1 Introduction

The purpose of these notes is to present the discrete time analogs of the results in Markov Loops and Renormalization by Le Jan [1]. A number of the results appear in Chapter 9 of Lawler and Limic [2], but there are additional results. We will tend to use the notation from [2] (although we will use [1] for some quantities not discussed in [2]), but our Section heading will match those in [1] so that a reader can read both papers at the same time and compare

2 Symmetric Markov processes on finite spaces

We let $\mathcal{X}$ denote a finite or countably infinite state space and let $q(x,y)$ be the transition probabilities for an irreducible, discrete time, Markov chain X_n on $\mathcal{X}$. Let A be a nonempty, finite, proper subset of $\mathcal{X}$ and let $Q = [q(x,y)]_{x,y\in A}$ denote the corresponding matrix restricted to states in A. For everything we do, we may assume that $\mathcal{X} \setminus A$ is a single point denoted ∂ and we let

$$\kappa_x = q(x,\partial) = 1 - \sum_{x\in A} q(x,y).$$

We say that Q is *strictly subMarkov* on A if for each $x \in A$ with probability one the chain eventually leaves A. Equivalently, all of the eigenvalues of Q have absolute value stricly less than one. We will call such weights *allowable*. Let $N = \#(A)$ and $\alpha_1,\ldots,\alpha_N$ the eigenvalues of Q all of which have absolute value strictly less than one. We let X_n^* denote the path

$$X_n^* = [X_0, X_1, \ldots, X_n].$$

We will let ω denote paths in A, i.e., finite sequences of points

$$\omega = [\omega_0, \omega_1, \ldots, \omega_n], \quad \omega_j \in A.$$

We call n the *length* of ω and sometimes denote this by $|\omega|$. The weight q induces a measure on paths in A,

$$q(\omega) = \mathbb{P}^{\omega_0}\{X_n^* = \omega\} = \prod_{j=0}^{n-1} q(\omega_j, \omega_{j+1}).$$

The path is called a *(rooted) loop* if $\omega_0 = \omega_n$. We let η^x denote the trivial loop of length 0, $\eta^x = [x]$. By definition $q(\eta^x) = 1$ for each $x \in A$.

♣ We have not assumed that Q is irreducible, but only that the chain restricted to each component is strictly subMarkov. We do allow $q(x,x) > 0$.

Since q is symmetric we sometimes write $q(e)$ where e denotes an edge. Let

$$\Delta f(x) = (Q - I) f(x) = \sum_{y\in\mathcal{X}} q(x,y)\,[f(y) - f(x)].$$

Unless stated otherwise, we will consider Δ as an operator on functions f on A which can be considered as functions on $\mathcal{X}$ that vanish on $\mathcal{X} \setminus A$. In this case, we can write

$$\Delta f(x) = -\kappa_x \, f(x) + \sum_{y \in A} q(x,y) \, [f(y) - f(x)].$$

♣ [1] uses $C_{x,y}$ for $q(x,y)$ and calls these quantities conductances. This paper does not assume that the conductances are coming from a transition probability, and allows more generality by letting κ_x be anything and setting

$$\lambda_x = \kappa_x + \sum_y q(x,y).$$

We do not need to do this — the major difference in our approach is that we allow the discrete loops to stay at the same point, i.e., $q(x,x) > 0$ is allowed. The important thing to remember when reading [1] is that under our assumption

$$\lambda_x = 1 \text{ for all } x \in A,$$

and hence one can ignore λ_x wherever it appears.

Two important examples are the following.

- Suppose $A = \{x\}$ with $q(x,x) = q \in (0,1)$. We will call this the *one-point example.*
- Suppose q is an allowable weight on A and $A' \subset A$. We can consider a Markov chain Y_n with state space $A' \cup \{\partial\}$ given as follows. Suppose $X_0 \in A'$. Then $Y_n = X_{\rho(n)}$ where $\rho_0 = 0$ and

  $$\rho_j = \min\left\{n > \rho_{j-1} : X_n \in A' \cup \{\partial\}\right\}.$$

  The corresponding weights on A' are given by the matrix $\hat{Q}_{A'} = [\hat{q}_{A'}(x,y)]_{x,y \in A'}$ where

  $$\hat{q}_{A'}(x,y) = \mathbb{P}^x\left\{X_{\rho(1)} = y\right\}, \quad x,y \in A'.$$

  We call this the chain *viewed at* A'. This is not the same as the chain induced by the weight

  $$q(x,y), \quad x,y \in A',$$

  which corresponds to a Markov chain killed when it leaves A'. Let $G_{A'}$ denote the Green's function restricted to A'. Then

  $$\hat{Q}_{A'} = I - [G_{A'}]^{-1}.$$

  Note that $[G_{A'}]^{-1}$ is not the same matrix as G^{-1} restricted to A'.

♣ We will be relating the Markov chain on A with random variables $\{Z_x : x \in A\}$ having joint normal distribution with covariance matrix G. One of the main properties of the joint normal distribution is that if $A' \subset A$, the marginal distribution of $\{Z_x : x \in A'\}$ is the joint normal with covariance matrix $G_{A'}$. We have just seen that this can be considered in terms of a Markov chain on A' with a particular matrix $\hat{Q}_{A'}$. Note that even if Q has no positive diagonal entries, the matrix $\hat{Q}_{A'}$ may have positive diagonal entries. This is one reason why it is useful to allow such entries from the beginning.

We let S_t denote a continuous time Markov chain with rates $q(x, y)$. Since q is a Markov transition probability (on $A \cup \{\partial\}$), we can construct the continuous time Markov chain from a discrete Markov chain X_n as follows. Let $T_1, T_2, \ldots$ be independent $Exp(1)$ random variables, independent of the chain X_n, and let $\tau_n = \tau_1 + \cdots + \tau_n$ with $\tau_0 = 0$. Then

$$S_t = X_n \text{ if } \tau_n \leq t < \tau_{n+1}.$$

We write S_t^* for the discrete path obtained from watching the chain "when it jumps", i.e.,

$$S_t^* = [X_0, \ldots, X_n] = X_n^* \text{ if } \tau_n \leq t < \tau_{n+1}.$$

If ω is a path with $\omega_0 = x$ and $\tau_\omega = \inf\{t : S_t^* = \omega\}$, then one sees immediately that

$$\mathbb{P}^x\{\tau_\omega < \infty\} = q(\omega). \tag{1}$$

♣ We allow $q(x, x) > 0$ so the phrase "when it jumps" is somewhat misleading. Suppose that $X_0 = x, X_1 = x$ and t is a time with $\tau_1 \leq t < \tau_2$. Then

$$S_t^* = [x, x].$$

If we only observed the continuous time chain, we would not observe the "jump" from x to x, but in our setup we consider it a jump. It is useful to consider the continuous time chain as the pair of the discrete time chain and the exponential holding times. We are making use of the fact that q is a transition probability and hence the holding times can be chosen independently of the position of the discrete chain.

2.1 Energy

The Dirichlet form or energy is defined by

$$\mathcal{E}(f, g) = \sum_e q(e)\, \nabla_e f\, \nabla_e \overline{g},$$

where $\nabla_e f = f(x) - f(y)$ where $e = \{x, y\}$. (This defines ∇_e up to a sign but we will only use it in products — in $\nabla_e f\, \nabla_e \overline{g}$ we take the same orientation of e for both differences.) We will consider this as a form on functions in A, i.e., on functions on $\mathcal{X}$ that vanish on $\mathcal{X} \setminus A$. In this case we can write

$$\begin{aligned}\mathcal{E}(f,g) &= \sum_{e \in e(A)} q(e)\,\nabla_e f \nabla_e \overline{g} + \sum_{e \in \partial_e A} q(e) \nabla_e f \nabla_e \overline{g} \\ &= \frac{1}{2} \sum_{x,y \in A} q(x,y)\,[f(x)-f(y)]\,[\overline{g}(x)-\overline{g}(y)] + \sum_{x \in A} \kappa_x\, f(x)\,\overline{g}(x) \\ &= \sum_{x,y \in A} f(x)\,\overline{g}(x) - \sum_{x,y \in A} q(x,y)\, f(x)\,\overline{g}(y).\end{aligned}$$

We let $\mathcal{E}(f) = \mathcal{E}(f,f)$.

