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FRACTAL AND MULTIFRACTAL PROPERTIES OF SLE

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Introduction

This is a slightly expanded version of my lectures at the 2010 Clay Mathematics Institute summer (winter) school in Buzios, Brazil. The theme is the fine properties of Schramm-Loewner evolution (SLE) curves with an emphasis on recent work I have done with a number of co-authors on fractal and multifractal properties. I assume basic knowledge of SLE at the level of foundational course presented by Vincent Beffara. I will try to discuss both results and ideas of proofs. Although discrete models motivate SLE, I will focus only on SLE itself and will not discuss convergence of discrete models.

The basic theme tying the results together is the SLE curve. Fine analysis of the curve requires estimates of moments of the derivatives, and in turn leads to studying martingales and local martingales. In the process, I will discuss existence of the curve, Hausdorff dimension of the curve, and a number or more recent results that I have obtained with a number of co-authors.

The five sections correspond roughly to the five lectures that I gave. Here is a quick summary.

- Section 1 proves a basic result of Rohde and Schramm [**15**] on the existence of the SLE curve for $\kappa \neq 8$. Many small steps are left to the reader; one can treat this as an exercise in the deterministic Loewner equation and classical properties of univalent functions such as the distortion theorem. Two main ingredients go into the proof: the modulus of continuity of Brownian motion and an estimate of the moments of the derivative of the reverse map. By computing the moment, we can determine the optimal Hölder exponent and see why $\kappa = 8$ is the delicate case. The estimation of the moment is left to the next section.
- Section 2 discusses how to use the reverse Loewner flow to estimate the exponent. This was the tool in [**15**] to get their estimate. Here we expand signficantly on their work because finer analysis is needed to derive "two-point" or "second moment" estimates which are required to establish fractal and multifractal behavior with probability one. Although there is a fair amount of calculation involved, there are a few general tools:

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  - Use scaling and a good choice of reparmetrization to reduce the problem to analysis of a one-variable SDE.
  - Find an appropriate martingale and use the Girsanov theorem to understand the measure obtained by weighting by the martingale.
  - For exceptional events on the path, find an event of high probability *in the weighted measure* that is contained in the exceptional event. Choose it appropriately so that two-point estimates can be obtained.

  These are standard methods in stochastic analysis. One of the most fundamental techniques in large deviation theory is to study a new measure (sometimes called a "tilting") on a space on which an exceptional set has large probability. If this new measure arises from a martingale, then the Girsanov theorem is the tool to studying probabilities in the new measure.

  The latter part of this section, starting with Section 2.6, contains some more advanced topics that were not covered in the lectures. I have included them in these notes because they are part of the reverse flow picture, but the material from this part is not used later. Readers should feel free to skip these and move to Section 3.

- Section 3 is essentially independent of Section 2 and considers the forward Loewner flow. The Hausdorff dimension of the SLE curve was analyzed in [**15**] and [**2**]. The basic questions are: how close does the SLE curve get to a $z \in \mathbb{H}$ and what does the path look like if it does get close to z? There is a fundamental local martingale in terms of the SLE Green's function, and if one uses a radial parametrization (depending on z), one gets a simple one-variable SLE. By weighting by this local martingale, one gets another process, two-sided radial SLE, which corresponds to SLE conditioned to hit a point. Here we use the Girsanov theorem to give very sharp estimates of the probability that the SLE gets near z. Finally, we discuss why trying to prove lower bounds for the Hausdorff dimension leads to studying a two-point estimate for the probability of getting close to two different points.
- Section 4 continues Section 3 by discussing the two-point estimate first proved by Beffara [**2**]. We only give a sketch a part of the argument as rederived in [**9**] and then we define an appropriate multi-point Green's function and corresponding two-point local martingale. The estimate and the two-point local martingale are used in the next section.
- Section 5 is devoted to the natural parametrization or length for $SLE_\kappa, \kappa < 8$. The usual parametrization for an SLE_κ is by capacity which does not correspond to the "natural" scaled parametrization one would give to discrete models. I start by giving the intuition for a definition, which leads to an expression of the type analyzed in Section 2, and then give a precise definition as developed in [**8, 11**]. The proof of existence in [**11**] uses the ideas from Section 4.

There are many exercises interspersed throughout the notes. I warn you that I use facts from the exercises later on. Therefore, a reader should read them whether or not he or she chooses to actually do them. I have an additional section at the end with one more exercise on Brownian motion which can be considered as an easier example of some of the ideas from Sections 4 and 5. I assume the reader knows basics of *SLE* and univalent function theory. Possible references are the notes from Beffara's course and my book [**5**].

These notes have been improved by questions and remarks by the participants of the school, and I thank all the participants. A particular thanks goes to Brent Werness for his work as a TA and his comments on these notes.

0.1. Basic definitions and notation. To set some basic definitions, I let g_t denote the conformal maps of chordal SLE_κ from 0 to ∞ in $\mathbb{H}$ parametrized so that the half-plane capacity grows at rate

$$a = \frac{2}{\kappa}.$$

I will use a throughout these notes because it makes formulas somewhat easier. It is always equal to $2/\kappa$ and the reader can make this replacement at any time! Under this parametrization, g_t satisfies the chordal Loewner equation

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{1}$$

where $U_t = -B_t$ is a standard Brownian motion. (We choose this parametrization so that the driving function has variance parameter 1.) The equation is valid for all $z \in \mathbb{C} \setminus \{0\}$ up to time $T_z \in (0, \infty]$ and g_t is the unique conformal transformation of

$$H_t := \{z \in \mathbb{H} : T_z > t\}$$

onto $\mathbb{H}$ with $g_t(z) = z + o(1)$ as $z \to \infty$. Note that $T_{\overline{z}} = T_z$ and $g_t(\overline{z}) = \overline{g_t(z)}$, so we restrict to z in the upper half plane $\mathbb{H}$ or its closure $\overline{\mathbb{H}}$. If $z \in \overline{\mathbb{H}} \setminus \{0\}$, and we let

$$Z_t = Z_t(z) = g_t(z) - U_t,$$

then the Loewner equation (1) can be written as the SDE

$$dZ_t = \frac{a}{Z_t}\, dt + dB_t, \quad Z_0 = z.$$

If $z \in \mathbb{H}$, one should note that the process Z_t takes values in $\mathbb{H}$, but the Brownian motion B_t is a real Brownian motion. If $z \in \mathbb{R} \setminus \{0\}$, this equation becomes the usual real-valued Bessel SDE.

The word *curve* in these notes always means a continuous function of time. If $\gamma : [0, \infty) \to \mathbb{C}$ is a curve, we write γ_t for the image or trace up to time t,

$$\gamma_t = \{\gamma(s) : 0 \le s \le t\}.$$

1. The existence of the *SLE* curve

In this section, we will present a proof of the following theorem first proved by Rohde and Schramm [**15**]. We start with a definition.

Definition The conformal maps g_t are *generated by the curve* γ if $\gamma : [0, \infty) \to \mathbb{H}$ is a curve such that for each t, H_t is the unbounded component of $\mathbb{H} \setminus \gamma_t$.

THEOREM 1. [**15**] *If $\kappa \neq 8$, then with probability one, the conformal maps g_t of chordal SLE_κ are generated by a curve.*

This theorem is also true for $\kappa = 8$, but the only current proof in this case comes from taking a limit of discrete processes [**12**]. We will consider only the $\kappa \neq 8$ case and along the way explain why $\kappa = 8$ is the hardest case. The curve is called the *SLE_κ curve*.

Let

$$\tilde{f}_t = g_t^{-1}, \qquad f_t = \tilde{f}_t(z + U_t) = g_t^{-1}(z + U_t). \tag{2}$$

(In some earlier work what we call $\tilde{f}_t$ is denoted f_t and what we call f_t is denoted $\hat{f}_t$. I have chosen the notation in (2) because f_t will be used more often than $\tilde{f}_t$ in this paper, and hence it will make the formulas nicer.) Heuristically, we would like to define

$$\gamma(t) = \tilde{f}_t(U_t) = f_t(0) = \lim_{y \to 0} f_t(iy), \tag{3}$$

so that $g_t(\gamma(t)) = U_t$. However, all that we know at the moment is that f_t is a conformal transformation of $\mathbb{H}$ onto H_t. One can give examples of conformal transformations such that the limit in (3) does not exist. In fact [**14**], one can give examples for solutions of the Loewner equation (1) with continuous U_t.

It is also possible to give examples for which the limit in (3) exists for all t, but for which the function γ is not continuous in t. However, if the limit (3) exists and γ is continuous, then it is not too difficult to see that H_t is the unbounded component of $\gamma(0,t]$. Indeed, since $\gamma(t) \in \partial H_t$, we know that $\gamma(0,t] \cap H_t = \emptyset$. Since H_t is simply connected (it is a conformal image of $\mathbb{H}$ under g_t^{-1}), it is connected and hence the bounded components of $H_t \setminus \gamma_t$ cannot intersect H_t. Also, if we define $\hat{H}_t$ to be the unbounded component of $\mathbb{H} \setminus \gamma_t$ and $\hat{g}_t$ the conformal transformation of $\hat{H}_t$ onto $\mathbb{H}$ with $\hat{g}_t(z) - z = o(1)$ as $z \to \infty$, one can show that $\hat{g}_t$ satisfies (1) for $z \in \hat{H}_t$ and hence $\hat{g}_t(z) = g_t(z)$. In particular, $T_z > t$ and $z \in H_t$.

Notational convention. We will use ψ to denote a *(continuous, increasing) subpower function*, that is an continuous, increasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$\lim_{t \to \infty} \frac{\log \psi(t)}{\log t} = 0.$$

Different occurences of ψ indicate different subpower functions. Note that if ψ_1, ψ_2 are subpower functions, so are $\psi_1 \, \psi_2$, $\psi_1 + \psi_2$, and $\psi(t) = \psi_1(t^r)$ for $r > 0$.

Lévy's theorem on the modulus of continuity of Brownian motion shows that with probability one, the driving function is *weakly Hölder*-1/2 by which we mean that for some subpower function ψ,

$$|U_{t+s} - U_t| \leq \sqrt{s}\, \psi(1/s), \quad 0 \leq t \leq 1, 0 \leq s \leq 1. \tag{4}$$

(In fact, we can choose $\psi(x) = c\sqrt{\log x}$ but we do not need this.) This condition is not sufficient to show existence of the curve. In fact, there are examples [**14**] with ψ constant for which the curve does not exist. To guarantee existence of the curve, we will bound $|f_t'(iy)|$ for y near zero. Let

$$\mathcal{D}_n = \left\{ \frac{j}{2^n} : j = 0, 1, \ldots, 2^n \right\}$$

denote the set of dyadic rationals in $[0,1]$ at level n.

Lemma 2. *For $SLE_\kappa, \kappa \neq 8$, there exists $\theta = \theta_\kappa > 0$ such that with probability one there exists $C < \infty$ such that*

$$|f_t'(2^{-n}\, i)| \leq C\, 2^{n(1-\theta)}, \quad t \in \mathcal{D}_{2n}. \tag{5}$$

PROOF. By the Borel-Cantelli lemma, it suffices to show that

$$\mathbb{P}\left\{|f_t'(2^{-n}\, i)| \leq C\, 2^{n(1-\theta)}\right\} \leq c\, 2^{-n(2+\epsilon)}$$

for some c, ϵ. See Theorem 11 and the comments following for the proof of this estimate. □

In this section, we will use a series of exercises to conclude the following deterministic result.

THEOREM 3. *Suppose $U_t, 0 \leq t \leq 1$ is a driving function satisfying* (4) *and* (5). *Then the corresponding maps are generated by a curve γ. Moreover,*

$$|\gamma(t+s) - \gamma(t)| \leq s^{\theta/2}\, \psi(1/s), \quad 0 \leq t < s+t \leq 1,$$

for some subpower function ψ.

♦In particular, it follows that γ is Hölder continuous of order α for all $\alpha < \theta/2$. In other words, γ is weakly Hölder-$(\theta/2)$

♦We recall our convention that the subpower function ψ takes different values in different places. The function ψ in Theorem 3 is not meant to be the same ψ as in (4). A careful reader can go through the proof and find how the ψ in the theorem depends on the ψ in (4).

♦Lemma 2 is not true for $\kappa = 8$. I would expect that one can give a direct proof of the existence of the curve for $\kappa = 8$, but it would require very careful analysis. In particular, we could not get away with being so cavalier about the subpower functions ψ.

We use the distortion theorem to construct the function γ. The first ingredient of the proof is a version of the distortion theorem that we leave as an exercise.

EXERCISE 4. *There exist C, r such if $f : \mathbb{H} \to \mathbb{C}$ is a conformal transformation, then for all $x \in \mathbb{R}, y > 0$,*

$$C^{-1}\,|f'(iy)| \leq |f'(iys)| \leq C\,|f'(iy)|, \quad \frac{1}{2} \leq s \leq 2,$$

$$C^{-1}\,(x^2+1)^{-r}\,|f'(iy)| \leq |f'(xy+iy)| \leq C\,(x^2+1)^{r}\,|f'(iy)|,$$

Hint: The distortion and growth theorems (see, e.g., [**5**]*) solve the equivalent problem in the unit disk $\mathbb{D}$. Although we do not need it here, you may wish to find the smallest possible r such that this holds.*

To extend the estimate to times that are not dyadic, we use the Loewner equation for the inverse. If g_t satisfies (1) and $\tilde{f}_t = g_t^{-1}$, then using $\tilde{f}_t(g_t(z)) = z$, we get the equation

$$\partial_t \tilde{f}_t(z) = \tilde{f}_t'(z)\, \frac{a}{U_t - z}. \tag{6}$$

Differentiating this, we get

$$\partial_t \tilde{f}_t'(z) = \tilde{f}_t''(z)\, \frac{a}{U_t - z} + \tilde{f}_t'(z)\, \frac{a}{(U_t - z)^2}.$$

Hence, if $z = x + iy$,

$$|\partial_t \tilde{f}_t'(z)| \leq a\, \left[|\tilde{f}_t''(z)|\, y^{-1} + |\tilde{f}_t'(z)|\, y^{-2}\right].$$

EXERCISE 5. *Show that there exists* $c < \infty$ *such that if* $f : \mathbb{H} \to \mathbb{C}$ *is a conformal transformation, then*

$$|f''(z)| \leq \frac{c}{\operatorname{Im}(z)} \, |f'(z)|.$$

Hint: Look up Bieberbach's theorem on the second coefficient of univalent functions on the disk. If you do this, you will find the optimal c.

Using the exercise, we now have

$$|\partial_t \tilde{f}_t'(x+iy)| \leq \frac{c}{y^2} \, |\tilde{f}_t'(x+iy)|. \tag{7}$$

It is now not difficult to show that the limit in (3) exists for all $t \in [0,1]$. We leave the steps as exercises.

