GEOMETRIC AND FRACTAL PROPERTIES OF SCHRAMM-LOEWNER EVOLUTION (SLE)

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CRITICAL PHENOMENA IN STATISTICAL PHYSICS

- Critical phenomena is the study of system at or near the point at which a phase transition occurs. For example, when the temperature of water varies there are three distinct phases (solid, liquid, gas) with sharp transitions.
- There are many simply stated lattice models depending on one or more parameters that have this sharp behavior as the parameters change.
- The behavior of these models depends strongly on the spatial dimension.
- In the case of spatial dimension 2, the systems have scaling limits that exhibit conformal invariance. This was discovered first by physicists which led them to many predictions (nonrigorous results) about these processes.

- One example well known to probabilists — in two dimensions, simple random walk has a scaling limit of Brownian motion which is conformally invariant in $\mathbb{R}^2 = \mathbb{C}$.
- There has been much work in the last fifteen years by mathematicians solving open questions about other conformally invariant models.
- There are a number of new techniques, but I will focus on one important object, the Schramm-Loewner evolution (SLE).
- In this talk, I will introduce some discrete models, define SLE, and then discuss some recent work about SLE.

SELF-AVOIDING WALK (SAW)

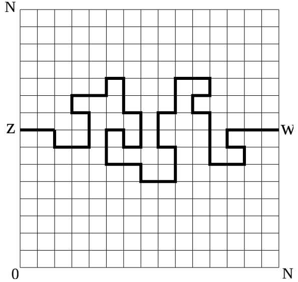
- Model for polymer chains — polymers are formed by monomers that are attached randomly except for a self-avoidance constraint.

$$\omega = [\omega_0, \ldots, \omega_n], \quad \omega_j \in \mathbb{Z}^2, \quad |\omega| = n$$

$$|\omega_j - \omega_{j-1}| = 1, \quad j = 1, \ldots, n$$

$$\omega_j \neq \omega_k, \quad 0 \leq j < k \leq n.$$

- Critical exponent ν: a typical SAW has diameter about $|\omega|^\nu$.
- If no self-avoidance constraint $\nu = 1/2$; for 2-d SAW Flory predicted $\nu = 3/4$.

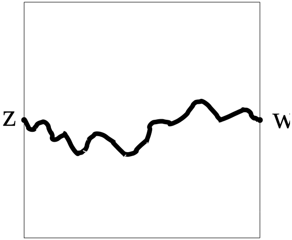


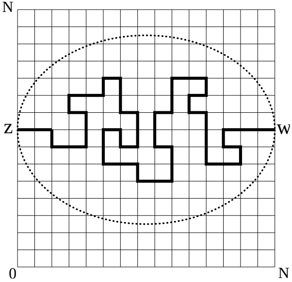
Each SAW from z to w gets measure $e^{-\beta|\omega|}$. Partition function

$$Z = Z(N, \beta) = \sum e^{-\beta|\omega|}.$$

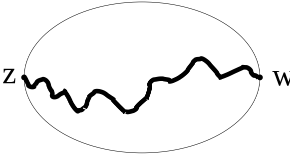
β small — typical path is two-dimensional
β large — typical path is one-dimensional
β_c — typical path is $(1/\nu)$-dimensional

Choose $\beta = \beta_c$; let $N \to \infty$; divide by $Z(N, \beta)$ and hope to get a probability measure on curves connecting boundary points of the square.



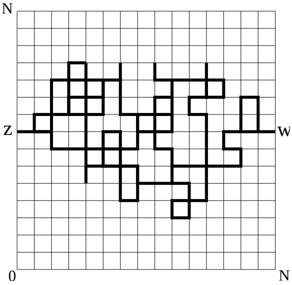


Similarly, if we fix $D \subset \mathbb{C}$, we can consider walks restricted to the domain D



Predict that these probability measures are conformally invariant.

