The probability that planar loop-erased random walk uses a given edge

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Abstract

We give a new proof of a result of Rick Kenyon that the probability that an edge in the middle of an $n \times n$ square is used in a loop-erased walk connecting opposite sides is of order $n^{-3/4}$. We, in fact, improve the result by showing that this estimate is correct up to multiplicative constants.

1 Introduction

Loop-erased random walk is a process obtained by erasing loops from simple random walk. Although the process can be defined for arbitrary Markov chains, we will discuss the process only on the planar integer lattice $\mathbb{Z}^2 = \mathbb{Z} + i\mathbb{Z}$. We start this paper by stating our main result which is an improvement of a result of Rick Kenyon [2].

Let

$$A_n = \{j + ik \in \mathbb{Z} + i\mathbb{Z} : -n < j < n+1, -n < k < n\},$$

$$\partial A_n = \{z \in \mathbb{Z}^2 : \mathrm{dist}(z, A_n) = 1)\}.$$

Let $\mathcal{K}_n$ denote the set of nearest neighbor paths $\omega = [\omega_0, \ldots, \omega_k]$ with $\mathrm{Re}[\omega_0] = -n, \mathrm{Re}[\omega_k] = n+1$ and $\{\omega_1, \ldots, \omega_{k-1}\} \subset A_n$. We write $|\omega| = k$ for the number of steps, and let $p(\omega) = 4^{-|\omega|}$ be the simple random walk measure. Let

$$f(n) = \sum_{\omega \in \mathcal{K}_n} p(\omega).$$

It is known that $\lim_{n\to\infty} f(n) = c_1 \in (0, \infty)$ (see, e.g., [5, Proposition 8.1.3]), where the constant c_1 can be given in terms of the Green's function of Brownian motion on a domain bounded by a square.

A path in $\mathcal{K}_n$ is a *self-avoiding walk (SAW)* if it does not visit any lattice point more than once. Let $\mathcal{W}_n$ denote the set of SAWs $\eta = [\eta_0, \ldots, \eta_k] \in \mathcal{K}_n$. For each $\omega \in \mathcal{K}_n$ there is a

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unique self-avoiding walk $L(\omega) \in \mathcal{W}_n$ obtained by chronological loop-erasing (see [5, Chapter 9] for appropriate definitions). The loop-erased measure $\hat{p}_n(\eta)$ is defined by

$$\hat{p}_n(\eta) = \sum_{\omega \in \mathcal{K}_n, L(\omega)=\eta} p(\omega).$$

Note that

$$\sum_{\eta \in \mathcal{W}_n} \hat{p}_n(\eta) = f(n).$$

Let $\mathcal{W}_n^+$ denote the set of $\eta \in \mathcal{W}_n$ that contain the directed edge $[0,1]$ and $\mathcal{W}_n^-$ those that contain $[1,0]$. Let $\mathcal{W}_n^* = \mathcal{W}_n^- \cup \mathcal{W}_n^+$ be the set of $\eta \in \mathcal{W}_n$ that contain the edge $[0,1]$ in either direction. We write $a_n \asymp b_n$ to mean that a_n/b_n and b_n/a_n are uniformly bounded. In this paper we prove the following theorem.

Theorem 1.1. *As $n \to \infty$,*

$$\sum_{\eta \in \mathcal{W}_n^*} \hat{p}_n(\eta) \asymp n^{-3/4}. \tag{1}$$

With a little more work, we could establish the existence of the limit

$$\lim_{n\to\infty} n^{3/4} \sum_{\eta \in \mathcal{W}_n^*} \hat{p}_n(\eta),$$

but we will not do it here. We plan to do this as part of a more general result in a future paper [1]. (Our argument would not give the value of the limit. While we believe we might be able to some of the relevant asymptotic constants, we definitely do not know how to compute the value of the limit in (16) below.) Our result is a strengthening of a result of Kenyon [2] who proved that

$$\sum_{\eta \in \mathcal{W}_n^*} \hat{p}_n(\eta) \approx n^{-3/4}, \tag{2}$$

where $\approx$ indicates that the logarithms of both sides are asymptotic. His proof used the relationship between loop-erased walks and two other models, dimers and uniform spanning trees. Another proof of (2) was given by Masson [11] using the relationship between loop-erased walk and the Schramm-Loewner evolution (SLE). We do not need to make reference to any of these models in our proof of (1). There are two main steps.

