1 + \left(\frac{\log R}{R}\right)\right], \tag{18}$$

$$m(w;\mathcal{O},D) = \frac{1-q}{2\,\log R} + O\left(\frac{q\log R + 1}{R\log R}\right). \tag{19}$$

Proof. By definition,

$$m(w;D,A_R) = \pi\int_{C_R} h_{\partial A_R}(w,z)\,h_D(z,w)\,|dz|.$$

By (11), we know that

$$h_{\partial A_R}(w,z) = \frac{1}{2\pi\,R\,\log R}\left[1 + O\left(\frac{\log R}{R}\right)\right].$$

By (7), we know that

$$h_D(z,w) = \frac{1}{2\pi}\,h_D(z,C_1)\left[1 + O\left(\frac{\log R}{R}\right)\right].$$

Combining these gives (18), and (19) follows from Lemma 3.1 and

$$m(w;\mathcal{O},A_R) = m(w;D,A_R) + m(w;\mathcal{O},D).$$

□

Corollary 3.3. *There exists $c < \infty$ such that the following is true. Suppose $R \geq 2$ and D is a domain with $A_R \subset D \subset \mathcal{O}$. Suppose $\partial D \cap \mathcal{O}_R$ is nonpolar and hence*

$$p = p_{R,D} := \sup_{|z|=R} h_{D \setminus \mathcal{O}_{R/2}}(z, C_{R/2}) < 1.$$

Then, if $|w| = 1$,

$$\left| m(w; \mathcal{O}, D) - \frac{1}{2 \log R} \right| \leq \frac{c}{(1-p) \log^2 R}.$$

Proof. Let q be as in the previous lemma. By (12) we see that

$$q \leq \frac{p \log 2}{(1-p) \log R}.$$

and hence the result follows from (19). □

Remark We will used scaled versions of this corollary. For example, if D is a nonpolar domain containing $\mathbb{D}$, $r < 1/2$, $D_r = D \cap \mathcal{O}_r$, and $|w| = r$,

$$\left| r^2 \, m(w; \mathcal{O}_r, D_r) - \frac{1}{2 \log(1/r)} \right| \leq \frac{c}{(1-p) \log^2(1/r)},$$

where

$$p = \sup_{|z|=1} h_{D_{1/2}}(z, C_{1/2}).$$

Proposition 3.4. *Suppose V is a nonpolar closed set, $z \neq 0$, and $z, 0 \notin V$. For $0 < r, s < \infty$, let*

$$D_{s,r} = \mathcal{O}_s \cap \mathcal{O}_r(z).$$

Then as $s, r \to 0+$, if $|w| = s$,

$$\frac{1}{\pi} \, m(w; D_{s,r}, D_{s,r} \setminus V) = \frac{1}{2\pi s^2 \log(1/s)} \, \frac{\log r}{\log(rs)} \, \left[1 + O\left(\delta_{r,s}\right)\right],$$

where $\delta_{r,s} = (\log(1/r))^{-1} + (\log(1/s))^{-1}$.

Remark The implicit constants in the $O(\cdot)$ term depend on V, z but not on w.

Proof. We will use (17) and write $\delta = \delta_{r,s}$. By scaling we may assume $z = 2$ and let $d = \min\{2, \mathrm{dist}(0, V), \mathrm{dist}(2, V)\}$. We will only consider $r, s \leq d/2$. By (6),

$$\mathcal{E}(w, C_d; A_{s,d}) = s^{-1}\, \mathcal{E}(w/s, C_{d/s}; A_{d/s}) = \frac{1}{s\, \log(1/s)}\, [1 + O(\delta)]\,.$$

Using this and (5) we can see that

$$\mathcal{E}(w, V; D_{s,r} \setminus V) = \frac{1}{s\, \log(1/s)}\, [1 + O(\delta)]\,.$$

For $\zeta \in V$, (15) gives

$$h_{D_{s,r}}(\zeta, C_s) = \frac{\log r}{\log(rs)}\, [1 + O(\delta)]\,. \tag{20}$$

Therefore. by (7)

$$h_{D_{s,r}}(\zeta, w) = \frac{\log r}{2\pi s \log(rs)}\, [1 + O(\delta)]\,.$$

□

The next proposition is the analogue of Proposition 3.4 with $z = \infty$.

