1 Part I

Exercise 1.1. *Let C_n denote the number of self-avoiding random walks starting at the origin in $\mathbb{Z}^2$ of length n.*

1. *Show that*

$$\lim_{n\to\infty} C_n^{1/n} = \inf_{n\to\infty} C_n^{1/n} := \rho.$$

   (Hint: Use $C_{n+m} \leq C_n C_m$.)

2. *Show that $2 < \rho < 3$.*

Exercise 1.2. *Let A denote a finite, connected subset of $\mathbb{Z}^2$ and $\partial A = \{z \in \mathbb{Z}^2 : \operatorname{dist}(z, A) = 1\}, \overline{A} = A \cup \partial A$. We consider $\overline{A}$ as an undirected graph whose edges E_A are those lattice edges connecting either two points in A or one point in A and one point in ∂A. The (discrete) Laplacian of a function $f : \overline{A} \to \mathbb{R}$ is defined by*

$$\Delta f(x) = \frac{1}{4} \sum_{|y-x|=1} [f(y) - f(x)],$$

and f is (discrete) harmonic on A if $\Delta f(x) = 0$ for all $x \in A$.

1. *Show that for each $F : \partial A \to \mathbb{R}$, there is a unique extension f_F of F to $\overline{A}$ that is harmonic on A.*

2. *If $f, g : \overline{A} \to \mathbb{R}$ are given we define*

$$\mathcal{E}(f, g) = \sum_{(x,y)\in E_A} [f(x) - f(y)]\,[g(x) - g(y)].$$

   Show that

$$\mathcal{E}(f_F, f_F) = \inf_f \mathcal{E}(f, f),$$

   where the infimum is over all functions f on $\overline{A}$ that agree with F on ∂A.

3. *Suppose f, g are functions on $\overline{A}$ such that f is harmonic on A and $g \equiv 0$ on ∂A. Show that $\mathcal{E}(f, g) = 0$.*

4. *The set of functions on A can be considered as $\mathbb{R}^A$. If $F : \partial A \to \mathbb{R}$, define*

$$\mathcal{C}(F) = (2\pi)^{-\#(A)} \int \exp\left\{-\frac{\mathcal{E}(f^{(F)}, f^{(F)})}{2}\right\}.$$

   Here the integral is over all functions f on A (using Lebesgue measure on $\mathbb{R}^A$) and $f^{(F)}$ denotes the function that equals f on A and F on ∂A. Show that for all F,

$$\mathcal{C}(F) = e^{-\mathcal{E}(f_F, f_F)/2}\, \mathcal{C}(0).$$

2 Brownian motion

Exercise 2.1. *Let B_t be a standard one dimensional Brownian motion. Show that with probability one, for every $\alpha > 1/2$,*

$$\lim_{t\to\infty} \frac{B_t}{t^\alpha} = 0.$$

Exercise 2.2. *Let B_t be a standard one-dimensional Brownian motion and let $r > 0$.*

- *Let $Y_t = r^{-1/2}\, B_{rt}$. Show that Y_t is a standard Brownian motion.*
- *Let $Z_t = t\, B_{1/t}$. Show that Z_t is a standard Brownian motion. (We define $Z_0 = 0$.)*

Exercise 2.3. *Suppose B_t is a standard one-dimensional Brownian motion. Show that with probability one for all $\alpha < 1/2$,*

$$\sup\left\{\frac{|B_t - B_s|}{(t-s)^\alpha} : 0 \le s < t \le 1\right\} < \infty,$$

but

$$\sup\left\{\frac{|B_t - B_s|}{(t-s)^{1/2}} : 0 \le s < t \le 1\right\} = \infty,$$

Exercise 2.4. *Let B_t be a standard one-dimensional Brownian motion. Find for each $r > 1$,*

$$\mathbf{P}\left\{B_t = 0 \text{ for some } 1 \le t \le r\right\}.$$

3 Martingales and stochastic integrals

Exercise 3.1. *Let B_t be a standard Brownian motion and let $a < 0 < b$. Let*

$$\tau = \inf\{t : B_t = a \text{ or } B_t = b\}.$$

We assume that $B_0 = 0$.