EXERCISE 6. *Use* (7) *to prove the following. There exists* $c < \infty$ *such that if* $\tilde{f}_t$ *satisfies* (6) *and* $s \leq y^2$*, then*

$$c^{-1} \, |\tilde{f}_t'(x+iy)| \leq |\tilde{f}_{t+s}'(x+iy)| \leq c \, |\tilde{f}_t'(x+iy)|,$$

$$|\tilde{f}_{t+s}(x+iy) - \tilde{f}_t(x+iy)| \leq c \, y^2 \, |\tilde{f}_t'(x+iy)|. \tag{8}$$

EXERCISE 7. *Suppose* $U_t, 0 \leq t \leq 1$ *is a driving function satisfying* (4) *and* (5)*. Then there exists a subpower function* ψ *such that for all* $0 \leq t \leq 1$ *and all* $0 < y \leq 1$,

$$|f_t'(iy)| \leq y^{\theta-1} \, \psi(1/y).$$

EXERCISE 8. *Suppose* $U_t, 0 \leq t \leq 1$ *is a driving function satisfying* (4) *and* (5)*. Then there exists a subpower function* ψ *such that for all* $0 \leq t \leq 1$*, the limit*

$$\gamma(t) = \lim_{y \to 0+} f_t(iy) \tag{9}$$

exists and for $0 < y \leq 1$,

$$|\gamma(t) - f_t(iy)| \leq y^{\theta} \, \psi(1/y). \tag{10}$$

We still have to show that γ is a continuous function of t and estimate its modulus of continuity. It suffices to estimate

$$|\gamma(t+s) - \gamma(t)|$$

where $t \in \mathcal{D}_{2n}$ and $0 \leq s \leq 2^{-2n}$. We use the triangle inequality. For every $y > 0$,

$$|\gamma(t+s) - \gamma(t)| \leq |\gamma(t+s) - f_{t+s}(iy)| + |f_{t+s}(iy) - f_t(iy)| + |\gamma(t) - f_t(iy)|.$$

Setting $y = 2^{-n}$ and using (10), we get

$$|\gamma(t+s) - \gamma(t)| \leq \; 2^{-n\theta} \, \psi(2^n) + |f_{t+s}(i2^{-n}) - f_t(i2^{-n})|.$$

We now write

$$|f_{t+s}(i2^{-n}) - f_t(i2^{-n})| = |\tilde{f}_{t+s}(U_{t+s} + i2^{-n}) - \tilde{f}_t(U_t + i2^{-n})|$$

$$\leq |\tilde{f}_{t+s}(U_{t+s} + i2^{-n}) - \tilde{f}_{t+s}(U_t + i2^{-n})| + |\tilde{f}_{t+s}(U_t + i2^{-n}) - \tilde{f}_t(U_t + i2^{-n})|.$$

The difference

$$|\tilde{f}_{t+s}(U_{t+s} + i2^{-n}) - \tilde{f}_{t+s}(U_t + i2^{-n})|$$

is bounded above by $|U_{t+s} - U_t|$ times the maximum of $|\tilde{f}_{t+s}'(z)|$ over all z on the interval connecting $U_{t+s} + i2^{-n}$ and $U_t + i2^{-n}$. Using Exercises 4 and 7, we see that this maximum is bounded above by $2^{n(1-\theta)}\,\psi(2^n)$ and (4) implies that

$$|U_{t+s} - U_t| \leq 2^{-n}\,\psi(2^n).$$

Therefore,

$$|\tilde{f}_{t+s}(U_{t+s} + i2^{-n}) - \tilde{f}_{t+s}(U_t + i2^{-n})| \leq 2^{-\theta n}\,\psi(2^n).$$

For the second term, we use (8) to get

$$\begin{aligned}|\tilde{f}_{t+s}(U_t + i2^{-n}) - \tilde{f}_t(U_t + i2^{-n})| &\leq c\,2^{-2n}\,|\tilde{f}_t'(U_t + i2^{-n})| \\ &\leq 2^{-n}\,2^{n(1-\theta)}\,\psi(2^n) \leq 2^{-n\theta}\,\psi(2^n).\end{aligned}$$

Combining all of the estimates, we have

$$|\gamma(s+t) - \gamma(t)| \leq s^{\theta/2}\,\psi(1/s), \qquad t \in \mathcal{D}_{2n}, \quad 0 \leq s \leq 2^{2n},$$

from which Theorem 3 follows.

1.1. Converse and Hölder continuity.

We have seen that (4) and (5) imply that the curve γ is weakly $(\theta/2)$-Hölder.

PROPOSITION 9. *Suppose U_t satisfies* (4). *Then there exists a subpower function ψ such that for $t \in \mathcal{D}_{2n}$,*

$$\max_{0 \leq s \leq 2^{-2n}} |\gamma(s+t) - \gamma(t)| \geq 2^{-n}\,|f_t'(i2^{-n})|\,\psi(2^n)^{-1}.$$

SKETCH OF PROOF. We write ψ for $\psi(2^n)$ and allow ψ to change from line to line. Using (4), we can see that the image of $\gamma(t, t+2^{-2n}]$ under g_t has diameter at most $2^{-n}\,\psi$. Since it has half-plane capacity $a\,2^{-n}$, it must include a least one point $z = x + iy = \gamma(s)$ with $|x| \leq 2^{-n}\,\psi$ and $y \geq 2^{-n}/\psi$. (Why?) Distortion estimates imply that $|f_t'(z)| \geq |f_t'(i2^{-n})|\,\psi^{-1}$. The Koebe-1/4 theorem applied to the map f_t on the disk of radius y about z shows that $\gamma(t)$, which is $f_t(0)$, is not in the disk of radius $2^{-n}\,|f_t'(i2^{-n})|\,\psi^{-1}$ about $f_t(z) = \gamma(t+s)$. □

The methods of the next section allow us to determine the critical value of θ for which there exist n, y with $|f_t'(i2^{-n})| \geq 2^{(1-\theta)n}$. This is the basic idea of the following theorem which we do not prove. One direction was proved in [**13**] and the other direction in [**6**]

THEOREM 10. *Let*

$$\alpha_* = \alpha_*(\kappa) = 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{8+\kappa}}.$$

If γ is an SLE_κ curve and $\epsilon > 0$, then with probability one $\gamma(t), \epsilon \leq t \leq 1$ is weakly α_*-continuous, but not Hölder continuous of any order $\alpha > \alpha_*$.

♦The behavior of the curve at $t = 0$ is different than for positive t because we are starting with the initial condition for H_t of $\mathbb{H}$. This is why we restrict the curve to times $\epsilon \leq t \leq 1$ in the statement of the theorem.

♦Note that the theorem implies that for $\kappa = 8$, the curve is not Hölder-α for any $\alpha > 0$. This indicates why $\kappa = 8$ is the hardest value to show the existence of the curve.

♦This is a statement about the modulus of continuity of $\gamma(t)$ as a function of t in the capacity parametrization. Sometimes "Hölder continuity of SLE" refers to the properties of the function $z \mapsto f_t(z)$ for fixed t. This is discussed in [**15**]. For this problem $\kappa = 4$ is the value for which the function is not Hölder continuous of any order $\alpha > 0$.

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2. The moments of $|f'|$

Using the existence of the curve as one motivation, we now proceed to discuss how one estimates

$$\mathbb{E}\left[|f_t'(z)|^\lambda\right],$$

where

$$f_t(z) = g_t^{-1}(z + U_t).$$

We summarize the main result here. Most of what we discuss here is proved in [**4**], but the final piece of the theorem as we state it here was done as Theorem 5.4 of [**6**]. A weaker form of this was in [**15**].

THEOREM 11. *If*

$$\lambda < \lambda_c = a + \frac{3}{16a} + 1 = \frac{2}{\kappa} + \frac{3\kappa}{32} + 1,$$

then as $t \to \infty$,

$$\mathbb{E}\left[|f_1'(i/t)|^\lambda\right] = \mathbb{E}\left[|f_{t^2}'(i)|^\lambda\right] \asymp t^{-\zeta},$$

where

$$\zeta = \zeta(\lambda) = \lambda + \frac{1}{2a}\sqrt{(2a+1)^2 - 4a\lambda} - 1 - \frac{1}{2a}.$$

Moreover, the expectation is carried on an event on which

$$|f_{t^2}'(i)| \approx t^\beta, \tag{11}$$

where

$$\beta = \beta(\lambda) = -\zeta'(\lambda) = \frac{1}{\sqrt{(2a+1)^2 - 4a\lambda}} - 1.$$

Roughly speaking,

$$\mathbb{P}\left\{|f_1'(i/t)| \approx t^\beta\right\} \approx t^{-(\zeta + \lambda\beta)}.$$

Note that $\zeta(\lambda_c) = a - \frac{1}{16a}, \beta(\lambda_c) = 1$ and hence

$$\zeta(\lambda_c) + \lambda_c\,\beta(\lambda_c) = 2a + \frac{1}{8a} + 1 = \frac{4}{\kappa} + \frac{\kappa}{16} + 1.$$

The right-hand side is minimized when $\kappa = 8$ at which it takes the value 2. For $\kappa \neq 8$, we can find $\beta < 1$ such that

$$\zeta(\lambda) + \lambda\beta(\lambda) > 2,$$

from which we can deduce Lemma 2 for $\theta = 1 - \beta$.

2.1. The reverse Loewner flow. We will use the *reverse Loewner flow* in studying the derivative. The reverse Loewner equation is the usual Loenwer equation run backwards in time. It takes the form

$$\partial_t h_t(z) = -\frac{a}{h_t(z) - V_t} = \frac{a}{V_t - h_t(z)}, \quad h_0(z) = z. \tag{12}$$

For each t, h_t is a conformal transformation of $\mathbb{H}$ onto a subdomain $h_t(\mathbb{H})$ satisfying $h_t(z) - z \to 0$ as $z \to \infty$. A relationship between the forward and reverse Loewner equations is given in the following exercise.

EXERCISE 12. *Suppose g_t, $0 \leq t \leq s$, is the solution to* (1) *and h_t is the solution to* (12) *with $V_t = U_{s-t} - U_s$. Then*

$$h_s(z) = f_s(z) - U_s.$$

If $U_t, 0 \leq t \leq s$ is a standard Brownian motion, then

$$V_t = U_{s-t} - U_s, \quad 0 \leq t \leq s,$$

is also a standard Brownian motion. Hence the following holds.

- If $g_t, 0 \leq t \leq s$ is the solution to (1) where U_t is a standard Brownian motion, and $h_t, 0 \leq t \leq s$ is the solution to (12) where V_t is a Brownian motion, then the random conformal transformations
$$z \longmapsto f_s(z) - U_s \quad \text{and} \quad z \longmapsto h_s(w)$$
have the same distribution. In particular, f_s' and h_s' are identically distributed and
$$\mathbb{E}\left[|f_s'(z)|^\lambda\right] = \mathbb{E}\left[|h_s'(z)|^\lambda\right].$$

The joint distribution of $\{f_t' : 0 \leq t \leq s\}$ is *not* the same as that of $\{h_s' : 0 \leq t \leq s\}$. However, we can give the joint distributions. For second moment estimates, we need to consider two times simultaneously. We state the relationship here; the interested reader may wish to verify this.

- Suppose $f_t, 0 \leq t \leq s+u$ is the solution to (1) where U_t is a standard Brownian motion. Let $h_t, 0 \leq t \leq s+u$ be the solution to (12) with $V_t = U_{s+u-t} - U_{s+u}$. Let $\tilde{h}_t, 0 \leq t \leq s$ be the solution to (12) with $V_t = U_{s-t} - U_s$. Let
$$Z_t(z) = h_t(z) - V_t.$$
Then,
  - $h_{s+u}(z) = \tilde{h}_s(Z_u(z))$.
  - h_u and $\tilde{h}_s$ are independent.
  - $f_s'(w)\, f_{s+u}'(z) = h_u'(z)\, \tilde{h}_s'(Z_u(z))\, \tilde{h}_s'(w)$
  - $f_{s+u}(z) - f_s(w) = \tilde{h}_s(Z_u(z)) - \tilde{h}_s(w)$.

2.2. Some computations. For this section we assume that h_t satisfies (12) with $V_t = -B_t$ being a standard Brownian motion.

EXERCISE 13. *Use the scaling property of Brownian motion to show that if $r > 0$, $h_t(z)$ has the same distribution as $h_{r^2t}(rz)/r$, and hence $h_t'(z)$ has the same distribution at $h_{r^2t}'(rz)$.*

Exercise 14. *Prove the following "parabolic Harnack inequality". For every compact $V \subset \mathbb{H}$ and every $s_0 \geq 1$, there exist c_1, c_2 (depending on V, s_0 but not on λ or κ) such that for all $t \geq 1$,*

$$c_1^\lambda \, \mathbb{E}\left[|h_t'(i)|^\lambda\right] \leq \mathbb{E}\left[|h_{st}'(z)|^\lambda\right] \leq c_2^\lambda \, \mathbb{E}\left[|h_t'(i)|^\lambda\right], \qquad s_0^{-1} \leq s \leq s_0, \quad z \in V.$$

Hint: Use scaling and the distortion theorem.

Let $z \in \mathbb{H}$ and define

$$Z_t = Z_t(z) = X_t + iY_t = h_t(z) - V_t = h_t(z) + B_t.$$

we will define a number of other quantities in this section. Even though we omit it in the notation, it is important to remember that there is a z dependence. The equation (12) can be written as

$$dX_t = -\frac{a\,X_t}{X_t^2 + Y_t^2}\,dt + dB_t, \qquad \partial_t Y_t = \frac{a\,Y_t}{X_t^2+Y_t^2}. \tag{13}$$

Differentiating (12) with respect to z gives

$$\partial_t [\log h_t'(z)] = \frac{a}{Z_t^2},$$

and by taking real parts, we get

$$\partial_t\,|h_t'(z)| = |h_t'(z)|\,\frac{a(X_t^2 - Y_t^2)}{(X_t^2+Y_t^2)^2}.$$

Let

$$S_t = \sin[\arg Z_t] = \frac{Y_t}{\sqrt{X_t^2+Y_t^2}}, \qquad \Upsilon_t = \frac{|h_t'(z)|}{Y_t}.$$

The chain rule gives

$$\partial_t \Upsilon_t = -\Upsilon_t\,\frac{2aY_t^2}{(X_t^2+Y_t^2)^2},$$

and an exercise in Itô's formula gives

$$dS_t^r = S_t^r\left[\frac{(2ar + \frac{r^2}{2} + \frac{r}{2})\,X_t^2 - \frac{r}{2}\,Y_t^2}{(X_t^2+Y_t^2)^2}\,dt - \frac{r\,X_t}{X_t^2+Y_t^2}\,dB_t\right]. \tag{14}$$

Proposition 15. *Suppose $r \in \mathbb{R}$ and*

$$\lambda = \lambda(r) = r\left(1 + \frac{1}{2a}\right) - \frac{r^2}{4a}, \quad \zeta = \zeta(r) = r - \frac{r^2}{a} = \lambda - \frac{r}{2a}.$$

If $z \in \mathbb{H}$, let

$$M_t = M_{t,r}(z) = |h_t'(z)|^\lambda\,Y_t^\zeta\,S_t^{-r}. \tag{15}$$

Then M_t is a martingale satisfying

$$dM_t = \frac{r\,X_t}{X_t^2+Y_t^2}\,M_t\,dB_t. \tag{16}$$

Exercise 16. *Verify as many of the calculations above as you want. Also, establish the following deterministic estimates if $z = x+iy$.*

$$y^2 \leq Y_t^2 \leq y^2 + 2at,$$

$$|h_t'(z)| \leq \frac{Y_t}{Y_0} \leq \sqrt{1 + 2a(t/y^2)}.$$

♦The parameters λ, ζ are the same parameters as in Theorem 11. However, it is useful to include the extra parameter r. There is only a "one real variable" amount of randomness (nontrivial quadratic variation) in the martingale M_t. For convenience we have written it in terms of the sine, S_t; earlier versions of these computations chose to write it in terms of $(X^2 + Y^2)$. Either way, one must choose the appropriate "compensator" terms which turn out to be in terms of Y_t and $|h_t'(z)|$, both of which are differentiable in t.

We can consider a new measure $\mathbb{P}^*$ obtained by weighting by the martingale M. To be more precise, if E is an event in the σ-algebra $\mathcal{F}_t = \sigma\{B_s : 0 \leq s \leq t\}$, then

$$\mathbb{P}^*(E) = M_0^{-1}\,\mathbb{E}\left[M_t\, 1_E\right].$$

The Girsanov theorem tells us that

$$dB_t = \frac{r\, X_t}{X_t^2 + Y_t^2}\, dt + dW_t,$$

where W_t is a Brownian motion with respect to $\mathbb{P}^*$. In other words,

$$dX_t = \frac{(r-a)X_t}{X_t^2 + Y_t^2}\, dt + dW_t. \tag{17}$$

♦Suppose M_t is a continuous, positive process satisfying

$$dM_t = A_t\, M_t\, dB_t.$$

Then M_t is a local martingale, but not necessarily a martingale. If one chooses stopping times τ_n by

$$\tau_n = \inf\{t : M_t \geq n \text{ or } |A_t| \geq n\},$$

then $M_t^{(n)} := M_{t\wedge\tau_n}$ is a martingale satisfying

$$dM_t^{(n)} = A_t\, 1_{\{\tau_n > t\}}\, M_t^{(n)}\, dB_t.$$

The Girsanov theorem tells us that it we weight by the martingale, then B_t satisfies

$$dB_t = A_t\, dt + dW_t, \quad t < \tau_n,$$

where W_t is a Brownian motion in the new measure which we denote by $\mathbb{P}^*$. At the moment, this is only valid for $t < \tau_n$. However, if

$$\lim_{n\to\infty} \mathbb{P}^*\{\tau_n \leq t\} = 0, \tag{18}$$

then the we can conclude that the process is actually a martingale.

