Random Fractals Coming from Two-Dimensional Statistical Physics

Gregory F. Lawler

Department of Mathematics
University of Chicago
5734 S. University Ave.
Chicago, IL 60637

lawler@math.uchicago.edu

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CRITICAL PHENOMENA IN STATISTICAL PHYSICS

- Study systems at or near parameters at which a phase transition occurs
- Parameter $\beta = C/T$ where T = temperature
- Large β (low temperature) — long range correlation.
- Small β (high temperature) — short range correlation
- Critical value β_c at which sharp transition occurs
- Belief: systems at criticality "in the scaling limit" exhibit fractal-like behavior (power-law correlations) with nontrivial critical exponents.
- The exponents depend on dimension.

TWO DIMENSIONS

- Belavin, Polyakov, Zamolodchikov (1984) — critical systems in two dimensions in the scaling limit exhibit some kind of "conformal invariance".
- A number of theoretical physicists (Nienhuis, Cardy, Duplantier, Saleur, ...) made predictions about critical exponents using nonrigorous methods — conformal field theory and Coulomb gas techniques.
- Exact rational values for critical exponents — predictions strongly supported by numerical simulations
- While much of the mathematical framework of conformal field theory was precise and rigorous (or rigorizable), the nature of the limit and the relation of the field theory to the lattice models was not well understood.

SELF-AVOIDING WALK (SAW)

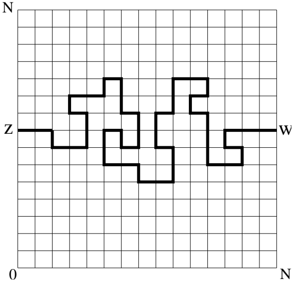
- Model for polymer chains — polymers are formed by monomers that are attached randomly except for a self-avoidance constraint.

$$\omega = [\omega_0, \ldots, \omega_n], \quad \omega_j \in \mathbb{Z}^2, \quad |\omega| = n$$

$$|\omega_j - \omega_{j-1}| = 1, \quad j = 1, \ldots, n$$

$$\omega_j \neq \omega_k, \quad 0 \leq j < k \leq n.$$

- Critical exponent ν: a typical SAW has diameter about $|\omega|^\nu$.
- If no self-avoidance constraint $\nu = 1/2$; for 2-d SAW Flory predicted $\nu = 3/4$.



Each SAW from z to w gets measure $e^{-\beta|\omega|}$. Partition function

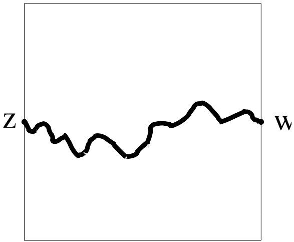
$$Z = Z(N, \beta) = \sum e^{-\beta|\omega|}.$$

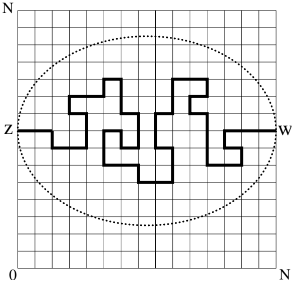
β small — typical path is two-dimensional
β large — typical path is one-dimensional
β_c — typical path is $(1/\nu)$-dimensional

Choose $\beta = \beta_c$; let $N \to \infty$. Expect

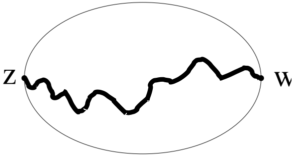
$$Z(N, \beta) \sim C(D; z, w)\, N^{-2b},$$

divide by N^{-2b} and hope to get a finite measure on curves connecting boundary points of the square of total mass $C(D; z, w)$ (can be made into probability measure by dividing by $C(D; z, w)$).



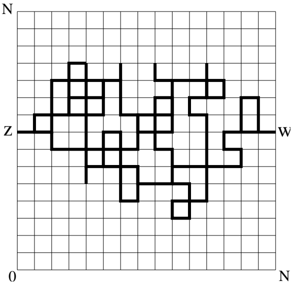


Similarly, if we fix $D \subset \mathbb{C}$, we can consider walks restricted to the domain D



Predict that these probability measures are conformally invariant.

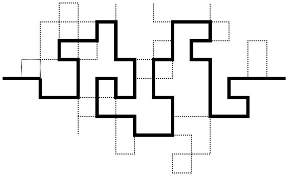
SIMPLE RANDOM WALK



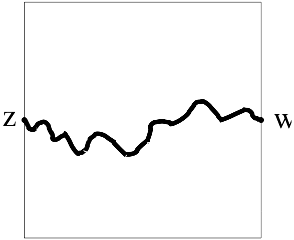
- Simple random walk — no self-avoidance constraint. Criticality: each walk ω gets weight $(1/4)^{|\omega|}$.
- Scaling limit is Brownian motion which is conformally invariant (Lévy).

