1 Brownian measures

1.1 Introduction

Probabilists generally like to view Brownian motion as a random process. Indeed, many of the powerful tools from probability require this viewpoint. However, there are other times, e.g., when considering models from statistical physics, to consider Brownian motion and related processes as a measure on paths, not necessarily a probability measure. We will look at Brownian motion this way although we will take advantage of the fact that the reader already has a good feeling for the probabilistic construct. We will consider a number of path measures but the most important will be the *Brownian loop measure* and the corresponding *Brownian bubble measure*.

It will be necessary to set up some notation. We will need a metric on the space of measures in order to topologize the space. This will allow us to consider continuous measure-valued functions and to integrate these functions. This can be skipped at a first reading.

- Let $\mathcal{K}$ denote the set of curves, that is, the set of continuous functions $\gamma : [0, t_\gamma] \to \mathbb{C}$. We do not make any further assumptions of γ of either smoothness or number of self-intersections.

- We call t_γ the *time duration* and and $\gamma(0), \gamma(t_\gamma)$ the *initial and terminal points*, respectively, of γ.

- If D is a domain, we will abuse notation slightly and write $\gamma \subset D$ if $\gamma(0, t_\gamma) \subset D$. The initial and terminal points may be in D or on ∂D. We write $\mathcal{K}_D = \{\gamma \in \mathcal{K} : \gamma \subset D\}$.

- If $\gamma \in \mathcal{K}_D$ and $f : D \to f(D)$ is holomorphic such that

$$t_{f\circ\gamma} := \int_0^{t_\gamma} |f'(\gamma(s))|^2 \, ds < \infty, \tag{1}$$

  and the limits

$$\lim_{t\downarrow 0} f(\gamma(0)), \quad \lim_{t\uparrow 0} f(\gamma(t_\gamma)) \tag{2}$$

  exist, then we define $f \circ \gamma \in \mathcal{K}_{f(D)}$ by

$$f \circ \gamma(t) = f[\gamma(\sigma(t))],$$

  where σ is defined by

$$\int_0^{\sigma(t)} |f'(\gamma(s))|^2 \, ds = t.$$

  The conditions in (1) and (2) obviously hold if the endpoints of γ are in D but need to be checked if one of the endpoints is a boundary point.

- For any $\gamma \in \mathcal{K}$, we define the oscillation by

$$\operatorname{osc}(\gamma, \delta) = \max\{|\gamma(s) - \gamma(t)| : 0 \le s, t \le t_\gamma, |s - t| \le \delta\}.$$

- We can associate to each curve γ an order pair $(t_\gamma, \tilde{\gamma})$ where $t_\gamma \in [0, \infty)$ and $\tilde{\gamma} \in \mathbb{C}[0, 1]$ given by

$$\tilde{\gamma}(t) = \gamma(t/t_\gamma).$$

We define the metric on curves

$$\rho(\gamma, \gamma_1) = |t_\gamma - t_{\gamma_1}| + \|\gamma - \gamma_1\|,$$

where $\|\cdot\|$ denotes the supremum norm.

There is another natural metric to put on curves. Let

$$\rho_1(\gamma, \tilde{\gamma}) = \inf_\phi \left[\|\mathrm{id} - \phi\| + \|\gamma - \tilde{\gamma} \circ \phi\|\right],$$

where id denotes the identity function, $\|\cdot\|$ the supremum norm, and the infimum is over all increasing homeomorphism $\phi : [0, t_\gamma] \to [0, t_{\tilde{\gamma}}]$. If we choose ϕ to be the linear homeomorphism $\phi(s) = (st_{\tilde{\gamma}}/t_\gamma)$ then

$$\|\mathrm{id} - \phi\| + \|\gamma - \tilde{\gamma} \circ \phi\| = \rho(\gamma, \tilde{\gamma}).$$

Hence $\rho_1 \leq \rho$. We claim that the two metrics give the same topology.

Lemma 1.1. *For every $\gamma \in \mathcal{K}$ and $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\{\eta \in \mathcal{K} : \rho_1(\gamma, \eta) < \delta\} \subset \{\eta \in \mathcal{K} : \rho(\gamma, \eta) < \epsilon\},$$

Moreover, we can choose

$$\delta = \min\left\{\frac{u}{2 \vee t_\gamma}, \frac{\epsilon}{4}\right\},$$

where u satisfies $\mathrm{osc}(\gamma, u) \leq \epsilon/4$.

Proof. Let u, δ be as above. We will show that if $\rho(\gamma, \eta) \geq \epsilon$ then $\rho_1(\gamma, \eta) \geq \delta$. Note that for every ϕ,

$$\|\mathrm{id} - \phi\| \geq |t_\gamma - \phi(t_\gamma)| = |t_\gamma - t_\eta|.$$

Hence if $|t_\gamma - t_\eta| \geq \delta$, then $\rho_1(\gamma, \delta) \geq |t_\gamma - t_\eta| \geq \delta$. Hence, we may assume that $|t_\gamma - t_\eta| \leq \delta \leq \epsilon/4$.

Since $\rho(\gamma, \eta) \geq \epsilon$ and $|t_\gamma - t_\eta| < \epsilon/2$, there exists $s \in [0, 1]$ such that

$$|\gamma(st_\gamma) - \eta(st_\eta)| \geq \epsilon/2.$$

Note that if $|s' - st_\gamma| \leq u$, then

$$|\gamma(s') - \eta(st_\eta)| \geq |\gamma(st_\gamma) - \eta(st_\eta)| - |\gamma(s') - \gamma(st_\gamma)| \geq \epsilon/4.$$

Since $|t_\gamma - t_\eta| \leq u/2t_\gamma$,

$$|st_\gamma - st_\eta| \leq t_\gamma |t_\gamma - t_\eta| \leq u/2,$$

and hence

$$|\gamma(s') - \eta(st_\eta)| \geq \epsilon/4, \quad |s' - st_\eta| \leq u/2.$$

Suppose $\phi : [0, t_\gamma] \to [0, t_\eta]$ is an increasing homeomorphism and suppose that $\phi(t) = st_\eta$. Then either $|t - \phi(t)| \geq u/2$ or $|\gamma(t) - \eta(\phi(t))| \geq \epsilon/4$. Therefore, $\rho_1(\gamma, \eta) \geq \delta$.

□

The metric ρ induces a metric on probability measures $\mathcal{M}^{\#}$ on $\mathcal{K}$ called the Prokhorov metric which we also denote by ρ. There are several equivalent definitions, If μ_1, μ_2 are probability measures we say that $\rho(\mu_1, \mu_2)$ is the infimum of all ϵ such that we can define (X, Y) on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ so that X has marginal distribution μ_1, Y has marginal distribution μ_2 and

$$\mathbb{P}\{\rho(X, Y) \geq \epsilon\} \leq \epsilon.$$

There is also a corresponding Prokhorov metric ρ_1 and the two metrics give the same topology on $\mathcal{M}^{\#}$.

We will want to consider measures on $\mathcal{K}$ that are not probability measures. Let $\mathcal{M}$ denote the set of measures on $\mathcal{K}$ of total mass strictly between 0 and ∞. We write such a measure μ as $\mu = (\mu^{\#}, q)$ where $q = \|\mu\| = \mu(\mathcal{K})$ is the total mass and $\mu^{\#} = \mu/q$ is the probability measure obtained by normalization. The metric on $\mathcal{M}$ is the induced measure on $\mathcal{M}^{\#} \times (0, \infty)$. In another words, $\mu_n \to \mu$ if and only if

$$\mu_n(\mathcal{K}) \to \mu(\mathcal{K}), \quad \text{and} \quad \mu_n^{\#} \to \mu^{\#}.$$

The zero measure is not in $\mathcal{M}$, but we will say that μ_n converges to the zero measure if $\mu_n(\mathcal{K}) \to 0$. In this case we do not require that $\mu_n^{\#}$ converge.

If $E \subset \mathcal{K}$ and μ is a measure we write $\mu|_E$ for the measure restricted to the subset E. If $D \subset \mathbb{C}$, we write μ_D for the measure restricted to the event $\{\gamma \subset D\}$.

1.2 Some easy lemmas

Here we collect some easy facts about the metrics ρ, ρ_1.

- If $\gamma \in \mathcal{K}$, then the reverse path γ^R is defined by

  $$\gamma^R(t) = \gamma(t_\gamma - t), \quad 0 \leq t \leq t_\gamma.$$

  Note that $\rho(\gamma, \eta) = \rho(\gamma^R, \eta^R), \operatorname{osc}(\gamma) = \operatorname{osc}(\gamma^R)$.

- If $\gamma \in \mathcal{K}$ and $z \in \mathbb{C}$, we define γ_z by $\gamma_z(t) = \gamma(t) + z,\ 0 \leq t \leq t_\gamma$.

- If $\gamma, \eta \in \mathcal{K}$ with $\eta(0) = \gamma(t_\gamma)$ we define the *concatenation* $\gamma \oplus \eta$ by $t_{\gamma \oplus \eta} = t_\gamma + t_\eta$ and

  $$\gamma \oplus \eta(t) = \begin{cases} \gamma(t) & 0 \leq t \leq t_\gamma \\ \gamma(t_\gamma) + \eta(s - t_\gamma) & t_\gamma \leq s \leq t_\gamma + t_\eta \end{cases}$$

- If $\gamma, \eta \in \mathcal{K}$, we define another concatenation $\gamma \otimes \eta$ by

  $$\gamma \otimes \eta = \gamma \oplus \eta_{\gamma(t_\gamma) - \eta(0)}.$$

  This is defined for all curves. It is a concatentation where we translate the initial point of η so that if matches up with the terminal point of γ.

- If $\gamma \in \mathcal{K}$ define

  $$\|\gamma\| = t_\gamma + \max_{0 \leq t \leq t_\gamma} |\gamma(t) - \gamma(0)|.$$

  We have defined this so that $\|\gamma_z\| = \|\gamma\|$.

- If $\gamma, \eta \in \mathcal{K}$, then
$$\rho_1(\gamma, \gamma \otimes \eta) \le \|\eta\|, \quad \rho_1(\gamma \otimes \eta, \gamma) \le \|\eta\|.$$
Indeed, for the first inequality we can consider the parametrizations
$$\phi_\delta(t) = \begin{cases} (1-\delta)\, t, & t \le t_\gamma, \\ (1-\delta)\, t_\gamma + (\delta/t_\eta)\,(t - t_\gamma), & t_\gamma \le t \le t_\gamma + t_\eta \end{cases},$$
as $\delta \to 0$.

- If $\gamma \in \mathcal{K}$, and $s < t_\gamma$, we define $\Phi_s \gamma$ to be the curve with time duration s with $\gamma(t) = \Phi_s\gamma(t), 0 \le t \le s$. Then using concatenation, we see that
$$\lim_{s \uparrow t_\gamma} \rho_1(\gamma, \Phi_s \gamma) = 0,$$
and hence this also holds for ρ.

1.3 Brownian measures

In this section we will introduce many measures in $\mathcal{M}$ that arise from complex Brownian motions. We will use ν for all of these measures with extra parameters included. Let us start easy by defining $\nu(z, \cdot; t)$ to be the probability measure which is the distribution of the curve $B_s, 0 \le s \le t$ where B_s is a standard Brownian motion starting at z. This is a probability measure. The $\cdot$ indicates that that variable is being integrated over. We will write
$$\nu(z, \cdot; t) = \int_C \nu(z, w; t)\, dA(w),$$
for appropriate measures $\nu(z, w; t)$ that we now describe. Here and throughout dA represents integration with respect to area. The integral can be considered a Riemann integral. In order to justify the expression we will show that $w \mapsto \nu(z, w; t)$ is a continuous function.