In our particular case, one can see from (17) and the bounds on Y_t that X_t does not blow up in finite time. Since $|h_t'(z)|$ is expressed in a differential equation involving X_t, Y_t, it also does not blow up. This is how one verifies (18) and shows that M_t is a martingale.

Let us now choose $z = i$ so that

$$M_0 = 1.$$

Since M_t is a martingale, we have for all r,

$$\mathbb{E}[M_t] = \mathbb{E}\left[|h_t'(i)|^{\lambda}\, Y_t^{\zeta}\, S_t^{-r}\right] = 1.$$

Typically we expect for large t that

$$Y_t \asymp t^{1/2}, \quad S_t \asymp 1, \tag{19}$$

and hence we might want to conclude that

$$\mathbb{E}\left[|h_t'(z)|^\lambda\right] \asymp t^{-\zeta/2}.$$

This is a hand-waving argument, and, in fact, it is not valid for all values of λ. As we will see below, the values of λ for which it will be valid are those values for which (19) holds typically *when we weight by the martingale* M_t. These values which we call *good* r satisfy

$$r < r_c := 2a + \frac{1}{2} = \frac{4}{\kappa} + \frac{1}{2}.$$

Let

$$q = r_c - r = 2a + \frac{1}{2} - r > 0.$$

Then the good values of r are those for which $q > 0$. Note that

$$\lambda(r_c) = a + \frac{3}{16a} + 1, \quad \zeta(r_c) = a - \frac{1}{16a}.$$

For $-\infty < r < r_c, -\infty < \lambda < \lambda_c = a + \frac{3}{16a} + 1$, the relationship $r \longleftrightarrow \lambda$ is a bijection and

$$r = 2a + 1 - \sqrt{(2a+1)^2 - 4a\lambda},$$

$$\zeta = \lambda - \frac{r}{2a} = \lambda + \frac{1}{2a}\sqrt{(2a+1)^2 - 4a\lambda} - 1 - \frac{1}{2a}.$$

2.3. Imaginary part parametrization. We assume that $z = i$ and $r < r_c$, that is,

$$q = 2a + \frac{1}{2} - r > 0.$$

We will introduce a time change under which the logarithm of the imaginary part of Z grows linearly. Let

$$\sigma(t) = \inf\{s : Y_s = e^{at}\},$$

and define

$$\hat{Z}_t = Z_{\sigma(t)}, \quad \hat{X}_t = X_{\sigma(t)}, \quad \hat{Y}_t = Y_{\sigma(t)} = e^{at}, \quad \hat{h}_t = h_{\sigma(t)}.$$

We also define

$$K_t = e^{-at}\,\hat{X}_t, \quad \hat{S}_t = S_{\sigma(t)} = \frac{e^{at}}{|\hat{Z}_t|} = \frac{1}{\sqrt{K_t^2+1}}, \quad J_t = \sinh^{-1}(K_t).$$

Under this parametrization, the pair of equations (13) can be written as a single one-variable SDE in K_t or J_t. We will list some computation below, but we summarize the basic idea as follows:

- If $r < r_c$ and we weight by the martingale M_t, then in the weighted measure, J_t is a positive recurrent diffusion.

EXERCISE 17. *Verify the following deterministic relations.*

$$\partial_t \sigma(t) = |\hat{Z}_t|^2,$$

$$\sigma(t) = \int_0^t e^{2as}\,(K_s^2+1)\,ds = \int_0^t e^{2as}\,\cosh^2 J_s\,ds,$$

$$|\hat{h}_t'(i)| = e^{aL_t},$$

where

$$L_t = \int_0^t \left[1 - \frac{2}{K_s^2+1}\right] ds = t - \int_0^t \frac{2}{\cosh^2 J_s},ds.$$

$$e^{-at} \leq |\hat{h}_t'(i)| \leq e^{at}.$$

EXERCISE 18. *Show that there exists a standard Brownian motion $\tilde{B}_t$ such that*

$$dK_t = -2a\, K_t\, dt + \sqrt{K_t^2 + 1}\, d\tilde{B}_t,$$

$$dJ_t = -(q+r)\, \tanh J_t\, dt + d\tilde{B}_t.$$

Hint: This requires knowing how to handle time changes in SDEs.

EXERCISE 19. *If $N_t = M_{\sigma(t)}$ where M_t is the martingale in* (15)*, then*

$$N_t = e^{\nu L_t}\, e^{\xi t}\, [\cosh J_t]^r, \tag{20}$$

where

$$\nu = a\lambda = r\, \left(\frac{q}{2} + \frac{1}{4}\right) + \frac{r^2}{4},$$

$$\xi = a\zeta = r\, \left(\frac{q}{2} + \frac{1}{4}\right) - \frac{r^2}{4},$$

Moreover,

$$dN_t = r\, [\tanh J_t]\, N_t\, d\tilde{B}_t.$$

Using the last exercise, we see that we must analyze the SDE

$$dJ_t = -(q+r)\, [\tanh J_t]\, dt + \tilde{B}_t, \quad J_0 = 0.$$

Note that this equation is written in terms of q, r; the parameter a has disappeared. We consider the martingale N_t in (20) which satisfies

$$dN_t = r\, [\tanh J_t]\, N_t\, d\tilde{B}_t, \quad N_0 = 1.$$

Let $\mathbb{P}^*, \mathbb{E}^*$ denote probabilities and expectations with respect to the measure obtained by weighting by the martingale N_t. Then

$$d\tilde{B}_t = r\, [\tanh J_t]\, N_t\, dt + dW_t,$$

where W_t is a standard Brownian motion with respect to $\mathbb{P}^*$. In particular,

$$dJ_t = -q\, [\tanh J_t]\, dt + dW_t.$$

♦Time changes of martingales (under some boundedness conditions) give martingales. Weighting by a time change of a martingale produces the same probability measure (on the σ-algebra $\mathcal{F}_\infty$) on a space as that obtained by weighting by the martingale. (This is subtle — a time changed process is not the same as the original process; it is the underlying measure on the probability space that is the same.) This is why we use the same letter $\mathbb{P}^*, \mathbb{E}^*$ for weighting by N_t as for M_t.

2.4. The one-variable SDE.

♦Many of the one-variable SDEs that arise in studying SLE can be viewed as equations arising from the Girsanov theorem by "weighting Brownian motion locally" by a function. Suppose F is a positive C^2 function on $\mathbb{R}$. Suppose B_t is a standard one-dimensional Brownian motion. Then Itô's formula gives

$$dF(B_t) = F(B_t)\,[A_t\,dt + \Phi_t\,dB_t],$$

where

$$\Phi_t = [\log F(B_t)]' = \frac{F'(B_t)}{F(B_t)}, \quad A_t = \frac{F''(B_t)}{2\,F(B_t)}.$$

In other words, if

$$M_t = F(B_t)\,\exp\left\{-\int_0^t \frac{F''(B_s)}{2\,F(B_s)}\,ds\right\},$$

then M_t is a local martingale satisfying

$$dM_t = [\log F(B_t)]'\,M_t\,dB_t.$$

If F satisfies some mild restrictions, then M_t is a martingale. If we let $\mathbb{P}^*$ be the measure obtained by weighting by M_t, then the Girsanov theorem implies that

$$dB_t = [\log F(B_t)]'\,dt + dW_t, \tag{21}$$

where W_t is a $\mathbb{P}^*$-Brownian motion. Let $p_t(x,y)$ denote the transition probabilities for Brownian motion and $p_t^*(x,y)$ the transitions for B_t under $\mathbb{P}^*$, that is, the transitions for the equation (21). We know that $p_t(x,y) = p_t(y,x)$. In general, it is hard to give an expression for $p_t^*(x,y)$, however, if we consider a path $\omega(s), 0 \le s \le t$, from x to y of time duration t, then the Radon-Nikodym derivative of $\mathbf{P}^*$ with respect to $\mathbb{P}$ on this path is given by

$$\frac{F(y)}{F(x)}\,\exp\left\{-\int_0^t \frac{F''(\omega(s))}{2\,F(\omega(s))}\,ds.\right\}$$

The expression in the exponential may be complicated, but the key fact is that it is the same for the reversed path $\omega^R(s) = \omega(t-s)$ which goes from y to x. Hence we get the reversibility relation

$$p_t^*(x,y) = \frac{F(y)^2}{F(x)^2}\,p_t^*(y,x).$$

This implies that F^2 gives an invariant density for the SDE (21).

Exercise 20. *Use the ideas above to find the invariant probability for diffusions satisfying the following.*

$$dX_t = a\,[\cot X_t]\,dt + dW_t, \quad a \ge 1/2, \quad 0 < X_t < \pi.$$

$$dX_t = -q\,X_t\,dt + dW_t, \quad q > 0.$$

In both cases, try to find a function F such that the equations arise by starting with a Brownian motion X_t and then weighting locally by the function F.

Let us consider the SDE

$$dJ_t = -q\,[\tanh J_t]\,dt + dW_t, \quad J_0 = 0,$$

with $q > 0$. This is the equation obtained by weighting a Brownian motion locally by the function $f(x) = [\cosh x]^{-q}$ and using this (see note above) or other standard techniques, one can see that this is positive recurrent diffusion with invariant probability density

$$v_q(x) = \frac{C_q}{\cosh^{2q} x}, \quad C_q^{-1} = \int_{-\infty}^{\infty} \frac{dx}{\cosh^{2q} x} = \frac{\Gamma(\frac{1}{2})\,\Gamma(q)}{\Gamma(q+\frac{1}{2})}.$$

Consider the functional L_t that appears in Exercise 17:

$$L_t = t - \int_0^t \frac{2}{\cosh^2 J_s}\, ds.$$

Since J_t is a positive recurrent distribution, at large times t the distribution of J_t is very close to the invariant distribution. If we write J_∞ for a random variable with the invariant distribution, we get

$$\lim_{t\to\infty} \frac{\mathbb{E}[L_t]}{t} = 1 - \mathbb{E}\left[\frac{2}{\cosh^2 J_\infty}\right] = 1 - \int_{-\infty}^{\infty} \frac{2\,C_q\, dx}{\cosh^{2q+2} x} = \beta := \frac{1-2q}{1+2q}.$$

Indeed, one expects more than convergence in expectation. Assuming that an appropriate strong law of large numbers and central limit theorem hold, we would expect

$$L_t = \beta t + O(t^{1/2}).$$

Indeed, one can give exponential estimates using the martingales for values of $\tilde{r}$ near r to show that there exists b such that

$$\mathbb{E}\left[\exp\left\{\frac{b|L_t - \beta t|}{\sqrt{t}}\right\}\right] \le c < \infty.$$

This gives immediate bounds on probabilities

$$\mathbb{P}\left\{|L_t - \beta t| \ge u\sqrt{t}\right\} = \mathbb{P}\left\{\exp\left\{\frac{b|L_t - \beta\, t|}{\sqrt{t}}\right\} \ge e^{bu}\right\} \le c\, e^{-bu}.$$

2.5. Returning to the reverse flow. Here we will not give complete details. We assume $z = i$. Since $N_t = M_{\sigma(t)}$ is a martingale,

$$\mathbb{E}[N_t] = \mathbb{E}[N_0] = 1.$$

Moreover, if E is any event depending on $B_s, 0 \le s \le \sigma(t)$, then

$$\mathbb{E}\left[N_t\, 1_E\right] = \mathbb{E}^*\left[1_E\right] = \mathbb{P}^*(E),$$

where as before $\mathbb{P}^*$ denotes probabilities obtained by weighting by the martingale. To compute $\mathbb{P}^*(E)$, one only needs to consider the one-variable SDE of the previous section.

For example, for some subpower function ψ we might specify the event $E_{t,c}$ such that the following holds for $0 \le s \le t$:

$$J_s \le c\,\psi(s),$$

$$J_s \le c\,\psi(t-s), \tag{22}$$

$$|L_s - \beta s| \le c\,\psi(s)\,\sqrt{s+1},$$

$$|L_s - \beta s| \le c\,\psi(t-s)\,\sqrt{s+1}. \tag{23}$$

Using the one-variable SDE we can show that

$$\lim_{c\to\infty} \liminf_{t\to\infty} \mathbb{P}^*(E_{t,c}) = 1. \tag{24}$$

♦The subpower function ψ needs to grow to infinity sufficiently fast for our estimates. However, for everything we do here, we could choose

$$\psi(s) = 1 \vee \exp\left\{(\log s)^{1/2}\right\},$$

which grows faster than any power of $\log s$. For some of our estimates we do need that

$$J_s \leq c, \quad |L_s - \beta s| \leq c\sqrt{s},$$

for $s = t - O(1)$. This is why we include the conditions (22) and (23).

We will now establish one direction of Theorem 11. Note that

$$\beta(\lambda) = \frac{1-2q}{1+2q} = \frac{2r-4a}{2+4a-2r} = \frac{1}{\sqrt{(2a+1)^2 - 4a\lambda}} - 1 = -\zeta'(\lambda).$$

(If one wants to understand why $\beta(\lambda)$ should equal $-\zeta'(\lambda)$, see the next subsection.) Choose c sufficiently large so that if $E = E_{t,c}$, then $\mathbb{P}^*(E) \geq 1/2$. Then

$$\frac{1}{2} \leq \mathbb{E}\left[M_{\sigma(t)}\, 1_E\right] \leq 1.$$

On the event E,

$$\sigma(t) = \int_0^t e^{2as}\, \cosh^2 J_s \asymp e^{2at}.$$

Here $\asymp$ means up to constants, which is stronger than up to " a subpower function". This uses the conditions (22) and (23). Also recall that

$$|h'_{\sigma(t)}(i)| = e^{aL_t} = \exp\left\{a\beta t + O(t^{1/2}\psi(t))\right\}.$$

With a little argument using distortion theorem ideas and the Loewner equation, we get (letting $T = e^{at}$),

$$\mathbb{E}\left[|h'_{T^2}(i)|^\lambda\, Y_{T^2}^\zeta\, S_{T^2}^{-r}\, 1_E\right] \asymp 1,$$

on an event on which

$$Y_{T^2} \asymp T, \quad S_{T^2} \asymp 1, \quad |h'_{T^2}(i)| \approx T^\beta.$$

In particular,

$$\mathbb{E}\left[|h'_{T^2}(i)|^\lambda\right] \geq c\, T^{-\zeta}.$$

The other direction takes a little more work which we do not do here although it uses some of the same ideas. For the upper bound, we need to control the terms for which Y_t and S_t are far from their typical values. This was done for many values of r in [**4**] and for all $r < r_c$ in [**6**].

♦The remainder of this section will discuss more advanced topics relating to the reverse Loewner flow. This will not be needed in the later sections, so readers should feel free to skip now to Section 3.