Proposition 3.5. *Suppose V is a nonpolar compact set, with $0 \notin V$. For $0 < s, r < \infty$, let*

$$D_{s,r} = \mathcal{O}_s \cap \mathbb{D}_{1/r}.$$

Then, for as $s, r \to 0+$, if $|w| = s$,

$$\frac{1}{\pi}\, m(w; D_{s,r}, D_{s,r} \setminus V) = \frac{1}{2\pi s^2\, \log(1/s)}\, \frac{\log r}{\log(rs)}\, [1 + O\,(\delta_{r,s})]\,,$$

where $\delta_{r,s} = (\log(1/r))^{-1} + (\log(1/s))^{-1}$.

Proof. The proof is the same as the previous proposition. In fact, it is slighly easier because (20) is justified by (5). □

4 Brownian loop measure

The Brownian loop measure is a measure on unrooted loops. It is this measure that is conformally invariant. For computational purposes it is useful to write the measure in terms of the bubble measure. The following expression is obtained by assigning to each unrooted loop the root closest to the origin, see [2].

$$\mu = \mu_{\mathbb{C}} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} m_{\mathcal{O}_r}(re^{i\theta})\, r\, dr\, d\theta. \tag{21}$$

To be precise, we are considering the right hand side as a measure on unrooted loops. If D is a subdomain, then μ_D is defined by restriction. If $\overline{\mathbb{D}}_r \subset D$, then the Brownian loop measure in D restricted to loops that intersect $\overline{\mathbb{D}}_r$ can be written as

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^{r} m_{D_s}(se^{i\theta})\, s\, dr\, d\theta, \tag{22}$$

where $D_s = D \cap \mathcal{O}_s$. If $r_1 < r$, then the Brownian loop measure of loops in $\mathcal{O}_{r_1}$ that intersect $\overline{\mathbb{D}}_r$ is given by

$$\frac{1}{\pi} \int_0^{2\pi} \int_{r_1}^{r} m_{D_s}(se^{i\theta})\, s\, dr\, d\theta.$$

Using this and appropriate properties of the bubble measure we can conclude the following.

Lemma 4.1. *For every $0 < s < r < \infty$ and $d > 0$, the loop measure of the set of loops in $\mathcal{O}_s$ of diameter at least d that intersect $\mathbb{D}_r$ is finite.*

Remark This result is not true for $s = 0$. The Brownian loop measure of loops in $\mathbb{C}$ of diameter greater than d that intersect the unit disk is infinite. See, e.g., Lemma 4.3 below.

Conformal invariance implies that the Brownian loop measure of loops in $A_{r,2r}$ that separate the origin from infinity is the the same for all r. It is easy to see that this measure is positive and the last lemma shows that it is finite. It follows that the measure of the set of loops that surround the origin is infinite.

If $V_1, V_2, \ldots$ are closed subsets of the Riemann sphere and D is a nonpolar domain, then

$$\Lambda(V_1, V_2, \ldots, V_k; D)$$

is defined to be the loop measure of the set of loops in D that intersect all of $V_1, \ldots, V_k$. Note that

$$\Lambda(V_1, V_2, \ldots, V_k; D) =$$
$$\Lambda(V_1, V_2, \ldots, V_{k+1}; D) + \Lambda(V_1, V_2, \ldots, V_k; D \setminus V_{k+1}). \tag{23}$$

If $V_1, V_2, \ldots, V_k$ are the traces of simple curves that include the origin, then the comment in the last paragraph shows that for all $r > 0$,

$$\Lambda(V_1, V_2, \ldots, V_k; \mathbb{D}_r) = \infty.$$

Lemma 4.2. *Suppose V_1, V_2 are closed sets and D is a domain. Let*

$$V^j = \overline{A_{e^{j-1}, e^j}}, \quad \mathcal{O}^j = \mathcal{O}_{e^j}, \quad D^j = D \cap \mathcal{O}^j.$$

Then

$$\Lambda(V_1, V_2; D) = \sum_{j=-\infty}^{\infty} \Lambda(V_1, V_2, V^{j+1}; D^j). \tag{24}$$

Proof. For each unrooted loop, consider the point on the loop closest to the origin. The measure of the set of loops for which the distance to the origin is exactly e^j for some integer j is 0. For each loop, there is a unique j such that the loop is in $\mathcal{O}^j$ but not in $\mathcal{O}^{j+1}$. Except for a set of loops of measure zero, such a loop intersects V^{j+1} but does not intersect V^k for $k < j+1$, and hence each loop is counted exactly once on the right-hand side of (24). ☐