Our reason to discuss topologies on measures on curves was to allow us to integrate measure-valued functions. Measurability of the functions is with respect to the Borel σ-algebra generated by the topology. We will consider continuous or piecewise continuous functions and so the integrals can be considered as Riemann integrals.

Let us describe the measure $\nu(z, w; t)$; for ease we will choose $z = 0$. Let
$$p_s(z, w) = (2\pi s)^{-1} \exp\left\{-|z - w|^2/2s\right\}$$
be the transition probabilities for complex Brownian motion. For each $\epsilon > 0$, let $\mu_\epsilon = \mu_\epsilon(0, w; t)$ be $\nu(0, \cdot; t)$ restricted to the event $\{|\gamma(t) - w| \le \epsilon\}$. We define
$$\nu(z, w; t) = \lim_{\epsilon \downarrow 0} (\pi\epsilon^2)^{-1}\, \mu_\epsilon.$$

Of course, we need to show that the limit exists. Since

$$\mu_\epsilon(\mathcal{K}) = \int_{|w-\zeta|\le\epsilon} p_t(0,\zeta)\, dA(\zeta),$$

it is easy to see that

$$\lim_{\epsilon\downarrow 0} (\pi\epsilon^2)^{-1}\, \mu_\epsilon(\mathcal{K}) = p_t(0,w).$$

We will show that the limit

$$\lim_{\epsilon\downarrow 0} \mu_\epsilon^{\#} = \nu^{\#}(0,w)$$

exists and in the process we will describe the limit.

Fix $\epsilon > 0$ and let B_t be a Brownian motion starting at 0. For each $0 \le s \le t$, let

$$F_\epsilon(s,z) = F_\epsilon(s,z;w) = \int_{|w-\zeta|\le\epsilon} p_{t-s}(z,\zeta)\, dA(\zeta).$$

$$J_s = F_\epsilon(s,B_s).$$

Note that $F_\epsilon(z,s)$ is C^2 in z and C^1 in s for $s < t$. Hence we can write

$$\begin{aligned} dJ_s &= \left[\dot{F}_\epsilon(s,B_s) + \frac{\Delta F_\epsilon(s,B_s)}{2}\right] ds + \nabla F_\epsilon(s,B_s)\, \cdot dB_s \\ &= J_s \left[\frac{\dot{F}_\epsilon(s,B_s)}{F_\epsilon(s,B_s)} + \frac{\Delta F_\epsilon(s,B_s)}{2F_\epsilon(s,B_s)}\, ds + \frac{\nabla F_\epsilon(s,B_s)}{F_\epsilon(s,B_s)}\, \cdot dB_s\right] \end{aligned}$$

Let $t_1 < t$. Then

$$M_s = J_s \exp\left\{-\int_0^s \left[\frac{\dot{F}_\epsilon(r,B_r)}{F_\epsilon(r,B_r)} + \frac{\Delta F_\epsilon(r,B_r)}{2F_\epsilon(r,B_r)}\right] dr\right\}$$

is a positive local martingale satisfying

$$dM_s = \nabla[\log F_\epsilon(s,B_s)] \cdot dB_s.$$

If we use the Girsanov theorem to tilt by the local martingale M_s, we see that

$$dB_t = \nabla[\log F_\epsilon(s,B_s)]\, dt + dW_t,$$

where W_t is a complex Brownian motion with respect to the new measure. This drift always points toward w, and from this we can see that there is no blowup of the process or the local martingale by time t_1.

In our new notation, Let $\mu_{\epsilon,t_1}^{\#}$ denote the measure on paths of time duration t_1 that is obtained from $\mu_\epsilon^{\#}$ by truncation. The Girsanov theorem tells us that $\mu_{\epsilon,t_1}^{\#}$ is absolutely continuous with respect to $\nu(0;\cdot;t_1)$ with Radon-Nikodym derivative

$$\frac{d\mu_{\epsilon,t_1}^{\#}}{d\nu(0,\cdot;t_1)}(\gamma) = F_\epsilon(t_1,\gamma(t_1)) \exp\left\{-\int_0^{t_1} \left[\frac{\dot{F}_\epsilon(s,\gamma(s))}{F_\epsilon(s,\gamma(s))} + \frac{\Delta F_\epsilon(s,\gamma(s))}{2F_\epsilon(s,\gamma(s))}\right] ds\right\}.$$

With this expression, we can take the limit and see that

$$\frac{d\mu^{\#}_{0,t_1}}{d\nu(0,\cdot;t_1)}(\gamma) = F(t_1,\gamma(t_1))\exp\left\{-\int_0^{t_1}\left[\frac{\dot F(s,\gamma(s))}{F(s,\gamma(s))}+\frac{\Delta F(s,\gamma(s))}{2F(s,\gamma(s))}\right]ds\right\},$$

where $F(s,z) = \lim_{\epsilon\downarrow 0} F_\epsilon(s,z) = p_{t-s}(z,w)$.

As $\epsilon \downarrow 0$, the process approaches Brownian motion weighted by $p_{t-s}(z,w)$. Note that

$$\nabla \log p_{t-s}(z,w) = \left(\frac{w-X_s}{t-s}, \frac{w-Y_s}{t-s}\right). \tag{3}$$

Having seen how to take the limit, it is useful to describe the process $\nu(0,w;t)$ directly. There are a number of equivalent ways of writing this. Let us write this as a process (W_s^1, W_s^2) in $\mathbb{R}^2$ beginning at the origin and ending at $w = (w^1, w^2)$ at time t.

- Let B_s be a standard Brownian motion in $\mathbb{R}^2$ and let
$$W_s = B_s + (s/t)\,(w - B_t).$$
- Take a Brownian motion B_s and "tilt locally" by $p_{t-s}(B_s, w)$.
- Equivalently, consider a process satisfying the diffusion equation
$$dX_s = \frac{(w^1 - X_t^1) + (w^2 - X_t^2)}{t-s}\,dt + dB_t,$$
where B_t is a standard Brownian motion.

The process is often called the *Brownian bridge* (for Brownian motion conditioned to be at w at time t). The bridge construction shows that $w \mapsto \nu(z,w;t)$ is continuous in w and using this we can see that

$$\int_D \nu(z,w;t)\,dA(w)$$

is the same as the measure $\nu(z,\cdot;t)$ restricted to the event that $\gamma(t) \in D$. We also can see that $t \mapsto \nu(z,w;t)$ is continuous for $t > 0$, and we can define

$$\nu(z,w;[0,T])) = \int_0^T \nu(z,w;t)\,dt.$$

We can also define

$$\nu(z,w) = \int_0^\infty \nu(z,w;t)\,dt,$$

although this is now an infinite measure.

Suppose that D is a regular domain. then $\nu_D(z,w;t)$ and $\nu_D(z,w)$ are defined to be the measures anove restricted to curves $\gamma \subset D$. Let us denote the total mass of $\nu_D(z,w;t)$ by $p_t^D(z,w)$, and the total mass of $\nu_D(z,w)$ is

$$\|\nu_D(z,w)\| = \int_0^\infty p_t^D(z,w)\,dt = \frac{1}{\pi}\,G_D(z,w).$$

Here G_D is the Green's function normalized so that $G_{\mathbb{D}}(0,z) = -\log|z|$.

Here is one way to check the constant for $\|\nu_D(z,w)\|$. Assume that we know that

$$\|\nu_D(z,w)\| = c_0\, G(z,w)$$

for some constant c_0. Consider the unit disk $\mathbb{D}$ with $\tau = \tau_D$. If B_t is a Brownian motion starting at the origin, then $M_t = |B_t|^2 - 2t$ is a martingale. Therefore,

$$0 = M_0 = \mathbb{E}[M_\tau] = \mathbb{E}[1 - 2\tau],$$

that is, $\mathbb{E}[\tau] = 1/2$. We should also have

$$\begin{aligned}
\mathbb{E}[\tau] &= \int_0^\infty \mathbb{P}\{\tau > t\}\, dt \\
&= \int_0^\infty \int_{\mathbb{D}} p_t(0,z)\, dA(z)\, dt \\
&= \int_{\mathbb{D}} c_0\, G(0,z)\, dA(z) \\
&= -2\pi c_0 \int_0^1 r\, \log r\, dr = \frac{c_0\, \pi}{2}.
\end{aligned}$$

Therefore, $c_0 = 1/\pi$.

If μ is a measure on $\mathcal{K}$, we let μ^R denote the reversal,

$$\mu^R(K) = \mu\{\gamma : \gamma^R \in K\}.$$

Proposition 1.2 (Reversibility)**.**

$$[\nu_D(z,w;t)]^R = \nu_D(w,z;t),$$

$$[\nu_D(z,w;t)]^R = \nu_D(w,z;t),$$

$$[\nu_D(z,w)]^R = \nu_D(w,z).$$

Proof. It is immedate that $p_t(z,w) = p_t(w,z)$. To show that $[\mu^\#(z,w;t)]^R = \mu^\#(w,z;t)$, we can use the Brownian bridge representation of this measure. For ease assume that $z = 0$. We know that $\mu^\#(0,w;t)$ is induced by

$$W_s = B_s + (s/t)\,(w - B_t), \quad 0 \le s \le t.$$

If $Y_s = W_{t-s} - w$, $Z_s = B_{t-s} - B_t$, then

$$Y_s = B_{t-s} - \frac{s}{t}\, w - \frac{t-s}{t}\, B_t = [B_{t-s} - B_t] + \frac{s}{t}\,[-w + B_t] = Z_s + \frac{s}{t}\,[-w - Z_t].$$

Since Z_t is a standard Brownian motion, we see that the distribution of the reversal of the bridge is the same as a bridge. This shows that $[\nu(z,w;t)]^R = \nu(w,z;t)$. Since $\gamma \subset D$ if and only if $\gamma^R \subset D$, we see that $[\nu_D(z,w;t)]^R = \nu_D(w,z;t)$, and by integrating over t, we get the second result. Although $\nu(z,w)$ is not a finite measure, we can write is as

$$\nu(z,w) = \lim_{s\to\infty} \nu_{\mathbb{D}_{-s}}(z,w).$$

$\square$

Suppose $f : D \to f(D)$ is a conformal transformation. The measures $\mu_D(z, w; t)$ do not transform very well because one needs to change time in the parametrization. However, the integrated versions do well.

Proposition 1.3. *If D is a regular domain and $f : D \to f(D)$ is a conformal transformation, then $f \circ \mu_D(z, w) = \mu_{f(D)}(f(z), f(w))$.*

Proof. We can consider this proposition as a combination of two results:

$$\|f \circ \mu_D(z, w)\| = \|\mu_{f(D)}(f(z), f(w))\|,$$

$$f \circ \mu_D^{\#}(z, w) = \mu_{f(D)}^{\#}(f(z), f(w)).$$

The first follows from $G_D(z, w) = G_{f(D)}(f(z), f(w))$, so let us concentrate on the second. It is useful to consider the measure $\mu_D^{\#}(z, w)$. Indeed, this measure can be obtained from Brownian motion in D tilted by the Green's function.