♣ If we write $\mathcal{E}_q(f,g)$ to denote the dependence on q, then it is easy to see for $a \in \mathbb{R}$,

$$\mathcal{E}_{a^2 q}(f,g) = \mathcal{E}_q(af, ag) = a^2\, \mathcal{E}_q(f,g).$$

The definition of $\mathcal{E}$ does not require q to be a subMarkov transition matrix. However, we can always find an a such that $a^2 q$ is subMarkov, so assuming that q is subMarkov is not restrictive.

♣ The set X in [1] corresponds to our A. [1] uses $z^x, x \in \mathcal{X}$ to denote a function on $\mathcal{X}$. [1] uses $e(z)$ for $\mathcal{E}(f)$; we will use e for edges.

Recall that $(-\Delta)^{-1} = (I-Q)^{-1}$ is the Green's function defined by

$$G(x,y) = \sum_{\omega : x \to y} q(\omega) = \sum_{n=0}^{\infty} \sum_{\omega : x \to y, |\omega| = n} \mathbb{P}^x\{X_n^* = \omega\} = \sum_{n=0}^{\infty} \mathbb{P}^x\{X_n = y\}.$$

This is also the Green's function for the continuous time chain.

Proposition 2.1.

$$G(x,y) = \int_0^\infty \mathbb{P}^x\{S_t = y\}\, dt = \sum_{\omega : x \to y} \int_0^\infty \mathbb{P}^x\{S_t^* = \omega\}\, dt.$$

Proof. The second equality is immediate. For any path ω in A, it is not difficult to verify that

$$q(\omega) = \int_0^\infty \mathbb{P}\{S_t^* = \omega\}\, dt.$$

This follows from (1) and

$$\mathbf{E} \int_s^\infty \mathbb{P}\{S_t^* = \omega \mid \tau_\omega = s\}\, dt = 1.$$

The latter equality holds since the expected amount of time spent at each point equals one. ∎

The following observation is important. It follows from the definition of the chain viewed at A'.

Proposition 2.2. *If q is an allowable weight on A with Green's function $G(x, y), x, y \in A$, and $A' \subset A$, then the Green's function for the chain viewed at A' is $G(x, y), x, y \in A'$.*

♣ In [1], Δ is denoted by L. There are two Green's functions discussed, V and G. These two quantities are the same under our assumption $\lambda \equiv 1$.

2.2 Feynman-Kac formula

The Feynman-Kac formula describes the affect of a killing rate on a Markov chain. Suppose q is an allowable weight on A and $\chi : A \to [0, \infty)$ is a nonnegative function.

2.2.1 Discrete time

We define another allowable weight q^χ by

$$q^\chi(x, y) = \frac{1}{1 + \chi(x)}\, q(x, y).$$

If $\omega = [\omega_0, \ldots, \omega_n]$ is a path, then

$$q^\chi(\omega) = q(\omega) \prod_{j=0}^{n-1} \frac{1}{1 + \chi(\omega_j)} = q(\omega) \exp\left\{ -\sum_{j=0}^{n-1} \log[1 + \chi(\omega)] \right\}. \tag{2}$$

We think of $\chi/(1 + \chi)$ as an additional killing rate to the chain. More precisely, suppose T is a positive integer valued random variable with distribution

$$\mathbb{P}\{T = n \mid T > n - 1, X_{n-1} = x\} = \frac{\chi(x)}{1 + \chi(x)}.$$

Then if $\omega_0 = x$,

$$\mathbb{P}^x\{S_n^* = \omega, T > n\} = q(\omega) \prod_{j=0}^{n-1} \frac{1}{1 + \chi(\omega_j)} = q^\chi(\omega).$$

This is the Feynman-Kac formula in the discrete case. we will compare it to the continuous time process with killing rate χ.

Let Q^χ denote the corresponding matrix of rates. Then we can write

$$Q^\chi = M_{1+\chi}^{-1}\, Q.$$

Here and below we use the following notation. If $g : A \to \mathbb{C}$ is a function, then M_g is the diagonal matrix with $M_g(x,x) = g(x)$. Note that if g is nonzero, $M_g^{-1} = M_{1/g}$. We let

$$G_\chi = (I - Q^\chi)^{-1} = (I - M_{1+\chi}^{-1} Q)^{-1} \tag{3}$$

be the Green's function for q^χ.

♣ Our G_χ is not the same as G_χ in [1]. The G_χ in [1] corresponds to what we call $\tilde{G}_\chi$ below.

2.2.2 Continuous time

Now suppose T is a continuous killing time with rate χ. To be more precise, T is a nonnegative random variable with

$$\mathbb{P}\{T \le t + \Delta t \mid T > t, S_t = x\} = \chi(x)\,\Delta t + o(\Delta t).$$

In particular, the probability that the chain starting at x is killed before it takes a discrete step is $\chi(x)/[1 + \chi(x)]$. We define the corresponding Green's function $\tilde{G}$ by

$$\tilde{G}(x,y) = \int_0^\infty \mathbb{P}^x\{S_t = y\}\,dt$$

There is an important difference between discrete and continuous time when considering killing rates. Let us first consider consider the case without killing. Let S_t denote a continuous time random walk with rates $q(x,y)$. Then S waits an exponential amount of time with mean one before taking jumps. At any time t, there is a corresponding discrete path obtained by considering the process when it jumps (this allows jumps to the same site). Let S_t^* denote the discrete path that corresponds to the random walk "when it jumps". For any path ω in A, it is not difficult to verify that

$$q(\omega) = \int_0^\infty \mathbb{P}\{S_t^* = \omega\}\,dt.$$

The basic reason is that if $\tau_\omega = \inf\{t : S_t^* = \omega\}$, then

$$\mathbf{E} \int_s^\infty \mathbb{P}\{S_t^* = \omega \mid \tau_\omega = s\}\,dt = 1.$$

since the expected amount of time spent at each point equals one. From this we see that the Green's function for the continuous walk which is defined by

$$\tilde{G}_\chi(x,y) = \int_0^\infty \mathbb{P}^x\{S_t = y, T > t\}\,dt.$$

Proposition 2.3.

$$\tilde{G}_\chi = G_\chi \, M_{1+\chi}^{-1}. \tag{4}$$

Proof. This is proved in the same was as 2.1 except that

$$\int_0^\infty \mathbb{P}\{S_t^* = \omega, T > t\}\, dt = \frac{q^\chi(\omega)}{1+\chi(y)}.$$

The reason is that the time until one leaves y (by either moving to a new site or being killed) is exponential with rate $1 + \chi(y)$. □

♣ By considering generators, one could establish in a different way

$$\tilde{G}_\chi = (1 - Q + M_\chi)^{-1},$$

which follows from (3) (4). This is just a matter of personal preference as to which to prove first.

In particular,

$$\det[\tilde{G}_\chi] \prod_x [1 + \chi(x)] = \det[G_\chi], \tag{5}$$

and

$$\tilde{G}_\chi = [I - Q + M_\chi]^{-1} = (I - Q)^{-1}\,(I + GM_\chi)^{-1} = G\,(I + GM_\chi)^{-1}. \tag{6}$$

Example Let us consider the one-point example. Then

$$G(x,x) = 1 + q + q^2 + \cdots = \frac{1}{1-q}.$$

For the discrete time walk with killing rate $1 - \lambda = \chi/(1+\chi)$,

$$G_\chi(x,x) = 1 + q\lambda + [q\lambda]^2 + \cdots = \frac{1}{1-q\lambda} = \frac{1-\chi}{1+\chi-q}.$$

For the continuous time walk with the same killing rate χ, we start the path and we consider an exponential time with rate $1+\chi$. Then the expected time spent at x before jumping for the first time is $(1+\chi)$. At the first jump time, the probability that we are not killed is $q/(1+\chi)$. (Here $1/1+\chi$ is the probability that the continuous time walk decides to move before being killed.) Therefore

$$\tilde{G}_\chi(x,x) = \frac{1}{1+\chi} + \frac{q}{1+\chi}\, G_\chi(x,x),$$

which gives

$$\tilde{G}_\chi(x,x) = \frac{1}{1-q+\chi} = \frac{G_\chi}{1+\chi}.$$

3 Loop measures

3.1 A measure on based loops

Here we expand on the definitions in Section (2) defining (discrete time) unrooted loops and continuous time loops and unrooted loops.

