Multifractal analysis of the reverse flow for the Schramm-Loewner evolution

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Abstract. The Schramm-Loewner evolution (SLE) is a one-parameter family of conformally invariant processes that are candidates for scaling limits for two-dimensional lattice models in statistical physics. Analysis of SLE curves requires estimating moments of derivatives of random conformal maps. We show how to use the Girsanov theorem to study the moments for the reverse Loewner flow. As an application, we give a new proof of Beffara's theorem about the dimension of SLE curves.

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1. Introduction

The Schramm-Loewner evolution (SLE) was introduced by Oded Schramm [11] as a candidate for scaling limits of models in statistical physics. It has led to a much greater rigorous understanding of scaling limits of critical models in two-dimensional statistical physics.

Here we give a brief introduction to SLE. See [5] for more details. Chordal SLE_κ in the upper half plane $\mathbb{H}$ is defined in terms of conformal maps g_t defined by

$$\partial_t g_t(z) = \frac{a}{g_t(z) - V_t}, \quad g_0(z) = z, \tag{1}$$

where $a = 2/\kappa$ and V_t is a one-dimensional Brownian motion. There is a corresponding random curve $\gamma(0,t]$. The relation between the two is that for a fixed time t, if H_t denotes the unbounded component of $\mathbb{H}\setminus\gamma(0,t]$, then g_t is a conformal

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transformation of H_t onto $\mathbb{H}$ satisfying

$$g_t(z) = z + \frac{at}{z} + O(|z|^{-2}), \quad z \to \infty.$$

We let $f_t = g_t^{-1}$, which is a conformal transformation of $\mathbb{H}$ onto H_t, and $\hat{f}_t(z) = f_t(z + V_t)$. (We have chosen a particular parametrization of the SLE path. In the original definition, the Loewner equation was written with 2 replacing a, and then the function V_t was a Brownian motion with variance $\kappa = 2/a$. Our choice is a simple linear reparametrizaton, i.e., using a different unit of time. We write many of our formulas in terms of a but throughout the paper $a = 2/\kappa$.)

A number of problems in SLE lead to studying moments of derivatives. When considering moments of $|\hat{f}_t'(z)|$, one can instead consider a reverse-time Loewner flow. In this paper, we consider solutions of the time-reversed Loewner equation

$$\partial_t h_t(z) = \frac{a}{U_t - h_t(z)}, \quad h_0(z) = z, \tag{2}$$

where $U_t = -B_t$ is a standard Brownian motion. Using only the Loewner equation (1) and the time-reversibility of Brownian motion (see Section 10.3), one can show that the distribution of $h_t(z) - U_t$ is the same as the distribution of $\hat{f}_t(z)$. In many ways the reverse Loewner flow (2) is easier to analyze than the forward flow (1), see, e.g., [10, 4, 9].

In this paper we will study the moments of $|h_t'|$ and show how they can be studied using relatively standard methods of stochastic calculus. This builds on previous work, especially that of Rohde and Schramm [10] who first studied the moments in order to show that the curve γ exists. Moments have also been studied by a number of other authors, see, e.g., [2]. There are several reasons to include a self-contained treatment of these moments. First, the recent work of the author [7, 8] relies on these estimates. Second, this is a nice example of a multifractal analysis that can be done. It illustrates the general technique in SLE of trying to reduce problems of the flow to a one-variable SDE and then to analyze the SDE. A particular emphasis is the role of the Girsanov theorem in analyzing the SDE.

Our approach to studying the moments is to find an appropriate martingale and then to use the Girsanov theorem to understand the distribution when weighted by the martingale. We first note that the scaling properties of Brownian motion imply that for each $r > 0$, the distribution of the random function $z \mapsto r^{-1} h_{r^2 t}(rz)$ is the same as the distribution of $z \mapsto h_t(z)$. In particular, the distribution of $h'_{r^2 t}(rz)$ is the same as that of $h_t'(z)$.

As an application, in Section 10 we give a new proof of Beffara's theorem [1] that the Hausdorff dimension of SLE_κ curves is $1 + \frac{\kappa}{8}$ for $\kappa \leq 8$.

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2. Studying the flow

If $z = x_z + iy_z \in \mathbb{H}$, let

$$Z_t(z) = X_t(z) + iY_t(z) = h_t(z) - U_t.$$

Then (2) can be written as

$$dZ_t(z) = -\frac{a}{Z_t(z)}\,dt + dB_t, \qquad Z_0(z) = z,$$

or

$$dX_t(z) = -\frac{a\,X_t(z)}{|Z_t(z)|^2}\,dt + dB_t, \quad \partial_t Y_t(z) = \frac{a\,Y_t(z)}{|Z_t(z)|^2}. \tag{3}$$

We write d for stochastic differentials (with respect to time) and ∂_t for actual derivatives. We note the following properties.

- $Y_t(z)$ increases with t and hence the solution to (2) exists for all times. Moreover, $\partial_t[Y_t(z)^2] \leq 2a$ and hence
$$y_z^2 \leq Y_t(z)^2 \leq y_z^2 + 2at.$$
- For each $t \geq 0$, h_t is a conformal transformation of $\mathbb{H}$ onto a subdomain $h_t(\mathbb{H})$ satisfying
$$h_t(z) = z - \frac{at}{z} + O(|z|^{-2}), \quad z \to \infty.$$

We let

$$S_t(z) = \sin\left[\arg Z_t(z)\right] = \left[\frac{X_t(z)^2}{Y_t(z)^2} + 1\right]^{-1/2}, \quad \Psi_t(z) = \frac{|h_t'(z)|}{Y_t(z)}.$$

A calculation using Itô's formula gives

$$d\left[S_t(z)^{r/2}\right] =$$
$$S_t(z)^{r/2}\left[\frac{(2ar + \frac{r^2}{2} + \frac{r}{2})X_t(z)^2 - \frac{r}{2}\,Y_t(z)^2}{|Z_t(z)|^4}\,dt - \frac{rX_t(z)}{|Z_t(z)|^2}\,dB_t\right]. \tag{4}$$

By differentiating (2) with respect to z, we can see that

$$\partial_t[\log h_t'(z)] = \frac{a}{Z_t(z)^2}.$$

Therefore,

$$\partial_t|h_t'(z)| = |h_t'(z)|\,\mathrm{Re}\left[\frac{a}{Z_t(z)^2}\right] = |h_t'(z)|\,\frac{a\,[X_t(z)^2 - Y_t(z)^2]}{|Z_t(z)|^4}, \tag{5}$$

and

$$\partial_t\Psi_t(z) = \Psi_t(z)\,\frac{-2a\,Y_t(z)^2}{|Z_t(z)|^4}.$$

In particular, $\Psi_t(z)$ decreases with t which implies

$$|h_t'(z)| \leq \frac{Y_t(z)}{Y_0(z)} \leq \sqrt{1 + 2a(t/y_z^2)}. \tag{6}$$

The next proposition introduces the family of martingales indexed by $r \in \mathbb{R}$ that will be the main tool for estimating the moments of $|h_t'(z)|$.

Proposition 2.1. *If $r \in \mathbb{R}$, $z = x_z + iy_z \in \mathbb{H}$, and*

$$\lambda = \lambda(r) = r\left(1 + \frac{1}{2a}\right) - \frac{r^2}{4a}, \qquad \zeta = \zeta(r) = r - \frac{r^2}{4a} = \lambda - \frac{r}{2a} \tag{7}$$

then

$$M_t(z) = |h_t'(z)|^\lambda \, Y_t(z)^\zeta S_t(z)^{-r} \tag{8}$$

is a martingale satisfying

$$dM_t(z) = \frac{r\, X_t(z)}{|Z_t(z)|^2}\, M_t(z)\, dB_t. \tag{9}$$

In particular,

$$\mathbb{E}\left[M_t(z)\right] = M_0(z) = y_z^\zeta \left[(x_z/y_z)^2 + 1\right]^{r/2},$$

Proof. The product rule combined with (3), (4), and (5) shows that $M_t = M_t(z)$ is a nonnegative local martingale satisfying (9). (Note that $|h_t'(z)|^\lambda, Y_t(z)^\zeta$ are differentiable quantities so there are no covariation terms.) We can use the Girsanov theorem to conclude that M_t is a martingale for $z \notin \mathbb{R}$. We give the sketch of the argument here. Readers unfamiliar with the use of stopping times with the Girsanov theorem for continuous local martingales should consult the appendix for more details.

If we use Girsanov's theorem and weight B_t by the local martingale $M_t(z)$ then

$$dB_t = \frac{r\, X_t(z)}{|Z_t(z)|^2}\, dt + d\tilde{B}_t$$

where $\tilde{B}_t$ is a Brownian motion with respect to the new measure $\mathbf{Q}$. In other words,

$$dX_t(z) = \frac{(r-a)\, X_t(z)}{|Z_t(z)|^2}\, dt + d\tilde{B}_t. \tag{10}$$

Note that $Y_t(z)$ and $|h_t'(z)|^d$ are differentiable quantities so their equations do not change in the new measure. (Actually the equation of $X_t(z)$ also does not change. What changes is the distribution of the random process B_t. In order to write the equation for $X_t(z)$ in terms of a Brownian motion in the new measure, the drift term is changed.) By comparing to a Bessel equation, it is easy to check that if $X_t(z)$ satisfies (10) then there is no explosion in finite time. Using (9), we see that $M_t(z)$ also does not have explosion in finite time. □

3. Multifractal analysis

Multifractal analysis refers to the study of moments of a random variable and measures obtained by weighting by powers of random variables. There is a significant overlap between this and large deviation theory. Here we discuss a simple version of this that applies to our situation.

Suppose we have a collection of random variables $D_t, t > 0$. The start of large deviation analysis is to estimate the exponential moments, e.g., to find a function $\zeta(\lambda)$ such that

$$\mathbb{E}[e^{\lambda D_t}] \approx e^{-\zeta(\lambda)t},$$

where $\approx$ means

$$\zeta(\lambda) = -\lim_{t\to\infty} \frac{\log \mathbb{E}[e^{\lambda D_t}]}{t}. \tag{11}$$

Many papers in probability are devoted to finding the function ζ for some particular random variables. While one often can only prove a result such as (11), there are many cases where one can give a stronger estimate:

$$\mathbb{E}[e^{\lambda D_t}] \asymp e^{-\zeta(\lambda) D_t}, \tag{12}$$

where the implicit constants in the $\asymp$ notation can be chosen uniformly over λ in an interval. If we know (12) and can show that ζ is C^2, then, as the next proposition shows, it is easy to conclude that (roughly speaking) the expectation in (12) concentrates on an event on which

$$D_t = -\zeta'(\lambda)\, t + O(t^{1/2}).$$

Proposition 3.1. *Suppose $\lambda_0 \in \mathbb{R}, \epsilon > 0$, $0 < c_1 < c_2 < \infty$, and ζ is a differentiable function such that for all $t \geq 1/\epsilon^2$,*

$$c_1\, e^{-t\zeta(\lambda)} \leq \mathbb{E}[e^{\lambda D_t}] \leq c_2\, e^{-t\zeta(\lambda)}, \quad |\lambda - \lambda_0| \leq \epsilon.$$

Suppose there exists $\alpha < \infty$ such that

$$|\zeta(\lambda) - \zeta(\lambda_0) - \zeta'(\lambda_0)\,(\lambda - \lambda_0)| \leq \alpha\,(\lambda-\lambda_0)^2, \quad |\lambda - \lambda_0| \leq \epsilon.$$

Then for all $t \geq 1/\epsilon^2$ and all $k > 0$,

$$\mathbb{E}\left[e^{\lambda_0 D_t}; |D_t - \mu t| \geq k\, t^{1/2}\right] \leq c_*\, e^{-k}\, \mathbb{E}\left[e^{\lambda_0 D_t}\right].$$

where $\mu = -\zeta'(\lambda_0), c_* = 2e^{\alpha} c_2/c_1$.