Let us be more explicit. For ease, assume that $w = 0$ and write $G(z) = G_D(z, w)$. For each $\delta, \epsilon > 0$, let $\tau_\epsilon = \inf\{t : B_t \notin D \text{ or } |B_t| = \epsilon\}$. We can consider a probability measure $\mu_\epsilon = \mu_{\epsilon,\delta}$ given by the paths $B_t, 0 \le t \le \tau_\delta$ *conditioned* on the event $\{|B_{\tau_\epsilon}| = \epsilon\}$. If we fix δ and let $\epsilon \downarrow 0$, then by properties of the Green's function we see that the limit measure μ_0 is the measure on paths $B_t, 0 \le t \le \tau_\delta$ weighted by the Green's function G. We then can take $\delta \downarrow 0$ and get our measure $\nu_D^{\#}(z, 0)$

To phrase this in terms of Girsanov's theorem, we let $M_t = G(B_t)$ which satisfies

$$dM_t = \nabla L_D(B_t)\, M_t\, dB_t.$$

If we use tilt by this local martingale, then

$$dB_t = \nabla L_D(B_t)\, dt + dW_t, \tag{4}$$

where W_t is a standard Brownian motion in the new measure. Note that

$$df(B_t) = f'(B_t) \cdot \frac{\nabla G_D(B_t)}{G_D(B_t)}\, dt + f'(B_t) \cdot dW_t,$$

Conformal invariance of Brownian motion and the Green's function shows that the measures $f \circ \nu^{\#}(z, w) = \nu_{f(D)}^{\#}(f(z), f(z))$. □

Rather than going through the limiting process, we could use Itô's formula to show that solutions to (4) with appropriate change of parametrization are conformally invariant.

Suppose $D = \mathbb{D}$ and $w = 0$. Then $G(\zeta) = -\log|\zeta|$ and $\nabla \log G(\zeta)$ is a vector pointing radially towards the origin with norm $[r\log(1/r)]^{-1}$ where $r = |\zeta|$. We want to show that with probability one the path reaches the origin in finite time. Consider the visits to the circles $C_n = \{|z| = e^{-n}\}$. Suppose that $|B_0| = e^{-n}$ and let $\tau = \tau_m = \inf\{t : B_t \in C_{n-1} \text{ or } B_t \in C_m\}$. In the tilted measure, we have

$$\mathbb{P}^*\{B_\tau \in C_{n-1}\} = \frac{\mathbb{P}\{B_\tau \in C_{n-1} \log e^{n-1}}{\mathbb{P}\{B_\tau \in C_{n-1} \log e^{n-1} + \mathbb{P}\{B_\tau \in C_m\} \log e^m} =$$

Starting at C_r, for usual Brownian motion the probability is $1/2$ that the path reaches C_{n-1} before C_{n+1}. Therefore, in the weighted measure $\mathbb{P}^*$

$$\begin{aligned}\lim_{m\to\infty} \mathbb{P}^*\{\tau_{n-1} < \tau_m \mid B_0 = e^{-n}\} &= \lim_{m\to\infty} \frac{\mathbb{P}\{B_\tau \in C_{n-1}\} \log e^{n-1}}{\mathbb{P}\{B_\tau \in C_{n-1} \log e^{n-1} + \mathbb{P}\{B_\tau \in C_m\} \log e^m} \\ &= \frac{\frac{m-1-n}{m-n}\, n - 1}{\frac{m-1-n}{m-n}\, n - 1 + \frac{1}{m-n}\, m} = \frac{n-1}{n}.\end{aligned}$$

In particular, if we let J_n be the total number of excursions between C_n and C_{n-1}, then

$$\mathbb{P}\{J_n \geq n\} = (n - 1/n)^n \leq 1/e,$$

and hence there exists $c < \infty$ such that $\mathbb{P}\{J_n \geq c\, n \log n\} \leq n^{-2}$. By the Borel-Cantelli lemma, we see that with $\mathbb{P}^*$-probability one, except for a finite number of integers n, $J_n \leq c\, n \log n$. Using this (we omit the argument), we can see that with $\mathbb{P}^*$-probability one, the process reaches 0 in finite time and is continuous at the terminal time.

1.4 Radial and angular part

When studying Brownian motion in $\mathbb{C}$ it is often to consider the radial and angular parts to be independent. To make this precise, let $W_t = X_t + iY_t$ be a complex Brownian motion, and let

$$\tilde{B}_t = \exp\{-W_t\}.$$

Then $\tilde{B}_t$ is a time change of a standard Brownian motion. To be more precise, let $f(z) = e^{-z}$ and define $\sigma(t)$ by

$$t = \int_0^{\sigma(t)} |f'(W_s)|^2 ds = \int_0^{\sigma(t)} e^{-2X_s}\, ds.$$

Then $B_t = \tilde{B}_{\sigma(t)}$ is a complex Brownian motion. Note that

$$\dot{\sigma}(t) = |f'(W_{\sigma(t)})|^{-2} = e^{-2X_{\sigma(t)}} = |B_t|^{-2}.$$

Itô's formula shows that

$$d|\tilde{B}_t| = de^{-X_t} = -e^{-X_t}\, dX_t + \frac{e^{-X_t}}{2}\, dt = \frac{|\tilde{B}_t|}{2}\, dt - |\tilde{B}_t|\, dX_t.$$

Apply the time change rule, we see that if we write $B_t = R_t \exp\{i\Theta_t\}$,

$$dR_t = \frac{1}{2\, R_t}\, dt + d\hat{X}_t, \quad d\Theta_t = \frac{1}{R_t}\, d\hat{Y}_t, \tag{5}$$

for a standard independent Brownian motions $\hat{X}_t, \hat{Y}_t$

Let W_t be a complex Brownian motion and let

$$\mu_s = \inf\{t : |W_t| = e^{-s}\}.$$

Let ν_s denote the measure induced by $B_{t+\sigma_s}, 0 \le t \le \sigma - \sigma_s$. Then

$$\nu_{\mathbb{D}}(0, \partial D) = \lim_{s\to\infty} \mu_s.$$

The probability measure μ_s can be constructed as follows, Let $\hat{X}_t, \hat{Y}_t$ be independent Brownian motions and let Θ be an independent random variable uniform on $[0, 2\pi]$. Consider the solution to (5) with initial condition $R_0 = e^{-s}, \Theta_0 = \Theta$ stopped at $T = \inf\{t : R_t = 1\}$. Then μ_s is the measure generated by $\exp\{R_t + i\Theta_t\}, 0 \le t \le T$. Since the initial condition Θ is indpendent of the Brownian motions, we can see that Θ_T and T are independent random variables. Letting $s \to 0$, we get the following. We state this for the unit disk but a similar result holds for other disks.

Proposition 1.4. *Let B_t be a Brownian motion starting at the origin and let $\tau = \tau_{\mathbb{D}}$. Then the random variables τ and $\arg(B_\tau)$ are independent. The latter is uniform on $[0, 2\pi]$ while the form has the distribution of*

$$\inf\{t : R_t = 1\},$$

where R_t is a Bessel process satisfying

$$dR_t = \frac{dt}{2\,R_t} + dW_t, \quad R_0 = 0.$$

Let us emphasize that this proposition strongly uses the fact that the domain is a disk and that the Brownian motion starts at the center of the disk. If the Brownian motion started very close to the boundary, then the random variables σ_s and $\arg(B_{\sigma_s})$ would not be independent. Indeed, if we know that τ is very small, then the argument would more likely to be very close to the starting argument, while if τ is large, then the argument would be almost uniformly distributed. Not so obvious, but true, is that if D is a domain, $z \in D$, and τ_D and B_{τ_D} are independent random variables, then D is a disk and z is the center point.

1.5 Interior to boundary

Suppose that D is a regular point, B_t is a Brownian motion starting at $z \in D$, and, as usual, $\tau = \tau_D = \inf\{t : B_t \notin D\}$. Suppose $B_0 = z \in D$. Then there is a probability measure $\nu_D(z, \partial D)$ which is the distribution of the random curve $B_t, 0 \le t \le \tau$. If $V \subset \partial D$, then we write $\nu_D(z, V)$ to be the restriction of $\nu_D(z, \partial D)$ to curves that lie in V. We can write

$$\nu_D(z, V) = \|\nu_D(z, V)\|\, \nu_D^{\#}(z, V),$$

where

$$\|\nu_D(z, V)\| = \mathrm{hm}_D(z, V),$$

and if $\mathrm{hm}_D(z, V) > 0$, $\nu_D^{\#}(z, V)$ is the measure obtained by tilting by the harmonic function

$$h(\zeta) = \mathrm{hm}_D(\zeta, V).$$

In this case the conditioning is elementary, but we can use the Girsanov formula to write down the SDE satisfied by the conditioned process. If w is a locally analytic point of ∂D, then we can define $\nu_D(z,w)$ so that if V is analytic curve

$$\nu_D(z,V) = \int_V \nu_D(z,w)\,|dw|.$$

There are two equivalent definitions (both of which need to be verified but this is not so difficult).

$$\nu_D(z,w) = \frac{1}{2}\lim_{\epsilon\downarrow 0}\nu_D(z,w+\epsilon\,\mathbf{n}),$$

where $\mathbf{n} = \mathbf{n}_w$ denotes the inward normal at w. Equivalently,

$$\|\nu_D(z,w)\| = \frac{1}{\pi}\,H_D(z,w) = \frac{1}{2\pi}\lim_{\epsilon\downarrow 0} G_D(z,w+\epsilon\,\mathbf{n}),$$

and $\nu_D^{\#}(z,w)$ is the h-process obtained by tilting by the harmonic function $h(\zeta) = H_D(\zeta,w)$.

To help keep our constants straight, consider $D = \mathbb{D}$ and $w = 1$. Under our normalization, $H_{\mathbb{D}}(0,1) = 1/2$, $\|\nu_{\mathbb{D}}(0,1)\| = 1/2\pi$. $G_{\mathbb{D}}(0,1-\epsilon) = -\log[1-\epsilon] \sim \epsilon$.

1.6 h-process

The h-process for Brownian motion conditioned to leave a domain D can be done easiest for Brownian motion in the $\mathbb{D}$ starting at the origin. Indeed, in this case if we start with a Brownian motion and let $\tau = \tau_D$, then the process $W_t = \overline{B_\tau}\,B_t$ can be shown to be an h-process conditioned to leave the disk at 1. To get the h-process in other simply connected domains (or from other starting points in $\mathbb{D}$), we can use conformal invariance. This does not require smoothness at the boundary, but we will assume that the conformal map extends continuously to the boundary point.

Let h be any positive harmonic function on a domain D. We can define the Green's function for the h-process

$$G_{D,h}(w,z) = \lim_{r\to\infty} r\,\mathbb{P}_h^w\left\{\mathrm{dist}\,(z,B[0,\tau]) \le e^{-r}\right\},$$

where $\mathbb{P}_h$ denotes the tilted measure. Note that as $r\to\infty$,

$$\mathbb{P}_h^w\left\{\mathrm{dist}\,(z,B[0,\tau]) \le e^{-r}\right\} \sim \mathbb{P}_h^w\left\{\mathrm{dist}\,(z,B[0,\tau]) \le e^{-r}\right\}\,\frac{h(w)}{h(z)},$$

from which we derive

$$G_{D,h}(w,z) = G_D(w,z)\,\frac{h(z)}{h(w)}.$$

If $f: D\to f(D)$ is a conformal transformation and h is a positive harmonic function on D, let $h_f(z) = h(f^{-1}(z))$ he the corresponding harmonic function on $f(D)$. Then, connformal invariance of the Green's function implies that

$$G_{f(D),h_f}(f(w),f(z)) = G_{D,h}(w,z).$$

Lemma 1.5. *There exists $c < \infty$ such that for every simply connected regular domain D, every positive harmonic function h on D, and all $z, w \in D$,*

$$G_{D,h}(z,w) \le c\,[1 + G_D(z,w)]\,.$$

Proof. Without loss of generality assume that $z = 0$ and $h(0) = 1$. Let ζ be a point on ∂D closest to the origin. If $|w| \le |\zeta|/2$, then the Harnack inequality implies that $h(w) \asymp 1$ and hence $G_{D,h}(0,w) \asymp G_D(0,w)$.