A *(discrete time) unrooted loop* $\overline{\omega}$ is an equivalence class of rooted loops under the equivalence relation

$$[\omega_0, \ldots, \omega_n] \sim [\omega_j, \omega_{j+1}, \ldots, \omega_n, \omega_1, \ldots, \omega_{j-1}, \omega_j].$$

We define $q(\overline{\omega}) = q(\omega)$ where ω is any representative of $\overline{\omega}$.

A *nontrivial continuous time rooted loop* of length $n > 0$ is a rooted loop ω of length n combined with times $\overline{T} = (T_1, \ldots, T_n)$ with $T_j > 0$. We think of T_j as the time for the jump from ω_{j-1} to ω_j. We will write the loop in one of two ways

$$(\omega, \overline{T}) = (\omega_0, T_1, \omega_1, T_2, \ldots, T_n, \omega_n).$$

The continuous time loop also gives a function $\omega(t)$ of period $T_1 + \cdots + T_n$ with

$$\omega(t) = \omega_j, \quad \tau_{j-1} \leq t < \tau_j.$$

Here $\tau_0 = 0$ and $\tau_j = T_1 + \cdots + T_j$.

♣ We caution that the function $\omega(t)$ may not carry all the information about the loop; in particular, if $q(x, x) > 0$ for some x, then one does not observe the "jump from x to x" if one only observes $\omega(t)$.

A *nontrival continuous time unrooted loop* of length n is an equivalence class where

$$(\omega_0, T_1, \omega_1, T_2, \ldots, T_n, \omega_n) \sim (\omega_1, T_2, \ldots, T_n, \omega_n, T_1, \omega_1).$$

A *trivial continuous time rooted loop* is an ordered pair (η^x, T) where $T > 0$.

In both the discrete and continuous time cases, unordered trivial loops are the same as ordered trivial loops. A *loop functional* (discrete or continuous time) is a function on unordered loops. Equivalently, it is a function on ordered loops that is invariant under the time translations that define the equivalence relation for unordered loops.

3.1.1 Discrete time measures

Define q_x to the be measure q restricted to loops rooted at x. In other words, $q_x(\omega)$ is only nonzero for loops rooted at x and for such loops.

$$q_x(\omega) = \sum_{n=0}^{\infty} \mathbb{P}^x\{[X_0, \ldots, X_n] = \omega\}.$$

We let $q = \sum_x q_x$, i.e., the measure that assigns measure $q(\omega)$ to each loop.

♣ Although q can be considered also as a measure on *paths*, when considering loop measures one restricts q to loops, i.e., to paths beginning and ending at the same point.

We use m for the rooted loop measure and $\overline{m}$ for the unrooted loop measure as in [2]. Recall that these measures are supported on nontrivial loops and

$$m(\omega) = \frac{q(\omega)}{|\omega|}, \qquad \overline{m}(\overline{\omega}) = \sum_{\omega \sim \overline{\omega}} m(\omega),$$

Here $\omega \sim \overline{\omega}$ means that ω is a rooted loop that is in the equivalence class defining $\overline{\omega}$. If we let m_x denote m restricted to loops rooted at x, then we can write

$$m_x(\omega) = \sum_{n=1}^{\infty} \frac{1}{n} \, \mathbb{P}^x\{X_n^* = \omega\}. \tag{7}$$

As in [2] we write

$$F(A) = \exp\left\{\sum_{\omega} m(\omega)\right\} = \exp\left\{\sum_{\overline{\omega}} \overline{m}(\overline{\omega})\right\} = \frac{1}{\det(I-Q)} = \det G. \tag{8}$$

3.1.2 Continuous time measure

We now define a measure on loops with continuous time which corresponds to the measure introduced in [1]. For each nontrivial discrete loop

$$\omega = [\omega_0, \omega_1, \ldots, \omega_{n-1}, \omega_n],$$

we associate holding times

$$T_1, \ldots, T_n,$$

where $T_1, \ldots, T_n$ have the distribution of independent $Exp(1)$ random variables. Given ω and the values $T_1, \ldots, T_n$, we consider the continuous time loop of time duration $\tau_n = T_1 + \cdots + T_n$ (or we can think of this as period τ_n) given by

$$\omega(t) = \omega_j, \qquad \tau_j \le t < \tau_{j+1},$$

where $\tau_0 = 0, \tau_j = T_1 + \cdots + T_j$. We therefore have a measure $\tilde{q}$ on continuous time loops which we think of as a measure on

$$(\omega, \overline{T}), \qquad \overline{T} = (T_1, \ldots, T_n).$$

The analogue of m is the measure μ defined by

$$\frac{d\mu}{d\tilde{q}}\,(\omega, \overline{T}) = \frac{T_1}{T_1 + \cdots + T_n}.$$

Since $T_1, \ldots, T_n$ are identically distributed,

$$\mathbf{E}\left[\frac{T_1}{T_1 + \cdots + T_n}\right] = \frac{1}{n} \sum_{j=1}^{n} \mathbf{E}\left[\frac{T_j}{T_1 + \cdots + T_n}\right] = \frac{1}{n}.$$

Hence if we integrate out the $\overline{T}$ we get the measure m.

Note that this generates a well defined measure on continuous time unrooted loops which we write (with some abuse of notation since the vector $\overline{T}$ must also be translated) as

$$(\overline{\omega}, \overline{T}),$$

We let μ and $\overline{\mu}$ denote the corresponding measures on rooted and unrooted loops, respectively. They can be considered as measures on discrete time loops by forgetting the time.

This is the measure μ restricted to nontrivial loops. The measure gives infinite measure to trivial loops. More precisely, if ω is a trivial loop, then the density for (ω, t) is e^{-t}/t. We summarize.

Proposition 3.1. *The measure μ considered as a measure on discrete loops agrees with m when restricted to nontrivial loops. For trivial loops.*

$$\mu(\eta^x) = \infty, \qquad \hat{m}(\eta^x) = 1.$$

In other words to "sample" from μ restricted to nontrivial loops we can first sample from m and then choose independent holding times.

We can relate the continuous time measure to the continuous time Markov chain as follows. Suppose S_t is a continuous time Markov chain with rates q and holding times $T_1, T_2, \ldots$. Define the continuous time loop $\tilde{S}_t$ as follows. Recall that S_t^* is the discrete time path obtained from $\tilde{S}_t$ when it moves.

- If $t < T_1$, $\tilde{S}_t$ is the trivial continuous time loop (η^{S_0}, t) which is the same as (S_t^*, t).
- If $T_n \leq t < T_{n+1}$ then $\tilde{S}_t = (S_t^*, \overline{T})$ where $\overline{T} = (T_1, \ldots, T_n)$.

Let μ_x denote the measure μ restricted to loops rooted at x. Let $Q_t^{x,x}$ denote the measure on $\tilde{S}_t^*$ when $S_0 = x$ and restricting to the event $\{S_t = x\}$. Then

$$\mu_x = \int_0^\infty \frac{1}{t}\, Q_t^{x,x}\, dt.$$

One can compare this to (7).

3.1.3 Killing rates

We now consider the measures $m, \overline{m}, \mu, \overline{\mu}$ if subjected to a killing rate $\chi : A \to [0, \infty)$. We call the correspondng measures $m^\chi, \overline{m}^\chi, \mu^\chi, \overline{\mu}^\chi$. The construction uses the following standard fact about exponential random variables (we omit the proof). We write $Exp(\lambda)$ for the exponential distribution with rate λ, i.e., with mean $1/\lambda$.