Lemma 4.3. *There exists $c < \infty$ such that if $0 < s < 1, R \geq 2$,*

$$\left| \Lambda(C_1, C_R; \mathcal{O}_s) - \log \left[\frac{\log(R/s)}{\log R} \right] \right| \leq \frac{c}{R \log R}.$$

In particular, there exists $c < \infty$ such that if $R \geq 2/s > 2$,

$$\left| \Lambda(C_1, C_R; \mathcal{O}_s) - \frac{\log(1/s)}{\log R} \right| \leq \frac{c \log^2(1/s)}{\log^2 R}.$$

Proof. By (22), rotational invariance, and the scaling rule, we get

$$\Lambda(C_1, C_R; \mathcal{O}_s) = 2 \int_s^1 r\, m(r; \mathcal{O}_r, A_{r,R})\, dr = 2 \int_s^1 r^{-1}\, \rho(R/r)\, dr,$$

where ρ is as in Lemma 3.1. From that lemma, we know that

$$\rho(R/r) = \frac{1}{2\log(R/r)} + O\left(\frac{r}{R \log(R/r)} \right),$$

and hence

$$\Lambda(C_1, C_R; \mathcal{O}_s) = O\left(\frac{1}{R\log R}\right) + \int_s^1 \frac{1}{r\,(\log R - \log r)}\,dr.$$

The first assertion follows by integrating and the second from the expansion

$$\log\left[\frac{\log(R/s)}{\log R}\right] = \frac{\log(1/s)}{\log R} + O\left(\frac{\log^2(1/s)}{\log^2 R}\right).$$

□

Lemma 4.4. *Suppose V is a closed, nonpolar set with $0 \notin V$ and $\alpha > 0$. There exists $c = c_{V,\alpha} < \infty$ such that for $r < \min\{1, \operatorname{dist}(0,V)\}/4$,*

$$\left|\Lambda(V, \mathcal{O}_{\alpha r}\setminus\mathcal{O}_r; \mathcal{O}_r) - \frac{\log\alpha}{\log(1/r)}\right| \le \frac{c}{\log^2(1/r)}.$$

Proof. By scaling, we may assume that $\operatorname{dist}(0,V) = 1$. It suffices to prove the result for r sufficiently small. By (22), we have

$$\Lambda(V, \mathcal{O}_{\alpha r}\setminus\mathcal{O}_r; \mathcal{O}_r) = \frac{1}{\pi}\int_0^{2\pi}\int_r^{\alpha r} m(se^{i\theta}; \mathcal{O}_s, D_s)\, s\, ds\, dr,$$

where $D_s = \mathcal{O}_s \setminus V$. By Corollary 3.3, for $r \le s \le \alpha r$,

$$m(se^{i\theta}; \mathcal{O}_s, D_s) = \frac{1}{2s^2\,\log(1/s)}\left[1 + O\left(\frac{1}{\log(1/r)}\right)\right].$$

Therefore,

$$\Lambda(V, \mathcal{O}_{\alpha r}\setminus\mathcal{O}_r; \mathcal{O}_r) = \left[1 + O\left(\frac{1}{\log(1/r)}\right)\right]\int_r^{\alpha r}\frac{ds}{s\,\log(1/s)}.$$

Also,

$$\begin{aligned}\int_r^{\alpha r}\frac{ds}{s\,\log(1/s)} &= \log\log\left(\frac{1}{r}\right) - \log\log\left(\frac{1}{\alpha r}\right)\\ &= \frac{\log\alpha}{\log(1/r)} + O\left(\frac{1}{\log^2(1/r)}\right).\end{aligned}$$

□

Lemma 4.5. *Suppose V_1, V_2 are disjoint, nonpolar closed subsets of the Riemann sphere with $0 \notin V_1$. Then there exists $c = c_{V_1,V_2} < \infty$ such that for all $r \leq \operatorname{dist}(0, V_1)/2$,*

$$\Lambda(V_1, C_r; \mathbb{C} \setminus V_2) \leq \frac{c}{\log(1/r)}. \tag{25}$$

Proof. Constants in this proof depend on V_1, V_2. Without loss of generality assume $0 \notin V_2$ and let $D_r = \mathcal{O}_r \setminus V_2$. We will first prove the result for $r \leq r_0 = [\operatorname{dist}(0, V_1) \wedge \operatorname{dist}(0, V_2)]/2$. By (22), we have

$$\Lambda(V_1, C_r; \mathbb{C} \setminus V_2) = \frac{1}{\pi} \int_{|z| \leq r} m(z; D_{|z|}, D_{|z|} \setminus V_1)\, dA(z). \tag{26}$$

