Fractal Properties of the Schramm-Loewner Evolution (SLE)

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December 12, 2012

OUTLINE OF TALK

- The Schramm-Loewner evolution (SLE_κ) is a family of random fractal curves that arise as limits of models in statistical physics.
- One reason that they are interesting is that they give examples of nontrivial curves for which one can prove facts about the fractal and multifractal structure.
- In this talk I will give an introduction to the curves, starting with some discrete models and then giving the definition.
- Then I will discuss recent rigorous work on the curves themselves. Here we will concentrate on SLE and not on the discrete processes.

SELF-AVOIDING WALK (SAW)

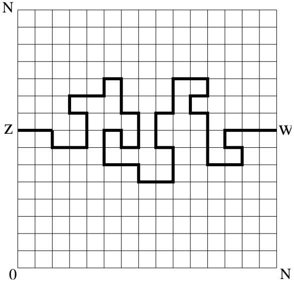
- Model for polymer chains — polymers are formed by monomers that are attached randomly except for a self-avoidance constraint.

$$\omega = [\omega_0, \dots, \omega_n], \quad \omega_j \in \mathbb{Z}^2, \quad |\omega| = n$$

$$|\omega_j - \omega_{j-1}| = 1, \quad j = 1, \dots, n$$

$$\omega_j \neq \omega_k, \quad 0 \leq j < k \leq n.$$

- Critical exponent ν: a typical SAW has diameter about $|\omega|^\nu$.
- If no self-avoidance constraint $\nu = 1/2$; for 2-d SAW Flory predicted $\nu = 3/4$.



Each SAW from z to w gets measure $e^{-\beta|\omega|}$. Partition function

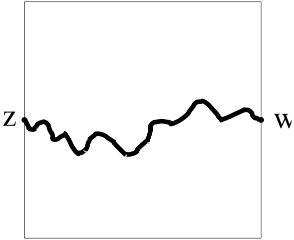
$$Z = Z(N, \beta) = \sum e^{-\beta|\omega|}.$$

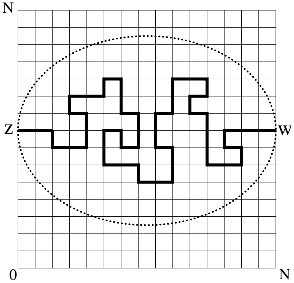
β small — typical path is two-dimensional
β large — typical path is one-dimensional
β_c — typical path is $(1/\nu)$-dimensional

Choose $\beta = \beta_c$; let $N \to \infty$. Expect

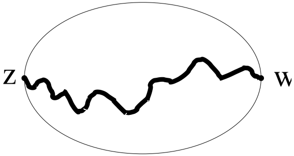
$$Z(N, \beta) \sim C(D; z, w)\, N^{-2b},$$

divide by $C(D; z, w)\, N^{-2b}$ and hope to get a probability measure on curves connecting boundary points of the square.



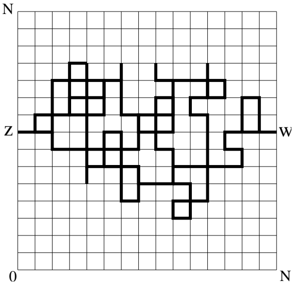


Similarly, if we fix $D \subset \mathbb{C}$, we can consider walks restricted to the domain D



Predict that these probability measures are conformally invariant.

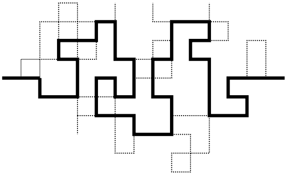
SIMPLE RANDOM WALK



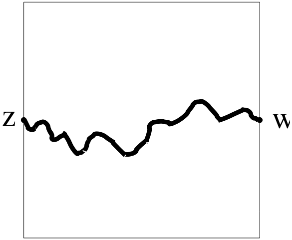
- Simple random walk — no self-avoidance constraint. Criticality: each walk ω gets weight $(1/4)^{|\omega|}$.
- Scaling limit is Brownian motion which is conformally invariant (Lévy).

LOOP-ERASED RANDOM WALK

Start with simple random walks and erase loops in chronological order to get a path with no self-intersections.

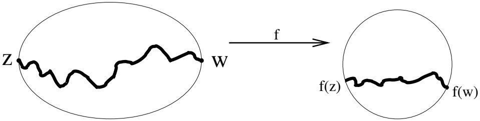


Limit should be a measure on paths with no self-intersections.