Proposition 3.2. *Suppose T_1, T_2 are independent with distributions $Exp(\lambda_1)$, $Exp(\lambda_2)$ respectively. Let $T = T_1 \wedge T_2, Y = 1\{T = T_1\}$. Then T, Y are independent random variables with $T \sim Exp(\lambda_1 + \lambda_1)$ and $\mathbb{P}\{Y = 1\} = \lambda_1/(\lambda_1 + \lambda_2)$.*

The definitions are as follows.

- m^χ is the measure on discrete time paths obtained by using weight

$$q^\chi(x, y) = \frac{q(x, y)}{1 + \chi(x)}.$$

- μ^χ restricted to nontrivial loops is the measure on continuous time paths obtained from m^χ by adding holding times as follows. Suppose $\omega = [\omega_0, \ldots, \omega_n]$ is a loop. Let $T_1, \ldots, T_n$ be independent random variables with $T_j \sim Exp(1 + \chi(\omega_{j-1}))$. Given the holding times, the continuous time loop is defined as before.

- $\hat{m}^\chi$ agrees with m^χ on nontrivial loops and $\hat{m}^\chi(\eta^x) = 1$.

- For trivial loops ω rooted at x μ^χ gives density $e^{-t(1+\chi(x))}/t$ for (ω, t).

- $\overline{m}^\chi, \overline{\mu}^\chi$ are obtained as before by forgetting the root.

There is another way of obtaining μ^χ on nontrivial loops. Suppose that we start with the measure μ on discrete loops. Then we define the conditional measure on $(\mu, \overline{T})$ by saying that the density on $(T_1, \ldots, T_n)$ is given by

$$f(t_1, \ldots, t_n) = e^{-(\lambda_1 t_1 + \lambda_n t_n)},$$

where $\lambda_j = 1 + \chi(\omega_{j-1})$. Note that this is not a probability density. In fact,

$$\int f(t_1, \ldots, t_n)\, d\mathbf{t} = \prod_{j=1}^{n} \frac{1}{1 + \chi(\omega_{j-1})} = \frac{m^\chi(\omega)}{m(\omega)}.$$

If we normalize to make it a probability measure, then the distribution of $T_1, \ldots, T_n$ is that of independent random variables, $T_j \sim Exp(1 + \chi(\omega_{j-1}))$.

The important fact is as follows.

Proposition 3.3. *The measure μ^χ, considered as a measure on discrete loops, restricted to nontrivial loops is the same as m^χ.*

We now consider trivial loops. If η^x is a trivial loop with time T with (nonintegrable) density $g(t) = e^{-t}/t$, then

$$\int_0^\infty [e^{-rt} - 1]\, g(t)\, dt = \int_0^\infty \frac{e^{-(1+r)t} - e^{-t}}{t}\, dt = \log \frac{1}{1 + r}. \tag{9}$$

Hence, although μ and μ^χ both give infinite measure to the trivial loop ω at x, we can write

$$\mu^\chi(\eta^x) - \mu(\eta^x) = \log \frac{1}{1+\xi(x)}.$$

Note that $\mu^\chi(\eta^x) - \mu(\eta^x)$ is not the same as $m^\chi(\eta^x) - m(\eta^x)$. The reason is that the killing in the discrete case does not affect the trivial loops but it does affect the trivial loops in the continuous case.

3.2 First properties

In [2, Proposition 9.3.3], it is shown that $F(A) = \det[(I-Q)^{-1}] = \det G$. Here we give another proof of this based on [1]. The key observation is that

$$m\{\omega : \omega_0 = x, |\omega| = n\} = \frac{1}{n}\, Q^n(x,x),$$

and hence

$$m\{\omega : |\omega| = n\} = \frac{1}{n}\, \mathrm{Tr}[Q^n].$$

Let $\alpha_1, \ldots, \alpha_N$ denote the eigenvalues of Q. Then the eigenvalues of Q^n are $\alpha_1^n, \ldots, \alpha_N^n$ and the total mass of the measure m is

$$\sum_{n=1}^\infty \frac{1}{n} \mathrm{Tr}[Q^n] = \sum_{j=1}^N \sum_{n=1}^\infty \frac{\alpha_j^n}{n} = -\sum_{j=1}^N \log[1-\alpha_j] = -\log[\det(I-Q)].$$

Here we use the fact that $|\alpha_j| < 1$ for each j.

♣ If we define the logarithm of a matrix by the power series

$$\log[I-Q] = -\sum_{n=1}^\infty \frac{1}{n}\, Q^n,$$

then the argument shows the relation

$$\mathrm{Tr}[\log(I-Q)] = \log\det(I-Q) = -\sum_{n=1}^\infty \frac{1}{n}\, \mathrm{Tr}[Q^n].$$

This is valid for any matrix Q all of whose eigenvalues are all less than one in absolute value.

3.3 Occupation field

3.3.1 Discrete time

For a nontrivial loop $\omega = [\omega_0, \ldots, \omega_n]$ define its (discrete time) occupation field by

$$N^x(\omega) = \#\{j : 1 \leq j \leq n : \omega_j = x\} = \sum_{j=1}^{n} 1\{\omega_j = n\}.$$

Note that $N^x(\omega)$ depends only on the unrooted loop, and hence is a loop functional. If $\chi : A \to \mathbb{C}$ is a function we write

$$\langle N, \chi \rangle(\omega) = \sum_{x \in A} N^x(\omega)\, \chi(x).$$

Proposition 3.4. *Suppose $x \in A$. Then for any discrete time loop functional Φ,*

$$m\,[N^x\,\Phi] = \overline{m}\,[N^x\,\Phi] = q_x[\Phi].$$

Proof. The first equality holds since $N^x\,\Phi$ is a loop functional. The second follows from the important relation

$$\sum_{\omega \sim \overline{\omega}, \omega_0 = x} q(\omega) = N^x(\omega)\,\overline{m}^*(\overline{\omega}). \tag{10}$$

To see this, assume $|\overline{\omega}| = n$ and $N^x(\overline{\omega}) = k > 0$. Let rn denote the number of distinct representatives of $\overline{\omega}$ and let $N^x(\overline{\omega}) = k$. Then it is easy to check that the number of distinct representatives of $\overline{\omega}$ that are rooted at x equals rk. Recall that

$$\overline{m}(\overline{\omega}) = r\,q(\overline{\omega}) = \frac{rk}{N^x(\overline{\omega})}\,q(\overline{\omega}) = \sum_{\omega \sim \overline{\omega}, \omega_0 = x} \frac{q(\omega)}{N^x(\overline{\omega})}.$$

□

Example

- Setting $\Phi \equiv 1$ gives
$$m\,[N^x] = G(x,x) - 1.$$
- Setting $\Phi = N^y$ with $y \neq x$ gives
$$\hat{m}\,[N^x N^y] = q_x(N^y).$$

  For any loop $\omega = [\omega_0, \ldots, \omega_n]$ rooted at x with $N^y(\omega) = k \geq 1$, there are k different wasy that we can write ω as

$$[\omega_0, \ldots, \omega_k] \oplus [\omega_k, \ldots, \omega_n],$$

with $\omega_k = y$. Therefore,

$$q_x(N^y) = \sum_{\omega_1,\omega_2} q(\omega_1)\, q(\omega_2)$$

where the sum is over all paths ω_1 from x to y and ω_2 from y to x. Summing over all such paths gives

$$q_x(N^y) = G(x,y)\, G(y,x) = G(x,y)^2.$$

- More generally, if $x_1, x_2, \ldots, x_k$ are points and $\Phi^{x_1,\ldots,x_k}$ is the functional that counts the number of times we can find $x_1, x_2, \ldots, x_k$ in order on the loop, then

$$\hat{m}\left[\Phi^{x_1,\ldots,x_k}\right] = G^*(x_1,x_2)\, G^*(x_2,x_3) \cdots G(x_{k-1},x_k)\, G^*(x_k,x_1),$$

where

$$G^*(x,y) = G(x,y) - \delta_{x,y}.$$

Consider the case $x_1 = x_2 = x$. Note that

$$\Phi^{x,x} = (N^x)^2 - N^x,$$

and hence

$$\hat{m}\left[(N^x)^2\right] = \hat{m}\left[\Phi^{x,x}\right] + \hat{m}\left[N^x\right] = [G(x,x)-1]^2 + G(x,x) = G(x,x)^2 - G(x,x) + 1.$$