Proof.

$$\begin{aligned}
&\mathbb{E}[e^{\lambda_0 D_t}; D_t \geq \mu t + kt^{1/2}] \\
&\qquad \leq \quad e^{-t^{-1/2}(\mu t + kt^{1/2})}\, \mathbb{E}[e^{(\lambda_0 + t^{-1/2})D_t}; D_t \geq \mu t + kt^{1/2}] \\
&\qquad \leq \quad c_2\, e^{-k}\, e^{-\mu t^{1/2}}\, \exp\{-t\zeta(\lambda_0 + t^{-1/2})\} \\
&\qquad \leq \quad c_2\, e^{-k}\, e^{\alpha}\, e^{-t\zeta(\lambda_0)} \leq (c_*/2)\, e^{-k}\, \mathbb{E}[e^{\lambda_0 D_t}],
\end{aligned}$$

Similarly, one shows that

$$\mathbb{E}[e^{\lambda_0 D_t}; D_t \leq \mu t - kt^{1/2}] \leq (c_*/2)\, e^{-k}\, \mathbb{E}[e^{\lambda_0 D_t}].$$

$\square$

Let $\mathbb{P}_t$ denote the probability measure with

$$\frac{d\mathbb{P}_t}{d\mathbb{P}} = \frac{e^{\lambda_0 D_t}}{\mathbb{E}[e^{\lambda_0 D_t}]}.$$

Then the conclusion of the proposition can be written as

$$\mathbb{P}_t\left\{\frac{|D_t - \mu t|}{\sqrt{t}} \geq k\right\} \leq c_*\, e^{-k}, \quad t \geq \epsilon^{-2}.$$

4. Moments of $|h'|$

In this section we fix $z = i$ and write $M_t, X_t, Y_t, \ldots$ for $M_t(i), X_t(i), Y_t(i), \ldots$ We will study the multifractal behavior of $D_t = \log|h'_{e^{2t}}(i)|$, i.e., we will find the function $\lambda \mapsto \zeta^*(\lambda)$ such that

$$\mathbb{E}[|h'_{t^2}(i)|^\lambda] \approx t^{-\zeta^*(\lambda)}.$$

Let

$$r_c = 2a + \frac{1}{2}, \quad \lambda_c := \lambda(r_c) = a + \frac{3}{16a} + 1, \quad \zeta_c := \zeta(r_c) = a - \frac{1}{16a}.$$

We show the significance of these values below; it is the value of λ such that (15) holds. If $r \leq r_c$, we can solve the quadratic equation (7), to get

$$r = r(\lambda) = 2a + 1 - \sqrt{(2a+1)^2 - 4a\lambda}, \quad \lambda \leq \lambda_c. \tag{13}$$

We can write ζ as a function of λ,

$$\zeta(\lambda) = \lambda - \frac{r}{2a} = \lambda + \frac{1}{2a}\sqrt{(2a+1)^2 - 4a\lambda} - 1 - \frac{1}{2a}. \tag{14}$$

As r increases from $-\infty$ to r_c, λ increases from $-\infty$ to λ_c. For $\lambda \leq \lambda_c$, we can write the martingale from Proposition 2.1 as

$$M_t = |h_t'(z)|^\lambda\, Y_t^{\zeta(\lambda)}\, S_t^{-r(\lambda)}.$$

Note that

$$\zeta'(\lambda) := \partial_\lambda \zeta(\lambda) = 1 - \frac{1}{\sqrt{(2a+1)^2 - 4a\lambda}}.$$

In particular, ζ is strictly concave with $\zeta'(-\infty) = 1$. The critical value λ_c satisfies

$$\zeta'(\lambda_c) = -1. \tag{15}$$

For $\lambda < \lambda_c$, we would like to show

$$\mathbb{E}\left[|h'_{t^2}(i)|^\lambda\right] \asymp t^{-\zeta(\lambda)}$$

and that the expectation is concentrated on an event for which

$$|h'_{t^2}(i)| \approx t^{-\zeta'(\lambda)}.$$

We will actually show a slightly weaker version (we will show the stronger version for a certain range of λ, see Section 9).

Let $I(t,m)$ denote the indicator function of the event

$$|X_t| \leq m\sqrt{t}, \qquad \frac{1}{m} \leq \frac{Y_t}{\sqrt{t}} \leq m.$$

Note that the "typical" values for X_t, Y_t are of order $\sqrt{t}$; in fact it is not difficult to show that

$$\lim_{m\to\infty} \mathbb{E}[I(t,m)] = 1.$$

For fixed m,

$$\mathbb{E}\left[|h_t'(i)|^\lambda \, I(t,m)\right] \asymp t^{-\zeta/2}\, \mathbb{E}\left[M_t\, I(t,m)\right] \leq t^{-\zeta/2}\, \mathbb{E}\left[M_t\right] = t^{-\zeta/2}.$$

(The implicit constants depend on m as well as ζ and κ.) One consequence of the next few sections is the following.

Proposition 4.1. *If $\lambda < \lambda_c$, there exist c, m such that*

$$\mathbb{E}[M_t\, I(m,t)] \geq c.$$

In particular, there exists c_1 such that

$$\mathbb{E}[|h_t'(i)|^\lambda] \geq \mathbb{E}[|h_t'(i)|^\lambda\, I(t,m)] \geq c_1\, t^{-\zeta/2}. \tag{16}$$

We will now explain why $\zeta^*(\lambda) \neq \zeta(\lambda)$ for $\lambda > \lambda_c$. Using the ideas of the previous section, we can show that for $\lambda < \lambda_c$ the expectation in $\mathbb{E}[|h_t'(i)|^\lambda\, I(t,m)]$ concentrates (roughly speaking) on an event on which

$$|h_{t^2}'(i)| \approx t^{-\zeta'(\lambda)}.$$

However, (6) tells us that

$$|h_{t^2}'(i)| \leq c\, t.$$

Hence, for $\lambda \geq \lambda_c$, the expectation is concentrated on an event with $|h_{t^2}'(i)| \asymp t$, and

$$\mathbb{E}[|h_{t^2}'(i)|^\lambda\, I(m,t)] \asymp t^{(\lambda-\lambda_c)-\zeta_c}, \quad \lambda \geq \lambda_c.$$

We can write

$$\mathbb{E}[|h_{t^2}'(i)|^\lambda\, I(m,t)] \asymp t^{-\zeta^*(\lambda)},$$

where

$$\zeta^*(\lambda) = \begin{cases} \zeta(\lambda) & \lambda \leq \lambda_c \\ \zeta(\lambda) + (\lambda_c - \lambda) & \lambda \geq \lambda_c. \end{cases}$$

The fact that the measure concentrates on (approximately) the same event for $\lambda \geq \lambda_c$ is reflected in the linearity of the function ζ^*.

Remark While there is a "phase transition" in the expectation at $\lambda = \lambda_c$ there is no corresponding transition as $\lambda \to -\infty$. Using either the Beurling projection theorem (see, e.g., [5]) or (19) it can be seen that $|h_{t^2}'(i)| \geq c\, t^{-1}$. This value is obtained if U_t is constant and $Z_t(i)$ goes deterministically upward. Since $\zeta'(\lambda) < 1$ for all λ, the weighted measure never concentrates on these paths.

Example When studying the Hausdorff dimension of an SLE path, one is led to study $|h'_{t^2}(i)|^d$, where

$$d = 1 + \frac{1}{4a} = 1 + \frac{\kappa}{8}.$$

For $\kappa < 8$, this turns out to be the Hausdorff dimension of the paths. Note that if $r = 1$, then $\lambda = d$. For $\kappa < 8$, we have $r < r_c, \lambda < \lambda_c$, and

$$\zeta^*(\lambda) = \zeta(\lambda) = 1 - \frac{1}{4a} = 2 - d.$$

Example Second moment arguments for Hausdorff dimension lead to studying the $2d$-moment, i.e.,

$$\lambda = 2 + \frac{1}{2a}.$$

There are two regimes to consider.

- $5/4 \leq a < \infty$. In this range $2d \leq \lambda_c$. We have

$$r = r(2d) = 2a + 1 - 2a\sqrt{1 - \frac{1}{a} - \frac{1}{4a^2}},$$

$$\zeta^*(2d) = \zeta(2d) = 1 + \sqrt{1 - \frac{1}{a} - \frac{1}{4a^2}}.$$

- $1/4 < a \leq 5/4$. In this range

$$2d = \lambda_c + \left[1 + \frac{5}{16a} - a\right],$$

where the term in brackets is nonnegative, and hence

$$\zeta^*(2d) = \zeta(\lambda_c) - \left[1 + \frac{5}{16a} - a\right] = 2a - \frac{3}{8a} - 1.$$

Note that $\zeta^*(2d) = 0$ if $a = 3/4$ and $\zeta^*(2d) < 0$ for $1/4 < a < 3/4$.

5. Change of time

In this section we fix $z \in \mathbb{H}$ and write X_t, Y_t, Z_t, S_t for $X_t(z), Y_t(z), Z_t(z), S_t(z)$ although it is important to remember that these quantities depend on the starting point z. Since Y_t is differentiable and strictly increasing, we can find a new parametrization (depending on z) such that $\log Y_t$ grows linearly. To be more precise, let

$$\sigma(t) = \inf\{s : Y_s = e^{at}\}, \quad \hat{Y}_t = Y_{\sigma(t)} = e^{at}, \quad \hat{X}_t = X_{\sigma(t)}, \quad \hat{Z}_t = Z_{\sigma(t)},$$

$$K_t = e^{-at}\,\hat{X}_t, \quad \hat{S}_t = S_{\sigma(t)} = \frac{e^{at}}{|\hat{Z}_t|} = (K_t^2 + 1)^{-1/2}, \quad \hat{h}_t = h_{\sigma(t)}.$$

Lemma 5.1. *If* $z = x + e^{at_0} i$,

$$\partial_t \sigma(t) = |\hat{Z}_t|^2, \quad \sigma(t) = \int_{t_0}^t |\hat{Z}_s|^2\, ds = \int_{t_0}^t e^{2as}\,(K_s^2 + 1)\, ds. \tag{17}$$

Proof. Since $\partial_t \hat{Y}_t = a\,\hat{Y}_t$, (3) and the chain rule imply

$$a\,\hat{Y}_t = \frac{a\,\hat{Y}_t}{|\hat{Z}_t|^2}\,[\partial_t \sigma(t)]\,.$$

□

Using (3) we get

$$d\hat{X}_t = -a\,\hat{X}_t\,dt + |\hat{Z}_t|\,d\tilde{B}_t,$$

$$dK_t = -2a\,K_t\,dt + \sqrt{K_t^2+1}\,d\tilde{B}_t, \tag{18}$$

where $\tilde{B}_t$ denotes the standard Brownian motion

$$\tilde{B}_t = \int_0^{\sigma(t)} \frac{1}{|Z_s|}\,dB_s.$$

From (5) and (17), we see that

$$\partial_t|\hat{h}_t'(z)| = |h_{\sigma(t)}'(z)|\,|Z_t|^2 = |h_{\sigma(t)}'(z)|\,\frac{a\,(\hat{X}_t^2 - \hat{Y}_t^2)}{|\hat{Z}_t|^2} = a\,|\hat{h}_t'(z)|\,\left[1 - 2\,\hat{S}_t^2\right],$$

and hence,

$$|\hat{h}_t'(x+i)| = \exp\left\{a\int_0^t [1 - 2\,\hat{S}_s^2]\,ds\right\}.$$

Note that this implies

$$e^{-at} \le |\hat{h}_t'(x+i)| \le e^{at}. \tag{19}$$

6. The SDE (18)

Let

$$q = 2a + \frac{1}{2} - r, \tag{20}$$

and note that if $r < r_c$, then $q > 0$. We will study the equation (18) which we write as

$$dK_t = \left(\frac{1}{2} - q - r\right)\,K_t\,dt + \sqrt{K_t^2+1}\,d\tilde{B}_t. \tag{21}$$

For this section, we consider q, r as the given parameters, and we define a by (20). A simple application of Itô's formula gives the following lemma.

Lemma 6.1. *Suppose J_t satisfies*

$$dJ_t = -\,(q+r)\,\tanh J_t\,dt + d\tilde{B}_t. \tag{22}$$

Then $K_t = \sinh J_t$ satisfies (21).

Let

$$L_t = t - \int_0^t \frac{2\,ds}{K_s^2+1} = t - \int_0^t \frac{2\,ds}{\cosh^2 J_s}.$$

Note that $-t \le L_t \le t$ and

$$\partial_t[e^{L_t}] = e^{L_t}\,\left[1 - \frac{2}{\cosh^2 J_t}\right]. \tag{23}$$

As in (17), we let

$$\sigma(t) = \int_0^t e^{2as}[K_s^2+1]\,ds = \int_0^t e^{2as}\cosh^2 J_s\,ds \ge \int_0^t e^{2as}\,ds = \frac{1}{2a}\,\left[e^{2at}-1\right]. \tag{24}$$

Although all the quantities above are defined in terms of J_t, it is useful to note that in the notation of the previous section,

$$|\hat{h}_t'(z)| = e^{aL_t}, \quad \hat{Y}_t = e^{at}, \quad \frac{\hat{X}_t}{\hat{Y}_t} = K_t, \quad [\hat{S}_t]^{-1} = \cosh J_t.$$

We let

$$N_t = e^{\nu L_t}\,e^{\xi t}\,[\cosh J_t]^r, \tag{25}$$

where

$$\nu = a\lambda = r\left(a + \frac{1}{2}\right) - \frac{r^2}{4} = r\left(\frac{q}{2} + \frac{1}{4}\right) + \frac{r^2}{4},$$

$$\xi = a\zeta = a\lambda - \frac{r}{2} = ar - \frac{r^2}{2} = r\left(\frac{q}{2} - \frac{1}{4}\right) + \frac{r^2}{4}.$$

In the notation of the previous section, $N_t = M_{\sigma(t)}$. We have written N_t and defined the exponents ν, ξ so they depend only on r, q and not on a. Note that

$$r(\nu) = -\left(q + \frac{1}{2}\right) + \sqrt{\left(q + \frac{1}{2}\right)^2 + 4\nu},$$

$$\xi(\nu) = \nu + \left(\frac{q}{2} + \frac{1}{4}\right) - \frac{1}{2}\sqrt{\left(q + \frac{1}{2}\right)^2 + 4\nu},$$

Since N_t is M_t sampled at an increasing family of stopping times, the next proposition is no surprise.