1. *Use the fact that $B_{t\wedge\tau}$ is a martingale to compute $\mathbf{P}\{B_\tau = b\}$.*
2. *Let $M_t = B_t^2 - t$. Show that M_t is a martingale.*
3. *By the optional sampling theorem, $M_{t\wedge\tau}$ is a martingale. Use this to compute $\mathbf{E}[M_\tau]$. (Note: there is a limit that has to be justified.)*
4. *Assume $b = -a = 1$. Let $N_t = e^{rB_t - (r^2/2)t}$. Show that r is a martingale. Use the optional sampling theorem on $N_{t\wedge\tau}$ to compute the moment generating function for τ, i.e., the function*

   $$\phi(s) = \mathbf{E}[e^{s\tau}].$$

   For which s is this finite?

Exercise 3.2. *Let B_t be a standard Brownian motion and let $a \in \mathbb{R}$, $r > 0$. Suppose that X_t is the solution to the Bessel SDE*

$$dX_t = \frac{a}{X_t}\, dt + dB_t, \qquad X_0 = 1.$$

We will only consider this up to the time $\tau = \inf\{t : X_t = 0\}$.

1. *For each $\epsilon > 0$, let*
   $$\tau = \inf\{t : X_t = \epsilon \text{ or } X_t = 1/\epsilon\}.$$
   Find $\mathbf{P}\{X_\tau = \epsilon\}$ by following this outline. Let
   $$f(x) = \mathbf{P}\{X_\tau = \epsilon | X_0 = x\}.$$
   Then $f(\epsilon) = 1, f(1/\epsilon) = 0$. Explain why $M_t = f(X_{t\wedge\tau})$ is a martingale. Assume for the moment that f is C^2 and use Itô's formula to find f. Then use the optional sampling theorem to justify.
2. *For which a is $\mathbf{P}\{\tau < \infty\} = 0$?*
3. *For which a is $\mathbf{P}\{X_{t\wedge\tau} \to \infty\} = 1$?*
4. *Let $\bar{B}_t = (B_t^1, \ldots, B_t^d)$ be a standard d-dimensional Brownian motion. Show that there is a one-dimensional Brownian motion W_t such that $X_t = |\bar{B}_t|$ satisfies*
   $$dX_t = \frac{\frac{d-1}{2}}{X_t}\, dt + dW_t.$$

4 Complex variables

Exercise 4.1. *Suppose $D \subset \mathbb{C}$ is a domain and f_n is a collection of harmonic (holomorphic) functions on D that are locally bounded, i.e., for each compact $K \subset D$ there is an $M_K < \infty$ such that $|f_n(z)| \le M_K$ for all n and all $z \in K$. Show that there exists a subsequence f_{n_j} that converges pointwise to a harmonic (holomorphic) function f. (Hint: Use diagonalization to find a subsequence that converges for all $z \in \mathbf{Q} + i\mathbf{Q}$. Then use derivative estimates to show that the function can be extended by continuity to a harmonic (holomorphic) function.)*

Exercise 4.2. *Let*

$$D = \{x + iy : 0 < x < \infty, 0 < y < \pi\}.$$

Use separation of variables to find a series formula for $H_D(x + iy, \tilde{y})$ for $0 < \tilde{y} < \pi$. Here is the outline:

- *Find a countable collection of harmonic functions of product form $h_n(x+iy) = \phi_n(x)\, g_n(y)$ that satisfy $h_n(x) = h_n(x + i\pi) = 0$ for all $x > 0$.*

- *Write*
  $$H_D(x+iy, y') = \sum_{n=1}^{\infty} b_n\, h_n(x+iy).$$
  Determine the coefficients b_n (that depend on $\tilde{y}$) by reasoning that
  $$\sum_{n=1}^{\infty} b_n\, \phi_n(0)\, g_n(y)$$
  equals the delta function at $\tilde{y}$.

Exercise 4.3.