- A combinatorial identity is proved which writes the left-hand side of (1) in terms of simple random walk quantities.
- The simple random walk quantities are estimated.

Our computation to obtain the exponent 3/4 uses the Brownian loop measure to estimate the random walk loop measure. This is in the spirit of Kenyon's calculations [2] since the loop measure is closely related to the determinant of the Laplacian.

Although the proof is self-contained (other than some estimates for simple random walk) it does use a key idea from Kenyon's work as discussed in [3, Section 5.7]. For each random walk path ω, we let $J(\omega)$ be the number of times that the path crosses any edge of the form $[-ki, -ki+1]$ or $[-ki+1, -ki]$ where k is a positive integer. Let $q(\omega) = (-1)^{J(\omega)}\, p(\omega)$. Let $Y_+(\omega)$ denote the number of times that ω uses the directed edge $[0,1]$, $Y_-(\omega)$ the number of times that ω uses the directed edge $[1,0]$, and $Y(\omega) = Y_+(\omega) - Y_-(\omega)$. The combinatorial identity is obtained by writing the quantity

$$\Lambda_n = \sum_{\omega \in \mathcal{K}_n} q(\omega)\, Y(\omega) = \sum_{\omega \in \mathcal{K}_n} p(\omega)\, (-1)^{J(\omega)}\, Y(\omega). \tag{3}$$

in two different ways.

The paper is written using the perspective of loop-erased walk in terms of the random walk loop measure as in [5, Chapter 9]. We start by reviewing this perspective in Section 2 and then we prove the identity in Section 3. Section 4 discusses the random walk estimates. One of the main motivations for doing the estimates in this paper is to show that the loop-erased random walk converges to SLE_2 in the *natural parametrization* [6, 8]. Up-to-constant estimates for the loop-erased walk probability can be viewed as a step in the program to establish this result.

2 Random walk loop measure

The *random walk loop measure* is a measure on unrooted random walk loops. A *rooted loop* is a nearest neighbor path

$$l = [l_0, l_1, \ldots, l_{2k}] \tag{4}$$

with $k \geq 0$ and $l_0 = l_{2k}$. We call l_0 the root of the loop. An *unrooted loop* $\bar{l}$ is an equivalence class of rooted loops with $k > 0$ under the equivalence

$$[l_j, l_{j+1}, \ldots, l_{2k}, l_1, l_2, \ldots, l_j] \sim [l_0, l_1, \ldots, l_{2k}]$$

for all j. Note that the orientation of the loops is maintained. The random walk loop measure m is defined by

$$m(\bar{l}) = 4^{-|\bar{l}|}\, \frac{d(\bar{l})}{|\bar{l}|},$$

where $d(\bar{l})$ is the number of rooted loops in the equivalence class of the unrooted loop $\bar{l}$. Note that $d(\bar{l})$ is always an integer dividing $|\bar{l}|$. In a slight abuse of notation, if l is a loop and $A \subset \mathbb{Z}^2$, we write $l \subset A$ to mean that the vertices of l are contained in A and $l \cap A$ for the set of vertices in A that l visits.

There is an equivalent way of defining this measure that we will also use. Enumerate $\mathbb{Z}^2 = \{v_1, v_2, \ldots, \}$ and let $V_n = \{v_1, \ldots, v_n\}$. We define a different measure on rooted loops by assigning to each (rooted) loop as in (4) with $k > 0, l \subset V_n$, and $l_0 = v_n$ measure $s^{-1}\, 4^{-2k}$ where $s = \#\{j : 1 \leq j \leq 2k, l_j = v_n\}$. This induces a measure on unrooted loops by summing over rooted loops that generate an unrooted loop. One can check that the induced measure on unrooted loops is the same as the loop measure above regardless of which enumeration of $\mathbb{Z}^2$ is chosen. (The factor s^{-1} compensates for the fact that several rooted loops give the same unrooted loop.) We will use an enumeration in which $|v_j|$ is nondecreasing.