Proposition 6.2. *Suppose $r \in \mathbb{R}$ and ξ, ν are defined as in Proposition 2.1. Then N_t as defined in (25) is a positive martingale satisfying*

$$dN_t = N_t\,r\,[\tanh J_t]\,d\tilde{B}_t. \tag{26}$$

In particular,

$$\mathbb{E}^x[e^{\nu L_t}\,[\cosh J_t]^r] = e^{-\xi t}\,\mathbb{E}^x[N_t] = [\cosh x]^r\,e^{-\xi t}.$$

Proof. Itô's formula gives (26). If we use Girsanov's theorem, the weighted paths satisfy

$$dJ_t = -q\,[\tanh J_t]\,dt + dW_t, \tag{27}$$

where

$$W_t = \tilde{B}_t - r\int_0^t \tanh J_s\,ds,$$

is a standard Brownian motion in the new measure. Since $|\tanh| \leq 1$, it is straightforward to show that this equation does not have explosion in finite time, and hence we can see that N_t is actually a martingale. □

7. The SDE (27) for $q > 0$

We now focus our discussion on the equation (27) which has only one parameter q that we will assume is positive. Recall from (20) that this corresponds to $r < r_c$.

Lemma 7.1. *Suppose $q > 0$ and J_t satisfies (27).*

- *J_t is a positive recurrent process with invariant density*

$$v_q(x) = \frac{C_q}{\cosh^{2q} x}, \quad -\infty < x < \infty,$$

where

$$C_q = \frac{\Gamma(q+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(q)}.$$

Moreover,

$$\int_{-\infty}^{\infty}\left[1 - \frac{2}{\cosh^2 x}\right] v_q(x)\,dx = \mu := \frac{1-2q}{1+2q}.$$

- *If*

$$F(x) = F_q(x) = \int_0^x [\cosh y]^{2q}\,dy,$$

then $F(J_t)$ is a local martingale.
- *There exists $c = c_q$ such that for $0 \leq y < x, k \geq 0$,*

$$\mathbb{P}^y\{|J_t| \geq x \text{ for some } k \leq t \leq k+1\} \leq c\left(\frac{\cosh y}{\cosh x}\right)^{2q}. \tag{28}$$

Proof. The computation of the invariant density is standard (but see the appendix for a derivation). Since

$$F''(x) = 2q\,[\tanh x]\,F'(x),$$

Itô's formula shows that $F(J_t)$ is a local martingale. Note that as $x \to \infty$, $F(x) \sim (2q)^{-1}\,[\cosh x]^{2q}$.

It suffices to prove (28) for $x \geq 1$. Assume first that $y = 0$. A coupling argument shows that the probability in (28) is bounded above by the corresponding probability where J_0 has the law of the invariant distribution. Suppose J_0 has density v_q, and let

$$Y = Y_{k,x} = \int_k^{k+2} 1\{|J_t| \geq x-1\}\, dt.$$

Then

$$\mathbb{E}[Y] = 2\int_{|y|\geq x-1} v_q(y)\, dy \leq \frac{c}{\cosh^{2q} x}.$$

Since the drift in (27) is bounded, we can see from the strong Markov property that for some $\delta > 0$,

$$\mathbb{E}[Y \mid |J_t| \geq x \text{ for some } k \leq t \leq k+1] \geq \delta,$$

and hence

$$\mathbb{P}\{|J_t| \geq x \text{ for some } k \leq t \leq k+1\} \leq \delta^{-1}\, \mathbb{E}[Y].$$

If $0 < y < x$, let $T = T(0,x)$ be the first time that $J_t \in \{0,x\}$. Using the strong Markov property we see that

$$\mathbb{P}^y\{|J_t| \geq x \text{ for some } k \leq t \leq k+1\ ;\ J_T = 0\} \leq \frac{c}{[\cosh x]^{2q}}.$$

Hence,

$$\mathbb{P}^y\{|J_t| \geq x \text{ for some } k \leq t \leq k+1\} \leq \frac{c}{[\cosh x]^{2q}} + \mathbb{P}^y\{J_T = x\}.$$

Applying the optional sampling theorem to the martingale $F(J_{T\wedge t})$, we see that

$$\mathbb{P}^y\{J_T = x\} = \frac{F(y)}{F(x)} \leq c\left(\frac{\cosh y}{\cosh x}\right)^{2q}.$$

□

Remark For $y = 0$, the estimate (28) is sharp for large t. For $y > 0$, the estimate is not sharp for large t; in fact, for large t one gets the same estimate as for $y = 0$. However, how large t needs to be depends on y, x and (28) is the best one can do if one wants a uniform estimate for all t, y, x.

As in the previous section, we let

$$L_t = \int_0^t \left[1 - \frac{2}{\cosh^2 J_s}\right] ds.$$

It follows from the previous lemma that as $t \to \infty$,

$$t^{-1}\, \mathbb{E}[L_t] \sim \int_{-\infty}^{\infty} \left[1 - \frac{2}{\cosh^2 x}\right] v_q(x)\, dx = \mu = \frac{1-2q}{1+2q}.$$

In fact, we claim that

$$L_t = \mu\, t + O(t^{1/2}).$$

In Section 10 we will need large (or, as sometimes called, moderate) deviations, i.e., probabilities that $|L_t - \mu t|$ is much larger than $t^{1/2}$. Let

$$\overline{L}_t = L_t - \mu t.$$

The standard way to obtain large deviation results is to obtain a bound on an exponential moment. The martingales allow us to do that rather easily here. The value b in the next proposition is not special; in fact, a small modification of the proof shows that the expectation is bounded for all b. However, we only need to use one value, and the proof simplifies slightly by restricting to this case. This proposition should be compared to Proposition 3.1.

Proposition 7.2. *Suppose J_t satisfies (27) with $q > 0$. Then there exists $c < \infty$ such that for all $0 \leq s < t$,*

$$\mathbb{E}\left[\exp\left\{\frac{b\,|\overline{L}_t - \overline{L}_s|}{\sqrt{t-s}}\right\}\right] \leq c, \tag{29}$$

where $b = 2q+1$.

Proof. Since $|\overline{L}_t - \overline{L}_s| \leq 2(t-s)$, it suffices to prove the bound for $t-s$ sufficiently large. For this proof, we will assume that $t-s \geq (4/q)^2$, i.e., $4/\sqrt{t-s} \leq q$.

We will first show that there is a c such that

$$\mathbb{E}\left[\exp\left\{\pm b\,\frac{(\overline{L}_t - \overline{L}_s)}{\sqrt{t-s}}\right\}\ [\cosh J_t]^{\pm\frac{4}{\sqrt{t-s}}}\right] \leq c. \tag{30}$$

For $\beta \in \mathbb{R}$, let

$$\delta = \delta(\beta) = \beta\left(\frac{q}{2} + \frac{1}{4}\right) - \frac{\beta^2}{4}, \quad \rho = \beta\left(\frac{q}{2} - \frac{1}{4}\right) - \frac{\beta^2}{4}.$$

Using Proposition 6.2 with (q,r) replaced with $(q-\beta,\beta)$, we see that

$$O_t := e^{\delta L_t}\, e^{\rho t}\, [\cosh J_t]^{\beta} = e^{\delta \overline{L}_t}\, e^{(\rho+\delta\mu)t}\, [\cosh J_t]^{\beta},$$

is a martingale. In particular, for $s < t$,

$$\mathbb{E}\left[e^{\delta(\overline{L}_t - \overline{L}_s)}\, e^{(\rho+\delta\mu)(t-s)}\, (\cosh J_t)^{\beta}\right] = \mathbb{E}\left[(\cosh J_s)^{\beta}\right]. \tag{31}$$

If $\beta \leq q$, then we can use (28) to see there is a $c < \infty$ such that

$$\mathbb{E}\left[(\cosh J_s)^{\beta}\right] < c.$$

If we apply this bound with $\beta = \pm 4/\sqrt{t-s} \leq q$, then

$$\delta = \pm\frac{2q+1}{\sqrt{t-s}} + \frac{1}{4(t-s)},$$

$$\rho = \pm\frac{2q-1}{\sqrt{t-s}} - \frac{1}{4(t-s)} = -\mu\delta + O\left(\frac{1}{t-s}\right).$$

Hence,

$$\exp\left\{\pm b\,\frac{\overline{L}_t - \overline{L}_s}{\sqrt{t-s}}\right\} \leq c\, e^{\delta(\overline{L}_t - \overline{L}_s)}\, e^{(\rho+\delta\mu)(t-s)},$$

and (30) follows from (31).

Clearly, (30) implies

$$\mathbb{E}\left[\exp\left\{b\,\frac{\overline{L}_t - \overline{L}_s}{\sqrt{t-s}}\right\}\right] < \infty.$$

To prove (29), we also need to show the corresponding inequality with b replaced with $-b$. By choosing y sufficiently large, we can see from (28) that

$$\mathbb{P}\{|J_t| \geq y\sqrt{t-s}\} \leq e^{-2b\sqrt{t-s}}.$$

Since $|\overline{L}_t - \overline{L}_s| \leq 2\sqrt{t-s}$, this implies

$$\mathbb{E}\left[\exp\left\{-b\,\frac{\overline{L}_t - \overline{L}_s}{\sqrt{t-s}}\right\}\;;\;|J_t| \geq y\sqrt{t}\right] \leq c.$$

However, if $|J_t| \leq y\sqrt{t}$, then $[\cosh J_t]^{-4/\sqrt{t}}$ is bounded below and hence (30) implies

$$\mathbb{E}\left[\exp\left\{-b\,\frac{\overline{L}_t - \overline{L}_s}{\sqrt{t-s}}\right\}\;;\;|J_t| \leq y\sqrt{t}\right] \leq c.$$

□

The next proposition makes precise the idea that $J_s = O(1)$ and

$$L_s = \mu\, s + O(\sqrt{s}), \quad L_t - L_s = \mu\,(t-s) + O(\sqrt{t-s}).$$

It is phrased in a way that is used in Section 10. In particular, it considers an event where the error is of order a constant times $\sqrt{s}$ or $\sqrt{t-s}$ when s is small but allows a somewhat larger error for other values of s. Let

$$F(s,t) = 2 + \min\{s, t-s\}.$$

Proposition 7.3. *Suppose J_t satisfies (27) with $q > 0$. For each $u, t > 0$, let $E_{t,u}$ be the event that the following holds for all $0 \leq s \leq t$:*

$$|J_s| \leq u\,\log F(s,t),$$

$$|\overline{L}_s| \leq u\,\sqrt{s}\,\log(s+2),$$

$$|\overline{L}_t - \overline{L}_s| \leq u\,\sqrt{t-s}\,\log(t-s+2),$$

Then

$$\lim_{u\to\infty}\inf_{t>0}\mathbb{P}(E_{t,u}) = 1.$$

Remark For future reference we note that on the event $E_{u,t}$,

$$\cosh^2 J_s \leq e^{2|J_s|} \leq F(s,t)^{2u}.$$

In particular, there exists C_u such that for all $t_1 < t$,

$$\sigma(t_1) := \int_0^{t_1} e^{2as}\,[\cosh^2 J_s]\,ds \leq C_u\,F(t_1,t)^u\,e^{2at_1}. \tag{32}$$

Proof. For ease we assume that t is a positive integer. From (28), we see that there is a c_1 such that

$$\begin{aligned}\mathbb{P}\{|J_s| \geq u\,\log(s+2) &\text{ for some } k \leq s \leq k+1\}\\ &\leq \mathbb{P}\{|J_s| \geq u\,\log(k+2) \text{ for some } k \leq s \leq k+1\}\\ &\leq c_1\,(k+2)^{-2qu}.\end{aligned}$$

Therefore,

$$\mathbb{P}\{|J_s| \geq u \, \log(s+2) \text{ for some } s \geq 0\} \leq c_1 \sum_{k=0}^{\infty} (k+2)^{-2qu}, \tag{33}$$

and the right-hand side goes to zero as $u \to \infty$. A similar argument shows that

$$\mathbb{P}\{|J_s| \geq u \, \log(t-s+2) \text{ for some } 0 \leq s \leq t\} \leq c_1 \sum_{k=0}^{\infty} (k+2)^{-2qu}. \tag{34}$$

Since $|\overline{L}_t - \overline{L}_s| \leq 2(t-s)$,

$$\mathbb{P}\{|\overline{L}_s| \geq u \sqrt{s} \, \log(s+2) \text{ for some } k \leq s \leq k+1\} \leq$$
$$\mathbb{P}\{|\overline{L}_k| \geq u \sqrt{k} \, \log(k+2) - 2\}.$$

Using (29) and Chebyshev's inequality, we see that the righthand side is bounded by a constant times $(k+2)^{-bu}$. Similarly,

$$\mathbb{P}\{|\overline{L}_t - \overline{L}_s| \geq u \sqrt{t-s} \, \log(t-s+2) \text{ for some } k \leq s \leq k+1\} \leq c\,(k+2)^{-bu}.$$

The argument proceeds as in (33) and (34).