LOOP-ERASED RANDOM WALK

Start with simple random walks and erase loops in chronological order to get a path with no self-intersections.

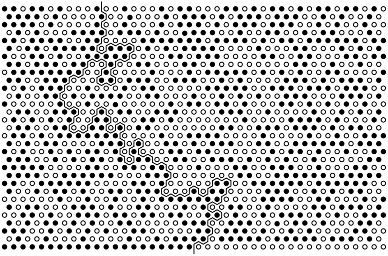


Limit should be a measure on paths with no self-intersections.



CRITICAL PERCOLATION

Color vertices of the triangular lattice in the upper half plane black or white independently each with probability 1/2.



Put a boundary condition of black on negative real axis and white on positive real axis. The percolation exploration process is the boundary between black and white.

STRATEGY

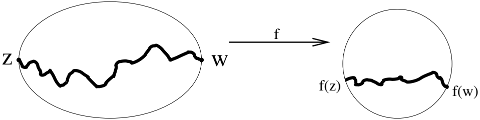
- Make precise the conformal invariance assumption and other properties expected of scaling limit.
- Find all possible limits satisfying these assumptions.
- For a given discrete process, identify which is the correct limit.
- Prove the discrete converges to continuous.
- Use conformal invariance or covariance to study fine properties of the continuous process and translate these back to the discrete process.

Nonrigorous approaches in mathematical physics using conformal field theory and renormalization group have some of the properties of this strategy.

ASSUMPTIONS ON SCALING LIMIT

Finite measure $\mu_D(z,w)$ and probability measure $\mu_D^{\#}(z,w)$ on curves connecting boundary points of a domain D.

$$\mu_D(z,w) = C(D;z,w)\,\mu_D^{\#}(z,w).$$



- Conformal invariance: If f is a conformal transformation

$$f \circ \mu_D^{\#}(z,w) = \mu_{f(D)}^{\#}(f(z),f(w)).$$

- Scaling rule

$$C(D;z,w) = |f'(z)|^b\,|f'(w)|^b\,C(f(D);f(z),f(w)).$$

- For simply connected D, $\mu_{\mathbb{H}}^{\#}(0,\infty)$ determines $\mu_D^{\#}(z,w)$ (Riemann mapping theorem).

What is meant by the image $f \circ \gamma$ of a curve $\gamma : [0, T] \to \mathbb{C}$?

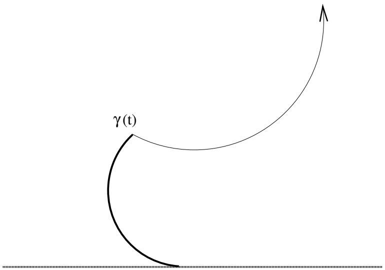
- One possibility is to consider curves modulo reparametrization so that we do not care how "fast" we traverse $f \circ \gamma$.
- If the curve γ has fractal dimension d, then the "natural" parametrization transforms as a d-dimensional measure. That is, the time to traverse $f \circ \gamma[r, s]$ is

$$\int_r^s |f'(\gamma(t))|^d \, dt.$$

- For Brownian motion, the fractal dimension of the paths is $d = 2$ and Lévy's result says uses that change the parametrization.
- We first consider paths modulo reparametrization and later discuss the correct parametrization.

- Domain Markov property Given $\gamma[0,t]$, the conditional distribution on $\gamma[t,\infty)$ is the same as

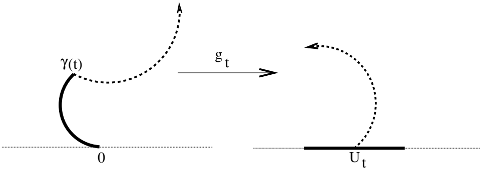
$$\mu_{\mathbb{H}\setminus\gamma(0,t]}(\gamma(t),\infty).$$



- Satisfied on discrete level by SAW, LERW, percolation exploration, ... (but not by simple random walk)

LOEWNER EQUATION IN UPPER HALF PLANE

- Let $\gamma : (0, \infty) \to \mathbb{H}$ be a simple curve with $\gamma(0+) = 0$ and $\gamma(t) \to \infty$ as $t \to \infty$.
- $g_t : \mathbb{H} \setminus \gamma(0, t] \to \mathbb{H}$



- Can reparametrize (by capacity) so that

$$g_t(z) = z + \frac{2t}{z} + \cdots, \qquad z \to \infty$$

- g_t satisfies

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \qquad g_0(z) = z.$$

Moreover, $U_t = g_t(\gamma(t))$ is continuous.