If $V = \{v_1, \ldots, v_k\} \subset A \subsetneq \mathbb{Z}^2$, we define

$$F_V(A) = \exp\left\{ \sum_{\bar{l} \subset A, \bar{l} \cap V \neq \emptyset} m(\bar{l}) \right\} = \prod_{j=1}^{k} G_{U_j}(v_j, v_j).$$

Here $U_j = A \setminus \{v_1, \ldots, v_{j-1}\}$ and G_U denotes the usual random walk Green's function in the set U. The second equality is obtained by associating to each unrooted loop, a rooted loop rooted at the vertex v_j of smallest index. If there are multiple choices, that is, if the loop visits the vertex of smallest index multiple times, the root is chosen uniformly over the possibilities. The loop-erased measure satisfies [5, Proposition 9.5.1]

$$\hat{p}_n(\eta) = p(\eta)\, F_\eta(A_n). \tag{5}$$

We can also define a loop measure using the signed weight $q(\omega) = (-1)^{J(\omega)}\, p(\omega)$. The quantities $J(l), Y(l)$ as defined in the introduction are functions of the unrooted loop $\bar{l}$. (Note that $Y(l)$ does depend on the orientation of l, so it is important that we are considering oriented, unrooted loops.) Let $\mathcal{J}_A$ denote the set of unrooted loops $\bar{l} \subset A$ such that $J(\bar{l})$ is odd. If $V \subset A$, let $\mathcal{J}_{A,V}$ denote the set of unrooted loops $\bar{l} \in \mathcal{J}_A$ that intersect V. Let

$$\begin{aligned} Q_V(A) &= \exp\left\{ \sum_{\bar{l} \subset A, \bar{l} \cap V \neq \emptyset} m(\bar{l})\, (-1)^{J(\bar{l})} \right\} \\ &= \exp\left\{ \sum_{\bar{l} \subset A, \bar{l} \cap V \neq \emptyset} m(\bar{l}) - 2 \sum_{\bar{l} \in \mathcal{J}_{A,V}} m(\bar{l}) \right\} = F_V(A)\, \exp\left\{-2m(\mathcal{J}_{A,V})\right\}. \end{aligned}$$

As in the case for F, if $V = \{v_1, \ldots, v_k\} \subset A$, then by associating to each unrooted loop a rooted loop of smallest index, we get

$$Q_V(A) = \prod_{j=1}^{k} g_{U_j}(v_j, v_j). \tag{6}$$

Here $U_j = A \setminus \{v_1, \ldots, v_{j-1}\}$ and

$$g_U(v_j, v_j) = \sum_l q(l) = \sum_l (-1)^{J(l)}\, p(l)$$

where the sum is over all (rooted) loops l from v_j to v_j staying in U. In particular, if $\eta \in \mathcal{W}_n$, then when the algebraic computation which gives (5) is applied to q, we get

$$\sum_{\omega \in \mathcal{K}_n, L(\omega) = \eta} q(\omega) = q(\eta)\, Q_\eta(A_n).$$

This implies that

$$\sum_{\omega\in\mathcal{K}_n, L(\omega)=\eta} (-1)^{J(\omega)-J(\eta)}\, p(\omega) = p(\eta)\, Q_\eta(A_n).$$

If $V=\{0\}$ is a singleton set, then

$$\lim_{n\to\infty} Q_V(A_n) = Q_V(\mathbb{Z}^2) = \sum_{k=0}^{\infty} s^k = (1-s)^{-1} > 0.$$