□

8. Lower bound

Having analyzed the one-variable equation for J_t we return to the original problem. Let $r < 2a + \frac{1}{2}$ and $q = 2a + \frac{1}{2} - r > 0$. We fix a u as in Proposition 7.3 such that for all t,

$$\mathbb{P}_*(E_{t,u}) \geq \frac{1}{2},$$

and we write just E_t for $E_{t,u}$. Here we write $\mathbb{P}_*$ for the probability measure to distinguish it from the probability measure under which the expectation $\mathbb{E}$ below is defined. Let $C = C_u$ be a constant such that (32) holds. We fix u and allow all constants in this section to depend on u. We let $z = i$ and write $X_t, Y_t, M_t, \ldots$ for $X_t(i), Y_t(i), M_t(i), \ldots$

Let M_t be is the martingale associated to r. which we can write as M_t as

$$M_t = |h_t'(i)|^\lambda \, Y_t^\zeta \, [\cosh \tilde{J}_t]^r,$$

where $\tilde{J}_t$ is defined by

$$\sinh \tilde{J}_t = \frac{X_t}{Y_t}.$$

Note that $J_t = \tilde{J}_{\sigma(t)}$. then Proposition 7.3 can be rewritten as

$$\begin{aligned} 1 &\geq \mathbb{E}\left[|h_{\sigma(t)}'(i)|^\lambda \, Y_{\sigma(t)}^\zeta \, (\cosh \tilde{J}_{\sigma(t)})^r \, 1_{E_t}\right] \\ &= e^{a\zeta t} \mathbb{E}\left[|h_{\sigma(t)}'(i)|^\lambda \, (\cosh \tilde{J}_{\sigma(t)})^r \, 1_{E_t}\right] \geq \frac{1}{2}. \end{aligned}$$

On the event E_u, J_t is uniformly bounded. Therefore, this implies

$$\mathbb{E}\left[|h'_{\sigma(t)}(i)|^\lambda \, 1_{E_t}\right] \asymp e^{-a\zeta t}.$$

We now derive some bounds that hold on the event E_t. Recall that

$$|h'_{\sigma(t)}(i)| = e^{aL_t} = e^{at\mu}\, e^{a\overline{L}_t}.$$

The proposition implies for all $0 \le s \le t$,

$$\cosh J_s \le F(s,t)^u,$$

$$\exp\left\{-u\sqrt{s}\log(s+2)\right\} \le e^{-as\mu}\,|h'_{\sigma(s)}(i)| \le \exp\left\{u\sqrt{s}\log(s+2)\right\},$$

$$\begin{aligned}\exp\left\{-u\sqrt{t-s}\log(t-s+2)\right\} &\le e^{-a(t-s)\mu}\,\frac{|h'_{\sigma(t)}(i)|}{|h'_{\sigma(s)}(i)|}\\ &\le \exp\left\{u\sqrt{t-s}\log(t-s+2)\right\}.\end{aligned}$$

The Loewner equation implies that

$$\sigma(s) \ge \frac{e^{2as}-1}{2a},$$

and the proposition gives the upper bound

$$\sigma(s) \le \int_0^s e^{2av}\cosh^2 J_v\,dv \le \int_0^s e^{2av}\,(v+2)^u\,J_v\,dv \le c\,(s+2)^{2u}\,e^{2as},$$

$$\sigma(s) \le \int_0^s e^{2av}\cosh^2 J_v\,dv \le \int_0^s e^{2av}\,(t-v+2)^{2u}\,J_v\,dv \le c\,(t-s+2)^u\,e^{2as},$$

i.e.,

$$\sigma(s) \le c\,F(s,t)^{2u}\,e^{2as}.$$

By inverting this, we get

$$s - c\min\{\log(s+2), \log(t-s+2)\} \le \frac{1}{a}\log Y_{e^{2as}} \le s+c, \quad 0 \le s \le t.$$

This yields

$$\cosh \tilde{J}_{e^{2as}} \le c\,F(s,t)^u$$

$$\exp\left\{-u\sqrt{s}\log(s+2)\right\} \le e^{-as\mu}\,|h'_{e^{2as}}(i)| \le \exp\left\{u\sqrt{s}\log(s+2)\right\},$$

$$\begin{aligned}\exp\left\{-u\sqrt{t-s}\log(t-s+2)\right\} &\le e^{-a(t-s)\mu}\,\frac{|h'_{\sigma(t)}(i)|}{|h'_{e^{2as}}(i)|}\\ &\le \exp\left\{u\sqrt{t-s}\log(t-s+2)\right\}.\end{aligned}$$

Once we have this, we can continue the process from time $\sigma(t)$ to time ce^{2at}. From this, one can deduce the following which is used in Section 10. The statement is rather cumbersome, but it essentially follows from what we have done.

Theorem 8.1. *Suppose* $r < 2a + \frac{1}{2}$ *and* λ, ζ *are defined as in Proposition* 2.1. *Let*

$$\mu = \frac{1-2q}{1+2q} = \frac{2(r-2a)}{1+2a-r}, \qquad F(s,t) = 2 + \min\{s, t-s\}.$$

For each b, u, *let* $A(b,u,t)$ *denote the indicator function of the event that the following holds for* $0 \le s \le t$:

$$b^{-1}\,\frac{e^{2as}}{F(s,t)^u} \le Y_{e^{2as}} \le b\,e^{as},$$

$$\cosh \tilde{J}_{e^{2as}} \le b\,F(s,t)^u,$$

$$b^{-1}\,\exp\left\{-u\sqrt{s}\log(s+2)\right\} \le e^{-as\mu}\,|h'_{e^{2as}}(i)| \le b\,\exp\left\{u\sqrt{s}\log(s+2)\right\},$$

$$b^{-1}\,\exp\left\{-u\sqrt{t-s}\log(t-s+2)\right\}\,|h'_{e^{2as}}(i)| \le$$

$$e^{-a(t-s)\mu}\,|h'_{e^{2at}}(i)| \le b\,\exp\left\{u\sqrt{t-s}\log(t-s+2)\right\}\,|h'_{e^{2as}}(i)|.$$

Then there exist b, u *such that for all* $t > 0$,

$$b^{-1} \le \mathbb{E}\left[|h'_{e^{2at}}(i)|^{\lambda}\,A(b,u,t)\right] \le b.$$

9. An upper bound

Here we prove a theorem which gives an upper bound on some of the moments of $|h'(z)|$ for a range of λ. The dependence on x is probably not optimal. Indeed, as remarked after Proposition 7.1, the estimates for large x used in the proof are not optimal.

Theorem 9.1. *Suppose*

$$0 < r < 6a - 2\sqrt{5a^2 - a}, \quad a \ge \frac{1}{4}, \tag{35}$$

$$0 < r < 2a + \frac{1}{2}, \quad a < \frac{1}{4}, \tag{36}$$

λ, ζ *are defined as in Proposition* 2.1 *and* $q = 2a + \frac{1}{2} - r$. *Then there exists a* $c < \infty$ *such that for all* $x \in \mathbb{R}$, $y > 0$,

$$\mathbb{E}[|h'_{(sy)^2}(xy+iy)|^{\lambda}] = \mathbb{E}[|h'_{s^2}(x+i)|^{\lambda}] \le\ c\,(x^2+1)^{\frac{r+\zeta}{2}}\,[\log(x^2+2)]^{2q}\,(s+1)^{-\zeta}.$$

In particular, if $a > 1/4$, *there exists a* $c < \infty$, *such that for all* $x \in \mathbb{R}$,

$$\mathbb{E}[|h'_{s^2}(x+i)|^{d}] \le\ c\,(x^2+1)^{1-\frac{1}{8a}}\,[\log(x^2+2)]^{4a-1}\,(s+1)^{d-2}.$$

The final assertion follows from the previous one by plugging in $r = 1$ which satisfies (35) for $a > \frac{1}{4}$.

Proof. By scaling we may assume $y = 1$ and without loss of generality, we assume $x = e^{al} \geq 0$. If $s \leq 1$, the Loewner equation implies that $|h'_{s^2}(x+i)| \asymp 1$, so we will assume $s = e^{at} \geq 1$. We write $X_s, Y_s, \ldots$ for $X_s(x+i), Y_s(x+i), \ldots$.

Consider the martingale

$$M_s = M_{s,r}(x+i) = |h'_s(x+i)|^\lambda \, Y_s^\zeta \, S_s^{-r}.$$

The conditions (35) and (36) imply that (36) holds for all a; $q > 0$; $r < 4a$; and

$$-2q < \zeta - 2q = r - \frac{r^2}{4a} - 2q < 0. \tag{37}$$

Recall that

$$\sigma(s) = \inf\{u : Y_u = e^{as}\},$$

and let $\hat{M}_s = M_{\sigma(s)}$. Let $\tau = \tau_t$ be the minimum of t and the smallest s such that

$$\hat{S}_s \leq (t - s + 1) \, e^{-a(t-s)}.$$

Let $\rho = \sigma(\tau)$ so that $S_\rho = \hat{S}_\tau$. (Note that τ is the time in the new parametrizaton, and $\rho = \sigma(\tau)$ is the corresponding amount of time in the original parameterization. If one considers curves modulo reparametrization, then ρ and τ represent the same stopping "time".) Note that

$$\rho = \sigma(\tau) = \int_0^\tau e^{2as} \, \hat{S}_s^{-2} \, ds \leq \int_0^t \frac{e^{2at}}{(t-s+1)^2} \, ds \leq e^{2at}.$$

For positive integer k, let $A_k = A_{k,t}$ be the event $\{t - k < \tau \leq t - k + 1\}$. Since M_t is a martingale, $\tau \leq e^{2at}$, and the event A_k depends only on $M_s, 0 \leq s \leq \tau$, the optional sampling theorem gives

$$\mathbb{E}[M_{e^{2at}} \, 1_{A_k}] = \mathbb{E}[M_\rho \, 1_{A_k}] = \mathbb{E}[\hat{M}_\tau \, 1_{A_k}].$$

Since Y_t increases with t, we know that on the event A_k,

$$Y_{e^{2at}} \geq Y_\rho \geq e^{at} \, e^{-ak}, \qquad S_\rho^2 \asymp e^{-2ak} \, k^2.$$

The Girsanov theorem implies that

$$\mathbb{E}[\hat{M}_\tau \, 1_{A_k}] = M_0 \, \mathbf{Q}(A_k) \leq c \, e^{alr} \, \mathbf{Q}(A_k),$$

where $\mathbf{Q}$ denotes the measure obtained by weighting by the martingale $\hat{M}$. From (28) we know that

$$\mathbf{Q}(A_k) \leq c \, e^{2aq(l-k)} \, k^{2q}.$$

Therefore, if we write $s = e^{at}$,

$$\begin{aligned} s^\zeta \, \mathbb{E}\left[|h'_{s^2}(x+i)|^\lambda \, 1_{A_k}\right] &\leq c \, e^{ak\zeta} \, \mathbb{E}\left[|h'_{s^2}(x+i)|^\lambda \, Y_{s^2} \, 1_{A_k}\right] \\ &\leq c \, e^{ak\zeta} \, \mathbb{E}[M_{s^2} \, 1_{A_k}] \\ &= c \, e^{ak\zeta} \, \mathbb{E}[M_\rho \, 1_{A_k}] \\ &\leq c \, e^{ak\zeta} \, e^{alr} \, \mathbf{Q}(A_k). \end{aligned}$$

Therefore,

$$s^{\zeta}\,\mathbb{E}\left[|h'_{s^2}(x+i)|^{\lambda}\right] \leq c\,e^{arl}\left[\sum_{k\leq l} e^{ak\zeta}\,\mathbf{Q}(A_k) + e^{al\zeta}\sum_{k>l} k^{2q}\,e^{a(k-l)(\zeta-2q)}\right].$$

Using (37), we can sum over k to get

$$s^{\zeta}\,\mathbb{E}\left[|h'_{s^2}(x+i)|^{\lambda}\right] \leq c\,e^{arl}\,l^{2q}\,e^{a\zeta l}.$$

□

10. Hausdorff dimension

We will prove that for $\kappa < 8$, the Hausdorff dimension of the paths is $d = 1+\frac{\kappa}{8}$. We will only prove the lower bound which is the hard direction; the upper bound was proved by Rohde and Schramm [10] and we sketch the proof in the next paragraph. Since Hausdorff dimension is preserved under conformal maps, it is easy to use the independence of the increments of Brownian motion to conclude that there is a d_* such that with probability one $\dim_h[\gamma[t_1,t_2]] = d_*$ for all $t_1 < t_2$. Using this and the upper bound, we can see that it suffices to prove that for all $\alpha < d$,

$$\mathbb{P}\{\dim_h(\gamma[1,2]) \geq \alpha\} > 0. \tag{38}$$

The computation (and rigorous upper bound) of the dimension was done by Rohde and Schramm who first noted that

$$M_t = M_t(z) = \Upsilon_t^{d-2}\,S_t^{4a-1}$$

is a local martingale, where

$$\Upsilon_t = \frac{\mathrm{Im} g_t(z)}{|g_t'(z)|}, \qquad S_t = [\sin\arg(g_t(z)-V_t)].$$

The Koebe (1/4)-theorem (see Section 10.4) shows that

$$\frac{1}{4}\,\Upsilon_t \leq \mathrm{dist}\,(0,\gamma[0,t]\cap\mathbb{R}) \leq 4\,\Upsilon_t.$$

If T_ϵ is the first time that $\Upsilon_t \leq \epsilon$, then the optional sampling theorem can be used to see that

$$M_0(z) = \mathbb{E}[M_{T_\epsilon}; T_\epsilon < \infty] = \epsilon^{d-2}\,\mathbb{E}[S_{T_\epsilon}^{4a-1}; T_\epsilon < \infty].$$

By using the Girsanov theorem [6], one can show that there is a c_* such that

$$\mathbb{E}[S_{T_\epsilon}^{4a-1}; T_\epsilon < \infty] \sim c_*\,\mathbb{P}\{T_\epsilon < \infty\}.$$

which shows that

$$\mathbb{P}\{\Upsilon_\infty \leq \epsilon\} \sim c_*^{-1}\,\epsilon^{2-d}\,M_0(z).$$

In particular,

$$\mathbb{P}\{\mathrm{dist}[\gamma(0,\infty),z] \leq \epsilon\} \asymp M_0(z)\,\epsilon^{2-d}.$$

From this the upper bound for the dimension follows easily.