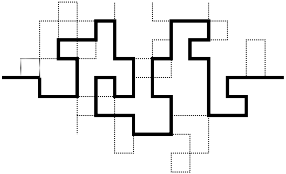
SIMPLE RANDOM WALK



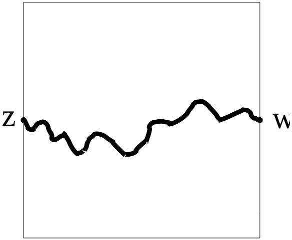
- Simple random walk — no self-avoidance constraint. Criticality: each walk ω gets weight $(1/4)^{|\omega|}$.
- Scaling limit is *Brownian motion* which is conformally invariant (Lévy).

LOOP-ERASED RANDOM WALK

Start with simple random walks and erase loops in chronological order to get a path with no self-intersections.

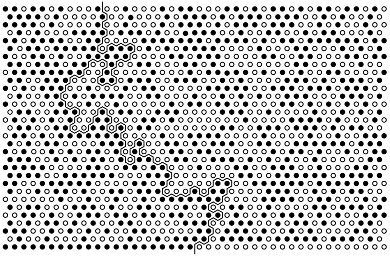


Limit should be a measure on paths with no self-intersections.



CRITICAL PERCOLATION

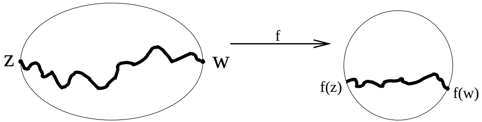
Color vertices of the triangular lattice in the upper half plane black or white independently each with probability 1/2.



Put a boundary condition of black on negative real axis and white on positive real axis. The *percolation exploration process* is the boundary between black and white.

ASSUMPTIONS ON SCALING LIMIT

Probability measure $\mu_D(z,w)$ on curves connecting boundary points of a domain D.



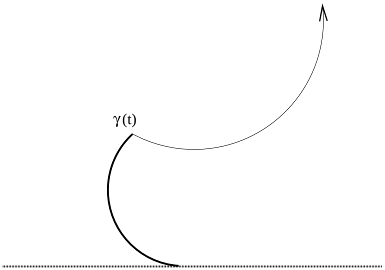
- **Conformal invariance**: If f is a conformal transformation

$$f \circ \mu_D(z,w) = \mu_{f(D)}(f(z), f(w)).$$

- For simply connected D, it suffices to know $\mu_{\mathbb{H}}(0,\infty)$ (Riemann mapping theorem).

- **Domain Markov property** Given $\gamma[0,t]$, the conditional distribution on $\gamma[t,\infty)$ is the same as

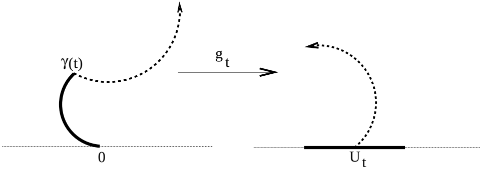
$$\mu_{\mathbb{H}\setminus\gamma(0,t]}(\gamma(t),\infty).$$



- Satisfied on discrete level by SAW, LERW, percolation exploration, ... (but not by simple random walk)

LOEWNER EQUATION IN UPPER HALF PLANE

- Let $\gamma : (0, \infty) \to \mathbb{H}$ be a simple curve with $\gamma(0+) = 0$ and $\gamma(t) \to \infty$ as $t \to \infty$.
- $g_t : \mathbb{H} \setminus \gamma(0, t] \to \mathbb{H}$



- Can reparametrize if necessary so that

$$g_t(z) = z + \frac{at}{z} + \cdots , \qquad z \to \infty$$

- g_t satisfies

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Moreover, $U_t = g_t(\gamma(t))$ is continuous.

(Schramm) Suppose γ is a random curve satisfying conformal invariance and Domain Markov property. Then U_t must be a random continuous curve satisfying

- For every $s < t$, $U_t - U_s$ is independent of $U_r, 0 \leq r \leq s$ and has the same distribution as U_{t-s}.

Therefore, U_t is a Brownian motion which is driftless by symmetry or by scaling. We can make the variance 1 by choosing a appropriately.

The *(chordal) Schramm-Loewner evolution with parameter* κ (SLE_κ) is the solution obtained by choosing $U_t = B_t$ and $a = 2/\kappa$.