Here $s=\mathbb{E}[J']$ where $J'=(-1)^{J(S[0,T_0])}$, S is a simple random walk starting at the the origin, and $T_0=\min\{j\ge 1 : S_j = 0\}$. Since $\mathbb{P}\{J'=1\}>0$ and $\mathbb{P}\{J'=-1\}>0$, we see that $|s|<1$, and hence the limit exists. A similar argument shows that if $\mathbb{Z}^2\setminus U$ is finite and non-empty, and v is in the unbounded component of U, then $g_U(v,v)$ is finite and strictly positive. Given this and (6), it is straightforward to show that if V is finite, then

$$Q_V = Q_V(\mathbb{Z}^2) = \lim_{n\to\infty} Q_V(A_n) \tag{7}$$

exists and is strictly positive. We will write $Q_{01}(A_n)$ for $Q_{\{0,1\}}(A_n)$

For the important computation of the random walk loop measure, we will use the Brownian loop measure as introduced in [10]. There are several equivalent definitions. We give one here that is convenient for computational purposes and is analogous to the second definition we gave for the random walk loop measure. We start with the Brownian (boundary) bubble measure in the upper half plane $\mathbb{H}$ started at the origin. It is the limit as $\epsilon\downarrow 0$ of a measure on paths from $i\epsilon$ to 0 in $\mathbb{H}$ which we now describe. Let $H_{\mathbb{H}}(z,x)=|\mathrm{Im}(z)|/|z|^2$ be π times the usual Poisson kernel in the upper half plane. In other words, the probability that a Brownian motion starting at z exits $\mathbb{H}$ at an interval $I\subset\mathbb{R}$ is

$$\frac{1}{\pi}\int_I H_{\mathbb{H}}(z,x)\,dx.$$

For each ϵ consider the measure of total mass $H_{\mathbb{H}}(\epsilon i,0)$ on paths whose normalized probability measure is that of a Brownian h-process to 0. (An h-process can be viewed roughly as a Brownian motion conditioned to leave $\mathbb{H}$ at 0.) As $\epsilon\downarrow 0$, the limit measure is a σ-finite measure $\nu_{\mathbb{H}}(0)$ on loops from 0 to 0 otherwise in $\mathbb{H}$. The normalization is such that the measure of bubbles that hit the unit circle equals one. This definition can be extended to simply connected domains with smooth boundaries either by the analogous definition or by the following conformal covariance rule: if $f:\mathbb{H}\to D$ is a conformal transformation, then

$$f\circ\nu_{\mathbb{H}}(0) = |f'(0)|^2\,\nu_D(f(0)).$$

(In the definition of $f\circ\nu_{\mathbb{H}}(0)$, we need to modify the parametrization of the curve using Brownian scaling, but the parametrization is not important in this paper.)

Given the Brownian bubble measure, the Brownian loop measure restricted to curves in the unit disk $\mathbb{D}$ can be written as

$$\frac{1}{\pi}\int_0^1\int_0^{2\pi} \nu_{r\mathbb{D}}(re^{i\theta})\,r\;d\theta\,dr, \tag{8}$$

To be more precise, the loop measure is the measure on *unrooted* loops induced by the above measure on rooted loops. (This representation of the measure on unrooted loops focuses on the rooted representative with root as far from the origin as possible.) The Brownian loop measure is the scaling limit of the random walk loop measure in a sense made precise in [9]. We discuss this more in Section 4.

3 A combinatorial identity

Let $\mathcal{K}_n'$ denote the set of nearest neighbor paths $\omega = [\omega_0, \omega_1, \ldots, \omega_k]$ with $\mathrm{Re}[\omega_0] = -n, \omega_k = 0$ and $\{\omega_1, \ldots, \omega_{k-1}\} \subset A_n \setminus [0, \infty)$. Let $\mathcal{K}_n''$ denote the set of nearest neighbor paths $\omega = [\omega_0, \omega_1, \ldots, \omega_k]$ with $\mathrm{Re}[\omega_0] = n+1, \omega_k = 1$ and $\{\omega_1, \ldots, \omega_{k-1}\} \subset A_n \setminus (-\infty, 1]$. There is a natural bijection between $\mathcal{K}_n'$ and $\mathcal{K}_n''$ obtained by reflection about the line $\{\mathrm{Re}(z) = 1/2\}$. Let

$$R_n = \sum_{\omega \in \mathcal{K}_n'} p(\omega) = \sum_{\omega \in \mathcal{K}_n''} p(\omega).$$