The lower bound follows from standard techniques provided that one has a "two-point" estmate

$$\mathbb{P}\{\mathrm{dist}[\gamma(0,\infty),z]\le\epsilon,\mathrm{dist}[\gamma(0,\infty),w]\ge\epsilon\}\asymp\epsilon^{2-d}\left(\frac{|z-w|}{\epsilon}\right)^{2-d}.$$

This was successfully established by Beffara [1] although the argument is somewhat complicated.

We take a different approach to proving the lower bound by using the reverse Loewner flow. As in Beffara's approach, we construct a measure on the curve that is in some sense a d-dimensionsal measure and use a version of Frostman's lemma.

10.1. A version of Frostman's lemma

The main tool for proving lower bounds for Hausdorff dimension is Frostman's lemma (see [3, Theorem 4.13]), a version of which we recall here: if $A\subset\mathbb{R}^m$ is compact and μ is a Borel measure with $\mu(\mathbb{R}^m\setminus A)=0$, $\mu(A)>0$, and

$$\mathcal{E}_\alpha(\mu):=\int\int\frac{\mu(dx)\,\mu(dy)}{|x-y|^\alpha}<\infty,\tag{39}$$

then the Hausdorff-α measure of A is infinite. In particular, $\dim_h(A)\ge\alpha$. The following two lemmas summarize a standard technique for proving lower bounds of dimensions of random sets.

Lemma 10.1. *Suppose $A\subset\mathbb{R}^m$ is compact and $0<d\le m$. Suppose $c>0$ and $r:(0,d)\to(0,\infty)$. Suppose there exists a decreasing sequence of compact sets A_n with $\cap_n A_n=A$ and a sequence of Borel measures μ_n with $\mu_n(\mathbb{R}^m\setminus A_n)=0$, $\mu(A_n)\ge c$, and $\mathcal{E}_\alpha(\mu_n)\le r(\alpha)$ for $0<\alpha<d$. Then $\dim_h(A)\ge d$.*

Proof. (sketch) There exists a subsequence μ_{n_k} such that μ_{n_k} converges to a measure μ. One needs only check that $\mu(\mathbb{R}^m\setminus A)=0$, $\mu(A)\ge c$ and $\mathcal{E}_\alpha(\mu)\le r(\alpha)$. □

Lemma 10.2. *Suppose $A\subset\mathbb{R}^m$ is a random compact subset of $\mathbb{R}^m$ and $0<d\le m$. Suppose $0<c_1<c_2<\infty$, $r:(0,d)\to(0,\infty)$, and $\delta_n\to 0$. Let $A_n=\{x\in\mathbb{R}^m:\mathrm{dist}(x,A)\le\delta_n\}$. Suppose there exists a sequence of random Borel measures μ_n and a sequence $\delta_n\to 0$ such that the following is true for each n and each $\alpha<d$:*

$$\mathbb{E}\left[\mu_n(A_n)\right]\ge c_1,$$

$$\mathbb{E}\left[\mu_n(A_n)^2\right]\le c_2,$$

$$\mathbb{E}\left[\mathcal{E}_\alpha(\mu_n)\right]\le r(\alpha),$$

$$\mu_n\left(\mathbb{R}^m\setminus A_n\right)=0.$$

Then,

$$\mathbb{P}\left\{\dim_h(A)\ge d\right\}\ge\frac{c_1^2}{c_2}.$$

Proof. Let $q = c_1^2/c_2$. Standard "second moment" arguments (see e.g. [5, Lemma A.15]) show that the first two inequalities imply that for every $p > 0$ there is an $\epsilon > 0$ such that

$$\mathbb{P}\{\mu_n(A_n) \geq \epsilon\} \geq q - p.$$

Since

$$\mathbb{P}\{\mathcal{E}_\alpha(\mu_n) \geq p^{-1}\, r(\alpha)\} \leq \frac{\mathbb{E}[\mathcal{E}_\alpha(\mu_n)]}{p^{-1}\, r(\alpha)} \leq p,$$

it follows that for each n,

$$\mathbb{P}\{\mu_n(A_n) \geq \epsilon, \mathcal{E}_\alpha(\mu_n) \leq p^{-1}\, r(\alpha)\} \geq q - 2p,$$

and hence

$$\mathbb{P}\{\mu_n(A_n) \geq \epsilon, \mathcal{E}_\alpha(\mu_n) \leq p^{-1}\, r(\alpha) \text{ infinitely often}\} \geq q - 2p.$$

On the event on the left-hand side, we have $\dim_h(A) \geq \alpha$ by the previous lemma. Therefore,

$$\mathbb{P}\{\dim_h(A) \geq \alpha\} \geq q - 2p.$$

Since this holds for all $\alpha < d$ and $p > 0$, the result follows. □

The next lemma is similar to many that have appeared before (see [5, A.3]), but the specific formulation may be new. For example, the assumptions (41) and (42) include subpower functions and are not quite as strong as if the functions were replaced by constants.

Definition We will call a function $\phi : [0, \infty) \to (0, \infty)$ a *subpower function* if it is increasing, continuous, and

$$\lim_{x\to\infty} \frac{\log \phi(x)}{\log x} = 0,$$

i.e., ϕ grows slower than x^q for all $q > 0$.

Throughout this paper we will use ϕ, ψ to denote subpower functions. Similarly to the way arbitrary constants are handled, we will allow the particular value of the function to vary from line to line. We will not try to find the optimal subpower function for the results in this paper. We will use the fact that if ϕ, ψ are subpower functions, so are $\phi + \psi, \phi\psi, \phi \wedge \psi, \phi^k$.

Lemma 10.3. *Suppose $\eta : [0, 1] \to \mathbb{R}^m$ is a random curve and*

$$\{F(j, n) : n = 1, 2, \ldots, j = 1, 2, \ldots n\}$$

are nonnegative random variables all defined on the same probability space. Suppose $1 < d \leq m$, and there exist a subpower function ψ, $0 < \xi < 1$, and $c < \infty$ such that the following holds for $n = 1, 2 \ldots$, and $1 \leq j \leq k \leq n$:

$$c^{-1} \leq \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}[F(j, n)] \leq c, \tag{40}$$

$$\mathbb{E}[F(j, n)F(k, n)] \leq \left(\frac{n}{k - j + 1}\right)^{\xi} \psi\left(\frac{n}{k - j + 1}\right), \tag{41}$$

and

$$\left|\eta\left(\frac{j}{n}\right)-\eta\left(\frac{k}{n}\right)\right| \geq \left(\frac{k-j}{n}\right)^{\frac{1-\xi}{d}} \psi\left(\frac{n}{|j-k|+1}\right)^{-1} 1\{F(j,n)F(k,n)>0\}. \tag{42}$$

Then

$$\mathbb{P}\{\dim_h(\eta[0,1]) \geq d\} > 0.$$

Remark The proof constructs a measure supported on the curve. The nth approximation is a sum of measures $\mu_{j,n}$ which are multiples of Lebesgue measure on small discs centered at $\eta(j/n)$. The multiple at $\eta(j/n)$ is chosen so that the total mass $\mu_{j,n}$ is $F(j,n)/n$. In particular, if $F(j,n)=0$, $\mu_{j,n}$ is the zero measure. To bound $\mathbb{E}\left[\mathcal{E}_\alpha(\mu_n)\right]$ we need to show that the measure is sufficiently spread out and (42) gives the necessary assumption. Note the assumption requires the inequality to hold only when $F(j,n)F(k,n)>0$. The assumption implies that if $j<k$ and $\eta(j/n)=\eta(k/n)$ (or are very close), then at most one of $\mu_{j,n}$ and $\mu_{k,n}$ is nonzero.

Proof. We fix ξ,ψ,d and constants in this proof depend on ξ,ψ,m,d. Let $\mu_{j,n}$ denote the (random) measure that is a multiple of Lebesgue measure on the disk of radius $r_n := n^{\frac{\xi-1}{d}}\,\psi(n)/4$ about $\eta(j/n)$ where the multiple is chosen so that $\|\mu_{j,n}\| = n^{-1}\,F(j,n)$. Here $\|\cdot\|$ denotes total mass. Let $\nu_n = \sum_{j=1}^n \mu_{j,n}$. From (40), we see that

$$\mathbb{E}\left[\|\nu_n\|\right] \geq c_1,$$

and from (41) we see that

$$\begin{aligned}\mathbb{E}\left[\|\nu_n\|^2\right] &= \frac{1}{n^2}\sum_{j=1}^n\sum_{k=1}^n \mathbb{E}[F(j,n)\,F(k,n)]\\ &\leq \frac{2}{n}\sum_{k=1}^n \left(\frac{n}{k}\right)^{\xi}\psi\left(\frac{n}{k}\right)\\ &\leq 2\int_0^1 \frac{\psi(1/x)\,dx}{x^\xi} < \infty.\end{aligned}$$

The last inequality uses $\xi<1$.

We will now show that for each $\alpha<d\leq m$, there is a C_α such that

$$\mathbb{E}[\mathcal{E}_\alpha(\nu_n)] = \sum_{j=1}^n\sum_{k=1}^n \mathbb{E}\left[\int\int \frac{\mu_{j,n}(dx)\,\mu_{k,n}(dy)}{|x-y|^\alpha}\right] \leq C_\alpha. \tag{43}$$

We will use the estimate

$$r^{-2m}\int_{|x-x_0|\leq r}\int_{|y-y_0|\leq r} \frac{d^m x\, d^m y}{|x-y|^\alpha} \leq c_\alpha\,\min\{r^{-\alpha}, |x_0-y_0|^{-\alpha}\}.$$

To estimate the terms with $j = k$, note that (41) with $j = k$ gives

$$\mathbb{E}\left[\int\int \frac{\mu_{j,n}(dx)\,\mu_{j,n}(dy)}{|x-y|^\alpha}\right] \leq c\,\frac{\mathbb{E}[F(j,n)^2]}{n^2}\,r_n^{-\alpha} \leq c\,\frac{n^\xi\,\psi(n)}{n^2}\,n^{\frac{(1-\xi)\alpha}{d}}\,\psi(n)^\alpha = o(n^{-1})$$

The last inequality uses $\alpha < d$. Therefore,

$$\mathbb{E}\left[\sum_{j=1}^n \int\int \frac{\mu_{j,n}(dx)\,\mu_{j,n}(dy)}{|x-y|^\alpha}\right] = o(1).$$

For $j < k$, we use the estimate

$$\int\int \frac{\mu_{j,n}(dx)\,\mu_{k,n}(dy)}{|x-y|^\alpha} \leq c\,\frac{F(j,n)\,F(k,n)}{n^2\,|\eta(j/n)-\eta(k/n)|^\alpha}.$$

Note that (40) and (41) combine to show that for each $\alpha < d$ there exist $c < \infty, \delta > 0$ (depending on α) such that

$$\mathbb{E}\left[\frac{F(j,n)\,F(k,n)}{|\eta(j/n)-\eta(k/n)|^\alpha}\right] \leq c\left(\frac{n}{k-j}\right)^{1-\delta}, \quad j < k. \tag{44}$$

Combining this with (44) gives,

$$\sum_{1\leq j<k\leq n} \mathbb{E}\left[\int\int \frac{\mu_{j,n}(dx)\,\mu_{k,n}(dy)}{|x-y|^\alpha}\right] \leq c\,\frac{C_3}{C_4\,n^2} \sum_{1\leq j<k\leq n}\left(\frac{n}{k-j}\right)^{1-\delta} \leq C_\alpha.$$

This gives (43). The lemma now follows from Lemma 10.2. □

Remark The usual form of this lemma chooses η to be the identity function and $d = 1-\xi$ in which case (42) is immediate. The condition (41) is often replaced with a stronger assumption where the subpower function ψ replaced with a constant.