(Rohde-Schramm) Solving the Loewner equation with a Brownian input gives a random curve.

The qualitative behavior of the curves varies greatly with κ

- $0 < \kappa \leq 4$ — simple (non self intersecting) curve
- $4 < \kappa < 8$ — self-intersections (but not crossing); not plane-filling
- $8 \leq \kappa < \infty$ — plane-filling

(Beffara) For $\kappa < 8$, the Hausdorff dimension of the paths is

$$1 + \frac{\kappa}{8}.$$

- $\kappa = 2$ loop-erased random walk [L,Schramm, Werner]
- $\kappa = 8/3$ self-avoiding walk. Strong theoretical reasoning why this is correct limit but scaling limit not proved. [LSW]
- $\kappa = 3, \kappa = 16/3$ Ising model and related cluster model interfaces. [Smirnov]
- $\kappa = 4$ interfaces of Gaussian free field and related harmonic explorer. [Schramm, Sheffield]
- $\kappa = 6$ percolation interfaces. [Smirnov, LSW]
- $\kappa = 8$ Peano curves arising from uniform spanning trees. [LSW]

QUESTIONS

How do we describe the evolution of a random curve $\gamma(t)$ which is fractal and has self-repulsion?

- Fractal dimension d of the path
- Hölder continuity of paths in given parametrization.
- Can we find a parametrization of the path that is Hölder continuous of order $\alpha < 1/d$?
- Can we find an analogue of the "derivative" of the path? In two dimensions, the local behavior of the conformal map is the key.
- Can we describe the exceptional points?

TOOLS

- Stochastic calculus (Brownian motion, Itô integral, (local) martingales, Girsanov theorem)
- Basics of conformal maps (Distortion theorems, extremal length, Beurling estimate). More generally, the bag of tricks for understanding boundary behavior of conformal transformations.
- Properties of two-dimensional Brownian motion (closely related to complex analysis).
- Simple parabolic PDE (heat equation and related).

The tricky thing is to learn how to study a curve by studying the "noncurve" — the curve is the boundary of a flow of conformal maps.

FRACTAL DIMENSION OF PATHS FOR $\kappa < 8$

- The first step in computing the dimension is to find $d, G(z)$ such that

$$\mathbf{P}\{\mathrm{dist}(z, \gamma(0,\infty)) \leq \epsilon\} \asymp G(z)\, \epsilon^{2-d}.$$

(Expected number of disks of radius ϵ needed to cover $\gamma(0,1]$ in a is of order ϵ^{-d}.)

- Using stochastic calculus, Rohde and Schramm (essentially) showed that the only possibility is

$$d = 1 + \frac{\kappa}{8}$$

$$G(re^{i\theta}) = r^{d-2} \sin^u \theta, \quad u = 8\kappa + \frac{1}{8\kappa} - 2 > 0.$$

They showed this gives an upper bound on the dimension.

- Easier to consider $\Upsilon_t = \Upsilon_t(z)$, (1/2 times) the conformal radius of z in H_t, the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$. By the Koebe-(1/4) Theorem or Schwarz lemma,

$$\Upsilon_t \asymp_2 \mathrm{dist}(z, \gamma(0, t] \cup \mathbb{R}).$$

  $\Upsilon = \Upsilon_\infty$.
- Using another technique, the Girsanov theorem, one can show as $\epsilon \to 0$,

$$\mathbf{P}\{\Upsilon < \epsilon\} \sim c_* \, G(z) \, \epsilon^{2-d}.$$

- $G(z)$ is called the Green's function for chordal SLE. Roughly speaking it gives the "probability that the SLE path goes through z".