- *Show that there exist positive constants c, α such that the following holds. Suppose B_t is a complex Brownian motion starting at $z \in \mathbb{D}$ and let $p(z)$ be the probability that the set $B[0, \tau_{\mathbb{D}}]$ does not disconnect the origin from the unit circle. Then*
  $$p(z) \leq c\,|z|^{\alpha}.$$
- *Suppose V is a connected set containing the origin but strictly bigger than a simple point. Let B_t be a complex Brownian motion starting at the origin and let*
  $$\rho = \inf\{t > 0 : B_t \in V\}.$$
  Show $\mathbf{P}\{\rho = 0\} = 1$.
- *Suppose $D \subset \mathbb{C}$ is a domain such that every connected component of ∂D is larger than a singleton set. Suppose $F : \partial D \to \mathbb{R}$ is a bounded, continuous function and let*
  $$f(z) = \mathbf{E}^z[F(B_{\tau_D})], \quad z \in D,$$
  and $f \equiv F$ on ∂D. Show that $f : \overline{D} \to \mathbb{R}$ is continuous.

Exercise 4.4. *Suppose D is a simply connected domain and $f : D \to \mathbb{C}$ is holomorphic and one-to-one. Show that D is simply connected. Give an example of a simply connected domain D and a function f that is locally one-to-one (i.e., $f'(z) \neq 0$ for all $z \in D$) such that $f(D)$ is not simply connected.*

Exercise 4.5. *Suppose*
$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$
is a univalent function on $\mathbb{D}$ with $f(0) = 0$. Show that
$$\mathrm{area}[f(\mathbb{D})] = \pi \sum_{n=1}^{\infty} n\,|a_n|^2.$$

Exercise 4.6. *Suppose γ is a smooth simple, closed curve in $\mathbb{C}$ and that D is the region bounded by the curve. Show that*

$$\mathrm{area}(D) = \frac{1}{2i}\int_\gamma \overline{z}\,dz.$$

Exercise 4.7. *Show there exist $0 < c_1 < c_2 < \infty$ such that the following holds. Suppose $\gamma : [0,1] \to \mathbb{C}$ is a (continuous) curve with $\gamma(0) \in \partial\mathbb{D}, \gamma(0,1] \subset \mathbb{D}$. Suppose B_t is a complex Brownian motion starting at the origin. Then,*

$$c_1\,\mathrm{diam}[\gamma[0,1]) \le \mathbf{P}\{B[0,\tau_D] \cap \gamma[0,1] \neq \emptyset\} \le c_2\,\mathrm{diam}(\gamma[0,1]).$$

Exercise 4.8. *Call D a slit domain(from infinity) if $D = \mathbb{C} \setminus \gamma[0,\infty)$ where γ is a simple curve $\gamma : [0,\infty) \to \mathbb{C}$ with $\gamma(t) \to \infty$ as $t \to \infty$ and $0 \notin \gamma[0,\infty)$. If D is a slit domain, let f_D denote the unique conformal transformation of $\mathbb{D}$ onto D with $f_D(0) = 0, f_D'(0) = 1$. Give the details of the proof that if $f \in \mathcal{S}$, then there exists a sequence of slit domains D_n such that $f_{D_n} \in \mathcal{S}$ and such that $f_{D_n} \to f$ uniformly on compact sets.*

5 Loewner differential equation

Exercise 5.1. *Suppose $D \subset \mathbb{H}$ is a domain with $\mathbb{H} \setminus D$ bounded. For $z \in D$ show that the following limit exists*

$$\phi_D(z) = \lim_{R\to\infty} R\,\mathbf{P}^z\{\mathrm{Im}[B_{\tau_R}] = R\},$$

where B_t is a complex Brownian motion and

$$\tau_R = \inf\{t : B_t \notin D \text{ or } \mathrm{Im}[B_t] \ge R\}.$$

Show that $\phi_D(z) > 0$ for all $z \in D$ and

$$\phi_D(z) = \mathrm{Im}[z] + O(|z|^{-1}), \quad z \to \infty.$$

(Note: we are not assuming that D is simply connected.) Prove that for all $z \in D$, $\phi_D(z) =$ $\mathrm{Im}(z)$ *if and only if for each $w \in \mathbb{C}$,*

$$\mathbf{P}^w\{B(0,\infty) \cap \mathbb{H} \setminus D \neq \emptyset\} = 0.$$

Exercise 5.2. *Show that if D is an in the previous example and D is simply connected, then if $g : D \to \mathbb{H}$ is a conformal transformation with $g_D(z) \sim z$ as $z \to \infty$, then $\phi = \mathrm{Im}[g_D]$.*