It will be useful for us to give a slight generalization of Lemma 10.3. Lemma 10.3 is the particular case of Corollary 10.4 with $\eta(j,n) = \eta(j/n)$.

Corollary 10.4. *Suppose $\eta : [0,1] \to \mathbb{R}^m$ is a random curve,*

$$\{F(j,n) : n = 1,2,\ldots, j = 1,2,\ldots n\}$$

are nonnegative random variables, and

$$\{\eta(j,n) : n = 1,2,\ldots, j = 1,2,\ldots n\}$$

are $\mathbb{R}^m$-valued random variables all defined on the same probability space. Suppose $0 < d \leq m$, and there exist a subpower function ψ, $0 < \xi < 1$, and $c > 0$ such that the following holds for $n = 1, 2\ldots,$ and $1 \leq j \leq k \leq n$:

$$\frac{1}{c} \leq \frac{1}{n}\sum_{j=1}^n \mathbb{E}[F(j,n)] \leq c, \tag{45}$$

$$\mathbb{E}[F(j,n)F(k,n)] \leq \left(\frac{n}{k-j+1}\right)^{\xi} \psi\left(\frac{n}{k-j+1}\right), \tag{46}$$

$$|\eta(j,n)-\eta(k,n)| \geq \left(\frac{|k-j|}{n}\right)^{\frac{1-\xi}{d}} \psi\left(\frac{n}{|j-k|+1}\right)^{-1} 1\{F(j,n)F(k,n)>0\}, \tag{47}$$

and such that with probability one

$$\lim_{n\to\infty} \max\{\mathrm{dist}[\eta(j,n),\eta[0,1]] : j=1,\ldots,n\} = 0. \tag{48}$$

Then

$$\mathbb{P}\{\dim_h(\eta[0,1]) \geq d\} > 0.$$

In particular, if it is known that there is a d_* *such that* $\mathbb{P}\{\dim_h(\eta[0,1]) = d_*\} = 1$, *then* $d_* \geq d$.

Proof. The proof proceeds as in Proposition 10.3. The measure $\mu_{j,n}$ in the proof is placed on the ball centered at $\eta(j,n)$ rather than $\eta(j/n)$. The key observation is that on the event (48) any subsequential limit of the measures ν_n must be supported on $\eta[0,1]$. □

10.2. Basic idea

In order to prove (38), we will show that the conditions of Corollary 10.4 are satisfied with

$$\xi = d(d-2)+1 = \frac{1}{16a^2} = \frac{\kappa^2}{64} \in (0,1).$$

Let

$$\beta = \frac{1}{4a} - \frac{1}{2} = \frac{\kappa}{8} - \frac{1}{2} = d - \frac{3}{2} = \frac{\xi-1}{d} + \frac{1}{2}.$$

For fixed positive integer n and integers $1 \leq j < k \leq n$, we let

$$S = S_{j,n} = 1 + \frac{j-1}{n}, \quad T = T_{j,k,n} = \frac{k-j}{n}, \quad S+T = 1+\frac{k-1}{n}, \tag{49}$$

$$\hat f_{j,n} = \hat f_{S_{j,n}},$$

$$\eta(t) = \gamma(1+t), \quad \eta(j,n) = \hat f_{j,n}(i/\sqrt{n}).$$

Note that $1 \leq S \leq S+T \leq 2, 0 \leq T \leq 1$. We will define an event $E_{j,n}$ with indicator function $I(j,n)$ and define

$$F(j,n) = n^{1-\frac{d}{2}}\, |\hat f'_{S_{j,n}}(i/\sqrt{n})|^d\, I(j,n).$$

The event $E_{j,n}$ will describe "typical" behavior when we weight the paths by $|\hat f'_{j,n}(i/\sqrt{n})|^d$; in particular, it will satisfy

$$\mathbb{E}\left[|\hat f'_{j,n}(i/\sqrt{n})|^d\, I(j,n)\right] \asymp \mathbb{E}\left[|\hat f'_{j,n}(i/\sqrt{n})|^d\right] \asymp n^{\frac{d}{2}-1}. \tag{50}$$

We define the event in Section 10.5. The typical value of $|\hat{f}'_{j,n}(i/\sqrt{n})|$ when weighted as above is n^β; more precisely, there exists a subpower function ϕ such that on the event $E_{j,n}$,

$$n^\beta\,\phi(n)^{-1} \le |\hat{f}'_{j,n}(i/\sqrt{n})|^d \le n^\beta\,\phi(n).$$

The "one-point" estimate (50) suffices to prove (45) in Corollary 10.4. One needs to prove the other two conditions as well. This is most easily done by considering the reverse time Loewner flow.

10.3. Reverse time

It is known (see, e.g., [10, 4]) that estimates for $\hat{f}'_t$ are often more easily derived by considering the reverse (time) Loewner flow. This is how the one-point estimate is derived. In this subsection, we review the facts about the Loewner equation in reverse time that we will need. Suppose that g_t is the solution to the Loewner equation

$$\partial_t g_t(z) = \frac{a}{g_t(z) - V_t}, \quad g_0(z) = z. \tag{51}$$

Here V_t can be any continuous function, but we will be interested in the case where V_t is a standard Brownian motion.

For fixed $T > 0$, let $F_t^{(T)}, 0 \le t \le T$, denote the solution to the time-reversed Loewner equation

$$\partial_t F_t^{(T)}(z) = -\frac{a}{F_t^{(T)}(z) - V_{T-t}} = \frac{a}{V_{T-t} - F_t^{(T)}(z)}, \quad F_0^{(T)}(z) = z. \tag{52}$$

Note that

$$F_{s+T}^{(S+T)}(z) = F_s^{(S)}(F_T^{(S+T)}(z)), \quad 0 \le s \le S.$$

Lemma 10.5. *If $t \le T$, then $F_t^{(T)} = f_{t,T-t}$. In particular, $F_T^{(T)} = f_T$.*

Proof. Fix T, and let $u_t = F_{T-t}^{(T)}$. Then (52) implies that u_t satisfies

$$\dot{u}_t(z) = \frac{a}{u_t(z) - V_t}, \quad u_T(z) = z.$$

By comparison with (51), we can see that $u_t(z) = g_t(f_T(z))$, and one can check that $g_t \circ f_T = f_{t,T-t}$. □

We will be using the reverse-time flow, to study the behavior of $\hat{f}$ at one or two times. We leave the simple derivation of the next lemma from the previous lemma to the reader. A primary purpose of stating this lemma now is to set the notation for future sections.

Lemma 10.6. *Suppose $S, T > 0$ and $V : [0, S+T] \to \mathbb{R}$ is a continuous function. Suppose $g_t, 0 \le t \le S+T$ is the solution to (51). As before, let $f_t = g_t^{-1}$ and $\hat{f}_t(z) = f_t(z + V_t)$. Let*

$$U_t = V_{S+T-t} - V_{S+T}, \quad 0 \le t \le S+T,$$

$$\tilde{U}_t = V_{S-t} - V_S = U_{T+t} - U_T, \quad 0 \le t \le S,$$

and let $h_t, 0 \le t \le S+T$, $\tilde h_t, 0 \le t \le S$, be the solutions to the reverse-time Loewner equations

$$\partial_t h_t(z) = \frac{a}{U_t - h_t(z)}, \quad h_0(z) = z,$$

$$\partial_t \tilde h_t(z) = \frac{a}{\tilde U_t - \tilde h_t(z)} = \frac{a}{U_{T+t} - U_T - \tilde h_t(z)}, \quad \tilde h_0(z) = z.$$

Then

$$\hat f_S(z) = \tilde h_S(z) - \tilde U_S, \quad \hat f_{S+T}(z) = h_{S+T}(z) - U_{S+T},$$
$$h_{S+T}(z) = \tilde h_S(h_T(z) - U_T) + U_T. \tag{53}$$

In particular,

$$\hat f_S'(w)\,\hat f_{S+T}'(z) = h_T'(z)\,\tilde h_S'(h_T(z) - U_T)\,\tilde h_S'(w),$$
$$\hat f_{S+T}(z) - \hat f_S(w) = \tilde h_S(h_T(z) - U_T) - \tilde h_s(w).$$

Proof. Note that

$$\partial_t[h_t(z) + V_{S+T}] = \frac{a}{V_{S+T-t} - (h_t(z) + V_{S+T})}, \quad h_0(z) + V_{S+T} = z + V_{S+T}.$$

Therefore, $f_{S+T}(z + V_{S+T}) = h_{S+T}(z) + V_{S+T} = h_{S+T}(z) - U_{S+T}$. Note also that

$$\partial_t[\tilde h_t(h_T(z) - U_T) + U_T] = \frac{a}{U_{T+t} - [\tilde h_t(h_T(z) - U_T) + U_T]},$$
$$\tilde h_0(h_T(z) - U_T) + U_T = h_T(z).$$

This gives (53).

□

Remark If V_t is a Brownian motion starting at the origin, then $U_t, \tilde U_t$ are standard Brownian motions starting at the origin. Moreover $\{U_t : 0 \le t \le T\}$ and $\{\tilde U_t : 0 \le t \le S\}$ are independent.

We let $\mathcal{F} = \mathcal{F}_S$ denote the σ-algebra generated by $\{V_s : s \le S\} = \{U_{T+s} - U_T : s \le S\}$ and $\mathcal{G} = \mathcal{G}_{S,T}$ the σ-algebra generated by $\{V_{S+t} - V_S : 0 \le t \le T\} = \{U_t : 0 \le t \le T\}$. Note that $\mathcal{F}$ and $\mathcal{G}$ are independent. We let $\mathcal{F} \vee \mathcal{G}$ be the σ-algebra generated by $\{U_t : 0 \le t \le S+T\}$.

Let us give an idea of the reason for including the $I(j,n)$ term in our definition of $F(j,n)$ for establishing the second bound in Corollary 10.4. Recall that

$$F(j,n) = n^{1-\frac{d}{2}}\,|\hat f_{j,n}'(i/\sqrt{n})|^d\,I(j,n)$$

where $I(j,n)$ is the indicator function of an event on which $|\hat f_{j,n}'(i/\sqrt{n})| \approx n^\beta$. To give "two-point" estimates, we need to consider $\mathbb{E}[F(j,n)F(k,n)]$. Suppose, for example, $j=k$. If we did not include the $I(j,n)$ term, then we would be estimating

$$\mathbb{E}\left[|\hat f_{j,n}'(i/\sqrt{n})|^{2d}\right],$$

which is not of the same order of magnitude as $(\mathbb{E}|\hat f_{j,n}'(i/\sqrt{n})|^d])^2$. Indeed, if we weight paths by $|\hat f_{j,n}'(i/\sqrt{n})|^{2d}$, we do not concentrate on paths with $|\hat f_{j,n}'(i/\sqrt{n})| \approx$

n^{β} but rather on paths with $|\hat{f}'_{j,n}(i/\sqrt{n})| \approx n^{\beta'}$ for some $\beta' > \beta$. However, when we include the $I(j,n)$ term, we can write roughly

$$\mathbb{E}[|\hat{f}'_S(i/\sqrt{n})|^{2d}\, I(j,n))] \approx n^{d\beta}\, \mathbb{E}[|\hat{f}'_S(i/\sqrt{n})|^{d}\, I(j,n)] \approx n^{2d\beta}\, \mathbb{P}(E_{j,n}).$$

10.4. Lemmas about conformal maps

In this section we collect some facts about conformal maps. They are consequences of the Koebe (1/4)-theorem and the distortion theorem (see, e.g., [5, Section 3.2]) which we now recall. Suppose $f : D \to f(D)$ is a conformal transformation, $z \in D$, and $d_{z,D} = \text{dist}(z, \partial D)$. The (1/4)-theorem states that $f(D)$ contains the open ball of radius $d_{z,D}\,|f'(z)|/4$ about $f(z)$ and the distortion theorem implies

$$\frac{1-r}{(1+r)^3}\,|f'(z)| \le |f'(w)| \le \frac{1+r}{(1-r)^3}\,|f'(z)|, \qquad |w-z| \le r d_{z,D}. \tag{54}$$

The immediately corollary of the (1/4)-theorem that we will need is the following: if $h : \mathbb{H} \to h(\mathbb{H})$ is a conformal transformation and $\text{Im}(z) = y_z > y_w = \text{Im}(w)$, then

$$|h(z) - h(w)| \ge \frac{|h'(z)|\,(y_z - y_w)}{4} \tag{55}$$

The form of the distortion theorem is the following lemma. One can values of c, α, but the actual values will not be important to us.