For any w, let V denote the closed disk of radius $\mathrm{dist}(w, \partial D)/2$ about w. Note that for $w' \in V$, $h(w) \asymp h(0,w'), G(w) \asymp G(0,w')$. Moreover, $G(0,w) \asymp \mathbb{P}^0\{B[0,\tau] \cap V \neq \emptyset\}$ and $\mathbb{P}^0_h\{B[0,\tau] \cap V \neq \emptyset\} \asymp \mathbb{P}^0\{B[0,\tau] \cap V \neq \emptyset\}\, h(w) \asymp G_D(0,w)\, h(w) = G_{D,h}(0,w)$. In particular, $G_{D,h}(0,w) \le c$. □

Corollary 1.6. *There exists $c < \infty$ such that if D is a bounded simply connected domain, and h is a positive harmonic function on D, then for every $w \in D$,*

$$\int_D G_{D,h}(w,z)\, dw < \infty.$$

In particular, with $\mathbb{P}_h$-probability one the time duration of an h-process is finite.

We are only assuming that D is bounded. This may be a bit surprising. Consider a saw tooth domain of the form $D = ((0,1) \times i(0,1)) \setminus V$ where $V = V_1 \cup V_2$ with

$$V_1 = \bigcup_{n=1}^{\infty} (2^{-2n}, 2^{-2n} + i(3/4)], \quad V_2 = \bigcup_{n=1}^{\infty} [2^{-2n+1} + i(1/4), 2^{-2n+1} + i).$$

We can find a positive harmonic function h whose boundary value is zero on $\partial D \setminus [0,i]$. Although the boundary behavior of the function f near the boundary is bad, we can still consider the h-process starting at, say, $(3/4) + i(3/4)$. Although the path must go "up and down" infinitely often to avoid the teeth of ∂D, the total time duration of the path is finite. However, in this case the path can not be extended to be continuous at its terminal time.

Corollary 1.7. *Suppose $f : \mathbb{D} \to D$ is a conformal transformation and suppose that f has a continuous extension to the boundary in a neighborhood of 1. Then we can write*

$$\nu^{\#}_D(f(0), f(1)) = f \circ \nu^{\#}_{\mathbb{D}}(0,1).$$

If f is analytic in a neighborhood of 1, then we can write

$$\nu_D(f(0), f(1)) = |f'(1)|^{-1}\, f \circ \nu^{\#}_{\mathbb{D}}(0,1).$$

The following will be used later in this chapter.

Proposition 1.8. *Suppose $D' \subset D$ are domains, $\zeta \in D$, and z is an analytic boundary point of ∂D. Let B_t be a standard Brownian motion starting at ζ under the measure $\mathbb{P}$ and let $\mathbb{P}_*$ be the measure under which B_t is an h-process in D from ζ to z. Let $T = \inf\{t : B_t \notin D'\}$ and $\tau = \tau_z = \inf\{t : B_t = z\}$. Then*

$$\mathbb{P}_*\{T < \tau\} = \frac{\mathbb{E}^{\zeta}\left[H_D(B_T, z); T < \tau\right]}{H(\zeta, z)}.$$

Proof. Let $M_t = H(\zeta, B_{t\wedge\tau})$ which is a local martingale for $t < \tau$. Then $\mathbb{P}_*$ denote the probability measure obtained by tilting by M_t. The result follows from the Girsanov theorem.

□

1.7 Upper half plane

The h-process is perhaps easiest seen in the upper half plane $\mathbb{H}$ with boundary point infinity. Here the path has infinite length and hence is not in $\mathcal{K}$. However, the SDE is rather nice. Indeed the Poisson kernel, that is, the positive harmonic function that vanishes on the boundary except for the target point, is $h(x+iy) = y$. When we tilt by this measure, in the new measure the real and imaginary components are independent. Indeed, the real part is a regular Brownian motion and the imaginary part satisfies the Bessel equation

$$dY_t = \frac{dt}{Y_t} + dB_t.$$

If we choose the boundary point on the real line, the formulas are not as nice but there are still not bad. Let us choose the origin for the boundary point in which case

$$h(x+iy) = \frac{y}{x^2+y^2},$$

with

$$\frac{\nabla h(x+iy)}{h}(x+iy) = \left(-\frac{2x}{y(x^2+y^2)}, \frac{x^2-y^2}{y(x^2+y^2)}\right).$$

Hence in the tilted measure we get

$$dX_t = \frac{-2X_t}{Y_t(X_t^2+Y_t^2)}\,dt + d\hat{X}_t,$$
$$dY_t = \frac{X_t^2-Y_t^2}{Y_t(X_t^2+Y_t^2)}\,dt + d\hat{Y}_t,$$

where $\hat{X}_t + i\hat{Y}_t$ is a Brownian motion in the tilted measure.

In the upper half plane, we can take advantage of the independence of the real and imaginary parts, to give an explicit form for the measures $\nu_{\mathbb{H}}(z,x;t)$ so that

$$\nu_{\mathbb{H}}(z,x) = \int_0^\infty \nu_{\mathbb{H}}(z,x;t)\,dt.$$

Lemma 1.9. *Let B_t be a one-dimensional Brownian motion starting at $y>0$ and let $T = \inf\{t : B_t = 0\}$. Then T has density*

$$\frac{y}{\sqrt{2\pi}\,t^{3/2}}\,\exp\left\{-y^2/2t\right\}. \tag{6}$$

Proof. Using symmetry and the reflection principle for Brownian motion,

$$\begin{aligned}
\mathbb{P}\{T \le t\} &= \mathbb{P}^y\left\{\min_{0\le s\le t} B_s \le 0\right\} \\
&= \mathbb{P}^0\left\{\max_{0\le s\le t} B_s \ge y\right\} \\
&= 2\mathbb{P}^0\left\{B_t \ge y\right\} \\
&= 1 - 2\,\mathbb{P}^0\{0 \le B_t \le y\} \\
&= 1 - 2\int_0^{y/\sqrt{t}} \frac{1}{\sqrt{2\pi}}\,e^{-x^2/2}\,dx
\end{aligned}$$

By differentiating, we get (6). □

Suppose we start a two-dimensional Brownian motion at yi and let $\tau = \tau_{\mathbb{H}}$. Then τ has the same distribution as T of the preceding lemma. We define

$$\|\nu_{\mathbb{H}}(yi, x; t)\| = \frac{y}{\sqrt{2\pi}\, t^{3/2}}\, e^{-y^2/2t}\, \frac{y}{\sqrt{2\pi t}}\, e^{-x^2/2t} = \frac{y}{2\pi t^2}\, e^{-(x^2+y^2)/2t}.$$

Note that

$$\int_0^\infty \|\nu_{\mathbb{H}}(yi, x; t)\|\, dt = \int_0^\infty \frac{y}{2\pi t^2}\, e^{-(x^2+y^2)/2t}\, dt = \frac{y}{\pi(x^2+y^2)} = \frac{1}{\pi}\, H_{\mathbb{H}}(yi, x).$$

The probability measure $\nu_{\mathbb{H}}^{\#}(yi, x; t)$ can be described by saying that the real and imaginary components are independent; the real component is a bridge of time duration t from 0 to x; and the imaginary part is a Brownian motion starting at y conditioned to hit 0 first at time t.

We now define

$$\nu_{\mathbb{H}}(0, x; t) = \lim_{y \downarrow 0} y^{-1}\, \nu_{\mathbb{H}}(yi, x; t),$$

and define $\nu_{\mathbb{H}}(x', x; t)$ similarly by translation. Then $\nu_{\mathbb{H}}(0, x; t)$ is a measure of total mass $(2\pi t^2)^{-1}\, e^{-x^2/2t}$ whose probability measure $\nu_{\mathbb{H}}^{\#}(0, x; t)$ can be described as follows:

- The real and imagainary parts are independent.
- The real part is a Brownian bridge from 0 to x of time duration t. We will write this probability measure as $\lambda_1^{\#}(x; t)$
- The imaginary part is the probability measure associated to a positive excursion of a Brownian motion conditioned to return to the origin at the first time at time t. We write this probability measure as $\lambda_2^{\#}(t)$.

Using the decomposition above, we also write

$$\lambda_1(x; t) = \frac{1}{\sqrt{2\pi t}}\, e^{-x^2/2}\, \lambda_1^{\#}(x; t),$$

$$\lambda_2(t) = \frac{1}{\sqrt{2\pi}\, t^{3/2}}\, e^{-y^2/2t}\, \lambda_2^{\#}(t).$$

Definition The *(Brownian) excursion measure* in the upper half-plane is defined by

$$\nu_{\mathbb{H}}(x, x') = \lim_{y \downarrow 0} y^{-1}\, \nu_{\mathbb{H}}(x + iy, x') = \int_0^\infty \nu_{\mathbb{H}}(x, x'; t)\, dt.$$

If $x \neq x'$, then $\nu_{\mathbb{H}}(x, x')$ is a finite measure with total mass

$$\|\nu_{\mathbb{H}}(x, x')\| = \frac{1}{\pi} H_{\partial\mathbb{H}}(x, x') = \frac{1}{\pi\,(x' - x)^2}.$$

If $x = x'$, then $\nu_{\mathbb{H}}(x, x')$ and the limit in the definition must be taken in an appropriate sense. For example $\nu_{\mathbb{H}}(x, x)$ restricted to loops of time duration at least t, is given by

$$\lim_{y \downarrow 0} y^{-1} \int_t^\infty \nu_{\mathbb{H}}(x + iy, x).$$

Definition Suppose D is a domain and $z, w \in \partial D$ such that ∂D is analytic in neighborhoods of z and w, Then the *(Brownian) excursion measure* $\nu_D(z, w)$ is defined by

$$\nu_D(z, w) = \lim_{\epsilon \downarrow 0} \epsilon^{-1} \, \nu_D(z + \epsilon \mathbf{n}_z, w),$$

where $\mathbf{n}_z$ denotes the unit interior normal at z. If V, V' are closed analytic arcs of ∂D, then we define

$$\nu_D(z, V) = \int_V \nu_D(z, w) \, |dw|,$$

$$\nu_D(V', V) = \int_{V'} \nu_D(z, V) \, |dz| = \int_{V'} \int_V \nu_D(z, w) \, |dw| \, |dz|.$$

If $z = w$, then the excursion measure $\nu_D(z, z)$ is an infinite measure while if $z \neq w$ it is a finite measure. Similarly, if $\text{dist}(z, V) > 0$, then $\nu_D(z, V)$ is a finite measure and if V, V' are compact disjoint arcs, $\nu_D(V', V)$ is a finite measure.