(Schramm) Suppose γ is a random curve satisfying conformal invariance and Domain Markov property. Then U_t must be a random continuous curve satisfying

- For every $s < t$, $U_t - U_s$ is independent of $U_r, 0 \leq r \leq s$ and has the same distribution as U_{t-s}.
- $c^{-1}\, U_{c^2 t}$ has the same distribution as U_t.

Therefore, $U_t = \sqrt{\kappa}\, B_t$ where B_t is a standard (one-dimensional) Brownian motion.

The (chordal) Schramm-Loewner evolution with parameter κ (SLE_κ) is the solution obtained by choosing $U_t = \sqrt{\kappa}\, B_t$.

(Rohde-Schramm) Solving the Loewner equation with a Brownian input gives a random curve.

The qualitative behavior of the curves varies greatly with κ

- $0 < \kappa \leq 4$ — simple (non self intersecting) curve
- $4 < \kappa < 8$ — self-intersections (but not crossing); not plane-filling
- $8 \leq \kappa < \infty$ — plane-filling

(Beffara) For $\kappa < 8$, the Hausdorff dimension of the paths is

$$1 + \frac{\kappa}{8}.$$

The fundamental tools for studying SLE are those of stochastic calculus (Itô integral and formula, martingales, Girsanov transformation)

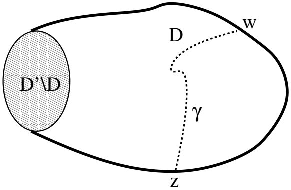
For which κ does SLE have double points?
Equivalent to ask, for which κ does SLE hit the real line?
Let $x > 0$ and $X_t = X_t(x) = g_t(x) - U_t$. Then SLE hits $[x, \infty)$ if and only if X_t reaches zero in finite time. X_t satisfies

$$dX_t = \frac{2}{X_t}\, dt + \sqrt{\kappa}\, dB_t.$$

Bessel equation. Well known that X_t reaches zero if and only if $\kappa > 4$.

WHICH κ FOR WHICH MODEL?

How does the $\mu_D(z,w)$ measure of a path change when we perturb the boundary? ($\kappa \leq 4$)



$$\frac{d\mu_D(z,w)}{d\mu_{D'}(z,w)} = 1\{\gamma \subset D\} \ \exp\left\{\frac{\mathbf{c}}{2}\Lambda(D';\gamma, D' \setminus D)\right\}$$

$\Lambda(D';\gamma, D' \setminus \gamma)$ is a conformal invariant given by the measure (using a certain Brownian loop measure) of loops in D' that intersect both $D' \setminus D$ and γ.

- For SAW, perturbing the domain does not change the measure. Expect $\mathbf{c} = 0$.
- For LERW, shrinking the domain loses some simple random walks whose loop-erasure is γ. Expect $\mathbf{c} < 0$.
- $\mathbf{c}$ is the central charge which is the parameter used in conformal field theory to distinguish models.

$$\mathbf{c} = \frac{(3\kappa - 8)\,(6 - \kappa)}{2\kappa}, \quad \kappa = \frac{(13 - \mathbf{c}) \pm \sqrt{(13 - \mathbf{c})^2 - 144}}{3}.$$

Each $\mathbf{c} < 1$ corresponds to two values κ, κ' with $\kappa\,\kappa' = 16$. $\mathbf{c} = 1$ corresponds to the double root $\kappa = \kappa' = 4$.

BROWNIAN PATHS

- The first major problem solved with *SLE* was the Brownian intersection exponents. One example goes back to a conjecture of Mandelbrot. Consider a Brownian island formed by taking taking a Brownian motion (random walk), conditioning to end at the same place it began, and filling in the bounded holds. Mandelbrot noted that simulations of this coastline indicated that it should have dimension $4/3$.
- L. showed that the dimension could be calculated in terms of a particular value of the intersection exponents.
- L, Schramm, and Werner showed how the locality property of SLE_6 could be used to calculate this exponent and verified Mandelbort's conjecture.

THE DIMENSION OF SELF-AVOIDING BROWNIAN MOTION. Having interpreted certain known relationships (to be quoted in Chapter 36) as implying that a self-avoiding random walk is of dimension 4/3, I conjecture that the same is true of self-avoiding Brownian motion.

