Randomness and Fractals

Why do so many physicists become traders?

Gregory F. Lawler

Department of Mathematics
Department of Statistics
University of Chicago

September 25, 2011

Mathematics and the "Real World"
Applied Mathematics

- The goal is to study some phenomenon (e.g., falling bodies, heat diffusion, stock prices, polymers)
- Start by making a mathematical model. This almost always entails some simplification of the real-world situation.
- Analyze the mathematics. Techniques to do this include
  - Rigorous proofs
  - Numerical approximations
  - For models with randomness, Monte Carlo simulations.
- Use the mathematical analysis to predict the real-world and check your predictions with reality. This may require estimating some parameters of the model.
- If there are discrepancies there are two things to check:
  - The mathematical analysis is incorrect (or at least incomplete)
  - The mathematical model is not correct (or some of the parameters are not correctly estimated).

- There are many examples where phenomena from very different areas produce the same mathematical model. This is why abstracting the mathematics is useful.
- Most of the research that is done in mathematics departments fits into the "mathematical analysis" part of the picture. Mathematicians study what one can conclude given certain assumptions.
- A large amount of the most interesting mathematical problems have come from modeling of physical phenomenon. In terms of mathematical sophistication, physics has been the biggest contributor to the forefront of mathematics although other areas (computer networks, biology, financial markets) have also contributed challenging interesting problems.
- The applier of mathematics might not understand everything that goes on in the mathematical analysis. However, it is critical that he or she understand the mathematical model.

Example: Newtonian physics

- The basic mathematical model of Newtonian physics can be written as
$$\text{force} = (\text{mass})\,(\text{acceleration})$$
- This equation is not exactly correct but is very close to "reality" when dealing with physics in the magnitudes that we can observe. (This equation is not good for describing motion on atomic scales or at very high speeds).
- If we assume this model, then we can use mathematical analysis to determine where a particle will be in the future. In this case, the mathematical tool is (differential and integral) calculus as developed by Leibniz, Newton, and many afterwards.
- Calculus studies the relationship between a quantity and the rate of change of the quantity. This abstracted idea has applications in numerous areas outside of physics.

- Suppose we are interested in a quantity $f(t)$ and we know that at time time t that the rate that it is changing is $G(t)$. In other words, if Δt denotes a small time interval

$$f(t+\Delta t) - f(t) \approx G(t)\,\Delta t.$$

- If we know $f(0)$ and the function G, then we can find a good approximation of $f(t)$ by choosing a small Δt and letting

$$f(t) = f(0) + G(0)\,\Delta t + G(\Delta t)\,\Delta t + \cdots + G(t-\Delta t)\,\Delta t.$$

- In calculus, we write the limit of this expression as Δt gets smaller and smaller,

$$f(t) = f(0) + \int_0^t G(s)\,ds.$$

- The rate is called the derivative and we write

$$f'(t) = \frac{df}{dt} = G(t).$$

- In applications, it is more common that one is given that the rate G depends on the current value of the function.

$$f'(t) = G(t, f(t)).$$

- For example if $f(t)$ denotes the value of a bond with interest rate r (compounded "continuously"), then

$$f'(t) = r\, f(t), \qquad G(t, f(t)) = r\, f(t).$$

- We can still write

$$f(t) = f(0) + \int_0^t G(s, f(s))\, ds,$$

but it can be harder to write the answer explicitly. This is an example of a differential equation. The example above can be solved explicitly

$$f(t) = f(0)\, e^{rt}.$$

- Even if the equation can not be solved explicitly, one can still approximate the solution (using a computer) by choosing small Δt and letting

$$f(\Delta t) = f(0) + G(0, f(0))\,\Delta t,$$

$$f(2\Delta t) = f(\Delta t) + G(\Delta t, f(\Delta t))\,\Delta t, \quad \text{etc.}$$

- This theory is deterministic. If one knows the initial value $f(0)$ and the rate function G, then the value $f(t)$ is determined even if it is difficult to find it exactly.
- In real-world applications, one often cannot determine $f(0)$ (or G) exactly. One would hope that if one can approximate $f(0)$ very well, then our $f(t)$ would also be close. This is not always true! Systems that do not have this property are often called chaotic.

- Systems that may theoretically be deterministic are effectively random because either one cannot solve the appropriate equations or one cannot estimate the parameters sufficiently accurately. For these systems one often uses a mathematical model with randomness.
- For example, when we flip a coin we can theoretically predict from the initial condition, the exact way we flip (and the wind conditions, and ...) whether it will come up heads or tails. From a practial perspective we cannot do this. We choose to model this as a random event which comes up heads with probability $1/2$ and tails with probability $1/2$.
- Systems which change with time where the change includes a random component is called stochastic processes. A major tool for studying (a class of such) systems is stochastic calculus. This is one of the basic subjects taught today in financial mathematics programs.