By considering the reversed path, we can see that $R_n = \mathbb{P}\{\mathrm{Re}(S_\tau) = -n\}$ where S is a simple random walk starting at the origin and $\tau = \min\{j > 0 : S_j \in \partial A_n \cup [0, \infty)\}$. It is known (see e.g., [5, Proposition 5.3.2]) that

$$R_n \asymp n^{-1/2}, \quad n \to \infty. \tag{9}$$

The goal of this section is to prove the following combinatorial identity which relates the probability that loop-erased walk uses the undirected edge $\{0, 1\}$ to some simple random walk quantities.

Theorem 3.1.

$$4 \sum_{\eta \in \mathcal{W}_n^*} \hat{p}_n(\eta) = Q_{01}(A_n)\, R_n^2\, \exp\left\{2m(\mathcal{J}_{A_n})\right\}.$$

Proof. We start by making the following topological observation:

$$(-1)^{J(\eta)}\, Y(\eta) = 1 \quad \text{if} \quad \eta \in \mathcal{W}_n^*. \tag{10}$$

To see this, consider the path η as a continuous path from $\{\mathrm{Re}(z) = -n\}$ to $\{\mathrm{Re}(z) = n+1\}$ in the domain $D = \{x + iy \in \mathbb{C} : -n < x < n+1, -n < y < n\}$. Then η is a crosscut of D such that $D \setminus \eta$ consists of two components, the "top" component D^+ and the "bottom" component D^-. Each ordered edge $[w, w']$ in η can be considered as subsets of ∂D^+ and ∂D^-. If we traverse the edge from w to w', the left-hand side of $[w, w']$ (considered as a prime end) is in ∂D^+ and the right-hand side is in ∂D^-. Let N_+ be the set of integers k such that the ordered edge $[ki, ki+1]$ is contained in η, N_- the set of integers k such that the ordered edge $[ki+1, ki]$ is contained in η, and $N = N_+ \cup N_-$. We claim that if $j \in N_+$ and k is the largest integer less than j with $k \in N$, then $k \in N_-$. Indeed, since $j \in N_+$, the open line segment from $ji + (1/2)$ to $ki + (1/2)$ is contained in D^- which implies that $k \in N_-$.

We now consider the smallest k such that $k \in N$. The line segment from $-ni + (1/2)$ to $ki + (1/2)$ is contained in D^- and hence $k \in N_+$. As we continue up the line $\{\mathrm{Re}(z) = 1/2\}$ we see that when we intersect edges in η, they alternate being in N_+ or N_-, with the first in N_+, the second in N_-, the third in N_+, etc. When we reach the unordered edge $\{0, 1\}$, we see that if $0 \in N_+$, then there have been an even number of edges before $\{0, 1\}$ and if $0 \in N_-$, there have been an odd number of edges. In other words, $(-1)^{J(\eta)} = 1$ if $\eta \in \mathcal{W}_n^+$ and $(-1)^{J(\eta)} = -1$ if $\eta \in \mathcal{W}_n^-$. This gives (10).