Proposition 1.10. *Suppose $f : D \to f(D)$ is a conformal transformation that can be extended locally in neighborhoods of $z, w \in \partial D$. Let V, V' be analytic arcs and that f is locally analytic at points of V, V'. Then*

$$f \circ \nu_D(z, w) = |f'(z)| \, |f'(w)| \, \nu_{f(D)}(f(z), f(w)),$$

$$f \circ \nu_D(z, V) = |f'(z)| \, \nu_{f(D)}(f(z), f(V)),$$

$$f \circ \nu_D(V', V) = \nu_{f(D)}(f(V'), f(V)).$$

In fact, from this we see that we do not need the assumption that the arcs V, V" be analytic. We write

$$\nu_{\partial D} = \nu_D(\partial D, \partial D),$$

and note that this is a conformally invariant measure.

Definition If z is a locally analytic boundary point of a domain D, the *(boundary) bubble measure* ${\nu^{\text{bub}}}_D(z)$ is defined by

$${\nu^{\text{bub}}}_D(z) = \pi \, \nu)D(z, z) == \pi \lim_{\epsilon \downarrow 0} \epsilon^{-1} \, \nu_D(z + \epsilon \mathbf{n}_z, z),$$

We include the factor π in order to agree with definitions in other places. Note that the bubble measure satisifes the conformal covariance rule

$$f \circ {\nu^{\text{bub}}}_D(z) = |f'(z)|^2 \, {\nu^{\text{bub}}}_{f(D)}(f(z)).$$

This assumes analtyicity of f in a neighborhood of z.

1.8 Concatenation formulas

When analyzing Brownian measures, it is useful to have formulas that break up a path into smaller pieces. Such path decompositions are standard for studying discrete processes. In the continuous situation, similar arguments are generally given in terms of the strong Markov property using stopping times for the process. When dealing with finite measures that are not probability measures it is useful to write down such expressions in terms of path decompositions. We will state some results here. The proofs are not difficult and will be omitted.

For ease, we will assume that the boundaries at which we split the path are locally analytic. One can give expressions without these assumptions, but they are not quite as nice. For practial purposes, one generally tries to split a path at a nice boundary.

Proposition 1.11. *Let D be a domain, let $D' \subset D$ be a subdomain, and let η denote the closure of $D \cap \partial D'$. Assume that η is a finite union of analytic curves, and let z, w be distinct analytic points in $\overline{D} \setminus \eta$.*

- *If $z \in \overline{D'}$,*
$$\nu_D(z,w) - \nu_{D'}(z,w) = \int_{D \cap \partial D_1} [\nu_{D'}(z,\zeta) \oplus \nu_D(\zeta,w)] \, |d\zeta|, \tag{7}$$
- *If $w \in \overline{D'}$,*
$$\nu_D(z,w) - \nu_{D'}(z,w) = \int_{D \cap \partial D'} [\nu_D(z,\zeta) \oplus \nu_{D'}(\zeta,w)] \, |d\zeta|, \tag{8}$$
- *If $z, w \in \overline{D'}$,*
$$\nu_D(z,w) - \nu_{D'}(z,w) = \int_{D \cap \partial D_1} \int_{D \cap \partial D'} [\nu_{D'}(z,\zeta_1) \oplus \nu_D(\zeta_1,\zeta_2) \oplus \nu_{D'}(\zeta_2,w)] \, |d\zeta_1| \, |d\zeta_2|. \tag{9}$$

We note that in (7), if $w \notin \overline{D_1}$, then $\nu_{D'}(z,w)$ is the zero measure, and hence the right-hand side equals $\nu_D(z,w)$. Similarly if either $z \notin \overline{D_1}$ or $w \notin \overline{D_1}$, the right-hand side of (9) equals $\nu_D(z,w)$.

Proof. These are all applications of the strong Markov property and reversibility. We will do (7) in the case $z \in D', w \in D$. Let $\tau = \tau_D, \tau' = \tau_{D'}$ and let $\sigma_s = \sigma_{\epsilon,w} = \inf\{t > \tau' : |B_t - w| \le \epsilon\}$. Let μ_ϵ denote the measure on paths $B_t, 0 \le t \le \sigma_\epsilon$ given by the Brownian measure restricted to the event $\{\sigma_\epsilon < \tau\}$. Then,
$$\nu_D(z,w) - \nu_{D'}(z,w) = \lim_{\epsilon \downarrow 0} (\pi\epsilon^2)^{-1} \, \mu_\epsilon.$$
We write $B[0,\sigma_\epsilon] = B[0,\tau'] \oplus B[\tau',\sigma_\epsilon]$. By the strong Markov property, the distribution of $B[\tau',\sigma_\epsilon]$ given $\mathcal{F}_{\tau'}$ is that of a Brownian motion starting at $B_{\tau'}$ stopped when it gets within distance ϵ of w. If we restrict to the event that this occurs before the path leaves D, multiply by $(\pi\epsilon^2)^{-1}$, and then take the limit, we get the measure $\nu_D(B_{\tau'},w)$. The expression (7) is obtained by integrating over all possible paths $B[0,\tau']$. If $z \in \partial D$ or $w \in \partial D$, we can do a similar argument or we can take limits as the points are approached from the inside.

We get (8) by path reversal,
$$\nu_D(z,w) - \nu_{D'}(z,w) = [\nu_D(w,z) - \nu_{D'}(w,z)]^R,$$
and we get (9) by combining (7) and (8).

□

2 Brownian loop measure

Let $\mathcal{K}_L$ denote the set of curves $\gamma \in \mathcal{K}$ with $\gamma(0) = \gamma(t_\gamma)$. We call elements of $\mathcal{K}_L$ *(rooted) loops.*

Let us define the measure ν on $\mathcal{K}_L$ by

$$\nu = \int_{\mathbb{C}} \nu(z)\, dA(z) = \int_C \int_0^\infty \nu(z;t)\, dt\, dA(z).$$

This is an infinite measure. Indeed, there are three places that the integral blows up. For fixed z, the integral over t blows up both near the origin and at infinity since $||\nu(z,t)\| = (2\pi t)^{-1}$. Also, there is a blowup in z because we are integrating over the unbounded set $\mathbb{C}$. If $D \subset \mathbb{C}$ is a domain, then the measure ν_D is defined by restriction,

$$\nu_D = \int_D \nu_D(z)\, dA(z) = \int_D \int_0^\infty \nu_D(z;t)\, dt\, dA(z).$$

If D is a bounded domain, then this measure is infinite, but there is only one source of blowup: the dt integral as $t \to 0$. In particular, if D is bounded, $\epsilon > 0$, and $V = V_{\epsilon,D}$, $V' = V'_{\epsilon,D}$ denote the set of loops $\gamma \subset D$ with $t_\gamma \geq \epsilon$ and $\mathrm{diam}(\gamma) \geq \epsilon$, respectively, then $\nu_D(V) < \infty$, $\nu_D(V') < \infty$. One way to view $\nu(z)$ in terms of finite measures is as a consistent collection of measures $\{\nu_D(z) - \nu_{D'}(z)\}$ where D, D' are bounded domains with $z \in D' \subset D$.

The measure $\nu_D(z)$ satisfies: if $w \to z$, then $\nu_D(w,z) \to \nu_D(z)$. More precisely, every subdomain $z \in D' \subset D$,

$$\lim_{w \to z} [\nu_D(w,z) - \nu_{D'}(w,z)] = \nu_D(z) - \nu_{D'}(z).$$

Using conformal invariance of the measures $\nu_D(w,z)$, we can see that the measures $\nu_D(z)$ are conformally invariant:

- If $f : D \to f(D)$ is a conformal transformation, then

$$f \circ \nu_D(z) = \nu_{f(D)}(f(z)).$$

However, the integrated measure ν_D is not conformally invariant.

Proposition 2.1. *If $f : D \to f(D)$ is a conformal transformation, and $g = f^{-1}$, then*

$$f \circ \nu_D = \int_{f(D)} |g'(w)|^2\, \nu_{f(D)}(w)\, dA(w).$$

In other words,

$$\frac{d[f \circ \nu_D]}{d\nu_{f(D)}}(\gamma) = |g'(\gamma(0))|^2.$$

Proof.

$$\begin{aligned} f \circ \nu_D &= \int_D f \circ \nu_D(z)\, dA(z) \\ &= \int_D \nu_D(f(z))\, dA(z) \\ &= \int_D |f'(z)|^{-2}\, \nu_{f(D)}(f(z))\, |f'(z)|^2\, dA(z) \\ &= \int_{f(D)} |g'(w)|^2\, \nu_{f(D)}(w)\, dA(w). \end{aligned}$$

□

We would like to define a conformally invariant measure. As a step towards this we define a new measure here.

Definition The *rooted Brownian loop measure* is the measure $\tilde{\mu}$ defined by

$$\frac{d\tilde{\mu}}{d\nu}(\gamma) = \frac{1}{t_\gamma}.$$

The measure $\tilde{\mu}_D$ is obtained by restriction.

This measure is not quite conformally invariant. However, if we restrict the measure to certain sets of curves, then we will get a conformally invariant measure. If γ is a loop, which we can view as a periodic function $\gamma : (-\infty, \infty) \to \mathbb{C}$, we define $\theta_s\gamma$ by $t_{\theta_s\gamma} = t_\gamma$ and

$$\theta_s\gamma(t) = \gamma(s+t), \quad -\infty < t < \infty.$$

Suppose $\mathcal{F}$ is the Borel σ-algebra of curves under the topology induced by the metric ρ. Let $\mathcal{F}^u$ be the sub σ-algebra of sets $E \in \sigma$ that satisfy

- If $\gamma \in E$, then $\theta_s\gamma \in E$ for all $0 \le s \le t_\gamma$.

It is easy to check that $\mathcal{F}^u$ is a σ-algebra. Examples of sets in $\mathcal{F}^u$ are:

$$\{\gamma : s \le t_\gamma \le t\},$$

$$\{\gamma : \gamma \subset D\},$$

$$\{\gamma : \gamma \cap V \neq \emptyset\}.$$

Also, if $E \in \mathcal{F}^u$ and $f : D \to f(D)$ is a conformal transformation, then $\{\gamma : f \circ \gamma \in E\} \subset \mathcal{F}^u$. Indeed, we can define an equivalence relation $\sim$ on curves by $\gamma \sim \gamma_1$ if $\gamma_1 = \theta_s\gamma$ for some s. Then, it is easy to see that if $\gamma \sim \gamma_1$, then $f \circ \gamma \sim f \circ \gamma_1$. The equivalence classes of loops γ under the equivalence relation $\sim$ are called *(oriented) unrooted loops*.

If ϕ is a continuous, nonnegative function on $\mathcal{K}$, let μ^ϕ denote the measure on $\mathcal{F}^u$ defined by

$$\frac{d\mu^\phi}{d\nu}(\gamma) = \phi(\gamma).$$

Note that if ϕ_1, ϕ_2 are two such functions such that for every $\gamma \in \mathcal{K}_L$,

$$\int_0^{t_\gamma} \phi_1(\theta_s\gamma)\, ds = \int_0^{t_\gamma} \phi_2(\theta_s\gamma)\, ds,$$

then $\mu^{\phi_1} = \mu^{\phi_2}$. We write μ_D^ϕ for the measures restricted to D.

Definition The *(unrooted) Brownian loop measure* μ is the measure $\tilde{\mu}$ restricted to the σ-algebra $\mathcal{F}^u$.