ASSUMPTIONS ON SCALING LIMIT

Probability measure $\mu_D^{\#}(z,w)$ on curves connecting boundary points of a domain D.



- Conformal invariance: If f is a conformal transformation

$$f \circ \mu_D^{\#}(z,w) = \mu_{f(D)}^{\#}(f(z), f(w)).$$

For simply connected D, $\mu_{\mathbb{H}}^{\#}(0,\infty)$ determines $\mu_D^{\#}(z,w)$ (Riemann mapping theorem).

What is meant by the image $f \circ \gamma$ of a curve $\gamma : [0, T] \to \mathbb{C}$?

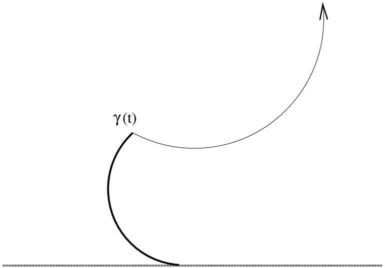
- One possibility is to consider curves modulo reparametrization so that we do not care how "fast" we traverse $f \circ \gamma$.
- If the curve γ has fractal dimension d, then the "natural" parametrization transforms as a d-dimensional measure. That is, the time to traverse $f \circ \gamma[r, s]$ is

$$\int_r^s |f'(\gamma(t))|^d \, dt.$$

- For Brownian motion, the fractal dimension of the paths is $d = 2$ and Lévy's result uses that change the parametrization.
- We first consider paths modulo reparametrization and later discuss the correct parametrization.

- Domain Markov property Given $\gamma[0,t]$, the conditional distribution on $\gamma[t,\infty)$ is the same as

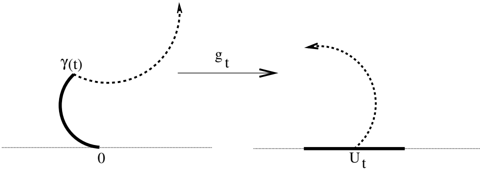
$$\mu_{\mathbb{H}\setminus\gamma(0,t]}(\gamma(t),\infty).$$



- Satisfied on discrete level by SAW and LERW, but not by simple random walk.

LOEWNER EQUATION IN UPPER HALF PLANE

- Let $\gamma : (0, \infty) \to \mathbb{H}$ be a simple curve with $\gamma(0+) = 0$ and $\gamma(t) \to \infty$ as $t \to \infty$.
- $g_t : \mathbb{H} \setminus \gamma(0, t] \to \mathbb{H}$



- Can reparametrize (by capacity) so that

$$g_t(z) = z + \frac{2t}{z} + \cdots , \qquad z \to \infty$$

- g_t satisfies

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \qquad g_0(z) = z.$$

Moreover, $U_t = g_t(\gamma(t))$ is continuous.

(Schramm) Suppose γ is a random curve satisfying conformal invariance and Domain Markov property. Then U_t must be a random continuous curve satisfying

- For every $s < t$, $U_t - U_s$ is independent of $U_r, 0 \le r \le s$ and has the same distribution as U_{t-s}.
- $c^{-1} U_{c^2 t}$ has the same distribution as U_t.

Therefore, $U_t = \sqrt{\kappa}\, B_t$ where B_t is a standard (one-dimensional) Brownian motion.

The (chordal) Schramm-Loewner evolution with parameter κ (SLE_κ) is the solution obtained by choosing $U_t = \sqrt{\kappa}\, B_t$.

(Rohde-Schramm) Solving the Loewner equation with a Brownian input gives a random curve.

The qualitative behavior of the curves varies greatly with κ

- $0 < \kappa \leq 4$ — simple (non self intersecting) curve
- $4 < \kappa < 8$ — self-intersections (but not crossing); not plane-filling
- $8 \leq \kappa < \infty$ — plane-filling

(Beffara) For $\kappa < 8$, the Hausdorff dimension of the paths is

$$1 + \frac{\kappa}{8}.$$

NATURAL PARAMETRIZATION/LENGTH

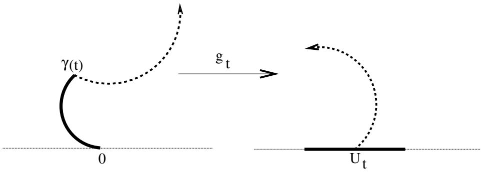
- Start with path
$$\omega = [\omega_0, \omega_1, \ldots]$$
in $\mathbb{Z}^2$. Assume it has "fractal dimension" d.
- Let
$$\gamma^{(n)}(t) = n^{-1}\,\omega_{n^d t}.$$
Hope to take limit as $n \to \infty$.
- For simple random walk, $d = 2$ and $\gamma(t)$ is Brownian motion (Donsker's theorem)
- Expect similar result for SAW ($d = 4/3$, $SLE_{8/3}$) and LERW ($d = 5/4$, SLE_2).