Let Λ_n be defined as in (3). We claim that

$$\Lambda_n = \sum_{\omega \in \mathcal{K}_n} q(\omega)\, Y(L(\omega)) = \sum_{\eta \in \mathcal{W}_n^*} \sum_{L(\omega)=\eta} p(\omega)\, (-1)^{J(\omega)-J(\eta)}. \tag{11}$$

To see this, suppose that $L(\omega) = \eta = [\eta_0, \ldots, \eta_k]$. Then we can write ω uniquely as

$$\omega = [\eta_0, \eta_1] \oplus l_1 \oplus [\eta_1, \eta_2] \oplus l_2 \oplus \cdots \oplus [\eta_{k-2}, \eta_{k-1}] \oplus l_{k-1} \oplus [\eta_{k-1}, \eta_k],$$

where l_j is a loop rooted at η_j that does not enter $\{\eta_1, \ldots, \eta_{j-1}\}$. We write

$$J(\omega) = J(\eta) + J_L(\omega), \quad Y(\omega) = Y(\eta) + Y_L(\omega),$$

where J_L, Y_L denote the contributions from the loops. Then

$$Y(\omega) = Y(\eta) + \sum_{j=1}^{k-1} Y(l_j).$$

For each loop l_j there is the corresponding reversed loop l_j^R for which $Y(l_j^R) = -Y(l_j)$. Since $J(l_j^R) = J(l_j)$ and $Y(l_j^R) = -Y(l_j)$, we get cancellation. Doing this for all the loops, we see that

$$\sum_{\omega \in \mathcal{K}_n, L(\omega)=\eta} q(\omega)\, [Y(\omega) - Y(\eta)] = 0.$$

This gives the first equality in (11). The second equality uses (10) and the fact that $Y(\eta) = 0$ if $\eta \notin \mathcal{W}_n^*$.

If $\eta \in \mathcal{W}_n^*$, then

$$\begin{aligned}
\sum_{L(\omega)=\eta} p(\omega)\, (-1)^{J(\omega)-J(\eta)} &= p(\eta) \sum_{L(\omega)=\eta} \frac{p(\omega)}{p(\eta)}\, (-1)^{J_L(\omega)} \\
&= p(\eta)\, Q_\eta(A_n) \\
&= p(\eta)\, \exp\left\{ \sum_{\bar{l} \subset A_n, \bar{l} \cap \eta \neq \emptyset} (-1)^{J(\bar{l})}\, m(\bar{l}) \right\} \\
&= p(\eta)\, F_\eta(A_n)\, \exp\left\{ -2 \sum_{\bar{l} \subset A_n, \bar{l} \cap \eta \neq \emptyset, J(\bar{l}) \text{ odd}} m(\bar{l}) \right\}.
\end{aligned}$$

If $J(\bar{l})$ is odd, then $\bar{l}$ must include at least one unordered edge $\{ki, ki+1\}$ with $k \geq 0$ and at least one unordered edge $\{ki, ki+1\}$ with $k < 0$. Therefore, topological considerations imply that if $\eta \in \mathcal{W}_n^*$, then $\eta \cap \bar{l} \neq \emptyset$. Hence

$$\sum_{\bar{l} \subset A_n, \bar{l} \cap \eta \neq \emptyset, J(\bar{l}) \text{ odd}} m(\bar{l}) = \sum_{\bar{l} \subset A_n, J(\bar{l}) \text{ odd}} m(\bar{l}) = m(\mathcal{J}_{A_n}).$$

Combining this with (11), we see that

$$\Lambda_n = \sum_{\eta \in \mathcal{W}_n^*} p(\eta)\, Q_\eta(A_n) = e^{-2m(\mathcal{J}_{A_n})} \sum_{\eta \in \mathcal{W}_n^*} p(\eta)\, F_\eta(A_n) = e^{-2m(\mathcal{J}_{A_n})} \sum_{\eta \in \mathcal{W}_n^*} \hat{p}_n(\eta). \tag{12}$$

We will now compute Λ_n as defined in (3) in a different way. Let $\omega = [\omega_0, \ldots, \omega_\tau] \in \mathcal{K}_n$. If ω does not visit 0 or ω does not visit 1, then $Y(\omega) = 0$. Hence, we only need to consider the sum over $\omega \in \mathcal{K}_n$ that visit both 0 and 1. For such ω, we define $T_0 = \min\{j : \omega_j = 0\}, T_0' = \max\{j < \tau : \omega_j = 0\}$, and we define T_1, T_1' similarly.