Equivalently, we can consider μ as a measure on unrooted loops, and perhaps this is more natural. Define an equivalence relation by $\gamma \sim \gamma'$ if $\gamma' = \theta_s \gamma$ for some s. The set of equivalence classes are the *unrooted loops*, and measures on unrooted loops can be defined by taking the quotient topology and then the corresponding Borel σ-algebra.

For the remainder, when we refer to the Brownian loop measure we are referring to μ. If we wish to refer to $\tilde{\mu}$, we will explicitly say the rooted loop measure. Note that $\mu = \mu^\phi$ where $\phi(\gamma) = 1/t_\gamma$ and hence

$$\int_0^{t_\gamma} \phi(\theta_s \gamma)\, ds = 1. \tag{10}$$

Using this we could define:

- The Brownian loop measure is the measure μ^ϕ where ϕ is any continuous, nonnegative function on $\mathcal{K}_L$ satisfying (10) for very loop $\gamma \in \mathcal{K}_L$.

Theorem 1. *Suppose $f : D \to f(D)$ is a conformal transformation. Then*

$$f \circ \mu_D = \mu_{f(D)}.$$

Proof. Let $g = f^{-1}$. By Proposition 2.1,

$$\frac{d(f \circ \nu_D)}{d\nu_{f(D)}}(\gamma) = |g'(\gamma(0))|^2,$$

and hence,

$$\frac{d(f \circ \tilde{\mu}_D)}{d\nu_{f(D)}}(\gamma) = \phi(\gamma),$$

where

$$\phi(\eta) = \frac{|g'(\eta(0))|^2}{t_{g\circ\eta}} = |g'(\eta(0))|^2 \left[\int_0^{t_\eta} |g'(\eta(s))|^2\, ds\right]^{-1}.$$

Note that

$$\phi(\theta_s \eta) = |g'(\eta(s))|^2 \left[\int_0^{t_\eta} |g'(\eta(s))|^2\, ds\right]^{-1},$$

and hence

$$\int_0^{t_\eta} \phi(\theta_s \eta)\, ds = 1.$$

Therefore, $\mu^\phi_{f(D)} = \mu_{f(D)}$.

□

It is worth noting that no topological constraints were put on the domain D. The domain can be multiply connected.

It will be useful to give an equivalent version of the measures $\nu(z)$ and $\mu(z)$. For ease assume that $z = 0$. To each loop $\gamma : [0, t_\gamma] \to \mathbb{C}$ with $\gamma(0) = \gamma(t_\gamma) = 0$, we associate the ordered pair $(t_\gamma.\tilde{\gamma})$ where $\tilde{\gamma} : [0, 1] \to \mathbb{C}$ is the scaled loop:

$$\tilde{\gamma}(s) = t_\gamma^{-1/2}\, \gamma(s\, t_\gamma), \quad 0 \le s \le 1.$$

We can write

$$\nu(0) = \left(\frac{1}{2\pi t}\, dt\right) \times \nu^{\#}(0, 0; 1),$$

$$\mu(0) = \left(\frac{1}{2\pi t^2}\, dt\right) \times \nu^{\#}(0, 0; 1),$$

Here we recall that $\nu^{\#}(0, 0; 1)$ is the probability measure associated to a Brownian bridge conditioned to return to the origin at time 1. Similarly we can consider the measures ν and μ as measures on triples $(z, t_\gamma, \tilde{\gamma})$, where z is the root of the loop. To be explicit, the triple $(z, t_\gamma, \tilde{\gamma})$ corresponds to the rooted loop

$$\gamma(t) = z + t_\gamma^{1/2}\, \tilde{\gamma}(t/t_\gamma), \quad 0 \le t \le t_\gamma.$$

and the measures ν and $\tilde{\mu}$ are given by

$$\nu = \text{area} \times \left(\frac{1}{2\pi t}\, dt\right) \times \nu^{\#}(0, 0; 1),$$

$$\tilde{\mu} = \text{area} \times \left(\frac{1}{2\pi t^2}\, dt\right) \times \nu^{\#}(0, 0; 1).$$

The Brownian loop measure by nature is defined up to an arbitrary multiplicative constant. It will be important for us to keep track of the constant we have chosen. Roughly speaking, in the measure μ, the density of loops of time duration t in a domain D looks like $(2\pi t^2)^{-1}\, \text{area}(D)$. This can be made precise if D goes to infinity for fixed t or t goes to zero for fixed D.

2.1 Brownian bubbles in $\mathbb{H}$

We have defined the measure Brownian loop measure as the measure obtained from a particular measure on rooted loops. This definition is best to prove certain properties such as conformal invariance and convergence of random walk loop measure to the Brownian loop measure, but it is not the best definition for computations. It is more convienient to choose roots of unrooted loops in a different way. This will lead us to *Brownian (boundary) bubbles*.

We will start by considering what happens when one chooses a point of smallest imaginary part as the root. Let $\mathcal{K}^{\text{bub}}$ be the set of $\gamma \in \mathcal{K}_L$ such that

$$\text{Im}(\gamma(s)) > \text{Im}(\gamma(0)), \quad 0 < s < t_\gamma.$$

Let $\mathcal{K}^{\text{bub}}(z)$ denote the set of $\gamma \in \mathcal{K}^{\text{bub}}$ with $\gamma(0) = z$.

We define the measure $\nu^{\text{bub}} = \nu^{\text{bub}}{}_{\mathbb{H}}(0)$ on $\mathcal{K}^{\text{bub}}(0)$ roughly as

$$\nu^{\text{bub}} = \pi\, \nu_{\mathbb{H}}(0,0) = \lim_{\epsilon \downarrow 0} \frac{\pi}{\epsilon}\, \nu_{\mathbb{H}}(\epsilon i, 0).$$

The factor π is just a conventional choice. This definition is imprecise, because we are taking a limit of measures whose total mass is going to infinity. To make it precise we will do the following. If $D \subset D'$, and D, D' agree in a neighborhood of z, we let $\Gamma_{D'}(D; z)$ be the $\nu^{\text{bub}}{}_{D'}(z)$ measure of loops that do not stay in D, that is, $\Gamma_{D'}(D; z)$ is the total mass of the measure $\nu^{\text{bub}}{}_{D'}(z) - \nu^{\text{bub}}{}_{D}(z)$. If $D \subset \mathbb{H}$ with $\text{dist}(0, \mathbb{H} \cap \partial D) > 0$, we write just $\Gamma(D)$ for $\Gamma_{\mathbb{H}}(D; 0)$. More generally, if D is a simply connected domain with analytic boundary point z, and z is an analytic boundary point and $f : D \to \mathbb{H}$ is a conformal transformation with $f(z) = 0$, then

$$\Gamma_D(D'; z) = |f'(z)|^2\, \Gamma[f(D')].$$

If $D \subset \mathbb{H}$ is a domain with $\text{dist}(0, \mathbb{H} \cap \partial D) > 0$, let $\Gamma(D)$ denote the ν^{bub} measure of loops that do not stay in D. Note that

$$\Gamma(D) = \int_{\mathbb{H} \setminus \partial D} H(w, 0)\, d\mathcal{E}_D(0, dw).$$

For example, if $D = \mathbb{D}_+$, then

$$\Gamma(\mathbb{D}_+) = \int_0^\theta [\sin\theta] \left[\frac{2}{\pi}\, \sin\theta\right] d\theta = 1.$$

Proposition 2.2. *Let $F(z) = -1/z$. Then*

$$\Gamma(D) = \text{hcap}\left[F(\mathbb{H} \cap \partial D)\right].$$

Proof. Let B_t be a Brownian motion in $\mathbb{C}$ and let $\tau = \tau_D = \inf\{t : B_t \notin D\}$. Then we can write

$$\Gamma(D) = \lim_{y \downarrow 0} \frac{1}{y} \mathbb{E}^{iy}\left[\text{Im}[H(B_\tau, 0)]; B_\tau \notin \mathbb{R}\right] = \lim_{y \downarrow 0} \frac{1}{y} \mathbb{E}^{iy}\left[\text{Im}[H(B_\tau, 0)]\right].$$

If $z = re^{i\theta}$, then $F(z) = r^{-1}\, e^{i\theta}$, and $H(z, 0) = \text{Im}\left[F(z)\right]$. Hence

$$\Gamma(D) = \lim_{y \to 0} \frac{1}{y}\, \mathbb{E}^{iy}[\text{Im} f(B_\tau)].$$

If $W = F \circ B$ and $\sigma = \inf\{t : W_t \notin F(D)\}$, then conformal invariance shows that

$$\Gamma(D) = \lim_{y \to \infty} y\, \mathbb{E}^{iy}[\text{Im} F(W_\sigma)].$$

□

Proposition 2.3. *Suppose D is a simply connected domain with* $\text{dist}(0, \mathbb{H} \cap \partial D) > 0$ *and let $f : D \to \mathbb{H}$ be a conformal transformation with $f(0) = 0$. Then*

$$\Gamma(D) = -\frac{1}{6}\, Sf(0), \tag{11}$$

where S denotes the Schwarzian derivative

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3f''(z)^2}{2f'(z)^2}.$$

Proof. Without loss of generality assume that $f'(0) = 1$. This determines f up to one real constant. Let $F(z) = -1/z$ and write $f = F \circ g \circ F$ where g is a conformal transformation of $F(D)$ onto $\mathbb{H}$ satisfying $F(z) \sim z$ as $z \to \infty$. By the previous proposition we know that

$$g(z) = z + x + \frac{\Gamma(D)}{z} + O(|z|^{-2})$$

for some $x \in \mathbb{R}$ (this is the one degree of freedom in the choice of f). Therefore,

$$f(x) = -\left[-\frac{1}{z} + x - \Gamma(D)z\right]^{-1} = z\,\left[1 - xz + \Gamma(D)\,z^2\right]^{-1} = z + xz^2 - \left[x^2 - \Gamma(D)\right]z^3,$$

and,

$$f''(0) = 2x, \quad f'''(x) = 6\,x^2 - 6\,\Gamma(D), \quad Sf'(0) = -6\Gamma(D).$$

□

Note that the left hand side of (11) is independent of the choice of conformal transformation f. Any other transformation would be of the form $T \circ f$ where T is a Möbius transformation of $\mathbb{H}$ fixing the origin. One could check directly that $S(T \circ f)(0) = Sf(0)$.

The definition of the bubbles is essentially the same as the excursion measure except for the factor of π and the fact that it is an infinite measure. Recall that if $f : \mathbb{H} \to D$ is a conformal transformation that is locally analytic at 0 and $x > 0$, then

$$f \circ \nu_{\mathbb{H}}(0, x) = |f'(0)|\,|f'(x)|\,\nu_D(f(0), f(x)).$$

We can do similarly for the Brownian bubbles and define ${\nu^{\text{bub}}}_D(z)$ at analytic boundary points z by

$$f \circ \nu^{\text{bub}}(0) = |f'(0)|^2\,{\nu^{\text{bub}}}_D(z).$$

Here $f : \mathbb{H} \to D$ is a conformal transformation with $f(0) = z$. It is not hard to check that this is the same as

$$\lim_{\epsilon \downarrow 0} \frac{\pi}{\epsilon} \nu_D(z + \epsilon \mathbf{n}_z, z),$$

where $\mathbf{n}_z$ is the inward unit normal. For this reason we will also use the excursion measure notation

$$\nu_D(z, z) = \frac{1}{\pi}\,{\nu^{\text{bub}}}_D(z) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \nu_D(z + \epsilon \mathbf{n}_z, z),$$

where we must remember this is not exactly a limit in $\mathcal{K}$ since the limit is an infinite measure. If $D' \subset D$, and z is an analytic boundary point, we define $\Gamma_D(D'; z)$ to be the ${\nu^{\text{bub}}}_D(z)$ measure of bubbles that intersect $D \cap \partial D'$. If f is as above and $D_1 \subset \mathbb{H}$, then we get the scaling rule

$$\Gamma(D_1) = |f'(0)|^2\,\Gamma_D(f(D'), f(0)).$$

There is a relationship between Brownian bubbles and the Brownian loop measure.