Important historical figures

- In 1827, Robert Brown described the jittery motion of particles from pollen grains which he observed under a microscope. From this the term Brownian motion was given to these kinds of motions. No precise theory was given at this time.
- Louis Bachelier (1900) in *Theory of Speculation* used the theory of Brownian motion to evaluate stock options. His work was not appreciated as much in his time as it is now.
- Albert Einstein (1905) considered the diffusion of a very large number of Brownian particles and then considered the behavior of a single particle. The theory combined with physical observation led to an accurate assessment of the size of atoms. Marian Smoluchowski (1906) did a similar analysis independently.
- The mathematical theory was developed later. It is sometimes called the Wiener process after Norbert Wiener. Mathematicians (like me in this talk) use the terms Brownian motion and Wiener process synonymously.

- The starting point for calculus is the study of lines (linear functions). To say $f'(t) = r$ is to say that at time t the function grows like a line with slope r,

$$f(t + \Delta t) = f(t) + r\,\Delta t.$$

- The study of line in one dimension comes in high school algebra. In many dimensions this subject becomes more difficult and is studied in linear algebra.
- For stochastic calculus the starting point is "random continuous motion" which is Brownian motion. It is a limit of random walk.

Random walk: the mathematical model

- In each time interval Δt we assume that the function $B(t)$ is equally likely to go up or down. The amount it goes up down we will call Δx. With probability $1/2$,

$$B(t + \Delta t) = B(t) + \Delta x,$$

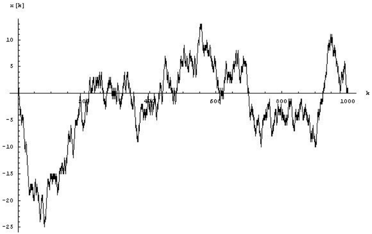
and with probability $1/2$,

$$B(t + \Delta t) = B(t) - \Delta x.$$

We can imagine at each time t we flip a fair come to decide $\pm$.

- We assume that the coin flips at different times are independent. In other words, at each new time we flip the coin again.

- If we choose Δt and Δx we can run simulations of the random walk.



- We need to choose Δx correctly (with respect to Δt) such that the picture is nontrivial.

A calculation

- Let $\Delta t = 1/n$, and
$$B(1) - B(0) = \sum_{j=1}^{n} X_j$$
where $X_j = \pm \Delta x$.
- Let $\mathbb{E}$ denote expectation (average value).
$$\begin{aligned}
\mathbb{E}\left[(B(1) - B(0))^2\right] &= \mathbb{E}\left[(\sum_{j=1}^{n} X_j)^2\right] \\
&= \mathbb{E}\left[\sum_{j=1}^{n}\sum_{k=1}^{n} X_j X_k\right] \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} \mathbb{E}[X_j X_k].
\end{aligned}$$

- If $j = k$, $\mathbb{E}[X_j X_k] = (\Delta x)^2$.
- If $j \neq k$, $X_j X_k$ is equally likely to be $(\Delta x)^2$ or $-(\Delta x)^2$ and hence $\mathbb{E}[X_j X_k] = 0$. This gives

$$\mathbb{E}\left[(B(1) - B(0))^2\right] = n\,(\Delta x)^2.$$

- If we want this expectation to equal one, then we must choose $\Delta x = 1/\sqrt{n} = \sqrt{\Delta t}$.
- From an abstract perspective, the calculation is the same as one of the basic facts from high school — the Pythagorean theorem

$$a^2 + b^2 = c^2.$$

- The basic study of stochastic calculus (or stochastic differential equations) is of processes that evolve using an equation of the form

$$dX_t = m\,dt + \sigma\,dB_t.$$

  m is the drift and is the same as the derivative from calculus. The number σ is called volatility in the finance literature.
- A standard model for a stock price is

$$dS_t = m\,S_t\,dt + \sigma\,S_t\,dB_t.$$

  This is often called geometric Brownian motion.
- Stock prices do not really follows such a law, but it is hoped that analysis assuming this will be close enough to correct. The fancy word for this hope is robustness of the model.

Option pricing: a toy example

- Assume there is a stock which currently sells at \$5 share. In a year the price will be \$10 or \$2. The probabilities are p and $1 - p$ for the two possibilities.
- We want to sell an option that allows the owner to buy a share in a year at \$8. Let us call the price of the option P.
- Assuming no inflation this option next year will be worth either \$2 (if the stock is selling at \$10) or zero (if the stock is selling at \$2).
- If we sell the option today at price P we will hedge by immediately buying q shares of the stock (for \$5q) and keepint the rest of the money \$(P - 5q) in the bank.