Let us derive this in a different way by computing $q_x(N^x)$. for the trivial loop η^x, we have $N^x(\eta^x) = 1$. The total measure of the set of loops with $N^x(\omega) = k \geq 1$ is given by r^k, where

$$r = \frac{G(x,x)-1}{G(x,x)}.$$

Hence,

$$q_x(N^x) = 1 + \sum_{k=1}^{\infty} k\, r^k = 1 + \frac{r}{(1-r)^2} = 1 + G(x,x)^2 - G(x,x).$$

3.3.2 Resticting to a subset

Suppose $A' \subset A$ and $\hat{q} = \hat{q}_{A'}$ denotes the weights associated to the chain when it visits A' as introduced in Section 2. For each loop ω in A rooted at a point in A', there is a corresponding loop which we will call $\Lambda(\omega; A')$ in A' obtained from removing all the vertices that are not in A'. Note that for

$$N^x(\Lambda(\omega; A')) = N^x(\omega)\, 1\{x \in A'\}.$$

By construction, we know that if ω' is a loop in A',

$$\hat{q}(\omega') = \sum_{\omega} q(\omega)\, 1\{\Lambda(\omega; A') = \omega'\}.$$

We can also define $\Lambda(\overline{\omega}; A')$ for an unrooted loop $\overline{\omega}$. Note that $\omega \sim \overline{\omega}$ if and only if $\Lambda(\omega; A') \sim \Lambda(\overline{\omega}; A')$. However, some care must be taken, since it is possible to have two different representatives ω^1, ω^2 of $\overline{\omega}$ with $\Lambda(\omega^1; A') = \Lambda(\omega^2; A)$. Let $m_{A'}, \overline{m}_{A'}$ denote the measures on rooted loops and unrooted loops, respectively, in A' generated by $\hat{q}$. The next proposition follows from (10).

Proposition 3.5. *Let $A' \subset A$ and let $m_{A'}$ denote the measure on loops in A generated by the weight $\hat{q}$. Then for every loop ω' in A',*

$$\hat{m}_{A'}(\overline{\omega}') = \sum_{\omega} m(\omega)\, 1\{\Lambda(\overline{\omega}; A') = \overline{\omega}'\}.$$

3.3.3 Continuous time

For a nontrivial continuous time loop $(\omega, \overline{T})$ of length n, we define the (continuous time) occupation field by

$$\ell^x(\omega, \overline{T}) = \int_0^{T_1+\cdots+T_n} 1\{\omega(t) = x\}\, dt = \sum_{j=0}^{n-1} 1\{\omega_j = x\}\, T_j.$$

For trivial loops, we define

$$\ell^x(\eta^y, T) = \delta_{x,y}\, T.$$

Note that ℓ is a loop functional. We also write

$$\langle \ell, \chi\rangle(\omega, \overline{T}) = \sum_{x \in A} \ell^x(\omega, \overline{T})\, \chi(x) = \int_0^{T_1+\cdots+T_n} \chi(\omega(t))\, dt.$$

The second equality is valid for nontrivial loops; for trivial loops $\langle \ell, \chi\rangle(\eta^x, T) = T\, \chi(x)$.

The continuous time analogue requires a little more setup.

Proof. We first consider $\overline{\mu}$ restricted to nontrivial loops. Recall that this is the same as $\overline{m}$ restricted to nontrivial loops combined with independent choices of holding times $T_1, \ldots, T_n$. Let us fix a discrete loop ω of length $n \geq 1$. Assume that $N^x(\omega) = k > 0$. Then (with some abuse of notation)

$$\ell^x(\overline{\omega}, \overline{T}) = \sum_{\omega \sim \overline{\omega}, \omega_0 = x} T_1(\omega).$$

We write $T_1(\omega)$ to indicate the time for the jump from ω_0 to ω_1. Therefore,

$$\overline{\mu}[\ell^x\, \Phi\, J_\omega] = \sum_{\omega \sim \overline{\omega}, \omega_0 = x} q(\omega)\, \mathbf{E}\left[T_1\, \Phi \mid \omega\right].$$

Here $\mathbf{E}\left[T_1\, \Phi \mid \omega\right]$ denotes the expected value given the discrete loop ω, i.e., the randomness is over the holding times $T_1, \ldots, T_n$. Summing over nontrivial loops gives

$$\overline{\mu}[\ell^x\, \Phi; \overline{\omega} \text{ nontrivial}] = \sum_{|\omega| > 0, \omega_0 = x} q(\omega)\, \mathbf{E}\left[T_1\, \Phi \mid \omega\right].$$

Also,

$$\overline{\mu}[\ell^x\,\Phi; \overline{\omega} = \eta^x] = \int_0^\infty \Phi(\eta^x, t)\, e^{-t}\, dt.$$

□

Example

- Setting $\Phi \equiv 1$ gives
$$\mu(\ell^x) = G(x,x).$$
- Let $\Phi = (\ell^x)^k$.

3.3.4 More on discrete time

Let

$$N_{x,y}(\omega) = \sum_{j=0}^{n} 1\{\omega_j = x, \omega_{j+1} = y\}, \quad N_x(\omega) = \sum_y N_{x,y}(\omega) = \#\{j < |\omega| : \omega_j = x\}.$$

We can also write $N_{x,y}(\overline{\omega})$ for an unrooted loop.

Let $V(x,k)$ be the set of loops ω rooted at x with $N_x(\omega) = k$ and

$$r(x,k) = \sum_{\omega \in V(x,k)} q(\omega),$$

where by definition $r(x,0) = 1$. It is easy to see that $r(x,k) = r(x,1)^k$, and standard Markov chain or generating function show what

$$G(x,x) = \sum_{k=0}^{\infty} r(x,k) = \sum_{k=0}^{\infty} r(x,1)^k = \frac{1}{1 - r(x,1)}.$$

Note also that

$$m[V(x,k)] = \frac{1}{k}\, r(x,k).$$

To see this we consider any unrooted loop ω that visits x k times and choose a representative rooted at x with equal probability for each of the k choices.[1] Therefore,

$$\overline{m}[\{\overline{\omega} : N_x(\overline{\omega}) \geq 1] = \sum_{n=1}^{\infty} \frac{1}{n} r(x,1)^n = -\log[1 - r(x,1)] = -\log G(x,x).$$

[1] Actually, it is slighly more subtle than this. If an unrooted loop ω of length n has rn representatives as rooted loops then $\overline{m}(\omega) = r\, q(\omega)$ and the number of these representatives that are rooted at x is $N_x(\overline{\omega})\, r$. Regardless, we can get the unrooted loop measure by giving measure $q(\overline{\omega})/k$ to the k representatives of $\overline{\omega}$ rooted at x.