- Beffara proved the more difficult two-point estimate

$$\mathbf{P}\{\Upsilon(z) < \epsilon, \Upsilon(w) < \epsilon\} \leq c\,\epsilon^{2-d}\left(\frac{\epsilon}{|z-w|}\right)^{2-d},$$

  at least for z, w restricted to a bounded domain bounded away from $\mathbb{R}$.
- With this estimate one can use standard techniques (Frostman's lemma) to show that Hausdorff dimension of path is d.
- With Brent Werness we reproved the estimate and also established the existence of a "two-point" Green's function, that is, a function $G(z, w)$ such that

$$\mathbf{P}\{\Upsilon(z) < \epsilon, \Upsilon(w) < \delta\} \sim c_*^2\, G(z,w)\,\epsilon^{2-d}\,\delta^{2-d}, \quad \epsilon, \delta \to 0.$$

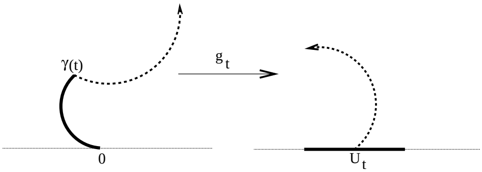
  Unfortunately, the proof does not give an explicit form for $G(z, w)$. One can write a PDE that $G(z, w)$ must satisfy.

How should an *SLE* curve be parametrized?

- *SLE* is defined using the Loewner equation. In order to make $g_t(z)$ differentiable with respect to t, one parametrizes the curve by capacity.
- If γ were a smooth curve, we could also consider the "natural" parametrization by arc length. In this case, capacity parametrization is a smooth reparametrization.
- For random d-dimensional curves, one might hope to find a "natural" parametrization.
- For discrete models such as SAW or LERW, there is a natural length of the path which corresponds to the number of steps taken — can we use a normalized version of this for the parametrization of the limit?
- There is a way to define the "natural parametrization" or "natural length" of an SLE curve. It turns out to be singular with respect to the capacity parametrization.

NATURAL PARAMETRIZATION OR LENGTH

- Suppose $\gamma : [0,1] \to \mathbb{R}^2$ is a curve such that for each $a < b$, $\dim_h(\gamma[a,b]) = d$.
- There are a number of properties that can describe a d-dimensional parametrization of a curve:
  - Hölder continuous of order $\alpha < 1/d$. Roughly speaking,
  $$|\gamma(s) - \gamma(t)| \approx |s - t|^{1/d}.$$
  - If $f : D \to f(D)$ is a conformal transformation, then the time needed to traverse $f(\gamma[0,t])$ is
  $$\int_0^t |f'(\gamma(s))|^d \, ds.$$
- Brownian motion in $\mathbb{C}$ has a 2-dimensional parametrization.
- The capacity parametrization of SLE is not a d-dimensional parametrization.



Let

$$\tilde{\gamma} = g_t\left(\gamma[t, t + \Delta t]\right).$$

- The cap param is defined so that the time to traverse $\tilde{\gamma}$ is Δt, the same as the time to traverse $\gamma[t, t + \Delta t]$
- In natural param, the time to traverse $\tilde{\gamma}$ should be

$$\int_0^{\Delta t} \left|g_s'(\gamma(s))\right|^d \, ds.$$

- Important to understand $|g_t'|$ or $|(g_t^{-1})'|$.

SOME POSSIBLE (UNPROVED) DEFINITIONS

- *d*-variation

$$\Theta_t = \lim_{n\to\infty} \sum_{j=1}^{n} \left|\gamma(\frac{j}{n}t) - \gamma(\frac{j-1}{n}t)\right|^d .$$

- *d*-dimensional Minkowski content of Θ_t,

$$\Theta_t = \lim_{\epsilon\to 0+} \epsilon^{d-2} \, \mathrm{Area}\, \{\mathrm{z} : \mathrm{dist}(\mathrm{z}, \gamma(0, \mathrm{t}]) \leq \epsilon\} .$$

- Hausdorff measure with appropriate gauge of Θ_t. (The usual *d*-dimensional Hausdorff measure of the path is probably zero — this is a characteristic of random fractals and makes them somewhat different than deterministic curves such as snowflakes.)