Suppose that $T_0 < T_1, T_0' > T_1'$. In this case we write

$$\omega = \omega^- \oplus l \oplus \omega^+, \tag{13}$$

where l is the loop $[\omega_{T_0}, \ldots, \omega_{T_0'}]$. Note that $Y(\omega) = Y(l)$. For any such loop l, there is the corresponding reversed loop $l^R = [\omega_{T_0'}, \omega_{T_0'-1}, \ldots, \omega_{T_0}]$ for which $Y(l^R) = -Y(l)$. These terms cancel and hence the sum in (3) over ω with $T_0 < T_1, T_0' > T_1'$ is zero. Similarly, the sum over ω with $T_1 < T_0 \leq T_0' < T_1'$ is zero.

Suppose that $T_0 > T_1, T_0' > T_1'$. Then we can write ω uniquely as

$$\omega = \omega^- \oplus l_1 \oplus \omega' \oplus l_0 \oplus \omega^+,$$

with the following conditions. Here l_0 is a loop in A_n rooted at 0, l_1 is a loop in $A_n \setminus \{0\}$ rooted at 1, ω' is a path from 1 to 0 whose other vertices are in $A_n \setminus \{0,1\}$, ω^- is a path from $\{\mathrm{Re}(z) = -n\}$ to 1 whose other vertices are in $A_n \setminus \{0,1\}$, and ω^+ is a path from 0 to $\{\mathrm{Re}(z) = n+1\}$ whose other vertices are in $A_n \setminus \{0,1\}$. Let $\tilde{\omega}^-$ be the reflection of ω^- about the real axis, and $\tilde{\omega} = \tilde{\omega}^- \oplus l_1 \oplus \omega' \oplus l_0 \oplus \omega^+$. Then $J(\omega^-) + J(\tilde{\omega}^-)$, and hence $J(\omega) + J(\tilde{\omega})$, are odd and these terms will cancel in the sum. Hence the sum over all ω with $T_0 > T_1, T_0' > T_1'$ is zero.

Let $\mathcal{K}_n^1$ be the set of paths in $\mathcal{K}_n$ that visit both 0 and 1 and satisfy $T_0 < T_1, T_0' < T_1'$. We have shown that

$$\Lambda_n = \sum_{\omega \in \mathcal{K}_n^1} q(\omega)\, Y(\omega).$$

If $\omega \in \mathcal{K}_n^1$, let $\rho = \min\{j > T_0' : \omega_j = 1\}$. Then we can write ω uniquely as

$$\omega = \omega^- \oplus l_0 \oplus \omega' \oplus l_1 \oplus \omega^+. \tag{14}$$

Here $l_0 = [\omega_{T_0}, \ldots, \omega_{T_0'}]$ is a loop in A_n rooted at 0, $l_1 = [\omega_\rho, \ldots, \omega_{T_1'}]$ is a loop in $A_n \setminus \{0\}$ rooted at 1, $\omega' = [\omega_{T_0'}, \ldots, \omega_\rho]$ is a path from 0 to 1, $\omega^- = [\omega_0, \ldots, \omega_{T_0}]$ is a path from

$\{\mathrm{Re}(z) = -n\}$ to 0, $\omega^+ = [\omega_{T_1'}, \ldots, \omega_\tau]$ is a path from 1 to $\{\mathrm{Re}(z) = n+1\}$. All of the vertices of $\omega', \omega^-, \omega^+$ other than the endpoints are in $A_n \setminus \{0,1\}$. Note that $Y(\omega) = Y(l_0) + Y(\omega')$. As in the previous arguments, we can replace l_0 with the reversed loop l_0^R, to see that

$$\sum_{\omega \in \mathcal{K}_n^1} (-1)^{J(\omega)} \, Y(l_0) \, p(\omega) = 0.$$