Lemma 10.7. *There exist $c_2, \alpha < \infty$ such that if $h : \mathbb{H} \to h(\mathbb{C})$ is a conformal transformation, $r \ge 1$, and*

$$z, w \in \mathcal{R}(r) := [-r, r] \times [1/r, r] = \{x + iy : -r \le x \le r, \quad 1/r \le y \le r\},$$

then

$$|h'(z)| \le c_2\, r^{\alpha}\, |h'(w)|.$$

Proof. The map $F(z) = (z-i)/(z+i)$ maps $\mathbb{H}$ conformally onto $\mathbb{D}$, so we can write

$$h(z) = f(F(z)),$$

where $f : \mathbb{D} \to h(D)$. We can apply the distortion theorem to f. Details are omitted. □

10.5. Defining the $F(j,n)$

In this section, we will define the event $E_{j,n}$, which will be $\mathcal{F}$-measurable, such that

$$F(j,n) = n^{1-\frac{d}{2}}\, \left|\hat{f}'_S(i/\sqrt{n})\right|^{d}\, 1_{E_{j,n}}. \tag{56}$$

We write

$$E_{j,n} = E_{j,n,1} \cap \cdots \cap E_{j,n,6},$$

for events that we define below. We define $F(k,n)$ similarly; it will be $(\mathcal{F} \vee \mathcal{G})$-measurable.

The event $E_{j,n}$ is defined in terms of the solution of the time-reversed Loewner equation. Let $h_t, \tilde{h}_t$ as in Lemma 10.6. We write $Z_t = h_t(i/\sqrt{n}) - U_t = X_t + iY_t, \tilde{Z}_t = \tilde{h}_t(i/\sqrt{n}) - \tilde{U}_t = \tilde{X}_t + i\tilde{Y}_t$. In particular, for $0 \leq s \leq S$,

$$h_{s+T}(i/\sqrt{n}) = \tilde{h}_s(Z_T) + U_T,$$
$$h'_{s+T}(i/\sqrt{n}) = \tilde{h}'_s(Z_T)\ h'_T(i/\sqrt{n}).$$

Remark Note that the transformation h_T is $\mathcal{G}$-measurable and the transformation $\tilde{h}_S$ is $\mathcal{F}$-measurable. The random variable Z_T is $\mathcal{G}$-measurable. The random variable $\tilde{h}'_S(Z_T)$ is neither $\mathcal{F}$-measurable nor $\mathcal{G}$-measurable. The key to bounding correlations at times S and $S+T$ is handling this random variable.

The six events will depend on a subpower function ϕ_0 to be determined later. Given ϕ_0 we define the following events. Recall that $1 \leq S \leq S + T \leq 2$,

$$E_{k,n,1} = \left\{ Y_t \geq t^{\frac{1}{2}}\, \phi_0(1/t)^{-1} \text{ for } 1/n \leq t \leq S+T \right\}. \tag{57}$$

$$E_{k,n,2} = \left\{ Y_t \geq t^{\frac{1}{2}}\, \phi_0(nt)^{-1} \text{ for } 1/n \leq t \leq S+T \right\}. \tag{58}$$

$$E_{k,n,3} = \left\{ |X_t| \leq t^{\frac{1}{2}}\, \phi_0(1/t) \text{ for } 1/n \leq t \leq S+T \right\},$$

$$E_{k,n,4} = \left\{ |X_t| \leq t^{\frac{1}{2}}\, \phi_0(nt) \text{ for } 1/n \leq t \leq S+T \right\}.$$

$$E_{k,n,5} = \left\{ \frac{(nt)^\beta}{\phi_0(nt)} \leq \left| h'_t(i/\sqrt{n}) \right| \leq (nt)^\beta\, \phi_0(nt) \text{ for } 1/n \leq t \leq S+T \right\}$$

$$E_{k,n,6} = \left\{ \frac{t^{-\beta}}{\phi_0(1/t)} \leq \left| \frac{h'_{S+T}(i/\sqrt{n})}{h'_t(i/\sqrt{n})} \right| \leq t^{-\beta}\, \phi_0(1/t) \text{ for } 1/n \leq t \leq S+T \right\}. \tag{59}$$

$E_{j,n,\cdot}$ are defined in the same way replacing $h_t, U_t, Z_t, S+T$ with $\tilde{h}_t, \tilde{U}_t, \tilde{Z}_t, S$.

Remark What we would really like to do is define is an event of the form

$$Y_t \asymp t^{\frac{1}{2}}, \quad |X_t| \leq c\, t^{\frac{1}{2}}, \quad |h'_t(i/\sqrt{n})| \asymp (nt)^\beta,$$

for all $0 \leq t \leq S + T$. However, this is too strong a restriction if we want the event to have positive probability (in the weighted measure). What we have done is modify this so that quantities are comparable for times near zero and for times near $S+T$ but the error may be larger for times in between (but still bounded by a subpower function).

Theorem 10.8. *There exist c_1, c_2 such that for all $t \geq 1/n$,*

$$\mathbb{E}\left[\left| h'_t(i/\sqrt{n}) \right|^d \right] = \mathbb{E}\left[|h'_{tn}(i)|^d \right] \leq c_2\, (tn)^{\frac{d}{2}-1}. \tag{60}$$

Moreover there exists a power function ϕ_0 such that if $E_{j,n}$ is defined as above, then

$$\mathbb{E}\left[\left| \tilde{h}'_S(i/\sqrt{n}) \right|^d\ I(j,n) \right] \geq c_1\ n^{\frac{d}{2}-1}. \tag{61}$$

Proof. See Theorems 8.1 and 9.1. . □

Remark The equality in (60) follows immediately from scaling. Since $|\tilde{h}'_S(i/\sqrt{n})|^d \approx n^\beta$ on $E_{j,n}$, we can see that

$$n^{\frac{d}{2}-1-d\beta}\,\phi(n)^{-1} \leq \mathbb{P}(E_{j,n}) \leq n^{\frac{d}{2}-1-d\beta}\,\phi(n)^{-1}.$$

Before proceeding further, we note that the Loewner equation gives

$$dX_t = -\frac{a\,X_t}{|Z_t|^2}\,dt - dU_t,$$

which implies

$$|U_t + X_t| \leq \int_0^t \frac{a\,|X_s|\,ds}{|Z_s|^2}.$$

Using this we can show that on the event $E_{k,n}$,

$$|U_t| \leq t^{\frac{1}{2}}\,\phi_0(1/t) \text{ for } 1/n \leq t \leq S+T, \tag{62}$$

$$|U_t| \leq t^{\frac{1}{2}}\,\phi_0(nt) \text{ for } 1/n \leq t \leq S+T,$$

with perhaps a different choice of subpower function ϕ_0. Hence, we may assume that the function ϕ_0 is chosen so that the last two inequalities hold as well.

10.6. Handling the correlations

Theorem 10.8 discusses the function $\tilde{h}_t$ and the corresponding processes $\tilde{X}_t, \tilde{Y}_t$ for a fixed value of S. In this section we assume Theorem 10.8 and show how to verify the hypotheses of Corollary 10.4 for ξ as defined earlier and some subpower function ϕ. Here $F(j,n)$ is defined as in (56). The first hypothesis (45) follows immediately from (60) so we will only need to consider (46)–(48). Throughout this subsection ϕ will denote a subpower function, but its value may change from line to line.

10.6.1. The estimate (46)**.** We first consider $j = k$. Then

$$\mathbb{E}[F(j,n)^2] = n^{2-d}\,\mathbb{E}\left[\left|\tilde{h}'_S(i/\sqrt{n})\right|^{2d}\,I(j,n)\right].$$

On the event $E_{j,n}$ we know that $|\tilde{h}'_S(i/\sqrt{n})| \leq n^\beta\,\phi(n)$. Therefore, using (60),

$$\mathbb{E}[F(j,n)^2] \leq n^{2-d+\beta d}\,\mathbb{E}\left[\left|\tilde{h}'_S(i/\sqrt{n})\right|^{d}\right]\,\phi(n) \leq n^{\xi}\,\phi(n).$$

We now assume $j < k$. We need to give an upper bound for

$$\mathbb{E}[F(j,n)\,F(k,n)] = n^{2-d}\,\mathbb{E}\left[\,1_{E_{j,n}}\,|\tilde{h}'_S(i/\sqrt{n})|^d\,1_{E_{k,n}}\,|h'_{S+T}(i/\sqrt{n})|^d\,\right].$$

Let $\tilde{E}_{k,n} = \tilde{E}_{k,n,1} \cap \tilde{E}_{k,n,3}$ where $\tilde{E}_{k,n,j}$ is defined as $E_{k,n,j}$ except that $1/n \leq t \leq S+T$ is replaced with $1/n \leq t \leq T$. Then $\tilde{E}_{k,n}$ is $\mathcal{G}$-measurable and $E_{k,n} \subset \tilde{E}_{k,n}$. Using (53), we can write

$$h'_{T+S}(i/\sqrt{n}) = h'_T(i/\sqrt{n})\,\tilde{h}'_S(Z_T).$$

Therefore,

$$n^{d-2}\,\mathbb{E}[F(j,n)\,F(k,n)] \leq$$

$$\mathbb{E}\left[\, 1_{E_{j,n}} \, |\tilde{h}_S'(i/\sqrt{n})|^d \, |\tilde{h}_S'(Z_T)|^d \, 1_{\tilde{E}_{k,n}} \, |h_T'(i/\sqrt{n})|^d \,\right]. \tag{63}$$

This is the expectation of a product of five random variables. The first two are $\mathcal{F}$-measurable and the last two are $\mathcal{G}$-measurable. The middle random variable $|\tilde{h}_S'(Z_T)|$ uses information from both σ-algebras: the transformation $\tilde{h}_S$ is $\mathcal{F}$-measurable but it is evaluated at Z_T which is $\mathcal{G}$-measurable.

We claim that it suffices to show that on the event $E_{j,n} \cap \tilde{E}_{k,n}$,

$$\left|\tilde{h}_S'(Z_T)\right|^d \leq T^{-\beta d}\, \phi(1/T), \tag{64}$$

for some subpower function ϕ. Indeed, once we have established this we can see that the expectation in (63) is bounded above by

$$T^{-\beta d}\, \phi(1/T)\, \mathbb{E}\left[|\tilde{h}_S'(i/\sqrt{n})|^d \, \left|h_T'(i/\sqrt{n})\right|^d \,\right],$$

which by independence equals

$$T^{-\beta d}\, \phi(1/T)\, \mathbb{E}\left[|\tilde{h}_S'(i/\sqrt{n})|^d\right] \, \mathbb{E}\left[\, |h_T'(i/\sqrt{n})|^d \,\right].$$

Using (60), we then have that this is bounded by

$$T^{-\beta d}\, \phi(T)\, n^{\frac{d}{2}-1}\, (nT)^{\frac{d}{2}-1} = T^{-\xi}\, \phi(1/T) = \left(\frac{n}{k-j}\right)^{\xi} \phi\left(\frac{n}{k-j}\right).$$

Hence, we only need to establish (64).

Let $w = Z_T$. By the definition of $\tilde{E}_{k,n}$, there is a subpower function ϕ such that $w \in T^{1/2}\, \mathcal{R}(\phi(1/T))$, where $\mathcal{R}(r)$ is as defined in Lemma 10.7. Using the Loewner equation and the fact that the imaginary part increases in the reverse flow, we can see that

$$\tilde{h}_T(w) \in T^{1/2}\, \mathcal{R}(\phi(1/T)), \quad |\tilde{h}_T'(w)| \leq \phi(1/T).$$

for perhaps a different ϕ (we will change the value of ϕ from line to line). By the definition of $E_{j,n}$ and (62), we know that

$$\tilde{Z}_T \in T^{1/2}\, \mathcal{R}(\phi(1/T)), \quad |\tilde{U}_T| \leq T^{1/2}\, \phi(1/T).$$

Hence also,

$$\tilde{h}_T(w) - \tilde{U}_T \in T^{1/2}\, \mathcal{R}(\phi(1/T)).$$

If we define $\tilde{h}_{T,S}$ by $\tilde{h}_S(z_1) = \tilde{h}_{T,S}(\tilde{h}_T(z_1) - \tilde{U}_T)$, then we know by the definition of $E_{j,n}$ that

$$|\tilde{h}_{T,S}'(\tilde{Z}_S)| \leq T^{-\beta}\, \phi(1/T).$$

Hence, by Lemma 10.7,

$$|\tilde{h}_{T,S}(\tilde{h}_T(w) - \tilde{U}_T)| \leq T^{-\beta}\, \phi(1/T).$$

10.6.2. The estimate (47)**.** Assume that $j < k$ and that $E_{j,n}$ and $E_{k,n}$ both occur. Then,

$$\eta(k,n) = \hat{f}_{S+T}(i/\sqrt{n}) = h_{S+T}(1/\sqrt{n}) + U_{T+S} = \tilde{h}_S(Z_T) + U_S.$$

Therefore,

$$\eta(k,n) - \eta(j,n) = \tilde{h}_S(Z_T) - \tilde{h}_S(i/\sqrt{n}).$$

Using the Loewner equation, we see that there is a c_1 such that

$$Y_T - n^{-1/2} > c_1\, Y_T. \tag{65}$$

(Since Y_T is increasing we only need to check this for $T = 1/n$.) Using (55), we get

$$|\eta(k,n) - \eta(j,n)| = \left|\tilde{h}_S(Z_T) - \tilde{h}_S(1/\sqrt{n})\right| \geq \frac{c_1}{4}\, Y_T\, |\tilde{h}'_S(Z_T)|.$$

Hence on our event,

$$|\eta\,(k,n) - \eta\,(j,n)|^d \geq T^{\frac{d}{2}+d\beta}\, \phi\left(\frac{n}{k-j}\right)^{-1} = T^{\xi-1}\, \phi\left(\frac{n}{k-j}\right)^{-1}.$$

Remark Note that we do not expect that last estimate to hold for all k, j, especially for $\kappa > 4$ for which SLE_κ has double points. The restriction to the event $E_{j,n} \cap E_{k,n}$ is a major restriction.