An empirical test of this conjecture provides an excellent opportunity to test also the length-area relation of Chapter 12. The plate is covered by increasingly tight square lattices, and we count the numbers of squares of side G intersected by a) the hull, standing for G-area, and b) its boundary, standing for G-length. Graphs relating G-length to G-area, using doubly logarithmic coordinates, were found to be remarkably straight, with a slope indistinguishable from $D/2=(4/3)/2=2/3$.

The resemblance between the curves in Plates 243 and 231, and their dimensions, is worth stressing.

NOTE. In Plate 243, the maximal open domains that $B(t)$ does not visit are seen in gray. They can be viewed as tremas bounded by fractals, hence the loop is a net in the sense of Chapter 14.

◄ The question arises, of whether the loop is a gasket or a carpet from the viewpoint of the order of ramification. I conjectured that the latter is the case, meaning that Brown nets satisfy the Whyburn property, as described on p. 133. This conjecture has been confirmed in Kakutani & Tongling (unpublished). It follows that the Brown trail is a universal curve in the sense defined on page 144. ► ■

CRITICAL PERCOLATION

- Stas Smirnov proved that the scaling limit of critical percolation on the triangular lattice satisfies a crossing formula predicted by physicist John Cardy. Using this and the work of LSW he established that the scaling limit of percolation is SLE_6.
- $\kappa = 6$ is the only value of κ for which SLE satisfies the locality property — something that would be expected of the scaling limit of percolation. (Schramm had already identified $\kappa = 6$ as the correct candidate for the scaling limit.)
- Smirnov's proof is particular to the triangular lattice. It is an open problem to establish this limit for other lattices, e.g., critical bond or site percolation in $\mathbb{Z}^2$

LOOP-ERASED RANDOM WALK

- LSW proved that the scaling limit of LERW is SLE_2 (Schramm had already identified $\kappa = 2$ as the appropriate candidate and the physics literature had $\mathbf{c} = -2$.)
- In particular, the paths has dimension $5/4$. Rick Kenyon had previously used a relationship with domino tilings and dimers to prove a discrete analogue of this statement. Recently, L. has given a different version of this argument that does not used domino tilings or dimers.
- The LERW is closely related to the uniform spanning tree. LSW shows that the scaling limit of the uniform spanning tree is SLE_8.
- One can obtain information about the loop-erased random walk directly from the SLE_2 result (Masson).

GAUSSIAN FREE FIELD

- Schramm and Sheffield have shown that the level lines of the Gaussian free field correspond to $\kappa = 4$ (as does a similar model called the harmonic explorer.) This is $\mathbf{c} = 1$.
- Much exciting work is being done by Sheffield (including joint work with others, Duplantier, Miller) on a mathematical model of quantum gravity which can be thought of as a model of random fractals in a random geometry. The random geometry (metric) comes from the Gaussian free field.

ISING MODEL

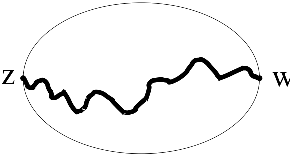
- The Ising model is a model for ferromagnets.
- It is one of a large class of models called Potts models which are related to random cluster models.
- The scaling limit of the interfaces for the Ising model should satisfy conformal invariance and the domain Markov property — hence SLE_κ for some κ. In fact $\kappa = 3, \mathbf{c} = 1/2$.
- Smirnov and collaborators have established the scaling limit — exciting work in progress.

SELF-AVOIDING WALK

- The scaling limit for SAW should have $\mathbf{c} = 0$ (restriction property). Assuming that the limit is on simple curves ($\kappa \leq 4$), this gives $\kappa = 8/3$.
- $SLE_{8/3}$ curves have dimension $4/3$ which gives the prediction $\nu = 3/4$.
- Simulations (Tom Kennedy) strongly support the conjecture that the limit is $SLE_{8/3}$.
- It is still an open question to prove that the scaling limit of SAW is $SLE_{8/3}$. (In fact, almost nothing is known rigorously about SAWs in two dimensions although Duminil-Copin and Smirnov have proven Nienhuis's predicition of the connective constant on the honeycomb lattice.)

NATURAL PARAMETRIZATION

- If $\kappa < 8$, then SLE_κ paths have Hausdorff dimension $d = 1 + \frac{\kappa}{8} \in (1, 2)$.
- The capacity parametrization is not the the natural parametrization that one should get from the scaling limit. In fact the two parametrizations are singular with respect to each other!
- Can we define the natural parametrization? Consider SLE_κ connecting boundary points z, w in a bounded domain D.