RIGOROUS DEFINITION (L - Sheffield)

- Let $\Theta_t(D)$ denote the "amount of natural time spent in domain D up to capacity time t". This is currently undefined. For ease, assume D is a bounded domain.
- Expect (up to multiplicative constant)

$$\mathbf{E}\left[\Theta_\infty(D)\right] = \int_D G(z)\, dA(z).$$

- Let $\gamma_t = \gamma(0,t]$.
- $$\mathbf{E}[\Theta_\infty(D) \mid \gamma_t] = \Theta_t(D) + \mathbf{E}[\Theta_\infty(D) - \Theta_t(D) \mid \gamma_t]$$
- Using conformal invariance, one can show that the last term equals

$$\Psi_t(D) = \int_D |g_t'(z)|^{2-d}\, G(Z_t(z))\, dA(z).$$

- $N_t = \mathbf{E}[\Theta_\infty(D) \mid \mathcal{F}_t]$ is a martingale. (This is the prototypical martingale — we have a final random variable and we take is conditional expectation given increasing information.)

- We have written

$$\Psi_t(D) = N_t - \Theta_t(D),$$

and $\Theta_t(D)$ is increasing.
- Doob-Meyer decomposition. Under certain assumptions, we can show that there exists a unique increasing process $\Theta_t(D)$ that makes

$$\Psi_t(D) + \Theta_t(D)$$

a martingale. This is the definition of $\Theta_t(D)$.
- There are technical issues in establishing the conditions to apply the Doob-Meyer theorem. These are essentially "two-point" or "second moment" estimates.
- The proof of the Doob-Meyer decomposition gives formula

$$\Theta_1 = \lim_{n\to\infty} \sum_{j=1}^{n} |f'_{\frac{j-1}{n}}(z)|^d \, \phi(z\sqrt{n}) \, G(z) \, dA(z).$$

where ϕ is a bounded function (almost of compact support) and $f_t(z) = g_t^{-1}(z + U_t)$.

- One either tries to get estimates for the "reverse Loewner flow" or uses a version of Beffara's estimate.

Theorem

(L, Sheffield) At least for $\kappa < 4(7-\sqrt{33}) = 5.021\cdots$, the natural parametrization defined as above exists.

- Proof uses reverse Loewner flow and does not need Beffara's estimate. It also gives bounds on Hölder continuity of Θ_t. The argument should work for all $\kappa < 8$ with corresponding bounds, but this has not been done.

Theorem

(L, Wang Zhou) The natural parametrization exists for all $\kappa < 8$.

- Proof uses Beffara's estimate and does not give the Hölder continuity bounds. It uses a two-point martingale which is expressed in terms of the two-point Green's function.

CONJECTURE The natural length is given (up to multiplicative constant) by the d-dimensional Minkowski content.

As a start to this, in work with Mohammad Rezaei, we have shown that the natural parametrization is local, in that the length of a curve does not depend on the domain. (This may seem obvious, but the definition of SLE in different domains uses conformal transformation.)

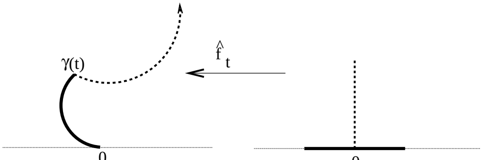
TIP MULTIFRACTAL SPECTRUM

(joint work with F. Johansson Viklund)

- To understand singularity of natural and capacity parametrizations, one analyzes exceptional points of the curves γ. The object of study is the derivative of the conformal map g_t near $\gamma(t)$.
- This is very closely related to the behavior of harmonic measure on the domain H_t near $\gamma(t)$.
- Consider the SLE_κ path $\gamma(t)$ with conformal maps g_t, driving function U_t and

$$f_t(z) = g_t^{-1}(z + U_t).$$

- More convenient to consider behavior of $f_t'(z)$ for z near the origin.