By (14),

$$h_{D_r}(w, z) \leq \frac{c}{r\, \log(1/r)}, \qquad w \in V_1, \quad |z| = r.$$

By comparison with an annulus, we get

$$\mathcal{E}_{D_{|z|} \setminus V_1}(z, V_1) \leq \frac{c}{|z|\, \log(1/|z|)}, \quad |z| = r.$$

Using (17), we then have

$$\frac{1}{\pi}\, m(z; D_{|z|}, D_{|z|} \setminus V_1) \leq \frac{c}{|z|^2\, \log^2(1/|z|)}.$$

By integrating, we get (25) for $r \leq r_0$.

Let $r_1 = \operatorname{dist}(0, V_1)/2$ and note that

$$\Lambda(V_1, C_{r_1}; \mathbb{C} \setminus V_2) \leq \Lambda(V_1, C_{r_0}; \mathbb{C} \setminus V_2) + \Lambda(V_1, C_{r_1}; \mathcal{O}_{r_0} \setminus V_2).$$

Using Lemma 4.1 we can see that

$$\Lambda(V_1, C_{r_1}; \mathcal{O}_{r_0} \setminus V_2) < \infty.$$

Therefore,

$$\Lambda(V_1, C_{r_1}; \mathbb{C} \setminus V_2) < \infty,$$

and we can conclude (25) for $r_0 \leq r \leq r_1$ with a different constant. □

Corollary 4.6. *Suppose V_1, V_2 are disjoint closed subsets of the Riemann sphere and D is a nonpolar domain. Then*

$$\Lambda(V_1, V_2; D) < \infty.$$

Proof. Assume $0 \notin V_1$. Lemma 4.5 shows that $\Lambda(V_1, \overline{\mathbb{D}}_s; D) < \infty$ for some $s > 0$. Note that

$$\Lambda(V_1, V_2; D) \leq \Lambda(V_1, \overline{\mathbb{D}}_s; D) + \Lambda(V_1, V_2; \mathcal{O}_s).$$

Since at least one of V_1, V_2 is compact, Lemma 4.1 implies that

$$\Lambda(V_1, V_2; \mathcal{O}_s) < \infty.$$

□

Theorem 4.7. *Suppose V_1, V_2 are disjoint, nonpolar closed subsets of the Riemann sphere. Then the limit*

$$\Lambda^*(V_1, V_2) = \lim_{r \to 0+} [\Lambda(V_1, V_2; \mathcal{O}_r) - \log\log(1/r)] \tag{27}$$

exists.

Proof. Without loss of generality, assume that $\mathrm{dist}(0, V_1) \geq 2$ and let $\mathcal{O}^k = \mathcal{O}_{e^{-k}}$. Let $\hat{V}_2 \subset V_2$ be a nonpolar closed subset with $0 \notin \hat{V}_2$. Constants in the proof depend on V_1, V_2. Since $\Lambda(V_1, V_2; \mathcal{O}_r)$ increases as r decreases to 0, it suffices to establish the limit

$$\lim_{k \to \infty} \left[\Lambda(V_1, V_2; \mathcal{O}^k) - \log k\right].$$

Repeated application of (23) shows that if $k \geq 1$,

$$\Lambda(V_1, V_2; \mathcal{O}^k) = \Lambda(V_1, V_2; \mathcal{O}^0) + \sum_{j=1}^{k} \Lambda(V_1, V_2, \mathcal{O}^{j-1} \setminus \mathcal{O}^j; \mathcal{O}^j).$$

Similarly, for fixed k, (23) implies

$$\begin{aligned}\Lambda(V_1, \mathcal{O}^{k-1} \setminus \mathcal{O}^k; \mathcal{O}^k) - \Lambda(V_1, V_2, \mathcal{O}^{k-1} \setminus \mathcal{O}^k; \mathcal{O}^k) \\ = \quad \Lambda(V_1, \mathcal{O}^{k-1} \setminus \mathcal{O}^k; \mathcal{O}^k \setminus V_2) \\ \leq \quad \Lambda(V_1, \mathcal{O}^{k-1} \setminus \mathcal{O}^k; \mathcal{O}^k \setminus \hat{V}_2).\end{aligned}$$