Theorem 2. *Considered as a measure on the unrooted σ-algebra $\mathcal{F}^u$,*

$$\mu = \frac{1}{\pi} \int_{\mathbb{C}} \nu^{\mathrm{bub}}{}_{\mathbb{H}+iy}(x+iy)\, dx\, dy = \int_{\mathbb{C}} \nu_{\mathbb{H}+z}(z,z)\, dA(z).$$

The factor of $1/\pi$ is to compensate for the π in the definition of the bubble. The basic idea is to attach to each unrooted loop the rooted loop obtained by rooting at the (unique up to a set of loops of measure zero) point of minimal imaginary part. Technically, it is easier to go the other way — we start with the measure

$$\int_{\mathbb{C}} \nu_{\mathbb{H}+z}(z,z)\, dA(z), \tag{12}$$

then assign the root of each loop uniformly, and see that we get the rooted Brownian loop measure. As before, let us write a rooted loop as a triple $(z, t_\gamma, \tilde{\gamma})$.

We give another argument here that splits the path into the real and imaginary parts. We start by giving an alternative construction of the measure

$$\nu_{\mathbb{H}}(0,0) = \lim_{y \downarrow 0} y^{-1}\, \nu_{\mathbb{H}}(iy, 0).$$

We recall that the limit on the right-hand side must be taken in some sense, that is, considered as a finite measure restricted to some nice set of curves. Another way to take the limit is as follows. Let use write

$$\nu_{\mathbb{H}}(iy, 0) = \int_0^\infty \nu_{\mathbb{H}}(iy, 0; t)\, dt,$$

where $\nu_{\mathbb{H}}(iy, 0; t)$ denotes the measure restricted to curves of time duration t. Then we define

$$\nu_{\mathbb{H}}(0,0) = \int_0^\infty \nu_{\mathbb{H}}(0, 0; t)\, dt.$$

where the measures $\nu_{\mathbb{H}}(0,0;t)$ can be defined by

$$\nu_{\mathbb{H}}(0, 0; t) = \lim_{y \downarrow 0} y^{-1}\, \nu_{\mathbb{H}}(iy, 0; t).$$

Since the real and imaginary parts of a Brownian motion are independent, we can write

$$\nu_{\mathbb{H}}(iy, 0; t) = \lambda_1(t) \times \lambda_2(t; y)$$

where $\lambda_1(t)$ is the one-dimensional Brownian loop measure at the origin for time t, and $\lambda_2(t;y)$ is the one-dimensional Brownian measure corresponding to paths starting at y and exiting the positive real axis at time t. Note that $\|\lambda_1(t)\| = 1/\sqrt{2\pi t}$ and $\lambda_1^{\#}(t)$ is the one-dimensional Brownian bridge measure.

The total mass $\|\lambda_2(t;y)\|$ is the density of the hitting time T of the origin for a one-dimensional Brownian motion starting at y, which is given by

$$\|\lambda_2(t;y)\| = \frac{y}{\sqrt{2\pi}\, t^{3/2}} \exp\left\{-y^2/2t\right\}. \tag{13}$$

This formula is derived using symmetry and the reflection principle for Brownian motion,

$$\begin{aligned}
\mathbb{P}\{T \le t\} &= \mathbb{P}^y \left\{ \min_{0 \le s \le t} B_s \le 0 \right\} \\
&= \mathbb{P}^0 \left\{ \max_{0 \le s \le t} B_s \ge y \right\} \\
&= 2\mathbb{P}^0 \{B_t \ge y\} \\
&= 1 - 2\,\mathbb{P}^0\{0 \le B_t \le y\} \\
&= 1 - 2 \int_0^{y/\sqrt{t}} \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx
\end{aligned}$$

By differentiating, we get (13). In particular, for $t > 0$,

$$\lim_{y \downarrow 0} y^{-1} \left\| \mu_{\mathbb{H}}(iy, 0; t) \right\| = \lim_{y \downarrow 0} y^{-1} \left\| \lambda_1(t) \times \lambda_2(t; y) \right\| = \frac{1}{2\pi t^2}.$$

The probability measure $\lambda_2^{\#}(t; iy)$ can be described as Brownian motion B_t starting at y tilted by the function

$$J_s = F(t - s, B_s),$$

where

$$F(s, x) = \frac{x}{\sqrt{2\pi}\, s^{3/2}} \exp\left\{-x^2/2s\right\}.$$

Note that

$$F'(s, x) = F(s, x) \left[\frac{1}{x} - \frac{x}{t - s} \right],$$

and hence we have

$$dB_s = \left[\frac{1}{B_s} - \frac{B_s}{t - s} \right] ds + dW_s,$$

where W_s is a Brownian motion in the tilted measure. By comparison with a Bessel process, we can see that this process does not hit the origin before time t. However, as $s \uparrow t$, the drift towards the origin is so strong to force $B_t = 0$ in the tilted measure. Using this equation, we can construct the limit

$$\lambda_2^{\#}(t; 0) = \lim_{y \downarrow 0} \lambda_2^{\#}(t; iy).$$

This measure on one-dimensional paths is essentially the same as what is known as Itô excursion measure for one-dimensional Brownian motion. We have described the measure on one-dimensional excursions in the positive half-line "conditioned" to return to the origin at time t. We put "conditioned" in quotes because we really multiply this measure by $[\sqrt{2\pi}\, t^{3/2}]^{-1}$

Putting this all together, we see that $\nu_{\mathbb{H}}(0, 0; t)$ is a measure of total mass $(2\pi t^2)^{-1}$ whose probability measure $\nu_{\mathbb{H}}^{\#}(0, 0; t)$ is a one-dimensional Brownian bridge in the first component and an independent one-dimensional excursion of in the second component, both of time duration t. Then

$$\nu_{\mathbb{H}}(0, 0) = \int_0^{\infty} \nu_{\mathbb{H}}(0, 0; t)\, dt.$$

In particular, if we consider the integrated measure

$$\int_{\mathbb{C}} \nu_{\mathbb{H}+z}(z,z)\, dA(z),$$

then the density of loops of time duration t is a domain D is asymptotic to $\mathrm{area}(D)\,(2\pi t^2)^{-1}$. This asymptotics can be taken as $D \uparrow \mathbb{C}$ for fixed t or $t \downarrow 0$ for fixed D. In particular, the constant is the same as for the Brownian loop measure μ.

It is not hard to show that Brownian scaling holds for the excursion measures $\nu^{\#}_{\mathbb{H}}(0,0;t)$, i.e., if γ is distributed according to $\nu^{\#}_{\mathbb{H}}(0,0;t)$ and $\tilde{\gamma}(s) = t^{-1/2}\,\gamma(st), 0 \le s \le t$, then $\tilde{\gamma}$ has the distribution $\nu^{\#}(0,0;1)$. Therefore, we can write the measure (12) as a measure on triples $(z, t_\gamma, \tilde{\gamma})$ as

$$\mathrm{area} \times \left(\frac{1}{2\pi t^2}\, dt \right) \times \nu^{\#}_{\mathbb{H}}(0,0,1).$$

We now use an important fact about one-dimensional bridges and one-dimensional excursions. If $\gamma(t), 0 \le t \le 1$ is a path with $\gamma(0) = \gamma(1) = 0$, considered as function of period one, we write $\Theta_s\gamma(t) = \gamma(s+t) - \gamma(s)$.

- If γ has the distribution $\lambda^{\#}_1(1)$ and U is an independent uniform random variable on $[0,1]$, then $\Theta_U\gamma$ has distribution $\lambda^{\#}_1(1)$.
- If γ has the distribution $\lambda^{\#}_2(0;1)$ and U is an independent uniform random variable on $[0,1]$, then $\Theta_U\gamma$ has distribution $\lambda^{\#}_1(1)$.

This can be proved in a number of ways. One way is to approximate by simple random walk for which the analogous facts are easy. Applying this to the real and imaginary parts of a two-dimensional process we get the following.

- If γ has the distribution $\nu^{\#}(0,0;1)$ and U is an independent uniform random variable on $[0,1]$, then $\Theta_U\gamma$ has distribution $\nu^{\#}(0,0;1)$.
- If γ has the distribution $\nu^{\#}_{\mathbb{H}}(0,0;1)$ and U is an independent uniform random variable on $[0,1]$, then $\Theta_U\gamma$ has distribution $\nu^{\#}(0,0;1)$.

It may be useful to consider a random walk analogue. Consider simple random walk on the integer lattice $\mathbb{Z}$. It is known that

$$\mathbb{P}\{S_{2n} = 0\} = \frac{1}{\sqrt{\pi n}} + O(n^{-3/2}).$$

Note that $1/\sqrt{\pi n}$ is $2p_{2n}(0,0)$ where p_t is the transition probability for Brownian motion. The factor of 2 comes from the bipartite nature of the random walk, that is, the fact that $\mathbb{P}\{S_{2n+1} = 0\}$. Let us fix an integer n and let $\omega = [\omega_0, \ldots, \omega_{2n}]$ be a random walk loop with $\omega_0 = \omega_{2n} = 0$. Let $X = X_{2n}$ be the largest integer x such that $-x$ is visited by ω and let $\sigma = \sigma_{2n}$ be the the smallest index s such that $\omega_s = -X$. Note that s and x (and, hence, also $2n - s$) must have the same parity. We have split $\omega = \omega^1 \oplus \omega^2$ in a unique fashion such that the the terminal point, say $-x$, of ω^1 is smaller than the other vertices of ω^1, the initial of ω^2 is

the same as the terminal point of ω^1, and no other vertex of ω^2 is smaller than its initial point. (However, $-x$ may appear more than once in ω^2.)

If S is a simple random walk starting at the origin and n is an integer, then

$$\sum_{x=0}^{\infty}\sum_{s=0}^{2n-1}\mathbb{P}\{S_{2n}=0, X=x, \sigma-s\} = \mathbb{P}\{S_{2n}=0\} \sim \frac{1}{\sqrt{\pi n}}.$$

Let us consider n large and let $x_n = x/\sqrt{2n}$. Recall that $W_t^{(n)} := S_{2nt}/\sqrt{2n}$ converges to a Brownian motion.