2.6. Multifractal spectra. The basic ideas of multifractal spectrum are the same as those in basic "large deviation" theory and uses a simple idea that sometimes goes under the name of the Legendre transform. Let us explain it heuristically. Suppose that Z_n is a sequence of random variables for which we know the asymptotics of the moment generating function,

$$\mathbb{E}\left[e^{\lambda Z_n}\right] \approx e^{-n\zeta(\lambda)}, \tag{25}$$

for λ in an open interval about the origin. Define $\rho(s)$ roughly by

$$\mathbb{P}\{Z_n \approx sn\} \approx e^{-n\rho(s)}.$$

Then,

$$\mathbb{E}\left[e^{\lambda Z_n}; Z_n \approx sn\right] \gtrsim e^{[\lambda s - \rho(s)]n}.$$

The exponent $\zeta(\lambda)$ can be obtained by maximizing the right-hand side in s,

$$\zeta(\lambda) = \inf_s \left[\lambda s - \rho(s)\right].$$

If ρ is smooth enough, the infimum is obtained at s_λ where

$$\rho'(s_\lambda) = \lambda.$$

Conversely, the Chebyshev inequality gives

$$\mathbb{P}\{Z_n \approx sn\} \lesssim \mathbb{E}[e^{\lambda Z_n}]\, e^{-\lambda sn},$$

and equality is obtained for the optimal s. In other words,

$$\rho(s) = \max_\lambda [-\zeta(\lambda) - s\lambda].$$

The maximizer is obtained at λ_s satisfying

$$\zeta'(\lambda_s) = -s.$$

The "multifractal regime" is the regime where different values of s give different values of λ. In this case we can say roughly:

- The expectation in (25) is carried on the event $Z_n \approx s_\lambda n$. This event has probability about

$$e^{-n\rho(s_\lambda)} \approx e^{-[\zeta(\lambda) + \lambda\, s_\lambda]n}.$$

♦Large deviation theory generally discusses events whose probabilities decay exponentially. In critical phenomena, one generally has events whose probability decays like a power law. It is easy to convert to exponential scales. For example, if we want to study $|h'_{t^2}(i)|$, the corresponding random variable might be

$$Z_n = \log\left|h'_{e^{2n}}(i)\right|.$$

2.7. The tip multifractal spectrum for *SLE*. Consider the number of times $t \in \mathcal{D}_{2n}$ such that

$$|f_t'(i2^{-n})| \approx 2^{n\beta}.$$

This is imprecise, but I will state some theorems below. If $\beta < 1$, the expected number of such times is

$$2^{n(2-\zeta-\lambda\beta)}, \tag{26}$$

where

$$\beta = \frac{1}{\sqrt{(2a+1)^2 - 4a\lambda}} - 1, \quad \zeta = \lambda + \frac{1}{2a}\sqrt{(2a+1)^2 - 4a\lambda} - 1 - \frac{1}{2a}.$$

The condition $\beta < 1$ corresponds to

$$\lambda < \lambda_c = 1 + a + \frac{3}{16a},$$

and in this range we can solve for λ as a function of β. Since there are 2^{2n} intervals of length 2^{-2n} in $[0,1]$, we can interpret (26) as saying that the "fractal dimension" of the set of times in $[0,1]$ at which $|f_t'(i2^{-n})| \approx 2^{n\beta}$ is $(2-\zeta-\lambda\beta)/2$. This fractal dimension is maximized when $r = \zeta = \lambda = 0$. In this case,

$$\beta = \beta(0) = -\frac{2a}{2a+1} = -\frac{4}{4+\kappa}.$$

♦This says that for small y, the typical value of $|f_t'(iy)|$, *when the curve is parametrized by capacity*, is $y^{\frac{4}{4+\kappa}}$. As one example, consider the limit as $\kappa \to 0$ (let's choose the capacity parametrization at rate 2). In this case, the curve grows deterministically, and by solving the Loewner equation one can check that $|f_t'(iy)| \approx y$.

Let us consider $\beta(0) \leq \beta < 1$. This represents larger than typical values of $|f_t'(iy)|$ and corresponds to $0 \leq \lambda < \lambda_c$. Let

$$K_\beta = \left\{ t \in [0,1] : \lim_{y \to 0+} \frac{\log |f_t'(iy)|}{-\log y} = \beta \right\}.$$

$$\overline{K}_\beta = \left\{ t \in [0,1] : \limsup_{y \to 0+} \frac{\log |f_t'(iy)|}{-\log y} \geq \beta \right\}.$$

THEOREM 21. [**7, 4**] *Suppose $\beta(0) \leq \beta < 1$ and let $\rho = \zeta + \lambda\beta$. The following hold with probability one.*

- *If $\rho > 2$, $\overline{K}_\beta$ is empty.*
- *If $\rho < 2$,*

$$\dim_h \left[\overline{K}_\beta\right] \leq \frac{2-\rho}{2}, \quad \dim_h \left[\gamma(\overline{K}_\beta)\right] \leq \frac{2-\rho}{1-\beta}.$$

- *If $\rho < 2$,*

$$\dim_h [K_\beta] = \frac{2-\rho}{2}, \quad \dim_h [\gamma(K_\beta)] = \frac{2-\rho}{1-\beta}.$$

Here $\dim_h$ denotes Hausdorff dimension.

- Brownian motion in $\mathbb{R}^d$, $d \geq 1$, has the property that with probability one, if $A \subset [0,1]$ then $\dim_h(B(A)) = 2\dim_h(A)$. SLE_κ in the capacity parametrization does not have the property that the $\dim_h(\gamma(A))$ depends only on $\dim_h(A)$.

- The relationship between $\dim_h[K_\beta]$ and $\dim_h[\gamma(K_\beta)]$ can be seen heuristically as follows. If $\dim_h[K_\beta] = \alpha$, then it takes about n^α intervals of length $1/n$ to cover K_β. The image of these intervals is dilated by a factor of about $|f'(1/\sqrt{n})| \approx n^{\beta/2}$. Hence it takes about $n^\alpha = n^{\frac{2\alpha}{1-\beta}\frac{2}{\beta-1}}$ sets of diameter $n^{2/(\beta-1)}$ to cover $\gamma(K_\beta)$. This suggests that the dimension of $\gamma(K_\beta)$ is $\frac{2\alpha}{1-\beta}$.
- The first two parts of the theorem can be deduced from the first moment bound Theorem 11. A similar bound was found in [**3**]. The lower bound is harder, and I will discuss this somewhat below.
- The maximum value of $(2-\rho)/(1-\beta)$ is $d = 1 + \frac{\kappa}{8}$ and occurs when
$$\beta_\# = \frac{\kappa}{\max\{4, \kappa - 4\}} - 1 > \beta(0).$$
A corollary of the last result is that the Hausdorff dimension of the SLE_κ path is d; this result was first proved in [**2**].

Exercise 22. *Use the conformal property of SLE_κ to show that for each β, the random variable* $\dim_h[K_\beta]$ *is constant with probability one.*

To prove the lower bound for a fixed $\beta(0) < \beta < 1$, we construct nontrivial (positive) measures ν, μ carried on $K_\beta, \gamma(K_\beta)$, respectively, such that
$$\int_0^1 \int_0^1 \frac{\nu(ds)\,\nu(dt)}{|s-t|^\alpha} < \infty, \quad \alpha < \frac{2-\rho}{2},$$
and
$$\int_{\mathbb{C}} \int_{\mathbb{C}} \frac{\mu(dz)\,\mu(dw)}{|z-w|^\alpha} < \infty, \quad \alpha < \frac{2-\rho}{\beta}. \tag{27}$$
We will focus on the latter which is slightly more difficult to handle. By the exercise above, it suffices to show that such a measure exists with positive probability.

We construct μ as a (subsequential) limit of a sequence μ_n of approximating measures. The form of the approximating measures is
$$\mu_n = \sum_{1/2 \le t \le 1, t \in \mathcal{D}_{2n}} \mu(n,t),$$
where $\mu(n,t)$ is a multiple of Lebesgue measure on the disk of radius r_n about $f_t(i2^{-n})$ where the multiple is chosen so that the total mass of $\mu(n,t)$ is
$$2^{n(\zeta-2)}\,|f_t'(i2^{-n})|^\lambda\, J(t,n).$$
Here λ is the exponent associated to β, r_n is a deterministic sequence decreasing to zero, and $J(t,n)$ is the indicator function of a nice event. We will not be precise, but the form of the event is
$$y^{-\beta}\,\psi(1/y)^{-1} \le |f_t'(iy)| \le y^{-\beta}\,\psi(1/y), \quad 2^{-n} \le y \le 1.$$
The subpower function ψ is chosen sufficiently large so that
$$\mathbb{E}\left[|f_t'(i2^{-n})|^\lambda\, J(t,n)\right] \ge c\,\mathbb{E}\left[|f_t'(i2^{-n})|^\lambda\right] \asymp 2^{-n\zeta},$$
which implies $\mathbb{E}[\mu_n] \ge c_1$. If one can show that
$$\mathbb{E}[\mu_n^2] \le c_2,$$
$$\mathbb{E}\left[\int\int \frac{\mu_n(z)\,\mu_n(w)}{|z-w|^\alpha}\right] \le c_\alpha < \infty,$$

then standard techniques (see Section 3.7) imply that with probability $p = p(c_1, c_2) > 0$, we can find a subsequential limit satisfying (27).

We will give only a sketch as to why one would hope to get such an estimate. Let us fix $t, t+s^2 \in \mathcal{D}_n$ with $1/2 \leq t < t+s^2 \leq 1$. We would like to write this as

$$\mathbb{E}\left[|f_t'(i2^{-n})|^\lambda \, |f_{t+s}'(i2^{-n})|^\lambda J(t,n)\, J(t+s,n)\right].$$

Using the relationship with the reverse Loewner flow as in Section 2.1, we consider an expectation of the form

$$\mathbb{E}\left[|\tilde{h}_t'(z)|^\lambda \, |h_{t+s^2}'(z)|^\lambda J(t,n)\, J(t+s^2,n)\right], \quad z = i2^{-n}.$$

(We abuse notation by using J for the corresponding events for the reverse flow.) Recall from Section 2.1 that

$$\tilde{h}_t'(z)\, h_{t+s^2}'(z) = \tilde{h}_t'(z)\, \tilde{h}_t'(Z_{s^2}(z))\, h_{s^2}'(z),$$

and the maps $\tilde{h}_t$ and h_{s^2} are independent. We also consider the map $\hat{h}$ which denotes the corresponding map at time $t - s^2$. Then we have

$$\tilde{h}_t'(z)\, h_{t+s^2}'(z) = \tilde{h}_{s^2}'(z)\, h_{2s^2}'(z)\, \hat{h}_{t-s^2}'(\tilde{Z}_{s^2}(z))\, \hat{h}_{t-s^2}'(Z_{2s^2}(z)).$$

Here we are writing

$$Z_r(z) = h_r(z) - V_r, \quad \tilde{Z}_r(z) = \tilde{h}_r(z) - \tilde{V}_r.$$

- The probability that $|h_{s^2}'(z)| \approx [s2^n]^\beta$ is the same as
$$\mathbb{P}\{|h_{s^2 2^{2n}}'(i)| \approx 2^{\beta n}\, s^\beta\} \approx 2^{-n\xi}\, s^{-\xi}.$$
- When we weight paths by $|h_{s^2}'(z)|^\lambda$, and let $Z_{s^2}(z) = X_{s^2} + iY_{s^2}$, then $Y_{s^2} \approx s, |X_{s^2}| \leq s\psi(1/s)$. Using the distortion theorem, if f any conformal transformation of $\mathbb{H}$,
$$|f'(Z_{s^2}(z))| \approx |f'(is)|.$$
- The probability that $|\tilde{h}_{s^2}'(z)| \approx [s2^n]^\beta$ is about $2^{-n\xi}\, s^{-\xi}$.
- Since $Y_{s^2} \approx s, |X_{s^2}| \leq s\psi(1/s)$.
$$|\tilde{h}_{s^2}'(Z_{s^2}(z))| \approx 1,$$
and hence
$$|h_{2s^2}'(z)| = |h_{s^2}'(z)\, \tilde{h}_{s^2}'(Z_{s^2}(z))| \approx [s2^n]^\beta,$$
and
$$|\tilde{h}_{s^2}'(z)\, h_{2s^2}'(z)| \approx [s2^n]^{2\beta}.$$
- When we weight by $|\tilde{h}_{s^2}'(z)\, h_{2s^2}'(z)|^\lambda$, then then the typical path is as above so that
$$|\hat{h}_{t-s^2}'(\tilde{Z}_{s^2}(z))| \approx |\hat{h}_{t-2s^2}'(Z_{2s^2}(z))| \approx |\hat{h}_{t-s^2}'(si)|.$$
- The probability that $|\hat{h}_{t-s^2}'(si)| \approx s^{-\beta}$ is about s^ζ.
- If we carry this argument out carefully, then we can choose appropriate events $J(t,n)$ such that

$$\mathbb{E}\left[|\tilde{h}_t'(z)|^\lambda \, |h_{t+s^2}'(z)|^\lambda J(t,n)\, J(t+s^2,n)\right] \leq 2^{-2n\xi}\, s^{-\zeta}\, \psi(1/s). \tag{28}$$

- Using the Koebe-1/4 theorem, we can see that on this event

$$|\tilde{Z}_{s^2}(z) - Z_{2s^2}(z)| \geq s\,\psi(1/s),$$

and hence

$$|\gamma(t) - \gamma(t+s^2)| \geq s^{1-\beta}\,\psi(1/s).$$

♦This is only a basic sketch of the argument. I am not putting in more details, but let me mention some of the reasons for defining the measure the way that I did.

- One might try to avoid the event $J(t,n)$ and define a measure proportional to $|f_t'(i2^{-n})|^\lambda$. It is probably true that this would concentrate on the correct set. However, the second moment estimates become tricky, because ones starts to estimate

$$\mathbb{E}\left[|f_t'(i2^{-n})|^\lambda\,|f_s'(i2^{-n})|^\lambda\right].$$

for s near t this starts looking like the 2λ power and the 2λ moment concentrates on a different event.
- One might try to put a measure proportional to the indicator function of an event. However, we do not have as sharp estimates for this probability. It is important in getting the second moment estimate that we an estimate as sharp as (28). In particular, for $1/s$ of order 1, the right-hand side is bounded by a constant times $2^{-n\zeta}$. It is because this is needed that the one-variable analysis leading to (24) was done.

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3. The forward flow and dimension

3.1. Some intuition. Suppose $z \in \mathbb{H}$ and γ is a chordal SLE_κ path from 0 to ∞ in $\mathbb{H}$. We will ask two related questions:

- Does the path hit z? If not, how close does it get?
- What is the Hausdorff dimension of the path $\gamma(0,\infty)$?

It is known, and we will give a derivation here, that the curve is plane-filling if and only if $\kappa \geq 8$. In other words,

$$\mathbb{P}\{z \in \gamma(0,\infty)\} = \begin{cases} 1 & \kappa \geq 8 \\ 0 & \kappa < 8 \end{cases}.$$

Suppose D is a bounded domain, bounded away from the real line. Suppose $\kappa < 8$. If the fractal dimension of $\gamma \cap D$ is d, then we expect that the number of disks of radius ϵ needed to cover the curve is of order ϵ^{-d}. If we divide D into ϵ^{-2} disks of radius ϵ, then the fraction of these disks needed to cover D is ϵ^{2-d}. In other words, the probability that a particular disk of radius ϵ is needed should be about ϵ^{2-d}. Using this as intuition, we expect as $\epsilon \to 0$,

$$\mathbb{P}\{\mathrm{dist}(\gamma,z) \leq \epsilon\} \approx \epsilon^{2-d}. \tag{29}$$

The goal of this section is to give a precise version of the relation (29). Rohde and Schramm [**15**] first showed that the correct value is

$$d = 1 + \frac{\kappa}{8}.$$

More precise estimates [**2**] are needed to give the result about the Hausdorff dimension which we state now.

THEOREM 23. [**2**] *If $\kappa < 8$, then with probability one, the Hausdorff dimension of γ_t for $t > 0$ is $1 + \frac{\kappa}{8}$.*

3.2. Basic definitions. Distance to the curve is not a conformal invariant or conformal covariant. A more useful, but similar, notion is conformal radius. Recall that $\mathbb{H}$ denotes the upper half plane, and let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk.

Definition If D is a (proper) simple connected domain and $z \in D$, we define $\Upsilon_D(z)$ to be one-half times the *conformal radius* of D with respect to z. In other words, if $f : \mathbb{D} \to D$ is a conformal transformation with $f(0) = z$, then

$$\Upsilon_D(z) = \frac{1}{2}\,|f'(z)|.$$

The factor of $1/2$ is a convenience so that

$$\Upsilon_{\mathbb{H}}(x + iy) = \mathrm{Im}(y).$$

It follows from the definition that the conformal radius is conformally *covariant* in the sense that if $F : D \to F(D)$ is a conformal transformation,

$$\Upsilon_{F(D)}(F(z)) = |F'(z)|\,\Upsilon_D(z). \tag{30}$$

By definition, we set $\Upsilon_{\mathbb{C}}(z) = \infty$. The conformal radius is closely related to the *inradius* defined by

$$\mathrm{inrad}_D(z) = \mathrm{dist}(z, \partial D).$$

Exercise 24. *Use the Koebe* $1/4$*-theorem, to show that for any simply connected domain* D *and* $z \in D$*,*

$$\frac{1}{2}\,\Upsilon_D(z) \le \mathrm{dist}(z, \partial D) \le 2\,\Upsilon_D(z). \tag{31}$$

Definition If D is a simply connected domain and w_1, w_2 are distinct points in ∂D, then

$$S_D(z; w_1, w_2) = \sin[\arg f(z)].$$

where $f : D \to \mathbb{H}$ is a conformal transformation with $f(0) = w_1, f(\infty) = w_2$.