From Lemma 4.4, we can see that

$$\Lambda(V_1, \mathcal{O}^{k-1} \setminus \mathcal{O}^k; \mathcal{O}^k) = \frac{1}{k} + O\left(\frac{1}{k^2}\right),$$

and hence the limit

$$\lim_{k\to\infty}\left[-\log k+\sum_{j=1}^{k}\Lambda(V_1,\mathcal{O}^{k-1}\setminus\mathcal{O}^k;\mathcal{O}^k)\right]$$

exists and is finite. By Lemma 4.5, we see that

$$\sum_{j=k}^{\infty}\Lambda(V_1,\mathcal{O}^{j-1}\setminus\mathcal{O}^j;\mathcal{O}^j\setminus\hat{V}_2)=\Lambda(V_1,\overline{\mathbb{D}}^k;D\setminus\hat{V}_2)\leq\frac{c}{k},$$

and hence

$$\sum_{j=k}^{\infty}\left[\Lambda(V_1,\mathcal{O}^{j-1}\setminus\mathcal{O}^j;\mathcal{O}^j)-\Lambda(V_1,V_2,\mathcal{O}^{j-1}\setminus\mathcal{O}^j;\mathcal{O}^j)\right]\leq\frac{c}{k},$$

where the constant c depends on V_1 and $\hat{V}_2$ but not otherwise on V_2.

□

Remark It follows from the proof that

$$\Lambda^*(V_1,V_2)=\Lambda(V_1,V_2;\mathcal{O}^k)-\log k+O\left(\frac{1}{k}\right),$$

where the $O(\cdot)$ terms depends on V_1 and $\hat{V}_2$ but not otherwise on V_2. As a consequence we can see that if $0\notin V_1$ and $V_{2,r}=V_2\cap\{|z|\geq r\}$, then

$$\lim_{r\to 0+}\Lambda^*(V_1,V_{2,r})=\Lambda^*(V_1,V_2). \tag{28}$$

The definition of Λ^* in (27) seems to make the origin a special point. Theorem 4.11 shows that this is not the case.

Lemma 4.8. *Suppose V is a nonpolar closed subset, $z\neq 0$ and $0\notin V$. Let $\alpha>0$. There exists c,r_0 (depending on z,V,α) such that if $0<r<r_0$,*

$$|\Lambda(V,\mathbb{C}\setminus\mathcal{O}_r;\mathcal{O}_{\alpha r}(z))-\log 2|\leq\frac{c}{\log(1/r)}.$$

Proof. We will first assume $z\notin V$. For $s\leq r$, let $D_s=\mathcal{O}_s\cap\mathcal{O}_{\alpha r}(z)$. As in (22),

$$\Lambda(V,\mathbb{C}\setminus\mathcal{O}_r;\mathcal{O}_{\alpha r}(z))=\frac{1}{\pi}\int_{|w|\leq r}m(w;D_{|w|},D_{|w|}\setminus V)\,dA(w).$$

By Lemma 3.4, if $|w| = s \le r$,

$$\frac{1}{\pi}\, m(w; D_s, D_s \setminus V) = \frac{1}{2\pi s^2 \log(1/s)}\, \frac{\log r}{\log(rs)} \left[1 + O\left(\frac{1}{\log(1/r)}\right)\right],$$

and therefore,

$$\Lambda(V, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z)) = \log r \int_0^r \frac{ds}{s\, \log(1/s)\, \log(rs)} \left[1 + O\left(\frac{1}{\log(1/r)}\right)\right].$$

A straightforward computation gives

$$\log r \int_0^r \frac{ds}{s\, \log(1/s)\, \log(rs)} = \log 2.$$

This finishes the proof for $z \notin V$.

If $z \in V$ and $t > 0$, let $V_1 \subset V$ be a closed nonpolar set with $z \notin V_1$. Then (23) implies

$$\Lambda(V, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z)) =$$
$$\Lambda(V_1, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z)) + \Lambda(V \setminus V_1, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z) \setminus V_1).$$

Since the previous paragraph applies to V_1 it suffices to show that

$$\Lambda(V \setminus V_1, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z) \setminus V_1) = O\left(\frac{1}{\log(1/r)}\right).$$

We can write

$$\Lambda(V \setminus V_1, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z) \setminus V_1) = \frac{1}{\pi} \int_{|w| \le r} m(w; D_s \setminus V_1, D_s \setminus V)\, dA(w).$$

By using (18) we can see that

$$m(w; D_s \setminus V_1, D_s \setminus V) \le \frac{c}{\log^2(1/s)},$$

and hence

$$\Lambda(V \setminus V_1, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z) \setminus V_1) \le c \int_0^r \frac{ds}{s\, \log^2(1/s)} \le \frac{c}{\log(1/r)}.$$

□

The following is the equivalent lemma for $z = \infty$. It can be proved similarly or by conformal transforamtion.