SCALING RULE

- Suppose γ has "natural parametrization".
- If $f : D \to f(D)$ is a conformal transformation, then the time needed to traverse $f(\gamma[0,t])$ is

$$\int_0^t |f'(\gamma(s))|^d \, ds.$$

- While SAW and LERW have limits that are SLE_κ, the capacity parametrization does not have this property.
- In fact, the capacity parametrization is singular with respect to the natural length.
- Problem: can we define the natural length for SLE_κ?



$$f_t(z) = g_t^{-1}(z + U_t)$$

- In capacity para., time to traverse $g_t(\gamma[t, t+\Delta t])$ is Δt.
- This should not be true for natural length.
- For natural length, need to understand $g_t'(w)$ near $\gamma(t)$ or $f_t'(z)$ near zero.

GREEN'S FUNCTION

- The *SLE* Green's function (for chordal *SLE* from w_1 to w_2 in D) is defined by

$$G_D(z; w_1, w_2) = \lim_{\epsilon \downarrow 0} \epsilon^{d-2} \, \mathbb{P}\{\mathrm{dist}(z, \gamma) \leq \epsilon\}.$$

  Defined up to multiplicative constant.
- This was computed by Rohde-Schramm and L. first showed the limit exists with distance replaced by conformal radius. More recently, L-Rezaei have proved the limit above exists.
- Let $G(z) = G_{\mathbb{H}}(z; 0, \infty)$. Then

$$G(z) = [\mathrm{Im}\, z]^{d-2} \, [\sin \arg z]^{\frac{8}{\kappa}-1}.$$

RIGOROUS DEFINITION

Let γ be SLE_κ in $\mathbb{H}$ parametrized by capacity. $\gamma_t = \gamma[0, t]$.

- Let Θ_t be the natural length of γ_t spent in a bounded domain D.
- Heuristic:

$$\mathbf{E}[\Theta_\infty] = \int_D G(z)\, dA(z).$$

-

$$\mathbf{E}[\Theta_\infty \mid \gamma_t] = \Theta_t + \Psi_t,$$

$$\Psi_t = \int_D G_{\mathbb{H}\setminus\gamma_t}(z; \gamma(t), \infty)\, dA(z).$$

- Θ_t is the increasing process that makes $\Psi_t + \Theta_t$ a martingale. (Doob-Meyer decomposition)

- (L-Sheffield) The natural length is well defined for $\kappa < 5.021\cdots$. It is Hölder continuous.
- (L-Zhou) Exists for all $\kappa < 8$. This proof relies on a slight generalization of a hard estimate of Beffara. It also uses a two-point Green's function (L-Werness). An improved version using a two-point time-dependent Green's function has been given (L-Rezaei)
- (L-Rezaei) The natural parametrization is given by the *d*-dimensional Minkowski content

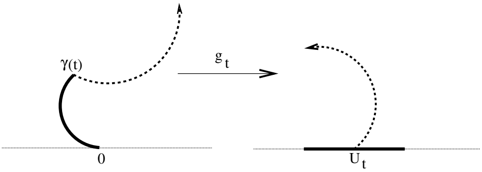
$$\Theta_t = c \lim_{\epsilon \downarrow 0} \epsilon^{d-2} \mathrm{Area}\{z : \mathrm{dist}(z, \gamma[0,t]) < \epsilon\}.$$

- (Rezaei) The *d*-dimensional Hausdorff measure of the path is zero.

TIP MULTIFRACTAL SPECTRUM

(work with F. Johansson Viklund)

- Study behavior of $|g_t'|$ near $\gamma(t)$ or $|f_t'|$ near 0.



- Let Λ_β denote the set of t such that as $y \downarrow 0$,

$$|f_t'(iy)| \approx y^{-\beta}.$$

- Closely related to behavior of harmonic measure near the tip of the curve.