This is [2, Proposition 9.3.2]. In [1], occupation times are emphasized. If Φ is a functional on loops we write $m(\Phi)$ for the corresponding expectation

$$m(\Phi) = \sum_{\omega} m(\omega)\, \Phi(\omega).$$

If Φ only depends on the unrooted loop, then we can also write $\overline{m}(\Phi)$ which equals $m(\Phi)$. Then

$$\overline{m}(N_x) = m(N_x) = \sum_{j=1}^{\infty} n\, r(x,n) = \sum_{j=1}^{\infty} r(x,1)^n = \frac{r(x,1)}{1 - r(x,1)} = G(x,x) - 1.$$

We can state the relationship in terms of Radon-Nikodym derivatives. Consider the measure on unrooted loops $\overline{\omega}$ that visit x given by

$$\overline{q}_x(\overline{\omega}) = \sum_{\omega \sim \hat{\omega}, \omega_0 = x} q(\omega),$$

where $\omega \sim \overline{\omega}$ means that ω is a rooted representative of $\overline{\omega}$. Then,

$$\overline{q}_x(\overline{\omega}) = N_x(\overline{\omega})\, \overline{m}(\overline{\omega}).$$

It is easy to see that

$$\sum_{|\omega|>0, \omega_0 = x} q(\omega) = G(x,x) - 1.$$

We can similarly compute $\overline{m}(N_{x,y})$. Let V denote the set of loops

$$\omega = [\omega_0, \omega_1, \ldots, \omega_n],$$

with $\omega_0 = x, \omega_1 = y, \omega_n = x$. Then

$$q(V) = q(x,y)\, G(y,x) = q(x,y)\, F(y,x)\, G(x,x),$$

where $F(y,x)$ denotes the first visit generating function

$$F(y,x) = \sum_{\omega} q(\omega),$$

where the sum is over all paths $\omega = [\omega_0, \ldots, \omega_n]$ with $n \geq 1, \omega_0 = y, \omega_n = x$ and $\omega_j \neq x$ for $0 < j < n$. This gives

$$\overline{m}(N_{x,y}) = q(x,y)\, G(y,x).$$

It is slighly more complicated to compute $\overline{m}(N_{x,y} \geq 1)$. The measure of the set of loops ω at x with $N_x = 1$ and such that $\omega_1 \neq y$ is given by

$$F(x,x) - q(x,y)\, F(y,x).$$

Note that $N_{x,y}(\omega) = 0$ for all such loops. Therefore the q measure of loops at x with $N_{x,y}(\omega) = 0$ is

$$\sum_{n=0}^{\infty} [F(x,x) - q(x,y)\, F(y,x)]^n = \frac{1}{1 - [F(x,x) - q(x,y)\, F(y,x)]}.$$

Therefore,

$$\sum_{\omega \in V; N_{x,y}(\omega)=1} q(\omega) = \frac{q(x,y)\, F(y,x)}{1 - [F(x,x) - q(x,y)\, F(y,x)]},$$

$$\sum_{\omega \in V; N_{x,y}(\omega)=k} q(\omega) = \left[\frac{q(x,y)\, F(y,x)}{1 - [F(x,x) - q(x,y)\, F(y,x)]} \right]^k.$$

To each unrooted loop ω with $N_{x,y}(\overline{\omega}) = k$ and $r|\overline{\omega}|$ different representatives we give measure $1/(rk)$ to the rk representatives ω with $\omega_0 = x, \omega_1 = y$. We then get

$$\begin{aligned} \overline{m}(N_{x,y} \geq 1) &= \sum_{k=1}^{\infty} \frac{1}{k} \left[\frac{q(x,y)\, F(y,x)}{1 + q(x,y)\, F(y,x) - F(x,x)} \right]^k \\ &= -\log \left[\frac{1 - F(x,x)}{1 + q(x,y)\, F(y,x) - F(x,x)} \right]. \end{aligned}$$

We will now generalize this. Suppose $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ are given points in A. For any loop

$$\omega = [\omega_0, \ldots, \omega_n]$$

define $N_{\mathbf{x}}(\omega)$ as follows. First define $\omega_{j+n} = \omega_j$. Then $N_{\mathbf{x}}$ is the number of increasing sequences of integers $j_1 < j_2 < \cdots < j_k < j_1 + n$ with $0 \leq j_1 < n$ and

$$\omega_{j_l} = x_l, \quad l = 1, \ldots, k.$$

Note that $N_{\mathbf{x}}(\omega)$ is a function of the unordered loop $\overline{\omega}$. Let $V_{\mathbf{x}}$ denote the set of loops rooted at x_1 such that such a sequence exists (for which we can take $j_1 = 0$). Then by concatentating paths, we can see that

$$q(V_{\mathbf{x}}) = G(x_1, x_2)\, G(x_2, x_3) \cdots G(x_{k-1}, x_k)\, G(x_k, x_1),$$

and hence as above

$$\overline{m}(N_{\mathbf{x}}) = G(x_1, x_2)\, G(x_2, x_3) \cdots G(x_{k-1}, x_k)\, G(x_k, x_1).$$

Suppose χ is a positive function on A. As before, let q^χ denote the measure with weights

$$q^\chi(x,y) = \frac{q(x,y)}{1 + \chi(x)}.$$

Then if $\omega = [\omega_0, \ldots, \omega_n]$, we can write

$$q^\chi(\omega) = q(\omega)\,\exp\left\{-\sum_{j=1}^n \log(1+\chi(\omega_j))\right\} = q(\omega)\,e^{-\langle \hat{\ell}, \log(1+\chi)\rangle}.$$

Here we are using a notation from [1]

$$\langle \hat{\ell}, f\rangle(\omega) = \sum_{j=1}^n f\omega_j) = \sum_{x\in A} N_x(\omega)\,f(\omega).$$

We have the corresponding measures $m^\chi, \overline{m}^\chi$ with

$$m^\chi(\omega) = e^{-\langle \hat{\ell}(\omega), \log(1+\chi)\rangle}\,m(\omega), \qquad \overline{m}(\overline{\omega}) = e^{-\langle \hat{\ell}(\overline{\omega}), \log(1+\chi)\rangle}\,m(\overline{\omega}).$$

As before, let G_χ denote the Green's function for the weight q^χ. The total mass of $\overline{m}^\chi$ is $\det G_\chi$.

Remark [1] discusses Laplace transforms of the measure m. This is just another way of saying total mass of the measure m^χ (as a function of χ). Proposition 2 in [1, Section 3.4] states

$$m(e^{-\langle \hat{\ell}, \log(1+\chi)\rangle} - 1) = \log\det G_\chi - \log\det G.$$

This is obvious since $\overline{m}(e^{-\langle \hat{\ell}, \log(1+\chi)\rangle})$ by defintion is the total mass of $\overline{m}^\chi$.

$$m(e^{-\langle \hat{\ell}, \log(1+\chi)\rangle}) = \sum_\omega m(\omega)\,\exp\left\{-\sum_x N^x(\omega)\,\log(1+\chi(x))\right\} = \sum_\omega m^\chi(\omega).$$

Moreover, using (9), we can see that

$$\hat{m}(e^{-\langle \hat{\ell}, \log(1+\chi)\rangle} - 1) = \log\det \tilde{G}_\chi - \log\det G.$$

3.3.5 More on continuous time

If $(\omega, \overline{T})$ is a continuous time loop we define the occupation field

$$\ell^x(\omega, \overline{T}) = \int_0^{T_1+\cdots+T_n} 1\{\omega(t) = x\}\,dt = \sum_{j=1}^n 1\{\omega_{j-1} = x\} T_j.$$

If χ is a function we write

$$\langle \ell, \chi\rangle = \langle \ell, \chi\rangle(\omega, \overline{T}) = \sum_x \ell^x(\omega, \overline{T})\,\chi(x).$$

Note the following.

- In the measure μ, the conditional expectation of $\ell^x(\omega; \overline{T})$ given ω is $N_x(\omega)$.
- In the measure μ^χ, the conditional expectation of $\ell^x(\omega; \overline{T})$ given ω is $N_x(\omega)/[1+\chi(\omega)]$.

Note that in the measure μ,

$$\mathbf{E}\left[\exp\{-\langle \ell, \chi\rangle\} \mid \omega\right] = \prod_{j=0}^{n-1} \mathbf{E}\left[e^{-\chi(\omega_j)\,T_j}\right] = \prod_{j=0}^{n-1} \frac{1}{1+\chi(\omega_j)} = \frac{m^\chi(\omega)}{m(\omega)}.$$

Using this we see that

$$\mu\left[(e^{-\langle \ell,\chi\rangle} - 1)\, 1\{|\overline{\omega}| \geq 1\}\right] = \log\det G_\chi - \log\det G. \tag{11}$$

Also (9) shows that

$$\mu\left[[e^{-\langle \ell,\chi\rangle} - 1]\; 1\{\text{discrete loop is } \eta^x\}\right] = -\log[1+\chi(x)]. \tag{12}$$

By (5) we know that

$$\log \tilde{G}_\chi = \log G_\chi - \sum_x \log[1+\chi(x)],$$

and hence we get the following.