The transformation f is unique up to a dilation, and hence the argument of $f(z)$ does not depend on the choice of f. By definition, S_D is a conformal *invariant*,

$$S_{F(D)}(F(z); F(w_1), F(w_2)) = S_D(z; w_1, w_2).$$

Let $\mathrm{hm}_D(z, \cdot)$ denote harmonic measure which is defined by saying that the probability that a Brownian motion starting at z exits D at $V \subset \partial D$ is given by $\mathrm{hm}_D(z, V)$.

Exercise 25.

- *Consider* $z = re^{i\theta} \in \mathbb{H}$*. Show that the probability that a Brownian motion starting at* z *exits* $\mathbb{H}$ *on* $(-\infty, 0)$ *equals* θ/π*. (Hint: what is the Poisson kernel in* $\mathbb{H}$*?)*
- *Find constants* $0 < c_1 < c_2 < \infty$ *such that the following holds. Suppose* D *is a simply connected domain and* w_1, w_2 *are distinct boundary points of* D*. Write*

$$\partial D = \{w_1, w_2\} \cup A_1 \cup A_2$$

where A_1, A_2 *are the two connected subarcs of* $D \setminus \{w_1, w_2\}$*. Then*

$$c_1\, S_D(z; w_1, w_2) \le \min\{\mathrm{hm}_D(z, A_1), \mathrm{hm}_D(z, A_2)\} \le c_2\, S_D(z; w_1, w_2). \tag{32}$$

♦When I discuss boundaries of simply connected domains D, I am using "prime ends". I will not give the precise definition, but the basic idea is that boundary points can be reached in different directions. For example, if $D = \mathbb{D} \setminus [0,1)$ and the boundary points are 0 and -1, then

$$A_1 = \{e^{i\theta} : 0 < \theta < \pi\} \cup (0,1]_+,$$
$$A_2 = \{e^{i\theta} : \pi < \theta < 2\pi\} \cup (0,1]_-,$$

where $(0,1]_+$ (resp., $[0,1]_-$) denotes the points in (0,1] reached from the upper half plane (lower half plane).

We now take a chordal SLE_κ path γ parametrized as in Section 0.1. For fixed $z \in \mathbb{H}$, we set $Z_t = Z_t(z) = X_t + iY_t = g_t(z) - U_t$, which satisfies

$$dZ_t = \frac{a}{Z_t}\, dt + dB_t.$$

3.3. Radial parametrization. Chordal SLE_κ uses the half-plane capacity. If $z \in \mathbb{H}$, we can choose a different parametrization such that $\log \Upsilon_{H_t}(z)$ decays linearly (this will be valid at least as long as $z \in H_t$.) Let us fix z, and let

$$\Upsilon_t = \Upsilon_t(z) = \Upsilon_{H_t}(z) = \frac{Y_t}{|g_t'(z)|}.$$

The last equality uses the scaling rule (30) with $F = g_t$. From the (deterministic) Loewner equation (1), we can compute

$$\partial_t |g_t'(z)| = \frac{a\,(Y_t^2 - X_t^2)}{(X_t^2 + Y_t^2)^2}, \qquad \partial_t \Upsilon_t = -\Upsilon_t\, \frac{2aY_t^2}{(X_t^2+Y_t^2)^2}.$$

Since Υ_t decreases with t we can define

$$\Upsilon_\infty = \lim_{t\to\infty} \Upsilon_t.$$

Definition If $z \in \mathbb{H}$, let

$$\sigma(t) = \sigma_z(t) = \inf\left\{s : \Upsilon_s \le e^{-2at}\right\},$$
$$\rho(t) = \rho_z(t) = \inf\left\{s : \mathrm{dist}(\partial H_t, z) \le e^{-2at}\right\}.$$

We call $\sigma(t)$ the *radial parametrization (with rate $2a$).*

In the radial parametrization $\log \hat\Upsilon_t := \log \Upsilon_{\sigma(t)}$ decays linearly. We choose rate $2a$ to makes some equations below a little nicer.

EXERCISE 26. *Suppose* $\mathrm{Im}(z) = 1$.

- *Find a constant c such that for all $t \ge 0$*

$$|\log \Upsilon_{\rho(t)} + 2at| \le c.$$

- *Show that there exists a $c > 0$ such that for all $t \ge 0$, $0 \le s \le 1$,*

$$\log \Upsilon_{\rho(t+s)} \le \log \Upsilon_{\rho(t)} + cs. \tag{33}$$

Hint: The first part is straightforward using (31)*, but the second part requires more argument.*

In the radial parametrization, the argument behaves in a relatively simple fashion. Let

$$\Theta_t = \arg Z_{\sigma(t)}.$$

The next exercise gives the equation that we will use.

EXERCISE 27. *Show that there is a standard Brownian motion $\hat{W}_t$ such that Θ_t satisfies*

$$d\Theta_t = (2a-1)\,[\cot \Theta_t]\,dt + d\hat{W}_t. \tag{34}$$

Hint: There are two parts to this. First, one uses Itô's formula to give an equation for $\arg Z_t$, and then one converts to the radial parametrization. The rate $2a$ is chosen so that $\hat{W}_t$ is a standard Brownian motion.

♦By comparison with the Bessel equation, one can show that solutions to the "radial Bessel equation"

$$dX_t = \beta\,[\cot X_t]\,dt + dW_t,$$

reach the origin in finite time if and only if $\beta < 1/2$. Hence solutions to the equation (34) reach the origin in finite time if and only if $a > 1/4, \kappa < 8$. Note that finite time in the radial parametrization corresponds to $\Upsilon_\infty > 0$ which corresponds to $z \notin \gamma(0,\infty)$. Hence from this we can see that SLE_κ is plane-filling if and only $\kappa \geq 8$. Another observation is that Θ_t is a martingale if and only if $\kappa = 4$.

♦When considering SLE_κ near an interior point z, the radial parametrization is very useful. However, the parametrization depends on the point z, so it is not as useful for considering two interior points simultaneously.

3.4. Green's function and one-point estimate. If $\kappa < 8$, let

$$d = 1 + \frac{\kappa}{8} = 1 + \frac{1}{4a}.$$

This will be the fractal dimension of the paths, but for the time being, let us consider this only as a notation.

Definition The *Green's function (for chordal SLE_κ from w_1 to w_2 in simply connected D)* is

$$G_D(z;w_1,w_2) = \Upsilon_D(z)^{d-2}\,S_D(z;w_1,w_2)^{4a-1} = \Upsilon_D(z)^{\frac{\kappa}{8}-1}\,S_D(z;w_1,w_2)^{\frac{8}{\kappa}-1}.$$

We set

$$G(z) = G_{\mathbb{H}}(z;0,\infty) = [\mathrm{Im} z]^{d-2}\,\sin^{4a-1}[\arg z].$$

The scaling rules for Υ_D and S_D imply the following scaling rule for G_D

$$G_D(z;w_1,w_2) = |F'(z)|^{2-d}\,G_{F(D)}(F(z);F(w_1),F(w_2)). \tag{35}$$

Roughly speaking, we think of $G_D(z;w_1,w_2)$ as representing the probability that the SLE_κ path gets close to z. A precise formulation of this comes in the following theorem which we prove in this section. If γ is an SLE_κ curve in D from w_1 to w_2, we write γ_t for $\gamma(0,t]$, D_t for the unbounded component of $D \setminus \gamma_t$ containing w_2 on its boundary, and if $z \in D$, $\Upsilon_t = \Upsilon_t(z) = \Upsilon_{D_t}(z;w_1,w_2)$ (if $z \notin D_t$, then $\Upsilon_t = 0$), and $\Upsilon_\infty = \lim_{t\to\infty} \Upsilon_t$. The next proposition shows that for all D, z, w_1, w_2,

$$\mathbb{P}\{\Upsilon_t \leq r\} \sim c_*\,G_D(z;w_1,w_2)\,r^{2-d}, \quad r \to 0+. \tag{36}$$

Using the scaling rule (35), we can write this as

$$\mathbb{P}\{\Upsilon_t \leq r\,\Upsilon_0\} \sim c\,(r\Upsilon_0)^{2-d}\,G_D(z;w_1,w_2) = c_*\,S_D(z;w_1,w_2)^{4a-1}\,r^{2-d}.$$

THEOREM 28. *For every $\kappa < 8$ there exists $u > 0$ such that the following holds. Suppose γ is an SLE_κ curve from w_1 to w_2 in D. Then for every $z \in D$,*

$$\mathbb{P}\{\Upsilon_\infty \le r\,\Upsilon_0\} = c_*\, S_D(z;w_1,w_2)^{4a-1}\, r^{2-d}\,[1+O(r^u)], \tag{37}$$

where

$$c_*^{-1} = \frac{1}{2}\int_0^\pi \sin^{4a} x\, dx.$$

In particular, if $r_0 < 1$, there exists $0 < c_1 < c_2 < \infty$ such that for all D, z, w_1, w_2 and all $0 < r \le r_0$,

$$c_1\, S_D(z;w_1,w_2)^{4a-1}\, r^{2-d} \le \mathbb{P}\{\Upsilon_\infty \le r\,\Upsilon_0\} \le c_1\, S_D(z;w_1,w_2)^{4a-1}\, r^{2-d}.$$

The relation (37) means the following. For each $r_0 < 1$, there exists $c < \infty$ which may depend on κ and r_0 but does not depend on r, D, w_1, w_2, z such that if $r \le r_0$,

$$\left|\mathbb{P}\{\Upsilon_\infty \le r\,\Upsilon_0\} - c_*\, S_D(z;w_1,w_2)^{4a-1}\, r^{2-d}\right| \le c\, S_D(z;w_1,w_2)^{4a-1}\, r^{(2-d)+u}.$$

We have not motivated why the function G_D or the value d sould be as given. We do so now. Suppose G_D, d exist satisfying (36). Let $\mathcal{F}_t$ denote the σ-algebra generated by $\{B_s : 0 \le s \le t\}$ (or, equivalently, by $\{\gamma(s), 0 \le s \le t\}$). If $\rho = \inf\{t : \Upsilon_t = r\}$, then

$$N_t = \mathbb{P}\{\Upsilon_\infty \le r \mid \mathcal{F}_t\}$$

should be a local martingale for $0 \le t < \rho$. If this is true, then

$$M_t = \mathbb{E}\,[G_D(z;w_1,w_2) \mid \mathcal{F}_t]$$

should be a local martingale. The domain Markov property of SLE_κ and (37) imply that

$$\begin{aligned} \mathbb{E}\,[G_D(z;w_1,w_2) \mid \mathcal{F}_t] &= G_{H_t}(z;\gamma(t),\infty) \\ &= |g_t'(z)|^{2-d}\, G_{\mathbb{H}}(g_t(z);U_t,\infty) \\ &= |g_t'(z)|^{2-d}\, G(Z_t(z)), \quad Z_t(z) = g_t(z) - U_t. \end{aligned}$$

The function G was first computed in [**15**] by essentially doing the following exercise.

EXERCISE 29. *Suppose $\kappa < 8$.*

- *Let*

  $$M_t = M_t(z) = G_{H_t}(z;\gamma(t),\infty) = |g_t'(z)|^{2-d} G(Z_t(z)).$$

  Then M_t is a local martingale satisfying

  $$dM_t = \frac{(1-4a)\,X_t}{X_t^2+Y_t^2}\, dB_t.$$

- *Suppose $\Phi : \mathbb{H} \to (0,\infty)$ is a C^2 function satisfying $\Phi(rz) = r^{\alpha-2}\Phi(z)$ for some $\alpha > 0$. Suppose also that for each z,*

  $$|g_t'(z)|^{2-\alpha}\, \Phi(Z_t(z))$$

  is a local martingale. Prove that $\alpha = d$ and $\Phi = cG$ for some c.

Unlike the similar local martingale M_t from the reverse Loewner flow in the previous section, this local martingale is not a martingale. It "blows up" on the event of probability zero that $z \in \gamma(0,\infty)$.

In the radial parametrization,

$$\hat{M}_t = M_{\sigma(t)} = e^{2at(2-d)}\,[\sin\Theta_t]^{4a-1}.$$

(If $\Upsilon_\infty > e^{-2at}$, then $\sigma(t) = \infty$ and $\hat{M}_t = 0$.) Recall that in the radial parametrization,

$$d\Theta_t = (1-2a)\,\Theta_t\,dt + d\hat{W}_t.$$

If

$$T = \inf\{t : \sin\Theta_t = 0\},$$

we can write

$$\hat{M}_t = e^{2at(2-d)}\,[\sin\Theta_{t\wedge T}]^{4a-1}.$$

This is a martingale satisfying

$$d\hat{M}_t = (4a-1)\,[\cot\Theta_t]\,\hat{M}_t\,d\hat{W}_t, \quad t < T.$$

(Itô's formula shows that this is a local martingale, and since $\hat{M}_t$ is uniformly bounded on every compact interval, we can see that it is actually a martingale.) If we weight by the martingale $\hat{M}_t$, then

$$d\hat{W}_t = (4a-1)\,\cot\Theta_t\,dt + dW_t,$$

where W_t is a Brownian motion in the new measure, which we denote by $\mathbb{P}^*$ (with expectations $\mathbb{E}^*$). In particular,

$$d\Theta_t = 2a\,\cot\Theta_t\,dt + dW_t.$$

Since $2a > 1/2$, with probability one with respect to the measure $\mathbb{P}^*$, the process never exits the open interval $(0,\pi)$. (Of course, since the martingale $\hat{M}_t$ equals zero when $\sin\Theta_t = 0$, it is obvious that if we weight by $\hat{M}_t$, the process should never leave $(0,\pi)$!)

♦Consider the SDE

$$dX_t = \beta\,\cot X_t\,dt + dB_t, \quad 0 < X_0 < \pi,$$

where $\beta > 1/2$. This is the equation obtained by starting with a Brownian motion X_t and weighting locally by the function $f(x) = [\sin x]^\beta$. By comparison with a Bessel equation, it is not hard to show that with probability one $0 < X_t < \pi$ for all times. The invariant probability distribution (see Exercise 20 and the comment above that), is

$$v(x) = C_{2\beta}\,[\sin x]^{2\beta}, \quad C_{2\beta}^{-1} = \int_0^\pi [\sin y]^{2\beta}\,dy.$$

If $p_t(x,y)$ denotes the density (as a function of y) of X_t given $X_0 = x$, it is standard to show that

$$p_t(x,y) = v(x)\,[1 + O(e^{-\alpha t})], \quad t \geq 1,$$

for some $\alpha = \alpha_\beta > 0$. In particular, if Φ is a nonnegative function on $(0,\pi)$,

$$\mathbb{E}\,[\Phi(X_t)] = \left[\int_0^\pi \Phi(x)\,v(x)\,dx\right]\,[1 + O(e^{-\alpha t})], \quad t \geq 1.$$

We can now prove Theorem 28. By conformal invariance, it suffices to consider SLE_κ from 0 to ∞ in $\mathbb{H}$ and $\mathrm{Im}(z) = 1$. Note that $\Upsilon_0 = 1$. Let $r = e^{-2at}$ and $v(x) = C_{4a}\,\sin^{4a} x$. Then,

$$\begin{aligned} \mathbb{P}\{\Upsilon_\infty \le r\} &= \mathbb{E}\left[1\{\Upsilon_\infty \le r\}\right] \\ &= \mathbb{E}^*\left[\hat{M}_t^{-1}; T > t\right] \\ &= \mathbb{E}^*\left[\hat{M}_t^{-1}\right] \\ &= r^{2-d}\left[\int_0^\pi [\sin x]^{1-4a}\, v(x)\, dx\right][1 + O(r^u)] \\ &= 2\, C_{4a}\, r^{2-d}\,[1 + O(r^u)]. \end{aligned}$$

Using (33), we get the following corollary in terms of distances.