- After a year our portfolio will be worth

$$P - 5q + 10q, \qquad \text{if } S = 10.$$

$$P - 5q + 2q, \qquad \text{if } S = 2.$$

- We need this to equal \$2 if $S = 10$ and nothing if $S = 2$. this gives

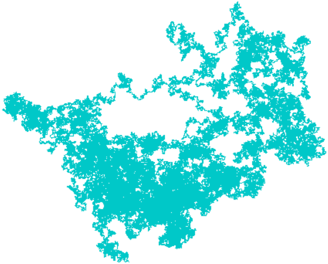
$$P + 5q = 2, \quad P = 3q.$$

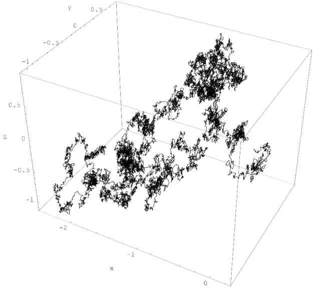
- Solving gives $P = 3/4, q = 1/4$. Note that we have found the fair price, and found the appropriate porfolio to hedge the option.
- If we can sell the option for more than $3/4$ we can make guaranteed money (under the assumption of the model).
- Our calculation used the current inflation rate (which we chose to be zero for convenience) and the values that the option can take in year one.
- The calculation did not use p, the probability that the stock goes up!

Black-Scholes Formula

- The Black-Scholes(-Merton) formula uses the same basic idea as the toy model to price options assuming that the stock price follows geometric Brownian motion.
- One needs to know the volatility and the bond interest rate but one does not need to know the drift term.
- It is very nice mathematics, but the assumptions in the model are far from correct.
- It is also hard to estimate the volatility.
- Merton and Scholes received the Nobel prize for this in 1997 (Black was deceased and hence ineligible).
- There is much more to financial mathematics than the Black-Scholes formula, we will change topics.

Brownian motion in two (or more) dimensions





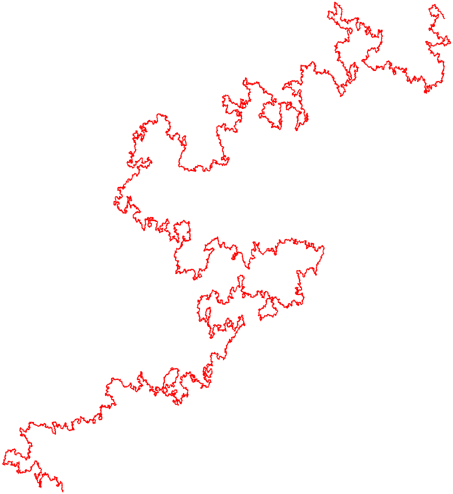
- Brownian motion moving in two or more dimensions traces out a set of fractal dimension two .
- Fractal dimension D means that it takes about N^D balls of diameter $1/N$ to cover the part of the curve in a box of size 1.
- $D = 2$ comes from the computation we did before

$$(\Delta x)^2 \approx \Delta t.$$

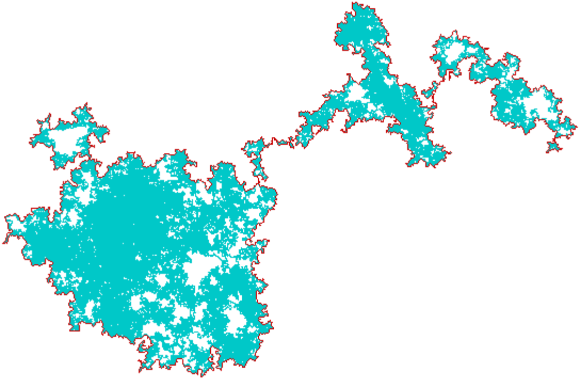
- This is an example of universality: completely random continuous processes give sets of fractal dimension two.
- There many examples of random fractals with other dimensions. We will discuss one that is used to model polymer chains.

- We start with the assumption that polymers are formed by monomers that join themselves in a long chain as randomly as possible with the constraint that the monomers must avoid each other.
- We consider random walk paths with the property that no point is visited more than once. This is called a self-avoiding random walk.
- A physical chemist, Paul Flory, gave a heuristic argument that can be interpreted to say that in two dimensions the walk should look like it has fractal dimension $D = 4/3$. (In three dimensions, he guessed $5/3$ which is close but probably not exactly correct.)

SAW in plane - 1,000,000 steps



- Benoit Mandelbrot in his book Fractal Geometry of Nature noted that the “coastline” of a two-dimensional Bronwian motion looked like a self-avoiding walk. In particular, it seemed to have fractal dimension $4/3$.



- With Oded Schramm and Wendelin Werner, I showed that the fractal dimension of the Brownian coastline is $4/3$.
- The proof uses an idea of "conformal invariance" that was first conjectured in the physics literature.
- This is just one piece of a much larger project.