- Let $G(\zeta) = G_D(\zeta; z, w)$ denote the (chordal *SLE*) Green's function defined by

$$G(\zeta) = \lim_{\epsilon \downarrow 0} \epsilon^{d-2} \, \mathbb{P}\{\mathrm{dist}(\gamma, \zeta) < \epsilon\}.$$

- This limit exists (Rohde-Schramm, L., L.-Rezaei), can be given explicitly, and gives a conformally covariant quantity

$$G_{f(D)}(f(\zeta); f(z), f(w)) = |f'(\zeta)|^{2-d} \, G_D(\zeta; z, w).$$

- Let $\gamma : (0, \infty) \to D$ be SLE_κ is some parametrization and let Θ_t denote the amount of "natural time" used by time t. Then we expect (in some choice of time unit),

$$\mathbf{E}\left[\Theta_\infty\right] = \int_D G(\zeta) \, dA(\zeta).$$

- More generally, we expect that if $\gamma_t = \gamma[0,t]$ and $D_t = D \setminus \gamma_t$,

$$\mathbf{E}\left[\Theta_\infty \mid \gamma_t\right] = \Theta_t + \int_{D_t} G_{D_t}(\zeta; \gamma(t), z)\, dA(\zeta).$$

- The left-hand side is a martingale and hence Θ_t is an increasing process such that

$$\Theta_t + \int_{D_t} G_{D_t}(\zeta; \gamma(t), z)\, dA(\zeta)$$

  is a martingale (Doob-Meyer decomposition).
- L-Sheffield used this characterization to define the natural parametrization and showed that it existed for $\kappa < 5.0\cdots$.
- L-Zhou extended this result to show that it is well defined for $\kappa < 8$.

- L-Rezaei have recently shown that the natural parametrization is given by the d-dimensional Minkowski content defined by

$$\Theta_t = \lim_{\epsilon \downarrow 0} \epsilon^{d-2} \, \mathrm{Area} \left\{ \zeta : \mathrm{dist}(\zeta, \gamma[0,t]) < \epsilon \right\}.$$

- Rezaei has shown that the d-dimensional Hausdorff measure of the path is zero. (This is not surprising for a random fractal, but the proof requires work.)
- It is an open problem to show that discrete curves converge to *SLE in the natural parametrization*.

COMMENTS AND FUTURE DIRECTIONS

- Overall *SLE* has been well received by the theoretical physics community even though most of the exponents predicted by *SLE* had already been predicted (or at least conjectured) in the physics community. There has already been some progress and more will occur using *SLE*, the Brownian loop measure, and other similar conformally invariant objects to construct fields.

- *SLE* is simply connected domains is well understood because conformal invariance and the Markov property determine the process (up to one parameter). Simply connected domains have the property that if one slits them from a boundary point, the slit domain is conformally equivalent to the original domain. This is not true for non-simply connected domains and conformal invariance and domain Markov property do not determine the measures.
- For these domains (as in the case of boundary perturbation) it is useful to consider finite measures that are not probability measures which are normalized limits of partition functions. The effect on the probability measure is obtained using Girsanov theorem.

- *SLE* decribes a path or interface by giving it a random dynamics. However, the path is not formed according to these dynamics — rather the description of the path using the Loewner equation is only a way of collecting information about conditional probabilities as we explore parts of the path/domain.
- This is why some "obvious" results are difficult to prove. Zhan (also Dubédat) has recently given a nice proof that $SLE_\kappa(\kappa \leq 4)$ from z to w in a domain D is the same as the path from w to z. This is immediate on the lattice level for most of the models we are considering but is not easy to proof for *SLE* directly.

- Extending *SLE* to nonsimply connected domains is an area of current research. This is strongly related to questions about the Gaussian free field and the determinant of the Laplacian in such domains.
- This talk has focused on *SLE*. There are a lot of other exciting aspects to studying conformally invariant systems in two dimensions.

WHAT ABOUT THREE DIMENSIONS?

- The use of conformal invariance to study these systems is essentially a two-dimensional phenomenon (although there are some applications to four-dimensional questions). It seems much harder to analyze the very important case of three dimensions.
- SAW and LERW are expected to give interesting, nontrivial, random fractal paths in three dimensions.
- Interfaces as in percolation or Ising model become random surfaces rather than random curves.
- In two dimensions, critical exponents tend to take on rational values. There is no reason to believe that this is true in three dimensions.
- For example, for SAW, Flory conjectured that a typical SAW of n steps in three dimensions would have diameter n^{ν} where $\nu = 3/5$. This is no longer believed to be the exact value. Numerical simulations suggest $\nu = .588\cdots$