COROLLARY 30. *If $\kappa < 8$ there exists $c < \infty$ such that for every D, z, w_1, w_2 and every $0 < r \le 1/2$, if γ is the path of SLE_κ from w_1 to w_2 in D, then*

$$\mathbb{P}\{\mathrm{dist}(\gamma, z) \le r\,\mathrm{inrad}_D(z)\} \le c\, S_D(z; w_1, w_2)^{4a-1}\, r^{2-d}. \tag{38}$$

The restriction $r \le 1/2$ is not required if $D = \mathbb{H}$. This can be seen from the following estimate that we will not prove here. One can prove this similarly to the proofs in this section, but another simple proof can be found in [**1**]. Note that

$$S_{\mathbb{H}}(x + x\epsilon i; 0, \infty) \sim \epsilon, \quad \epsilon \to 0+.$$

PROPOSITION 31. *If $\kappa < 8$, there exists $c < \infty$ such that if γ is the path of SLE_κ from 0 to ∞ in $\mathbb{H}$, $x, \epsilon > 0$, then*

$$\mathbb{P}\{\mathrm{dist}(\gamma, x) < \epsilon\, x\} \le c\, \epsilon^{4a-1}.$$

3.5. Two-sided radial and radial SLE_κ. Two-sided radial SLE_κ from 0 to ∞ through z can be thought of as chordal SLE_κ from 0 to ∞ conditioned to go through z. This is conditioning on an event of probability zero, so we need to be careful in the definition. A standard way to define events "conditioned on events of measure zero" is to consider a sequence of events of positive probability decreasing to the event, condition with respect to these events of positive probability, and hope to obtain a limit of the measures. It is more convenient to use the Girsanov theorem directly to define two-sided radial but our definition is equivalent to other natural ways of defining the measure (see Exercise 32). The term "two-sided radial" comes from thinking of the path as two (interacting) radial paths from z to 0 and ∞, respectively.

Definition If $\kappa < 8$, then *two-sided radial SLE_κ from 0 to ∞ through z (up to time T_z)* is chordal SLE_κ weighted by the local martingale

$$M_t = G_{H_t}(z; \gamma(t), \infty).$$

If we use the radial parametrization as above, then for two-sided radial SLE,

$$d\Theta_t = 2a\,\sin\Theta_t\, dt + dW_t,$$

where W_t is a Brownian motion. In the half-plane capacity parametrization, two-sided radial SLE_κ satisfies

$$dX_t = \frac{(1-3a)X_t}{X_t^2 + Y_t^2}\, dt + dW_t, \quad \partial_t Y_t = -\frac{aY_t}{X_t^2 + Y_t^2}. \tag{39}$$

EXERCISE 32. *Suppose $\kappa < 8$ and $z \in \mathbb{H}$. For $t < t'$ consider the following probability measures on paths $\gamma(s), 0 \leq s \leq \sigma(t)$. Here we use the half-plane capacity.*

- $\mathbb{P}_1 = \mathbb{P}_{1,t}$ *is chordal SLE_κ conditioned on the event $\{\sigma(t) < \infty\}$ stopped at time $\sigma(t)$.*
- $\mathbb{P}_2 = \mathbb{P}_{2,t}$ *is two-sided radial SLE_κ stopped at time $\sigma(t)$.*
- $\mathbb{P}_{3,t'} = \mathbb{P}_{3,t',t}$ *is chordal SLE_κ conditioned on the event $\{\sigma(t') < \infty\}$ stopped at time $\rho(t)$.*

Then,

- *Show that P_1, P_2 are mutually absolutely continuous and give the Radon-Nikodym derivative.*
- *Show that*
$$\lim_{t' \to \infty} \|P_2 - P_{3,t'}\| = 0,$$
where $\|\cdot\|$ denotes variation distance.

In the discussion on natural parametrization, we will need to consider the time duration in the half-plane capacity of radial $SLE_\kappa, \kappa < 8$. Suppose γ is two-sided radial SLE from 0 to ∞ in z and let $T_z = \inf\{t : \gamma(t) = z\}$. Then with probability one, $T_z < \infty$. We let $\phi(z;t)$ denote the distribution function,

$$\phi(z;t) = \mathbb{P}^*\{T_z \leq t\},$$

where $\mathbb{P}^*$ denotes probabilities using two-sided radial SLE_κ. This is also the distribution time for $\inf\{t : Y_t = 0\}$ where X_t, Y_t satisfy (39) with $X_0 + iY_0 = z$. Let $\phi(z) = \phi(z;1)$; scaling implies

$$\phi(z;t) = \phi(z/t^2).$$

We think of $\phi(z;t)$ as the probability that $z \in \gamma(0,t]$ given that $z \in (0,\infty)$. One can show that

$$\mathbb{E}\left[M_t(z)\right] = \left[1 - \phi(z;t)\right] M_0(z).$$

♦Two-sided radial can be considered as a type of "$SLE(\kappa,\rho)$" process. We will not define these processes here. In fact, most, if not all such processes, can be viewed as processes obtained from the Girsanov theorem by weighting by a local martingale. I find the Girsanov viewpoint more natural, because it generally can be seen as weighting locally by a function. The function may depend on a number of marked points, e.g., the SLE_κ Green's function.

3.5.1. *Radial SLE_κ.* Another example of a process that can be obtained from chordal SLE_κ by a local martingale is radial SLE_κ. If γ is an SLE_κ, process, let

$$\Phi_t = H_{\mathbb{H}}(g_t(z), U_t) = |Z_t|^{-1} \arg[Z_t].$$

Here $H_{\mathbb{H}}(z,x)$ denotes π times the Poisson kernel for Brownian motion in $\mathbb{H}$. We will not need radial SLE in these notes, so we leave this relationship as an exercise.

EXERCISE 33. *Let γ denote SLE_κ. Let*

$$b = \frac{3a-1}{2} = \frac{6-\kappa}{2\kappa}.$$

Show that

$$d\Phi_t^b = \Phi_t^b \left[A_t\, dt + \frac{(1-3a)\, X_t}{X_t^2 + Y_t^2}\, dB_t\right],$$

for some A_t. In particular,

$$M_t = \exp\left\{-\int_0^t A_s\,ds\right\}\Phi_t^b,$$

is a local martingale satisfying

$$dM_t = \frac{(1-3a)\,X_t}{X_t^2+Y_t^2}\,M_t\,dB_t.$$

The process that one obtains by weighting by M_t in this exercise is *radial* SLE_κ from 0 to z in $\mathbb{H}$. More precisely, it is radial SLE_κ defined up to the first time that the path separates z from ∞. For this process, in the radial parametrization one gets

$$d\Theta_t = a\,\sin\Theta_t\,dt + dW_t,$$

where W_t is a standard Brownian motion. Radial SLE_κ is usually described as a random curve growing from the boundary of the unit disk to the origin. It is usually parametrized using a radial parametrization (so that the logarithm of the conformal radius decays linearly). Note that this process reaches the origin in finite time if and only if $a < 1/2$ ($\kappa > 4$). If $\kappa > 4$, the path disconnects z from ∞ in finite (radial) time.

3.6. Beffara's two-point estimate. The first proof of Theorem 23 as well as one proof of existence of natural parametrization uses the following estimate.

PROPOSITION 34. *Suppose $\kappa < 8$ and D is a bounded domain bounded away from the real line. Then there exists $c < \infty$ such that for all $\epsilon, \delta > 0$ and $z, w \in D$,*

$$\mathbb{P}\{\Upsilon_\infty(z) \le \epsilon, \Upsilon_\infty(w) \le \delta\} \le c\,\epsilon^{2-d}\,\delta^{2-d}\,|z-w|^{d-2}.$$

This proposition is not easy to prove. The hard work is showing the estimate when $|z-w|$ is of order 1. In Section 4.2 we discuss the proof of the following.

PROPOSITION 35. *Suppose $\kappa < 8$, and $0 < u_1 < u_2 < \infty$ Then there exists $c < \infty$ such that for all $\epsilon, \delta > 0$ and all z, w with*

$$u_1 \le \mathrm{Im}(z), \mathrm{Im}(w) \le u_2,$$
$$u_1 \le |z-w| \le u_2,$$

then

$$\mathbb{P}\{\Upsilon_\infty(z) \le \epsilon, \Upsilon_\infty(w) \le \delta\} \le c\,\epsilon^{2-d}\,\delta^{2-d}.$$

Let us discuss how to get Proposition 34 from Proposition 35. It suffices to consider $\epsilon, \delta, |z-w|$ sufficiently small and without loss of generality assume

$$\delta \le \epsilon \le |z-w|.$$

(If $|z-w| \le \epsilon$, we can use the estimate on $\mathbb{P}\{\mathrm{Im}(w) \le \delta\}$.) Let ρ be the first time t that $\Upsilon_t(z) \le 10|z-w|$. Then

$$5\,|z-w| \le \mathrm{inrad}_{H_\rho}(z) \le 20\,|z-w|.$$

$$\mathbb{P}\{\rho < \infty\} \asymp |z-w|^{2-d}. \tag{40}$$

Applying distortion estimates to g_ρ on the disk of radius $5|z-w|$ about z, we see that

$$|g_\rho(z) - g_\rho(w)| \asymp |g_\rho'(z)|\,|z-w| \asymp |g_\rho'(w)|\,|z-w| \asymp \mathrm{Im}[g_\rho(z)] \asymp \mathrm{Im}[g_\rho(w)].$$

(The reader may wish to verify this. Note that we get both lower and upper bounds on $|g_\rho(z) - g_\rho(w)|$. This uses the univalence of g_ρ.) Therefore $\Upsilon_\rho(w) \asymp |z-w|$. Therefore, for some c,

$$\mathbb{P}\{\Upsilon_\infty(z) \leq \epsilon, \Upsilon_\infty(w) \leq \delta \mid \mathcal{F}_\rho\} \leq \mathbb{P}\left\{\Upsilon_\infty(z) \leq \frac{\epsilon\,\Upsilon_\rho(z)}{|z-w|}, \Upsilon_\infty(w) \leq \frac{c\,\delta\,\Upsilon_\rho(w)}{|z-w|} \mid \mathcal{F}_\rho\right\}.$$

By conformal invariance, the right-hand side equals

$$\mathbb{P}\left\{\Upsilon_\infty(g_\rho(z)) \leq \frac{\epsilon\,\mathrm{Im}(g_\rho(z))}{|z-w|}, \Upsilon_\infty(g_\rho(w)) \leq \frac{c\,\delta\,\mathrm{Im}(g_\rho(w))}{|z-w|}\right\}.$$

If we let $z_1 = g_\rho(z)/\mathrm{Im}(g_\rho(z)), w_1 = g_\rho(w)/\mathrm{Im}(g_\rho(z))$, then by conformal invariance, this equals

$$\mathbb{P}\left\{\Upsilon_\infty(z_1) \leq \frac{\epsilon}{|z-w|}, \Upsilon_\infty(w_1) \leq \frac{\tilde{c}\,\delta}{|z-w|}\right\},$$

where $\tilde{c} = c\,\mathrm{Im}(w)/\mathrm{Im}(z)$. Since $|z_1 - w_1| \asymp \mathrm{Im}(w_1) \asymp \mathrm{Im}(z_1) = 1$, Proposition 35 shows that this probability is bounded above by

$$c\,\epsilon^{2-d}\,\delta^{2-d}\,|z-w|^{2(d-2)}.$$

Combining this with (40), we get Proposition 34.

3.7. Hausdorff dimension. Proposition 34 was the hard step in Beffara's proof of the lower bound of the Hausdorff dimension of SLE_κ curves. In this subsection, we will sketch the basic technique to convert such two-point estimates into estimates on dimension.

THEOREM 36. *If $\kappa < 8$, then with probability one the Hausdorff dimension of $\gamma[0,\infty)$ is $d = 1 + \frac{\kappa}{8}$.*

It is not difficult to see that the value of the dimension is almost surely constant. Giving the upper bound is not difficult using the one-point estimate. We will only discuss the hard direction, the lower bound. It suffices to show that for some domain D and every $\alpha < d$, there is a positive probability that the Hausdorff dimension of $\gamma \cap D$ is at least α. The hard work is the two-point estimate Proposition 34. The rest follows from the proposition below which has appeared a number of places.

PROPOSITION 37. *Suppose $0 < \beta < m$ and A is a random closed subset of $[0,1]^m$. If $\mathbf{j} = (j_1, \ldots, j_m) \in S_n := \{1, \ldots, 2^n\}^m$. Let*

$$V_n(\mathbf{j}) = \left[\frac{j_1 - 1}{2^n}, \frac{j_1}{2^n}\right] \times \cdots \times \left[\frac{j_m - 1}{2^n}, \frac{j_m}{2^n}\right],$$

let $K_n(\mathbf{j})$ denote the indicator function of the event

$$\{A \cap V_m(\mathbf{j}) \neq \emptyset\}.$$

Assume there exist $0 < c_1 < c_2 < \infty$ and a subpower function ψ such that

$$c_1\,2^{(\beta-m)n} \leq \mathbb{E}\left[K_n(\mathbf{j})\right] \leq c_2\,2^{(\beta-m)n}, \tag{41}$$

$$\mathbb{E}\left[K_n(\mathbf{j}_1)\,K_n(\mathbf{j}_2)\right] \leq 2^{2(\beta-m)n}\left(\frac{|\mathbf{j}_1 - \mathbf{j}_2|}{2^n}\right)^{(\beta-m)n} \psi\left(\frac{2^n}{|\mathbf{j}_1 - \mathbf{j}_2|}\right). \tag{42}$$

Then with positive probability, $\dim_h(A) \geq \beta$.

Sketch of proof. Let

$$A_n = \bigcup_{K_n(\mathbf{j})=1} V_n(\mathbf{j}).$$

Then $A_1 \supset A_2 \supset A_3 \supset \cdots$ and

$$A = \bigcap_{n=1}^{\infty} A_n.$$

Let μ_n denote the (random) measure whose density with respect to Lebesgue measure on $\mathbb{R}^m$ is

$$f_n(x) = 2^{\beta n}\, 1\{x \in A_n\}.$$

The estimates (41) and (42) imply that there exist c_3, c_4 such that

$$c_3 \leq \mathbb{E}\left[\mu_n(A_n)\right]^2 \leq \mathbb{E}\left[\mu_n(A_n)^2\right] \leq c_4.$$

Moreover, for every $\alpha < \beta$, there exists $C_\alpha < \infty$ such that

$$\mathbb{E}\left[\mathcal{E}_\alpha(\mu_n)\right] \leq C_\alpha,$$

where

$$\mathcal{E}_\alpha(\mu) = \int\int \frac{\mu(dx)\,\mu(dy)}{|x-y|^\alpha}.$$

Using second moment methods and the Markov inequality, we see that there exists $c_5, c_6 > 0$ (independent of α) and $\tilde{C}_\alpha < \infty$ such that

$$\mathbb{P}\left\{c_5 \leq \mu_n(A_n) \leq c_6 \text{ and } \mathcal{E}_\alpha(\mu_n) \leq \tilde{C}_\alpha \text{ for infinitely many } n\right\} \geq c_5.$$

On the event on the left-hand side, we can take a subsequential limit and construct a measure μ supported on A with $\mu(A) \geq c_5$ and $\mathcal{E}_\alpha(A) \leq C_\alpha\}$. On this event, Frostman's lemma implies that $\dim_h(A) \geq \alpha$. □

4. Two-point estimates

In this section we sketch the main idea in the proof of Proposition 35 and discuss the multi-point Green's function for *SLE*.

4.1. Beurling estimate.

4.1. Beurling estimate. There is one standard estimate for Brownian motion (harmonic measure) that we will use in the next section. We state it here. A proof can be found in [**5**].