Proposition 3.6.

$$\mu[e^{-\langle \ell,\chi\rangle} - 1] = \log \tilde{G}_\chi - \log G_\chi.$$

Although we have assumed that χ is positive, careful examination of the argument will show that we can also establish this for general χ in a sufficiently small neighborhood of the origin.

4 Poisson process of loops

4.1 Definition

4.1.1 Discrete time

The loop soup with intensity α is a Poissonian realization from the measure m or $\overline{m}$. The rooted soup can be considered as an independent collection of Poisson processes $M_\alpha(\omega)$ with $M_\alpha(\omega)$ having intensity $m(\omega)$. We think of $M_\alpha(\omega)$ as the number of times ω has appeared by time α. The total collection of loops $\mathcal{C}_\alpha$ can be considered as a random increasing multi-set (a set in which elements can appear multiple times). The unrooted soup can be obtained from the rooted soup by forgetting the root. We will write $\mathcal{C}_\alpha$ for both the rooted and unrooted versions. Let

$$|\mathcal{C}_\alpha| = \sum_{\omega\in\mathcal{C}_\alpha} m(\omega) = \sum_{\overline{\omega}\in\mathcal{C}_\alpha} \overline{m}(\omega).$$

If Φ is a loop functional, we write

$$\Phi_\alpha = \sum_{\omega \in \mathcal{C}_\alpha} \Phi(\omega) := \sum_\omega M_\alpha(\omega)\, \Phi(\omega).$$

If $\chi : A \to \mathbb{C}$, we set

$$\langle \mathcal{C}_\alpha, \chi \rangle = \sum_{x \in A} \sum_{\omega \in \mathcal{C}_\alpha} N_\alpha^x(\omega)\, \chi(x)).$$

In the particular case $\chi = \delta_x$, we get the occupation field

$$L_\alpha^x = \sum_\omega M_\alpha(\omega)\, N_\alpha^x(\omega).$$

Using the moment generating function of the Poisson distribution, we see that

$$\mathbf{E}\left[e^{-\Phi_\alpha}\right] = \exp\left\{\alpha \sum_\omega m(\omega)\, [e^{-\phi(\omega)} - 1]\right\}.$$

In particular,

$$\begin{aligned} \mathbf{E}\left[e^{-\langle \mathcal{C}_\alpha, \log(1+\chi)\rangle}\right] &= \prod_\omega \mathbf{E}\left[e^{-M_\alpha(\omega)\langle \omega, \log(1+\chi)\rangle}\right] \\ &= \exp\left\{\sum_\omega \alpha m(\omega)\, [e^{\langle \omega, \log(1+\chi)\rangle} - 1]\right\} \\ &= \exp\left\{\alpha \sum_\omega [m^\chi(\omega) - m(\omega)]\right\} \\ &= \left[\frac{\det G_\chi}{\det G}\right]^\alpha. \end{aligned}$$

The last step uses (8) for the weights q^χ and q. Note also that

$$\mathbf{E}[\langle \mathcal{C}_t, \delta_x \rangle] = t\, [G(x,x) - 1].$$

Proposition 4.1. *Suppose $\mathcal{C}_\alpha$ is a loop soup using weight q on A and suppose that $A' \subset A$. Let*

$$\mathcal{C}'_\alpha = \{\Lambda(\omega; A') : \omega \in A\},$$

where $\Lambda(\omega; A')$ is defined as in Proposition 3.5. Then $\mathcal{C}'_\alpha$, is a loop soup for the weight $\hat{q}_{A'}$ on A'. Moreover, the occupations fields $\{L_\alpha^x : x \in A'\}$, are the same for both soups.

4.1.2 Continuous time

The continuous time loop soup for nontrivial loops can be obtained from the discrete time loop soup by choosing realizations of the holding times from the appropriate distributions. The trivial loops must be added in a different way. It will be useful to consider the loop soup as the union of two independent soups: one for the nontrivial loops and one for the trivial loops.

- Start with a loop soup $\mathcal{C}_\alpha$ of the discrete loop soup of nontrivial loops.
- For each loop $\omega \in \mathcal{C}_\alpha$ of length n we choose holding times $T_1, \ldots, T_n$ independently from an $Exp(1)$ distribution. Note that the times for different loops in the soup are independent as well as the different holding times for a particular loop. The occupation field is then defined by
$$\mathcal{L}_\alpha^x = \sum_{(\omega,\overline{T})\in\mathcal{C}_\alpha} \ell^x(\omega,\overline{T}).$$
- For each $x \in A$, take a Poisson point process of times $\{t_r(x) : 0 \le r < \infty\}\}$ with intensity e^{-t}/t. We consider a be a Poissonian realization of the trivial loops $(\eta^x, t_r(x))$ for all x and all $r \le \alpha$. With probability one, at all times $\alpha > 0$, there exist an infinite number of loops. We will only nee to consider the occupation field,
$$\tilde{\mathcal{L}}_\alpha^x = \sum_{(\eta^x, t_r(x))} t_r(x).$$
where the sum is over all trivial loops at x in the field by time α. In other words, Note that
$$\mathbf{E}[e^{-\tilde{\mathcal{L}}_\alpha^x\,\chi(x)}] = \exp\left\{\alpha\int_0^\infty [e^{-t\chi(x)} - 1]\,t^{-1}\,e^{-t}\,dt\right\} = \frac{1}{[1+\chi(x)]^\alpha}.$$
This shows that $\tilde{\mathcal{L}}_\alpha^x$ has a $Gamma(\alpha, 1)$ distribution.

Associated to the loop soups is the occupation field
$$\hat{\mathcal{L}}_\alpha^x = \mathcal{L}_\alpha^x + \tilde{\mathcal{L}}_\alpha^x = \sum_{(\omega,\overline{T})\in\mathcal{L}_\alpha} \ell^x(\omega,\overline{T}) + \sum_{(\eta^y,T)\in\tilde{\mathcal{L}}_\alpha} \delta_{x,y}T.$$

If we are only interested in the occupation field, we can contruct it by starting with the discrete occupation field and adding randomness. The next proposition makes this precise. We will call a process $\Gamma(t)$ a *Gamma process (with parameter 1)* if it has independent increments and $\Gamma(t+s) - \Gamma(t)$ has a $Gamma(s, 1)$ distribution. In particular, the distribution of $\{\Gamma(n) : n = 0, 1, 2, \ldots\}$ is that of the sum of independent $Exp(1)$ random variables. If

---

♣ Recall that a random variable Y has a $Gamma(s, 1), s > 0$ distribution if it has density
$$f_s(t) = \frac{t^{s-1}\,e^{-t}}{\Gamma(s)}, \quad t \ge 0.$$

Note that the moments are given by

$$\mathbf{E}[Y^\beta] = \frac{1}{\Gamma(s)} \int_0^\infty t^{\beta+s-1}\, e^{-t}\, dt = (s)_\beta := \frac{\Gamma(s+\beta)}{\Gamma(s)}.$$

For integer β,

$$\mathbf{E}[Y^\beta] = (s)_\beta = s\,(s+1)\,\cdots\,(s+\beta-1). \tag{13}$$

More generally, a random variable Y has a $Gamma(s,r)$ distribution if Y/r has a $Gamma(s,1)$ distribution. The square of a normal random variable with variance σ^2 has a $Gamma(1/2, \sigma^2/2)$ distribution.

Proposition 4.2. *Suppose on the same probability space, we have defined a discrete loop soup $\mathcal{C}_\alpha$ and Gamma process $\{Y^x(t)\}$ for each $x \in A$. Assume that the loop soup and all of the Gamma processes are mutually independent. Let*

$$L_\alpha^x = \sum_\omega M_\alpha(\omega)\, N^x(\omega)$$

denote the occupation field generated by $\mathcal{C}_\alpha$. Define

$$\hat{\mathcal{L}}_\alpha^x = Y^x(L_\alpha^x + \alpha). \tag{14}$$

Then

$$\{\hat{\mathcal{L}}_\alpha^x : x \in A\}$$

have the distribution of the occupation field for the continuous time soup.