- Let

$$\rho = \rho_\kappa(\beta) = \frac{\kappa}{8(\beta+1)} \left[\left(\frac{\kappa+4}{\kappa} \right) (\beta+1) - 1 \right]^2 .$$

- (L-Johansson Viklund) With probability one, if $\rho < 2$,

$$\dim_h(\Lambda_\beta) = \frac{2-\rho}{2}$$

$$\dim_h(\gamma(\Lambda_\beta)) = \frac{2\dim_h(\Lambda_\beta)}{1-\beta} = \frac{2-\rho}{1-\beta}.$$

- If $\rho > 2$, then $\Lambda_\beta = \emptyset$.

- $\dim_h(\Lambda_\beta)$ depends on the capacity parametrization of the path. The quantity $\dim_h(\gamma(\Lambda_\beta))$ is independent of the parametrization.
- Finding the formula for ρ requires analyzing $\mathbf{E}\left[|f_t'(i)|^\lambda\right]$ for large t.
- Computing the almost sure multifractal spectrum requires more work than just computing ρ. There are tricky second moment estimates involved.
- Nonrigorous ("physicist") treatments of multifractal spectrum may compute ρ but do not do the second moment work needed to make this an almost sure statement.

- ρ can be computed by analyzing $\mathbf{E}\left[|f_t'(i)|^\lambda\right]$ for large t. For $r < 2a + \frac{1}{2}$, $(a = 2/\kappa)$

$$r(\lambda) = 2a + 1 - \sqrt{(2a+1)^2 - 4a\lambda},$$

$$\zeta(\lambda) = \lambda - \frac{r}{2a}$$

$$-\beta(\lambda) = \zeta'(\lambda) = 1 - \frac{1}{\sqrt{(2a+1)^2 - 4a\lambda}},$$

$$\mathbf{E}\left[|f_t'(i)|^\lambda\right] \asymp t^{-\zeta(\lambda)/2}$$

  and a typical path when weighted by $|f_t'(i)|^\lambda$ has $|f_t'(i)| \approx t^{\beta(\lambda)/2}$,

$$\mathbb{P}\{|f_{t^2}'(i)| \approx t^{\beta/2}\} \approx t^{-\rho}, \quad \rho = \lambda\beta + \zeta.$$

- The technique is to find an appropriate martingale and use Girsanov theorem to analyze the paths in the measure tilted by the martingale.

- As an example, consider the natural parametrization. This corresponds to $\lambda = d = 1 + \frac{\kappa}{8}$.

$$r = 1, \;\; \lambda = d, \;\; \zeta = 2 - d, \;\; \frac{\beta}{2} = d - \frac{3}{2} = \frac{1}{4a} - \frac{1}{2}$$

$$\mathbf{E}\left[|\hat{f}_1'(i/\sqrt{n})|^d\right] = \mathbf{E}\left[|\hat{f}_n'(i)|^d\right] \asymp n^{\frac{d}{2}-1}$$

$$\mathbb{P}\{|\hat{f}_1'(i/\sqrt{n})| \approx n^{d-\frac{3}{2}}\} \approx n^{-(d^2-2d+1)}$$

- The Hausdorff dimension of the set of *times* $t \in [0,1]$ with $|\hat{f}_t'(i/\sqrt{n})| \approx n^{d-\frac{3}{2}}$ equals

$$1 - (d^2 - 2d + 1) = d(2-d) \in (0,1).$$

- The natural parametrization is carried on a set of **??** of dimension $d(2-d)$.
- The dimension of *points* $\gamma(t)$ satisfying this is d

HÖLDER CONTINUITY OF γ

- Consider $\gamma(t), \epsilon \leq t \leq 1$ (with capacity parametrization)
- $\gamma(t), \epsilon \leq t \leq 1$ is Hölder continuous of order $\alpha < \alpha_*$ but not $\alpha > \alpha_*$ where

$$\alpha_* = 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{8+\kappa}}.$$

- One direction shown by Joan Lind. Other direction by L-Johansson Viklund.
- $\alpha_* > 0$ unless $\kappa = 8$. Showing that the curve exists is much harder for $\kappa = 8$ than other values.

SOME OPEN PROBLEMS

- Show that for $\kappa < 8$, *SLE* with the natural parametrization is Hölder continuous for $\alpha < 1/d$.
- Find modulus of continuity for SLE_8 in capacity parametrization.
- Extend multifractal spectrum analysis to entire path, not just tip (the "first moment" calculations have been done but not the second moment analysis for almost sure behavior).
- Show that discrete processes converge to *SLE* in the natural parametrization. Work is being done on the loop-erased walk.
- Find a Hausdorff gauge function for which the Hausdorff measure of *SLE* paths is finite and positive.

THANK YOU!