Proposition 38 (Beurling estimate). *There is a $c < \infty$ such that the following is true. Suppose $\eta : [0,1] \to \overline{\mathbb{D}}$ is a curve with $|\eta(0)| = \epsilon, |\eta(1)| = 1$. Then the probability that a Brownian motion starting at the origin reaches the unit circle without hitting η is bounded above by $c\epsilon^{1/2}$.*

♦This estimate is a corollary of a stronger result, the Beurling projection theorem, which implies that for fixed ϵ, the radial line segment $\{re^{i\theta} : \epsilon \leq r \leq 1\}$ maximizes the probability. By finding an appropriate conformal transformation, one can see that the probability in this case is asymptotic to $c\,\epsilon^{1/2}$.

4.2. Proof of Proposition 35. We will only sketch some of the main ideas in the proof of Proposition 35 following [**9**]. For ease we will consider $z = -1 + i, w = 1 + i$ and $\epsilon, \delta < 1/4$ with $n \geq 2$. The main idea is to show that if a curve is going to get very close to z and very close to w, it does one of two things. Either it gets very close to z without having gotten very close to w and then gets close to w or vice versa. What is unlikely to happen is that the curves gets near z and then near w and then nearer to z and then nearer to w, etc. In order to keep track of this, we note that any time the path goes near z and then near w it must go through the imaginary axis $\mathcal{I} = \{iy : y > 0\}$. In fact, there are crosscuts contained in $\mathcal{I}$ that it must cross.

Some topological issues come up. One is to show that for each t, there exists unique $\mathcal{I}_t$ with $\mathcal{I}_0 = \mathcal{I}$ such that $\{\mathcal{I}_t : t \geq 0\}$ satisfies the following properties. Recall that H_t is the unbounded component of $\mathbb{H} \setminus \gamma_t$,

- Each $\mathcal{I}_s$ is an open connected subarc of $\mathcal{I}$ that divides H_s into two components, one containing z and the other containing w.
- $s < t$ implies $\mathcal{I}_s \supset \mathcal{I}_t$.
- If $s < t$ and $\gamma[s,t] \cap \mathcal{I} = \emptyset$, then $\mathcal{I}_s = \mathcal{I}_t$.

We define a sequence of stopping times τ_j and radii q_j, r_j as follows.

- $\tau_0 = 0, q_0 = 1, r_0 = 1,$
- If $\tau_j < \infty$, let
$$\gamma^j = \gamma[0, \tau_j], \quad H^j = H_{\tau_j},$$
$$q_j = \text{dist}\left[z, \gamma^j\right], \quad r_j = \text{dist}\left[w, \gamma^j\right].$$
$$\sigma_{j+1} = \inf\left\{t > \tau_j : |\gamma(t) - z| \leq q_j/2 \text{ or } |\gamma(t) - w| \leq r_j/2\right\},$$
$$\tau_{j+1} = \inf\{t > \sigma_{j+1} : \gamma(t) \in \overline{\mathcal{I}_{\sigma_{j+1}}}\}.$$
- Note that if $\tau_{j+1} < \infty$, then either
$$q_{j+1} \leq q_j/2 \text{ and } r_{j+1} > r_j/2$$
or
$$q_{j+1} > q_j/2 \text{ and } r_{j+1} \leq r_j/2.$$
We call the two cases z-excursions and w-excursions, respectively.

What makes the proof tricky is that there are many different combinations of z and w excursions that can occur for which eventually $\text{dist}(z, \gamma) \leq \epsilon$ and $\text{dist}(w, \gamma) \leq \epsilon$. Let us consider the case of z-excursions. What we need to show is something like the following.

- There exists $\alpha > 0$ such that the probability of a successful z-excursion at the $(j+1)$ step with $q_{j+1} \leq \delta q_j$ given γ^j is bounded above a constant times

$$\delta^{2-d}\, q_j^{\alpha}. \tag{43}$$

We will discuss the proof of (43); the proof actually gives a particular value of α but this is not important. With this estimate, one can sum over all possible combinations of z and w excursions and get the result. This last step is fairly straightforward and we will not do this here.

To prove (43), it is important to realize that a successful z-excursion requires two events to occur.

- There exists a first time $T > \tau_j$ such that $|\gamma(T) - z| \leq \delta\, q_j$.
- The SLE_κ path hits the crosscut $\mathcal{I}_T$ after time T.

There are different topological situations to consider but the basic idea is that those configurations for which the first event is relatively likely to occur are those for which the second event is unlikely.

To separate into two cases, consider the domain H^j and let A_1, A_2 be the two arcs in the boundary as in (32). Let

$$\hat{S}_j = S_{H^j}(z; \gamma(\tau_j), \infty).$$

Recall from that equation that

$$\hat{S}_j \asymp \min \{\mathrm{hm}_{H^j}(z, A_1), \mathrm{hm}_{H^j}(z, A_2)\}.$$

Case 1. Suppose $I = \mathcal{I}_{\tau_j}$ is an unbounded segment. Then the endpoint of this component is $\gamma(\tau_j)$. Since I disconnects z and w, we can see that one of the arcs A_1, A_2, say A_2, defined at time τ_j has the following property: any path from z to A_2 staying in H^j must go through I. This observation and the Beurling estimate (Proposition 38) give

$$h_{H^j}(z, A_2) \leq c\, q_j^{1/2}$$

and hence

$$\hat{S}_j \leq c\, q_j^{1/2}.$$

Using (38), we see that the probability for γ to get within distance δr_j of z is bounded above by a constant times $\delta^{2-d}\, q_j^{(4a-1)/2}$.

Case 2. Suppose $I = \mathcal{I}_{\tau_j}$ is a bounded segment and let A_1, A_2 be the arcs in ∂H^j. There is a radial line of length q_j from z to ∂H^j; it hits one of these arcs, let us assume it is A_1. Let Δ_j denote the infimum of all r such that there exists a curve in H^j from z to A_2 that stays in the disk of radius r about z. The Beurling estimate and (32) as above imply

$$\hat{S}_j \leq c\, (q_j/\Delta_j)^{1/2}.$$

We split into two possibilities: $\Delta_j < \sqrt{q_j}$ and $\Delta_j \geq \sqrt{q_j}$.

- If $\Delta_j \geq \sqrt{q_j}$, then we can use (38) to say that the probability for γ to get within distance δr_j of z is bounded above by a constant times $\delta^{2-d}\, q_j^{(4a-1)/4}$.
- If $\Delta_j \leq \sqrt{q_j}$, there exist curves η_1, η_2 in the intersection of H^j with the disk of radius $q_j^{1/2}$ about z from z to A_1, A_2, respectively. We can concatenate these curves to get a crosscut of H^j from A_1 to A_2 going through z, staying in the disk of radius $q_j^{1/2}$ of z. This curve disconnects I from ∞ in H^j. Let

  $$T = \inf\{t \geq \tau_j : |\gamma(t) - z| = \delta\, q_j\}.$$

  By (38), the probability that $T < \infty$ is bounded above by a constant time δ^{2-d}. The claim is that the conditional probability that γ intersects $I = \mathcal{I}_T$ after time T given $T < \tau_{j+1}$ is bounded above by a constant times q^{α}. To see this we take a radial line segment from $\gamma(T)$ to z. Combining this with a subset of η, we can see that there is a curve from $\gamma(T)$ to ∂H_T that disconnects I from ∞ in H_T and has diameter less than $2q_j^{1/2}$. Using this and estimates on Brownian excursion measure (details omitted

— the Beurling estimate is used again), we can see that if l is the image of I under g_T, then

$$\mathrm{diam}(l)/\mathrm{dist}(U_T, l) \leq c\, q_j^{1/2}.$$

The probability that SLE_κ in H_T from $\gamma(T)$ to ∞ hits I can now be bounded using Proposition 31.

4.3. Multi-point Green's function for SLE_κ. The techniques to prove the two-point estimate in [**9**] can be used to prove the following two-point version of (38).

THEOREM 39. [**9**] *For every $\kappa < 8$, there exists a function $G(z,w)$ such that if $z, w \in \mathbb{H}$, then*

$$\lim_{\epsilon,\delta \to 0} \epsilon^{d-2}\, \delta^{d-2}\, \mathbb{P}\{\Upsilon_\infty(z) \leq \epsilon, \Upsilon_\infty(w) \leq \delta\} = c_*^2\, G(z,w).$$

This theorem does not give an explicit form for G. We can write

$$G(z,w) = \tilde{G}(z,w) + \tilde{G}(w,z),$$

where $\tilde{G}$ represents the "ordered" Green's function. For example $\tilde{G}(z,w)$ represents the probability of visiting z and then later visiting w. We can represent $\tilde{G}(z,w)$ in terms of two-sided radial SLE_κ through z as we now show. The limit in the theorem is independent of how ϵ, δ go to zero. Let us stretch things a bit and let ϵ go to zero fixing δ. Recall that (38) implies

$$\lim_{\epsilon \to 0} \epsilon^{d-2}\, \mathbb{P}\{\Upsilon_\infty(z) \leq \epsilon\} = c_*\, G(z).$$

The limiting distribution on paths is that of two-sided radial SLE_κ going through z. The conditional probability that $\Upsilon_\infty(w) \leq \delta$ "after γ hits z given γ goes through z" should be

$$\mathbb{P}\{\Upsilon_\infty(w) \leq \delta\}$$

for SLE_κ in H_{T_z} from z to ∞. Using (38) again we see that

$$\lim_{\epsilon \to 0} \epsilon^{d-2}\, \mathbb{P}\{\Upsilon_\infty(w) \leq \epsilon\} = c_*\, G_{H_{T_z}}(w; z, \infty).$$

Therefore,

$$\tilde{G}(z,w) = \mathbb{E}^*\left[G_{H_{T_z}}(w; z, \infty)\right] = \mathbb{E}^*\left[|g_{T_z}'(w)|^{2-d}\, G(Z_{T_z}(w))\right],$$

where $\mathbb{E}^*$ denotes expectation with respect to two-sided radial SLE_κ going through z.

The basic idea is that if γ is going to get very close to both z and w, it either first gets very close to z and then gets very close to w or vice versa. It does not keep going back and forth. The estimate (43) is the critical step for making this rigorous.

With the two-point Green's function, we have a two-point local martingale. Let

$$M_t(z,w) = |g_t'(z)|^{2-d}\, |g_t'(w)|^{2-d}\, G(Z_t(z), Z_t(w)). \tag{44}$$

Recall that

$$\sigma_z(s) = \inf\left\{t : \Upsilon_t(z) \leq e^{-2as}\right\}.$$

It follows from Theorem 39 that for all s, $M_{t \wedge (\sigma_z(s) \wedge \sigma_w(s))}$ is a martingale and hence $M_t(z,w)$ is a local martingale.

♦Another possible approach to find the multi-point Green's function is to find a function such that $M_t(z,w)$ in (44) is a local martingale. Using Itô's formula and the product rule, this gives a differential equation in three real variables. (Two complex variables gives four real variables, but a scaling relation reduces the number of variables by one.) This is a possible approach to finding a closed form for this function. However, I suspect that one would need an estimate similar to (43) to prove Theorem 39.

5. Natural parametrization or length

5.1. Motivation and heuristics. The capacity parametrization for SLE is very useful for analyzing the process. In particular, it is the parametrization in which the maps g_t are differentiable in t. However, if one considers scaling limits of discrete processes, there are other parametrizations that one would choose. For example, if one scales self-avoiding walks on the lattice, it is standard to parametrize so that each lattice step is taken in the same amount of time. We can ask if we can find such a parametrization for SLE_κ.

Let us suppose for the moment, that such a parametrization exists. Suppose γ is the curve of SLE_κ parametrized by half-plane capacity; for convenience, let us choose the rate so that the half-plane capacity of $\gamma(0,t]$ is t.

Let Θ_t denote the amount of "natural time" needed to traverse the curve $\gamma(0,t]$. Here, we are starting γ in the half-plane capacity parametrization. We would hope that Θ_t is a continuous, increasing process. We can also view Θ_t as a measure supported on the path where the measure of $\gamma(0,t]$ is Θ_t. In the case $\kappa \geq 8$, one natural choice would be $\Theta_t = \text{area}(\gamma(0,t])$. We will restrict our consideration to $\kappa < 8$. In this case, we might want to define Θ_t some d-dimensional measure of $\gamma(0,t]$. Suppose Θ_t exists, and consider $\Theta_{t+\Delta t} - \Theta_t$ which is the amount of "natural time" needed to traverse $\gamma[t,t+\Delta t]$. Let $\eta = g_t(\gamma[t,t+\Delta t])$. The capacity parametrization is invariant under g_t in the sense that the half-plane capacity of η is Δt. Hence we expect that $\text{diam}\,\eta \approx \sqrt{\Delta t}$. If the natural parametrization is a d-dimensional measure then the natural time needed to traverse η is about

$$(\text{diam}\,\eta)^d \approx (\Delta t)^{d/2}.$$

Also, d-dimensional measures have the property that if one blows up a set by a factor of r, then the measure is multiplied by r^d. If we consider $U_t + i\sqrt{\Delta t}$ as a typical point on η, we might guess that the amount of natural time to traverse $\gamma(t,t+\Delta t]$ is

$$|f_t'(i\sqrt{\Delta} t)|^d\,(\Delta t)^{d/2}.$$

Using this as motivation and setting $\Delta t = n$ or 2^{-n}, we might conjecture the following:

$$\Theta_t = \lim_{n\to\infty} n^{-d/2} \sum_{j \leq tn} |f_{j/n}'(i/\sqrt{n})|^d,$$

$$\Theta_t = \lim_{n\to\infty} 2^{-nd/2} \sum_{j \leq t2^n} |f_{j2^{-n}}'(i2^{-n/2})|^d. \tag{45}$$

Much of the more intricate analysis in Section 2 was developed to try to understand the right-hand side. Although we still do not know that the limit exists, there are some things we can say.

In the notation of that section, note that if $r = 1$, then

$$\lambda = d, \quad \zeta = 2 - d, \quad q = 2a - \frac{1}{2}, \quad \frac{\beta}{2} = \frac{1}{4a} - \frac{1}{2} = d - \frac{3}{2}.$$

For $\kappa < 8$, $1 < r_c$, so the approach there works. In particular,

$$\mathbb{E}[|f'_{s/t^2}(i/t)|^d] = \mathbb{E}[|f'_s(i)|^d] \asymp s^{\frac{d}{2}-1}.$$

which gives, for example,

$$\mathbb{E}\left[n^{-d/2} \sum_{j \leq n} |f'_{j/n}(i/\sqrt{n})|^d\right] \asymp n^{-d/2} \sum_{j \leq n} j^{\frac{d}{2}-1} \asymp 1.$$

(Since we expect $\mathbb{E}[\Theta_1]$ to be positive and finite, this is consistent with our heuristics.) Also, the expectation in

$$\mathbb{E}[|f'_1(i/n)|^d]$$

is carried on an event on which

$$|f'_1(i/\sqrt{n})|^d \approx n^{\beta/2} = n^{d-\frac{3}{2}}.$$

The probability of this event is about $n^{-\alpha}$ where

$$n^{-\alpha}\, n^{d\beta/2} = n^{\frac{d}{2}-1}.$$

Therefore,

$$\alpha = d\beta + 2 - d = d^2 - 2d + 1$$

As we let $n \to \infty$, we get an exceptional set $A \subset [0,1]$ of times which can be covered by about $n^{1-\alpha}$ intervals of length $1/n$. Hence has fractal dimension

$$1 - \alpha = d(2-d).$$

However, the images of these exceptional intervals should have diameter of order

$$n^{-1/2} \cdot |f'_1(i/\sqrt{n})| \approx n^{d-2}.$$

Hence the image of A is covered by $n^{1-\alpha}$ intervals of diameter n^{d-2} which means the dimension d^* of the image satisfies

$$[n^{d-2}]^{-d^*} = n^{1-\alpha}.$$

This retrieves $d^* = d$.

There has been a lot of hand-waving in this argument, but one can make it rigorous [**4**] to show the following.

- With probability one, the Hausdorff dimension of the set of times t such that

$$|f'_t(iy)| \approx y^{-\beta}, \quad y \to 0+, \tag{46}$$

  is $d(2-d)$.
- With probability one, there is a subset A of times satisfying (46) such that $\gamma(A)$ has Hausdorff dimension d.