By comparison with Brownian motion, we expect that if $s+x$ is even,

$$\mathbb{P}\{S_s = -x; S_j > -x, j < x\} \sim 2\,F(s,x),$$

where $F(s,x)$ is the Brownian quantity,

$$F(s,x) = \frac{x}{\sqrt{2\pi}\,s^{3/2}}\,\exp\left\{-x^2/2s\right\}.$$

The factor of 2 comes from the bipartite nature of the walk. However, we expect

$$\mathbb{P}\{S_s = -x; S_j \geq -x, j < x\} \sim 4\,F(s,x).$$

The extra factor of two represents the expected number of visits to $-x$ before reaching $-x-1$. Therefore, if x is even,

$$\begin{aligned}
\mathbb{P}\{S_{2n}=0, \sigma=2s\} &= \sum_{k=0}^{\infty}\mathbb{P}\{S_{2n}=0, \sigma=2s, X=2k\} \\
&\sim \sum_{k=0}^{\infty} 2F(2r,2k)\,4\,F(2n-2r,2k) \\
&= \frac{8}{(2n)^2}\sum_{k=0}^{\infty} F\left(\frac{r}{n}, \frac{2k}{\sqrt{2n}}\right) F\left(\frac{n-r}{n}, \frac{2k}{\sqrt{2n}}\right) \\
&\asymp \frac{8}{(2n)^{3/2}}\int_0^{\infty} F(t,z)\,F(1-t,z)\,dz \\
&= \frac{8}{(2n)^{3/2}}\,\frac{1}{2\sqrt{2\pi}} = \frac{1}{2n\sqrt{\pi n}}.
\end{aligned}$$

Similarly, we get

$$\mathbb{P}\{S_{2n}=0, \sigma=2s+1\} \sim \frac{1}{2n\sqrt{\pi n}},$$

which is consistent wth

$$\sum_{s=0}^{2n-1}\mathbb{P}\{S_{2n}=0, \sigma=2s+1\} = \mathbb{P}\{S_{2n}=0\} \sim \frac{1}{\sqrt{\pi n}}.$$

Proposition 2.4. *Considered as a measure on the unrooted σ-algebra $\mathcal{F}^u$,*

$$\mu_{\mathbb{H}} = \frac{1}{\pi}\int_0^{\pi}\int_0^{\infty} \nu_{U_r}(re^{i\theta}, re^{i\theta})\,r\,dr\,d\theta, \tag{14}$$

where $U_r = \{z \in \mathbb{H} : |z| > r\}$.

Proof. Let $f(z) = e^z$ which is a conformal transformation of $D := \{x + iy \in \mathbb{H} : 0 < y < \pi\}$ onto $\mathbb{H}$. Therefore, if $D_y = D \cap (\mathbb{H} + y)$,

$$\begin{aligned} \mu_{\mathbb{H}} = f \circ \mu_D &= \int_0^\pi \int_0^\infty f \circ \nu_{D_y}(x+iy, x+iy)\, dx\, dy \\ &= \int_0^\pi \int_0^\infty e^{2x}\, \nu_{U_{e^x}}(e^{x+iy}, e^{x+iy})\, dx\, dy \\ &= \int_0^\pi \int_0^\infty r\, \nu_{U_r}(re^{iy}, re^{iy})\, dr\, dy \end{aligned}$$

□

Proposition 2.5. *There exists $c < \infty$ such that if $K \subset \mathbb{H}$ such that $D = \mathbb{H} \setminus K$ is a domain with* $\mathrm{dist}(0, K) \geq 1$, *and $q(\epsilon, D)$ denotes the $\mu_{\mathbb{H}}$ measure of loops that intersect both the circle of radius ϵ and K, then*

$$q(\epsilon, D) = \epsilon^2\, \Gamma(D)\, [1 + O(\epsilon)], \quad \epsilon \downarrow 0.$$

Proof. The $\nu^{\mathrm{bub}}{}_{U_r}(re^{i\theta})$ measure of loops that intersect K is given by $\Gamma_{U_r}(D \cap U_r; re^{i\theta})$. Then (18) implies that

$$q(\epsilon, D) = \frac{1}{\pi} \int_0^\pi \int_0^\epsilon \Gamma_{U_r}(D \cap U_r; re^{i\theta})\, r\, dr\, d\theta.$$

Let

$$f_r(z) = z + \frac{r^2}{z},$$

which is a conformal transformation of U_r onto $\mathbb{H}$. Then

$$\Gamma_{U_r}(D \cap U_r; re^{i\theta}) = |f_r'(re^{i\theta})|^2\, \Gamma_{\mathbb{H}}(f_r(D), f_r(re^{i\theta})).$$

Since $|f_r(re^{i\theta})| = O(r)$, we can use derivative estimates to see that

$$\Gamma_{\mathbb{H}}(f_r(D), f_r(re^{i\theta})) = \Gamma(f_r(D))\, [1 + O(r)].$$

Using $f_r(z) = f_r(1) + O(r^2)$ for $|z| = 1$ and
we see that

$$\Gamma(f(D)) = \Gamma(D)[1 + O(r)].$$

Also,

$$|f'(re^{i\theta})|^2 = \left|1 - e^{-2i\theta}\right|^2 = 4\, \sin^2 \theta.$$

Therefore,

$$\begin{aligned} q(\epsilon, D) &= \frac{\Gamma(D)}{\pi} \int_0^\pi \int_0^\epsilon 4[\sin^2 \theta]\, r\, [1 + O(r)] dr\, d\theta \\ &= \epsilon^2\, \Gamma(D)\, [1 + O(\epsilon)]. \end{aligned}$$

□

Let $\Lambda_D(V_1, V_2)$ denote the μ_D measure of loops that intersect both V_1 and V_2.

Proposition 2.6. *There exists $c < \infty$ such that the following holds. Suppose $D \subset \mathbb{H}$ is a domain with* $\mathrm{dist}(0, \partial D) \geq 1$. *Let $0 < r \leq 1/2$ and let V be a closed subset of $\mathbb{H}$ contained in $\{|z| \leq r\}$. Then,*

$$|\Lambda_{\mathbb{H}}(\mathbb{H} \setminus D, V) - \mathrm{hcap}(V)\,\Gamma(D)| \leq c\, r\, \mathrm{hcap}(V)\,\Gamma(D).$$

Proof. (Sketch) If V is the half disk of radius r, then this is the previous lemma. More generally, for any loop that hits both K and $r\mathbb{D}_+$, we consider the excursions from the unit circle to $r\mathbb{D}_+$. For each of these we can ask the probability that the excursion hits V.

We claim that for an excursion in $\mathbb{D}_+$ connecting two boundary points in $\mathbb{D}_+ \setminus \mathbb{H}$, the probability to hit V given that the excursion hits $r\mathbb{D}_+$ is $r^{-2}\,\mathrm{hcap}(V)\,[1+O(r)]$. To see this we start by recalling that the probability that the excursion from $z = e^{\beta i}$ to $w = e^{\psi i}$ hits $r\mathbb{D}_+$ is $H_{\partial\mathbb{D}_+}(z,w)^{-1}$ times

$$\begin{aligned}
\int_{|\zeta|=r} H_{\mathbb{D}_+}(\zeta, w)\, d\mathcal{E}_{\mathbb{D}_+ \setminus r\mathbb{D}_+}(z, d\zeta) &= [1 + O(r)]\, \frac{2r \sin\beta}{\pi} \int_0^\pi H_{\mathbb{D}_+}(re^{i\theta}, w)\, \sin\theta\, d\theta \\
&= [1 + O(r)]\, \frac{2r^2 \sin\beta\, \sin\psi}{\pi} \int_0^\pi \sin^2\theta\, d\theta \\
&= r^2\, \sin\beta\, \sin\psi\, [1 + O(r)]. \qquad (15)
\end{aligned}$$

Since K is contained in $r\mathbb{D}_+$, we can use the strong Markov property to see that

$$\int_{|\zeta|=r} H_{\mathbb{D}_+}(\zeta, w)\, d\mathcal{E}_{\mathbb{D}_+ \setminus V}(z, d\zeta) = [1 + O(r)]\, \frac{2r \sin\beta}{\pi} \int_0^\pi \mathbb{E}^{re^{i\theta}}[H_{\mathbb{D}_+}(B_\sigma, w)]\, \sin\theta\, d\theta,$$

where σ is the first time that a Brownian motion leaves $\mathbb{D}_+ \setminus V$. We can see that

$$\begin{aligned}
\mathbb{E}^{re^{i\theta}}[H_{\mathbb{D}_+}(B_\sigma, w)] &= \sin\psi\, \mathbb{E}^{re^{i\theta}}[\mathrm{Im}(B_\sigma); B_\sigma \in V]\, [1 + O(r)] \\
&= \sin\psi\, \mathbb{E}^{re^{i\theta}}[\mathrm{Im}(B_{\sigma'})]\, [1 + O(r)]
\end{aligned}$$

where σ' is the first time that the Brownian motion leaves $\mathbb{H} \setminus V$. Recall that

$$\mathrm{hcap}(V) = \frac{2r}{\pi} \int_0^\infty \mathbb{E}^{re^{i\theta}}[\mathrm{Im}(B_{\sigma'})]\, \sin\theta\, d\theta.$$

Therefore,

$$\int_{|\zeta|=r} H_{\mathbb{D}_+}(\zeta, w)\, d\mathcal{E}_{\mathbb{D}_+ \setminus V}(z, d\zeta) = \sin\beta\, \sin\psi\, \mathrm{hcap}(V)\, [1 + O(r)]. \qquad (16)$$

Comparing (15) and (16) gives the claim. □

By considering derivatives we get the following corollary.

Corollary 2.7. *Suppose that $D \subset \mathbb{H}$ is a subdomain, γ is a continuous curve in $\mathbb{H}$ with $\gamma(0+) = 0$, and let $\gamma_t = \gamma(0, t]$. Suppose that $\gamma_t \subset D$. Let $a(t) = \mathrm{hcap}[\gamma_t]$ and assume that a is C^1. Let g_t denote the unique conformal transformation of the unbounded component of $\mathbb{H} \setminus \gamma_t$ onto $\mathbb{H}$ with $g_t(z) - z \to 0$ and let $\beta_s = g_s(\gamma(s))$. Then*

$$\Lambda_{\mathbb{H}}(\gamma_t, \mathbb{H} \setminus D) = \int_0^t \dot{a}(s)\, \Gamma(g_s(D); \beta_s)\, ds. \qquad (17)$$

2.2 Brownian loop measure on cylinder

Let $\mathcal{C}yl$ denote the cylinder $[0, 2\pi] \times \mathbb{C}$ or equivalently the complex plane with the identification $z \sim z + 2\pi k$ for all integers k. The Brownian loop measure on $\mathcal{C}yl$ can be defined as before

$$\text{area} \times \left(frac12\pi t^2 \, \nu^{\#}_{\mathcal{C}yl,t} dt \right),$$

where now $\nu^{\mathcal{C}yl}$ denotes the appropriate Brownian bridge of time duration on the cylinder. This latter term can be considered in two independent parts: the vertical part does a one-dimensional Brownian motion on $\mathbb{R}$ and the horizontal part does a Brownian motion on the circle, that is, on the interval $[0, 2\pi]$ with periodic boundary conditions. Similarly, we can consider a Brownian bubble decomposition where we choose the y part from a Brownian excursion and again choose the x part from a Brownian motion on the circle.

Conformal invariance can be proved as before and this is most important when consider the exponential map $z \mapsto e^{iz}$ that conformally maps $\mathcal{C}yl$ onto $\mathbb{C} \setminus \{0\}$. The image of the Brownain loop measure is the Brownian loop measure.

Proposition 2.8. *Considered as a measure on the unrooted σ-algebra $\mathcal{F}^u$,*

$$\mu_{\mathbb{C}} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \nu_{O_r}(re^{i\theta}, re^{i\theta}) \, r \, dr \, d\theta = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \nu_{r\mathbb{D}}(re^{i\theta}, re^{i\theta}) \, r \, dr \, d\theta, \tag{18}$$

where $O_r = \{z \in \mathbb{C} : |z| > r\}$.