10.6.3. The estimate (48)**.** This estiamte was essentially proved by Rohde and Schramm [10] when they proved existence of the curve. In fact, we can give an argument here. On the event $E_{j,n}$, we have $|\hat{f}'_S(i/\sqrt{n})| \approx n^\beta$. Therefore, using the Koebe (1/4)-Theorem we can conclude on this event that for every $\epsilon > 0$, there is a c such that

$$\text{dist}\left[\hat{f}_S(i/\sqrt{n}), \gamma[0,s] \cap \mathbb{R}\right] \leq c\, n^{\beta+\epsilon}\, n^{-1/2} = c\, n^{d+\epsilon-2}.$$

Since $d < 2$, this goes to 0 for ϵ sufficiently small.

Appendix A. On the Girsanov theorem

In this section, we give a review of the Girsanov theorem. Assume B_t is a standard Brownian motion and $\mathcal{F}_t = \sigma\{B_s : 0 \leq s \leq t\}$ is the filtration generated by the Brownian motion. Suppose that M_t is a nonnegative solution of the stochastic differential equation

$$dM_t = M_t\, A_t\, dB_t, \quad M_0 = x_0 > 0. \tag{66}$$

Here, and throughout, A_t will denote a process such that A_t is $\mathcal{F}_t$-measurable and of the form

$$A_t = \tilde{A}_t\, 1\{T > t\},$$

for some stopping time T and continuous process $\tilde{A}_t$. More precisely, we require (66) to hold only for $M_t > 0$. and if $\tau = \inf\{s : M_s = 0\}$, then $M_t = M_\tau \equiv 0$ for $t \geq \tau$.

Solutions to (66) are not necessarily martingales because it is possible for mass to "escape to infinity in finite time". However solutions to (66) are nonnegative continuous *supermartingales*,

$$\mathbb{E}[M_t \mid \mathcal{F}_s] \leq M_s, \quad s \leq t.$$

In order to show that M_t is a martingale it suffices to show that for each t, $\mathbb{E}[M_t] = x_0$, for then for each $s < t$

$$\mathbb{E}\left[M_t - \mathbb{E}[M_t \mid \mathcal{F}_s]\right] = 0.$$

Since the integrand is nonnegative this implies that $M_t = \mathbb{E}[M_t \mid \mathcal{F}_s]$.

A sufficient condition for a continuous local martingale to be a martingale is boundedness in the following sense: for each t there exists $K_t < \infty$ such that with probability one $|M_s| \leq K_t, 0 \leq s \leq t$.

If M_t is a nonnegative continuous local submartingale, we can obtain a martingale by stopping the process. To be more precise, let $T_n = \inf\{t : M_t \geq n\}$. Then $M_{t \wedge T_n}$ is a martingale. Suppose we know that for each t,

$$\lim_{n\to\infty} \mathbb{E}\left[M_{t\wedge T_n} ; T_n < t\right] = 0. \tag{67}$$

Then the optional sampling theorem and the monotone convergence theorem imply

$$\mathbb{E}[M_t] = \mathbb{E}[M_0].$$

The condition (67) means that mass does not escape to infinity by time t. Therefore, to show that a nonnegative continuous local martingale is a martingale it suffices to prove (67) for each t.

The Girsanov theorem is a theorem about nonnegative *martingales* satisfying (66). Suppose M_t is such a martingale. Let $\mathbf{Q}_t$ be the probability measure on $\mathcal{F}_t$-measurable events given by

$$\mathbf{Q}_t(V) = M_0^{-1}\,\mathbb{E}\left[M_t\, 1_V\right].$$

If $s < t$ and V is $\mathcal{F}_s$-measurable, then it is also $\mathcal{F}_t$-measurable and $\mathbf{Q}_s(V) = \mathbf{Q}_t(V)$. This follows from properties of conditional expectation:

$$\mathbb{E}\left[M_t\, 1_V\right] = \mathbb{E}\left[\mathbb{E}(M_t\, 1_V \mid \mathcal{F}_s)\right] = \mathbb{E}\left[1_V\, \mathbb{E}(M_t \mid \mathcal{F}_s)\right] = \mathbb{E}\left[M_s\, 1_V\right].$$

(Note that this calculation uses the fact that M_t is a martingale.) Hence these measures are consistent and give rise to a measure on $\mathcal{F}_\infty$ which we call $\mathbf{Q}$.

Theorem A.1 (Girsanov). *Under the assumptions above, with respect to* $\mathbf{Q}$,

$$W_t = B_t - \int_0^t A_s\, ds \tag{68}$$

is a standard Brownian motion.

We can write (68) as

$$dB_t = A_t\, dt + dW_t. \tag{69}$$

We can state the theorem informally as: "*if we weight the paths by the martingale* M_t*, then in the weighted measure* B_t *satisfies* (69)".

The Girsanov theorem requires that M_t be a martingale. For many applications in SLE, one only knows that M_t satisfies (66), and hence only that M_t is a *local* martingale. However, we can still use Girsanov by using stopping times (this procedure is sometimes called localization). Assume for ease that $M_0 = 1$; otherwise, we consider M_t/x_0. Let $T = T_n = \inf\{t : M_t \geq n \text{ or } |A_t| \geq n\}$. Then $M_{t\wedge T}$ is a bounded martingale satisfying (66) which we write as

$$dM_{t\wedge T} = M_{t\wedge T}\, A_t\, dB_t, \quad 0 \leq t \leq T.$$

The Girsanov theorem applies to $M_{t\wedge T}$. Hence, if we weight by $M_{t\wedge T}$, then in the weighted measure B_t satisfies

$$dB_t = A_t\, dt + dW_t, \quad 0 \leq t \leq T. \tag{70}$$

As a slight abuse of notation, we can say that if we weight by the local martingale M_t, then in the weighted measure B_t satisfies (69). This is shorthand for saying that (70) holds for all $n < \infty$. To show that M_t is a martingale, it suffices to establish (67), and do this one can use (70). If we let $\mathbf{Q}$ denote the weighted measure, we can rewrite (67) as

$$\lim_{n\to\infty} \mathbf{Q}\{T_n < t\} = 0.$$

Remark The Girsanov theorem could be stated as a theorem about nonnegative local martingales in which case the measure $\mathbf{Q}_t$ on $\mathcal{F}_t$ is a subprobability measure. We find it more convenient to restrict the Girsanov theorem to martingales and to use stopping times.

As an example, suppose that B_t is a standard Brownian motion and $\phi(z) = e^{\psi(z)}$ is a positive C^2 function. Then Itô's formula shows that

$$M_t = \phi(B_t)\, \exp\left\{-\frac{1}{2}\int_0^t [\psi''(B_s) + \psi'(B_s)^2]\, ds\right\},$$

is a local martingale satisfying

$$dM_t = \psi'(B_t)\, M_t\, dB_t.$$

If we weight by the local martingale M_t, then

$$dB_t = \psi'(B_t)\, dt + dW_t, \tag{71}$$

where W_t is a Brownian motion in the new measure which we denote by $\mathbf{Q}$. This must be interpreted in terms of stopping times; however, if solutions to the SDE (71) do not blow up in finite time, then M_t does not blow up in finite time in the measure $\mathbf{Q}$ and hence M_t is a martingale. A sufficient condition for M_t to be a martingale is that ψ' is uniformly bounded.

Assume M_t is a martingale. Let $p_t(x, y)$ denote the transition density for the Brownian motion and let $q_t(x, y)$ denote the transition density for the SDE (71). We claim that for all t, x, y,

$$\phi(x)^2\, q_t(x, y) = \phi(y)^2\, q_t(y, x). \tag{72}$$

To see this from the perspective of the Girsnaov theorem, consider the set of paths $\gamma : [0,t] \rightarrow \mathbb{R}$ with $\gamma(0) = x, \gamma(t) = y$. Then $\mathbb{P}, \mathbf{Q}$ give two measure on the set of such paths with

$$\frac{d\mathbf{Q}}{d\mathbb{P}}\,(\gamma) = \frac{\phi(y)}{\phi(x)}\,\exp\left\{-\frac{1}{2}\int_0^t [\psi'' + (\psi')^2](\gamma(s))\,ds\right\}.$$

Here we have used the fact that M_t is a deterministic function of the path $\gamma(s), 0 \leq s \leq t$. Similarly, if we consider the reversed path γ^R with $\gamma^R(s) = \gamma(t-s)$, we get

$$\frac{d\mathbf{Q}}{d\mathbb{P}}\,(\gamma^R) = \frac{\phi(x)}{\phi(y)}\,\exp\left\{-\frac{1}{2}\int_0^t [\psi'' + (\psi')^2](\gamma^R(s))\,ds\right\}.$$

The integral inside the exponential is not easy to compute but it is the same for γ and γ^R, which gives

$$\phi(x)^2\,\frac{d\mathbf{Q}}{d\mathbb{P}}\,(\gamma) = \phi(y)^2\,\frac{d\mathbf{Q}}{d\mathbb{P}}\,(\gamma^R).$$

Combining this with the symmetry of $\mathbb{P}$ gives (72). In particular,

$$\int \phi^2(x)\,q_t(x,y)\,dx = \int \phi^2(y)\,q_t(y,x)\,dx = \phi^2(y).$$

This shows that ϕ^2 us an invariant density for (72).

Often one starts with the SDE (71) in which case one can find ϕ using

$$\phi = e^{\int \psi'}.$$

Example If $\psi(x) = bx$, we obtain the well known exponential martingale

$$M_t = e^{bB_t}\,e^{-b^2 t/2}.$$

In the weighted measure, which we denote by $\mathbf{Q}_b$, B_t satisfies

$$dB_t = b\,dt + dW_t,$$

i.e., B_t is a Brownian motion with drift. This has no explosion in finite time, so M_t is a martingale and

$$e^{-b^2 t/2} = \mathbb{E}[M_t\,e^{-bB_t}] = \mathbb{E}_{\mathbf{Q}_b}[e^{-bB_t}] = e^{-b^2 t}\,\mathbb{E}_{\mathbf{Q}_b}[e^{-bW_t}].$$

This is the standard martingale computation for the moment generating function for a normal.

Example Assume $B_0 > 0$ and let $\phi(x) = x^b$, i.e., $\psi(x) = b\log x$. Then

$$M_t = B_t^b\,\exp\left\{-\frac{b(b-1)}{2}\int_0^t \frac{ds}{B_s^2}\right\}.$$

In the weighted measure, B_t satisfies the Bessel equation

$$dB_t = \frac{b}{B_t}\,dt + dW_t.$$

If $b \geq 1/2$, then the process under the weighted measure does not reach 0 in finite time. For these values, we can say that M_t is a martingale.

Example If $\psi(x) = -bx^2/2$, then

$$M_t = e^{-bB_t^2/2}\, e^{-bt}\, \exp\left\{ b^2 \int_0^t B_s^2\, ds \right\}.$$

In the weighted measure, B_t satisfies the Ornstein-Uhlenbeck equation

$$dB_t = -b\, B_t\, dt + dW_t.$$

The invariant density is proportional to $\phi(x)^2 = e^{-bx^2}$ from which we see that the process is positive recurrent.

Example In the study of radial SLE, the case $\psi'(x) = (b/2)\cot(x/2)$ arises for which

$$\phi(x) = \exp\left\{ \int \psi' \right\} = \sin^b(x/2).$$

Assume $B_0 \in (0, 2\pi)$. In the weighted measure B_t satisfies

$$dB_t = \frac{b}{2}\, \cot(B_t/2)\, dt + dW_t.$$

By comparison with a Bessel process we see that this process stays in $(0, 2\pi)$ for all time provided that $b \geq 1/2$. The invariant density is $\sin^{2b}(x/2)$.

Example In Section 7, we consider the case

$$\phi(x) = [\cosh x]^b, \qquad \psi'(x) = b\, \tanh x.$$

Note that ψ' is bounded. The invariant density is $\phi(x)^2 = [\cosh x]^{2b}$.

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