An equivalent, and sometimes more convenient, way to define the occupation field is to take two independent Gamma processes at each site $\{Y_1^x(t), Y_2^x(t)\}$ and replace (14) with

$$\hat{\mathcal{L}}_\alpha^x = \mathcal{L}_\alpha^x + \tilde{\mathcal{L}}_\alpha^x := Y_1^x(L_\alpha^x) + Y_2^x(\alpha).$$

The components of the field $\{\tilde{\mathcal{L}}_\alpha^x : x \in A\}$ are independent and independent of $\{\mathcal{L}_\alpha^x : x \in A\}$. The components of the field $\{\mathcal{L}_\alpha^x : x \in A\}$ are not independent but are conditionally independent given the discrete occupation field $\{L_\alpha^x : x \in A\}$.

I

♣ If all we are interested in is the occupation field for the continuous loop soup, then we can take the construction in Proposition 4.2 as the definition.

♣If $A' \subset A$, then the occupation field restricted to A' is the same as the occupation field for the chain viewed at A'.

Proposition 4.3. *If $\hat{\mathcal{L}}_\alpha$ is the continuous time occupation field, then there exists $\epsilon > 0$ such for all $\chi : A \to \mathbb{C}$ with $|\chi|_2 < \epsilon$,*

$$\mathbf{E}\left[e^{-\langle \hat{\mathcal{L}}_\alpha, \chi\rangle}\right] = \left[\frac{\det \tilde{G}_\chi}{\det G}\right]^\alpha. \tag{15}$$

Proof. Note that

$$\mathbf{E}\left[e^{-\langle \hat{\mathcal{L}}, \chi\rangle} \mid L_\alpha\right] = \prod_x \left[\frac{1}{1+\chi(x)}\right]^{L_\alpha^x + \alpha} = \left[\prod_x \frac{1}{1+\chi(x)}\right]^\alpha \prod_x \prod_\omega \left[\frac{1}{1+\chi(x)}\right]^{N^x(\omega)\, M_\alpha(\omega)}.$$

Since the $M_\alpha(\omega)$ are independent,

$$\begin{aligned}
\mathbf{E}\left[\prod_x \prod_\omega \left[\frac{1}{1+\chi(x)}\right]^{N^x(\omega)\, M_\alpha(\omega)}\right] &= \prod_\omega \mathbf{E}\left[\prod_x \left[\frac{1}{1+\chi(x)}\right]^{N^x(\omega)\, M_\alpha(\omega)}\right] \\
&= \prod_\omega \mathbf{E}\left[e^{-\langle N(\omega), \log(1+\chi)\rangle\, M_\alpha(\omega)}\right] \\
&= \exp\left\{\alpha \sum_\omega m(\omega)\, [e^{-\langle N, \log(1+\chi)\rangle} - 1]\right\} \\
&= \left[\frac{\det G_\chi}{\deg G}\right]^\alpha.
\end{aligned}$$

□

♣ Although the loop soups for trivial loops are different in the discrete and continuous time settings, one can compute moments for the continuous time occupation measure in terms of moments for the discrete occupation measure.

For ease, let us choose $\alpha = 1$. Recall that

$$\tilde{G}_\chi = (I - Q + M_\chi)^{-1} = G\,(I + GM_\chi)^{-1}.$$

We can therefore write

$$\frac{\det \tilde{G}_\chi}{\det G} = \det(I + GM_\chi)^{-1} = \det(I + M_{\sqrt{\chi}} G M_{\sqrt{\chi}})^{-1}.$$

♣ To justify the last equality formally, note that

$$M_{\sqrt{\chi}}(I + GM_\chi)\, M_{\sqrt{\chi}}^{-1} = I + M_{\sqrt{\chi}}\, G\, M_{\sqrt{\chi}}^{-1}.$$

This argument works if χ is strictly positive, but we can take limits if χ is zero in some places.

4.2 Moments and polynomials of the occupation field

f k is a positive integer, then using (13) we see that

$$\mathbf{E}\left[(\mathcal{L}_\alpha^x)^k\right] = \mathbf{E}\left[\mathbf{E}[(\mathcal{L}_\alpha^x)^k \mid L_\alpha^x]\,\right] = \mathbf{E}\left[(L_\alpha^x + \alpha)_k\right].$$

More generally, if $A' \subset A$ and $\{k_x : x \in A'\}$ are positive integers,

$$\mathbf{E}\left[\prod (\mathcal{L}_\alpha^x)^{k_x}\right] = \mathbf{E}\left[\mathbf{E}\left(\prod (\mathcal{L}_\alpha^x)^{k_x} \mid L_\alpha^x, x \in A'\right)\,\right] = \mathbf{E}\left[\prod (L_\alpha^x + \alpha)_{k_x}\right]$$

Although this can get messy, we see that all moments for the continuous field can be given in terms of moments of the discrete field.

5 The Gaussian free field

Recall that the Gaussian free field (with Dirichlet boundary conditions) on A is the measure on $\mathbb{R}^A$ whose Radon-Nikodym derivative with respect to Lebesgue measure is given by

$$Z^{-1}\, e^{-\mathcal{E}(\phi)/2}$$

where Z is a normalization constant. Recall [2, (9.28)] that

$$\mathcal{E}(\phi) = \phi \cdot (I - Q)\phi,$$

so we can write the density as a constant times $e^{-\langle \phi, G^{-1}\phi\rangle/2}$. As calculated in [2] (as well as many other places) the normalization is given by

$$Z = (2\pi)^{\#(A)/2}\, F(A)^{1/2} = (2\pi)^{\#(A)/2}\, \exp\left\{\frac{1}{2}\sum_\omega m(\omega)\right\} = (2\pi)^{\#(A)/2}\, \sqrt{\det G}.$$

In other words the field

$$\{\phi(x) : x \in A\}$$

is a mean zero random vector with a joint normal distribution with covariance matrix G. Note that if $\mathbf{E}$ denotes expectations under the field measure,

$$\begin{aligned}
\mathbf{E}\left[\exp\left\{-\frac{1}{2}\sum_x \phi(x)^2\,\chi(x)\right\}\right] &= \frac{1}{(2\pi)^{\#(A)/2}\sqrt{\det(G)}}\int \exp\left\{-\frac{f\cdot(I - Q + M_\chi)f}{2}\right\} \\
&= \frac{\sqrt{\det \tilde{G}_\chi}}{\sqrt{\det G}}\int \frac{1}{(2\pi)^{\#(A)/2}\sqrt{\det(\tilde{G}_\chi)}}\exp\left\{-\frac{f\cdot \tilde{G}_\chi f}{2}\right\} \\
&= \frac{\sqrt{\det \tilde{G}_\chi}}{\sqrt{\det G}}. \qquad (16)
\end{aligned}$$

Here we use the relation $\tilde{G}_\chi = (I - Q + M_\chi)^{-1}$. The third equality follows from the fact that the term inside the integral in the second line is the normal density with covariance matrix $\tilde{G}_\chi$. Similarly, if $F : \mathbb{R}^A \to \mathbb{R}$ is any function,

$$\mathbf{E}\left[\exp\left\{-\frac{1}{2}\sum_x \phi(x)^2\,\chi(x)\right\}\,F(\phi)\right] = \frac{\sqrt{\det \tilde{G}_\chi}}{\sqrt{\det G}}\,\tilde{\mathbf{E}}\left[F(\phi)\right],$$

where $\tilde{\mathbf{E}} = \mathbf{E}_{\tilde{G}_\chi}$ denotes expectation assuming covariance matrix $\tilde{G}_\chi$.

Theorem 1. *Suppose q is a weight with corresponding loop soup $\mathcal{L}_\alpha$. Let ϕ be a Gaussian field with covariance matrix G. Then $\mathcal{L}_{1/2}$ and $\phi^2/2$ have the same distribution.*

Proof. By comparing (15) and (16) we see that the moment generating functions of $\mathcal{L}_{1/2}$ and $\phi^2/2$ agree in a neighborhood of the origin. □

References

[1] Le Jan

[2] Lawler and Limic