Example: Looking at Natural Length

- The Doob-Meyer theory gives a formula of the "natural length" of $\gamma[0,1]$ that looks like

$$\lim_{n\to\infty} n^{1-\frac{d}{2}} \sum_{j=1}^{n} |f'(i/\sqrt{n})|^d .$$

- $$\mathbf{E}\left[|f_1'(i/\sqrt{n})|^d\right] \asymp n^{\frac{d}{2}-1}$$

- This expectation is concentrated on paths for which $|f_1'(i/\sqrt{n})| \approx n^{d-\frac{3}{2}}$.

$$\mathbf{P}\{|f_1'(i/\sqrt{n})| \approx n^{d-\frac{3}{2}}\} \approx n^{-(d^2-2d+1)}$$

- The "dimension" of set of *times* $t \in [0,1]$ with $|\hat{f}_t'(i/\sqrt{n})| \approx n^{d-\frac{3}{2}}$ equals

$$1-(d^2-2d+1) = d(2-d) < 1.$$

- This is a set of dimension strictly less than one. However, these are the points on which the natural parameterization lives.
- From this we see that the natural param is singular with respect to capacity parametrization and is carried on a set of times Λ of dimension $d(2-d)$.
- However, the dimension of $\gamma(\Lambda) = d$, the full dimension of the path.
- If we replace d with other powers λ, we concentrate on different sets of times. This is an example of multifractal behavior.

MOMENTS FOR DERIVATIVES OF f

- To understand the "multifractal behavior" of $f_1'(iy)$ for small y, one studies the moment

$$\mathbf{E}[|f_1'(iy)|^\lambda] = \mathbf{E}\left[|f_{1/y^2}'(i)|^\lambda\right].$$

- To state the results we introduce some notation. For a range of λ,

$$r(\lambda) = 2a + 1 - \sqrt{(2a+1)^2 - 4a\lambda},$$

$$\zeta(\lambda) = \lambda - \frac{r}{2a}, \quad \beta(\lambda) = -\zeta'(\lambda).$$

$$\zeta'(\lambda) = 1 - \frac{1}{\sqrt{(2a+1)^2 - 4a\lambda}}, \quad \zeta'(\lambda_c) = -1,$$

$$\rho(\lambda) = -\lambda\beta + \zeta.$$

Theorem

(L-Johansson Viklund)
Let Λ_β denote the set of t such that

$$|\hat{f}_t'(iy)| \approx y^{-\beta}, \quad y \to 0+ .$$

Then if $\rho \leq 2$,

$$\dim_h(\Lambda_\beta) = \frac{2-\rho}{2},$$

$$\dim_h[\gamma(\Lambda_\beta)] = \frac{2\dim_h(\Lambda_\beta)}{1-\beta} = \frac{2-\rho}{1-\beta}.$$

- This is a statement about the almost sure Hausdorff dimension and not just a statement about expectations. This requires second moment estimates.

- If B_t is a two-dimensional Brownian motion and A is any (perhaps random) Borel subset of $[0,1]$, then

$$\dim_h [B(A)] = 2 \dim_h(A).$$

  For exceptional sets of SLE we don't get a simple rule like this.

- Instead we get the rule

$$\dim_h[\gamma(\Lambda_\beta)] = \frac{2}{\beta - 1} \dim_h[\Theta_\beta].$$

  This follows (roughly) from the fact that on the set Λ_β,

$$|\hat{f}_t'(\epsilon i)| \approx \epsilon^{-\beta}.$$

- Similar ideas allow one to prove the following.

Theorem

(L-Johansson Viklund) If γ is SLE$_\kappa$ (in capacity parametrization) with probability one $\gamma(t), \epsilon \leq t \leq 1$ is Hölder continuous of order $\alpha < \alpha_*$ but not $\alpha > \alpha_*$ where

$$\alpha_* = 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{8 + \kappa}}.$$

- $\alpha_* > 0$ except if $\kappa = 8$.
- This can be conjectured from moment estimates of Rohde and Schramm and one direction was proved by Joan Lind. The new direction requires second moment estimates.

SOME CLOSING COMMENTS

- The theory of SLE is a very powerful tool which allows for detailed analysis of conformally invariant fractals. It is a very active area of research
- The two-dimensional world of “critical phenomena” is particularly nice because of conformal invariance.
- There are also very interesting models of random curves in three dimensions, such as the self-avoiding walk and the loop-erased walk, but we do not have tools to analyze them.
- Interfaces such as in percolation are also interesting but these are now random surfaces.
- It would be good to construct continuous models for three-dimensional random fractals with self-repulsion even if we cannot show that they are the limits of known processes.