In particular, the Hausdorff dimension of the curve is at least d. This gives another proof of Beffara's theorem. In fact, this is one case of the analysis in Section 2. The details were carried out in this case is [**4**] and generalized to the tip multifractal spectrum in [**7**].

♦An important corollary of this multifractal analysis is that the natural parametrization is singular with respect to the capacity parametrization.

♦A comment about terminology: for smooth curves, the term natural parametrization is used in some circles for parametrization by arc length. We think of this as the d-dimensional analogue. It is also the scaling limit of parametrization by arc length and so the phrase natural *length* is also used. Natural length may be a better word, but one must remember that this is a d-dimensional quantity.

5.2. A rigorous definition of natural parametrization. Suppose γ is an SLE_κ curve with $\kappa < 8$. We have seen that if $z \in \mathbb{H}$, then

$$\lim_{\epsilon \to 0+} \epsilon^{d-2}\, \mathbb{P}\{\Upsilon(z) \leq \epsilon\} = c_*\, G(z).$$

We conjecture (and think it might not be too difficult to prove) that there exists a constant c_0 such that the following is true.

Conjecture. *There exist $c_0 > 0$, such that if D is a simply connected domain and γ is SLE_κ from w_1 to w_2, then for $z \in D$,*

$$\lim_{\epsilon \to 0+} \epsilon^{d-2}\, \mathbb{P}\{\mathrm{dist}(\gamma, z) \leq \epsilon\} = c_0\, G_D(z; w_1, w_2).$$

We will assume this conjecture. We list another conjecture, which we believe is more difficult.

Conjecture. *If D is a simply connected domain and γ is SLE_κ from w_1 to w_2, then for $z \in D$, $t > 0$, the limit*

$$\Theta_t = \lim_{\epsilon \to 0+} \epsilon^{d-2}\, c_0^{-1}\, \mathrm{area}\{z : \mathrm{dist}(\gamma_t, z) \leq \epsilon\}$$

exists.

Let us assume this conjecture and see what it implies. For ease, let us assume that D is a bounded domain, and γ is an SLE_κ curve from w_1 to w_2 in D. The choice of parametrization is not really important, but let us assume that it retains the capacity parametrization from the upper half plane. Then

$$\Theta_\infty = \lim_{\epsilon \to 0+} \epsilon^{d-2}\, c_0^{-1}\, \mathrm{area}\{z : \mathrm{dist}(\gamma, z) \leq \epsilon\}. \tag{47}$$

It seems reasonable from (47) that

$$\mathbb{E}[\Theta_\infty] = \int_D G_D(z; w_1, w_2)\, dA(z),$$

where we write dA for integrals with respect to area. Also,

$$\mathbb{E}[\Theta_\infty \mid \gamma_t] = \mathbb{E}[\Theta_t + (\Theta_\infty - \Theta_t) \mid \gamma_t] = \Theta_t + \mathbb{E}[\Theta_\infty - \Theta_t \mid \gamma_t].$$

But the conformal Markov property and (47) would imply that

$$\mathbb{E}[\Theta_\infty - \Theta_t \mid \gamma_t] = \int_{D_t} G_{D_t}(z; \gamma(t), w_2)\, dA(z),$$

where, as before, D_t is the unbounded component of $D \setminus \gamma_t$ containing w_2 on the boundary. But $\mathbb{E}[\Theta_\infty \mid \gamma_t]$ is a martingale. Hence we get the following characterization.

- Θ_t is the unique increasing, adapted process such that

$$N_t := \Theta_t + \int_{D_t} G_{D_t}(z; \gamma(t), w_2)\, dA(z) \tag{48}$$

  is a martingale.

We will use this to try to construct Θ_t.

Let us return to the upper half plane. There is a technical issue that we would expect $\mathbb{E}[\Theta_\infty] = \infty$. To avoid this problem, suppose D is a bounded subdomain of $\mathbb{H}$ and let, $\Theta_t = \Theta_t(D)$ be the amount of "natural time" that the process has spent in D up to capacity time t. Then using the reasoning above, we would expect

$$\mathbb{E}[\Theta_\infty] = \int_D G(z)\, dA(z)$$

$$\mathbb{E}[\Theta_\infty \mid \gamma_t] = \Theta_t + \Psi_t$$

where

$$\Psi_t = \Psi_t(D) = \int_D M_t(z)\, dA(z),$$

and

$$M_t(z) = G_{H_t}(z; \gamma(t), \infty) = |g_t'(z)|^{2-d}\, G(Z_t(z))$$

is the local martingale from Section 3.4.

Definition The *natural parametrization* Θ_t *(restricted to D)* is the unique, adapted, continuous, increasing process such that

$$N_t = \Psi_t + \Theta_t$$

is a martingale.

The definition implies the existence of such a process. Uniqueness is not difficult since the difference between any two candidates would be a continuous martingale with paths of bounded variation and hence must be zero. Existence is the issue. Since $M_t(z)$ is a positive local martingale, it is a supermartingale. We recall the function ϕ from Section 3.5 which satisfies

$$\mathbb{E}\left[M_t(z)\right] = \left[1 - \phi(z;t)\right] \mathbb{E}\left[M_t(z)\right], \qquad \phi(z) = \phi(z;1).$$

From this, one can see that Ψ_t is a supermartingale. The Doob-Meyer decomposition implies that there exists Θ_t such that Ψ_t is a local martingale. However, it is not immediate that Θ_t is nontrivial.

♦$M_t(z)$ is a positive supermartingale, but the increasing process Θ_t^* that makes $M_t(z)+\Theta_t^*$ a local martingale is the zero process. This shows that some work is needed to show our Θ_t is not identically zero. In particular, we need to show that Ψ_t is not a local martingale.

It turns out the process Θ_t is nontrivial for all $\kappa < 8$. This was proved for $\kappa < \kappa_0 = 4(7 - \sqrt{33}) = 5.021\cdots$. in [**8**] and for $\kappa < 8$ in [**11**]. The former approach, which should work for all $\kappa < 8$, yields some more information, so we will discuss both methods. Let us consider Θ_1.

The way to construct Θ_1, as in the proof of the Doob-Meyer theorem, is to discretize time. Let $\mathcal{D}_n$ denote the dyadics. Then for $t \in \mathcal{D}_n$, we let

$$\Theta_{t+2^{-n}}^{(n)} - \Theta_t^{(n)} = \mathbb{E}[\Psi_t - \Psi_{t+2^{-n}} \mid \gamma_t].$$

Using the conformal Markov property of SLE and by changing variables, one can show that

$$\mathbb{E}[\Psi_t - \Psi_{t+2^{-n}} \mid \gamma_t] = \int_{\mathbb{H}} |f_t'(z)|^d\, \phi(z; 2^{-n})\, G(z)\, 1\{f_t(z) \in D\}\, dA(z). \tag{49}$$

Hence

$$\Theta_1^{(n)} = \sum_{j=0}^{2^n-1} \int_{\mathbb{H}} |f'_{j2^{-n}}(z)|^d \, \phi(z;2^{-n}) \, G(z) \, 1\{f'_{j2^{-n}}(z) \in D\} \, dA(z). \tag{50}$$

♦The expression (49) is a little complicated, but let us approximate it. The function $\phi(z;2^{-n})$ is of order one for $|z| \leq 2^{-n/2}$ and near zero otherwise. For typical z in this disk, $G(z) \asymp (2^{-n/2})^{d-2} = 2^{-n(\frac{d}{2}-1)}$. If we choose $i2^{-n/2}$ as a typical point in the disk, then we can see that the expression is of the same order of magnitude as

$$2^{-n} \, 2^{-n(\frac{d}{2}-1)} \, |f'(i2^{-n/2})|^d = 2^{-nd/2} \, |f'(i2^{-n/2})|^d.$$

Given this we see the similarity between this expression and the one in (45).

It is easy to see that the process

$$N_t^{(n)} = \Psi_t + \Theta_t^{(n)}, \quad t \in \mathcal{D}_n,$$

is a discrete time martingale. The hard step is to take the limit. If the martingales have a uniform L^2 bound, then there one can take the limit easily. In [**8**], this bound was shown directly for $\kappa < \kappa_0$. It was conjectured that this would hold for all $\kappa < 8$. The method gives a bound on the Hólder continuity of Θ_t (with respect to the capacity parametrization).

The Doob-Meyer theorem says that one can take the limit provided that the collection of random variables $\{\Theta_T\}$ is uniformly integrable where T runs over all stopping times with $T \leq 1$. Using this, as in [**11**] adapting an argument from [**16**], we can show existence for all $\kappa < 8$. The argument uses the multi-point Green's function $G(z,w)$. Using Theorem 39, one can see that

$$M_t(z,w) = |g_t'(z)|^{2-d} \, |g_t'(w)|^{2-d} \, G(Z_t(z), Z_t(w))$$

is a positive local martingale and hence a supermartingale. The two-point estimate Proposition 34 combined with Theorem 39 gives

$$G(z,w) \leq c\,|z-w|^{d-2} \qquad z,w \in D.$$

In [**11**], it is shown that there exists $c < \infty$ such that

$$G(z)\,G(w) \leq c\,G(z,w).$$

From this we see that if T is a stopping time,

$$\mathbb{E}\left[M_T(z)\,M_T(w)\right] \leq c\,\mathbb{E}\left[M_T(z,w)\right] \leq c\,\mathbb{E}\left[M_0(z,w)\right] = c\,G(z,w) \leq c\,|z-w|^{d-2}.$$

Therefore,

$$\mathbb{E}\left[\Psi_T^2\right] \leq \int_D \int_D \mathbb{E}\left[M_T(z) M_T(w)\right] \, dA(z)\, dA(w) \leq c \int_D \int_D |z-w|^{d-2} \, dA(z)\, dA(w) \leq c.$$

The uniform bound on $\mathbb{E}\left[\Psi_T^2\right]$ shows the uniform integrability.

5.3. Other domains and other reference measures. We have defined the natural parametrization Θ_t for SLE_κ in $\mathbb{H}$ and given an expression for it as a limit (50). This can be taken as a limit with probability one at least if we take a subsequence. Suppose γ is an SLE_κ curve in $\mathbb{H}$ with the capacity parametrization. Define

$$S_t = \inf\{s : \Theta_s = t\} \quad \eta(t) = \gamma(S_t).$$

Then η is SLE_κ in $\mathbb{H}$ with the natural parametrization.

Suppose

$$F : \mathbb{H} \to D$$

is a conformal transformation with $F(0) = w_1, F(\infty) = w_2$. If γ is an SLE_κ curve in $\mathbb{H}$ with the capacity parametrization, then

$$\tilde{\gamma} = F \circ \gamma(t)$$

is SLE_κ in D from w_1 to w_2 with the capacity parametrization. Define Θ_t^D by

$$\Theta_t^D = \int_0^t |F'(\gamma(s))|^d \, d\Theta_s = \int_0^{\Theta_t} |F'(\eta(s))|^d \, ds.$$

It is not too difficult to show that Θ_t^D satisfies the characteristic property (48) and hence is the correct definition of the natural parametrizaton in D.

If D is a subdomain of $\mathbb{H}$ and $w_1 = 0, w_2 = \infty$, there are two ways to define the natural parametrization for a curve — by considering is as a curve in $\mathbb{H}$ or as a curve in D. If we prove something like (47), it would be obvious that the definitions agree. While we do not have (47), in [**10**] it is shown that the definitions agree. There are many questions which are open. For example, is the natural parametrization for a curve the same as the natural parametrization of the reversal? Dapeng Zhan [**17**] proved that SLE_κ is reversible at least for $0 < \kappa \leq 4$; if we have a result like (47), reversibility of the natural length would be immediate. Unfortunately, it is now an open problem.

Although we used area as our reference measure, we could also define natural parametrization $\Theta_{t,\mu}$ with respect to a different positive measure μ. In this case, we would set

$$\Psi_t = \Psi_{t,\mu} = \int_D G_{D_t}(z; \gamma(t), w_2) \, \mu(dz),$$

and require that $\Psi_t + \Theta_{t,\mu}$ be a martingale. If one follows the proof, one can see that a condition on μ sufficient in order for the $\Theta_{t,\mu}$ to be well defined is

$$\int\int \frac{\mu(dz)\mu(dw)}{|z-w|^{2-d}} < \infty.$$

In other words, if μ is an α-dimensional measure for $\alpha > 2 - d$, then it is well defined.

Exercise 40. *Suppose η is SLE_κ in $\mathbb{H}$ from 0 to ∞ in the natural parametrization. Suppose $F : \mathbb{H} \to D$ is a conformal transformation as above, and let $\tilde{\eta}(t) = F(\eta(t))$. Show that $\tilde{\eta}$ has the natural parametrization with respect to the measure μ with*

$$\mu(dz) = |(F^{-1})'(z)|^d \, dA(z).$$

An exercise: Brownian motion in three dimensions

An example which has many of the properties that we have seen in the last three sections is Brownian motion. We will discuss the case of three dimensions and give a sequence of exercises for the reader. This example is somewhat easier than SLE_κ but it contains many of the same ideas. Throughout this subsection, B_t will denote a standard Brownian motion in $\mathbb{R}^3$ and $G(x) = 1/|x|$ will denote the Green's function of three-dimensional Brownian motion. The important properties of G are radial symmetry and the fact that G is harmonic on $\mathbb{R}^3 \setminus \{0\}$. Let

$$M_t(z) = G(B_t - z),$$

$$\rho_z(r) = \inf\{t : |B_t - z| \le r\},$$

EXERCISE 41. *Show that for $z \neq 0$, $M_t(z)$ is a local martingale satisfying*

$$dM_t(z) = M_t \, \frac{\nabla G(B_t - z)}{G(B_t - z)} \cdot dB_t.$$

Use the local martingle to conclude

$$\mathbb{P}\{\rho_z(r) < \infty\} = \frac{r}{|z|} \wedge 1.$$

For each z, we can consider the measure $\mathbb{P}^*_z$ obtained by weighting by the local martingale $M_t(z)$. Under this measure

$$dB_t = J_t \, dt + dW_t, \quad J_t = \frac{\nabla G(B_t - z)}{G(B_t - z)},$$

where W_t is a standard three-dimensional Brownian motion with respect to $\mathbb{P}^*_z$. We claim that $M_t(z)$ is not a martingale. This can be derived from either of the next two exercises. Let

$$T_z = \inf\{t : B_t = z\}.$$

If $z \neq 0$, then $\mathbb{P}\{T_z = \infty\} = 1$.

EXERCISE 42. *Show that with probability one with respect to $\mathbb{P}^*_z$, $T_z < \infty$.*

EXERCISE 43. *Show that*

$$\lim_{t \to \infty} \mathbb{E}\left[M_t(z)\right] = 0.$$

The next exercise concerns the two-point Green function. This is the analogue of Theorem (39) although the exercise is significantly easier than the proof of that theorem.

EXERCISE 44. *Show that*

$$\lim_{\epsilon, \delta \to 0+} \epsilon^{-1} \, \delta^{-1} \, \mathbb{P}\left\{\rho_z(\epsilon) < \infty, \rho_w(\delta) < \infty\right\} = G(z, w),$$

where

$$G(z, w) = \left[G(z) + G(w)\right] G(z - w).$$

Show that $M_t(z, w) = G(B_t - z, B_t - w)$ is a local martingale.

EXERCISE 45. *Show there exist c_1, c_2 such that*

$$c_1 G(z) \, G(w) \le G(z, w) \le c_2 \, G(z) \, G(w) \, \frac{|z| + |w|}{|z - w|}.$$

Exercise 46. *Show there exists $c < \infty$ such that for every stopping time T,*

$$\mathbb{E}\left[M_T(z)\, M_T(w)\right] \leq c\, G(z)\, G(w)\, \frac{|z| + |w|}{|z - w|}.$$

Suppose D is a bounded domain and let

$$\Psi_t = \Psi_t(D) = \int_D M_t(z)\, d^3 z.$$

Exercise 47. *Show that Ψ_t is a supermartingale. Let*

$$l_t = \int_0^t 1\{B_s \in D\}\, ds.$$

Show that there exists c such that

$$\Psi_t + c\, l_t$$

is a martingale.

The take home message from the last exercise is that the usual parametrization of Brownian motion is (up to a multiplicative constant) its natural parametrization.

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