Lemma 4.9. *Suppose V is a nonpolar closed subset, and $0 \notin V$. Let $\alpha > 0$. There exists c, r_0 (depending on V, α) such that if $0 < r < r_0$,*

$$\left|\Lambda(V, \mathbb{C} \setminus \mathcal{O}_r; \mathbb{D}_{\alpha/r}) - \log 2\right| \leq \frac{c}{\log(1/r)}.$$

We extend this to k closed sets.

Lemma 4.10. *Suppose $V_1, \ldots, V_k$ are closed nonpolar subsets of $\mathbb{C} \setminus \{0\}$. Let $\alpha > 0$. There exists c, r_0 (depending on $z, \alpha, V_1, \ldots, V_k$) such that if $0 < r < r_0$,*

$$|\Lambda(V_1, \ldots, V_k, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z)) - \log 2| \leq \frac{c}{\log(1/r)}.$$

$$\left|\Lambda(V_1, \ldots, V_k, \mathbb{C} \setminus \mathcal{O}_r; \mathbb{D}_{\alpha/r}) - \log 2\right| \leq \frac{c}{\log(1/r)}.$$

Proof. If $k = 2$, inclusion-exclusion implies

$$\Lambda(V_1 \cup V_2, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z)) + \Lambda(V_1, V_2, , \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z)) =$$
$$\Lambda(V_1, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z)) + \Lambda(V_2, , \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z)).$$

Since Lemma 4.8 applies to $V_1 \cup V_2, V_1, V_2$, we get the result. The cases $k > 2$ and $z = \infty$ are done similarly. □

Theorem 4.11. *Suppose V_1, V_2 are disjoint, nonpolar closed subsets of the Riemann sphere and $z \in \mathbb{C}$. Then*

$$\Lambda^*(V_1, V_2) = \lim_{r \to 0+} \left[\Lambda(V_1, V_2; \mathcal{O}_r(z)) - \log\log(1/r)\right].$$

Moreover,

$$\Lambda^*(V_1, V_2) = \lim_{R \to \infty} \left[\Lambda(V_1, V_2; D_R) - \log\log R\right].$$

Proof. We will assume $0 \notin V_1$. Using (27), we see that it suffices to prove that

$$\lim_{r \to 0+} \left[\Lambda(V_1, V_2; \mathcal{O}_r(z)) - \Lambda(V_1, V_2; \mathcal{O}_r)\right] = 0.$$

Note that

$$\Lambda(V_1, V_2; \mathcal{O}_r(z)) - \Lambda(V_1, V_2; \mathcal{O}_r) =$$
$$\Lambda(V_1, V_2, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_r(z)) - \Lambda(V_1, V_2, \mathbb{C} \setminus \mathcal{O}_r(z); \mathcal{O}_r).$$

Lemma 4.10 implies

$$\Lambda(V_1, V_2, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_r(z)) = \log 2 + O\left(\frac{1}{\log(1/r)}\right), \tag{29}$$

where the constants in the error term depend on z, V_1, V_2. Similarly, using translation invariance of the loop measure, we can see that

$$\Lambda(V_1, V_2, \mathbb{C} \setminus \mathcal{O}_r(z); \mathcal{O}_r) = \log 2 + O\left(\frac{1}{\log(1/r)}\right).$$

□

If $V_1, V_2, \ldots, V_k$ are pairwise disjoint nonpolar closed sets of the Riemann sphere, we define similarly

$$\Lambda^*(V_1, \ldots, V_k) = \lim_{r \to 0+} \left[\Lambda(V_1, V_2, \ldots, V_k; \mathcal{O}_r) - \log\log(1/r)\right].$$

One can prove the existence of the limit in the same way or we can use the relation

$$\Lambda^*(V_1, \ldots, V_k) = \Lambda^*(V_1, \ldots, V_{k+1}) + \Lambda(V_1, \ldots, V_k; \mathbb{C} \setminus V_{k+1}).$$

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