Also $Y(\omega') \in \{0,1\}$ with $Y(\omega') = 1$ if and only if $T_0' + 1 = \rho$, that is, if $\omega' = [0,1]$. Therefore, if $\mathcal{K}_n^2$ denotes the set of paths in $\mathcal{K}_n^1$ with $\omega' = [0,1]$, then

$$\Lambda_n = \sum_{\omega \in \mathcal{K}_n^2} (-1)^{J(\omega)} \, p(\omega) = \sum_{\omega \in \mathcal{K}_n^2} (-1)^{J(\omega^-) + J(l_0) + J(l_1) + J(\omega^+)} \, p(\omega). \tag{15}$$

If $\omega \in \mathcal{K}_n^2$, let ξ be the smallest j such that ω_j is on the positive real axis. Suppose for the moment that $\xi < T_0$. Then we can write

$$\omega^- = \omega^{-,1} \oplus \omega^{-,2},$$

by splitting the path at time ξ. The path $\omega^{-,2}$ is a path from the positive real axis to 0 that does not go through the point 1. Hence, $J(\omega^{-,2}) + J(\tilde{\omega}^{-,2})$ is odd, where $\tilde{\omega}^{-,2}$ is the reflection of $\omega^{-,2}$ about the real axis. These terms will cancel in the sum (15), and hence it suffices to sum over ω^- such that $\omega^- \cap [1, \infty) = \emptyset$. For these ω^-, we can see by topological reasons that $(-1)^{J(\omega^-)} = 1$. By a similar argument, it suffices to sum over ω^+ satisfying $\omega^+ \cap (-\infty, 0] = \emptyset$, and for these walks $(-1)^{J(\omega^+)} = 1$. Therefore, if $\mathcal{K}_n^3$ denote the set of paths in $\mathcal{K}_n^2$ satisfying

$$\omega^- \cap [1, \infty) = \emptyset, \quad \omega^+ \cap (-\infty, 0] = \emptyset,$$

we see that

$$\Lambda_n = \sum_{\omega \in \mathcal{K}_n^3} (-1)^{J(l_0) + J(l_1)} \, p(\omega).$$

Let us write any $\omega \in \mathcal{K}_n^3$ as in (14). We must choose $\omega^- \in \mathcal{K}_n', (\omega^+)^R \in \mathcal{K}_n''$ and $\omega' = [0,1]$. Summing over all of these possibilities, gives a factor of $R_n^2/4$. The choices of l_0, l_1 are independent of the choices of ω^- and ω^+. The only restriction is that the loops lie in A_n and l_1 does not contain the origin. By our definition,

$$\sum_{l_0, l_1} (-1)^{J(l_0) + J(l_1)} \, p(l_0) \, p(l_1) = g_{A_n}(0,0) \, g_{A_n \setminus \{0\}}(1,1) = Q_{01}(A_n).$$

Therefore,

$$\Lambda_n = \sum_{\omega \in \mathcal{K}_n^3} (-1)^{J(l_0) + J(l_1)} \, p(\omega) = \frac{1}{4} \, R_n^2 \, Q_{01}(A_n).$$

Comparing this with (12) gives the theorem. $\square$

4 Estimate on the random walk loop measure

Using Theorem 3.1 and the estimates (7) and (9), we see that

$$\sum_{\eta \in \mathcal{W}_n^*} \hat{p}_n(\eta) \asymp n^{-1} \exp\{2m(\mathcal{J}_{A_n})\}.$$

The proof of (1) is finished with the following proposition.

Proposition 4.1. *There exists $c < \infty$ such that for all n,*

$$\left| m(\mathcal{J}_{A_n}) - \frac{1}{8} \log n \right| \leq c.$$

Proof. Let $C_n = \{z \in \mathbb{Z}^2 : |z| < e^n\}$. We will prove the stronger fact that the limit

$$\lim_{n \to \infty} \left[m(\mathcal{J}_{C_n}) - \frac{n}{8} \right] \tag{16}$$

exists by showing that

$$\sum_{n=1}^{\infty} \left| m(\mathcal{J}_{C_{n+1}} \setminus \mathcal{J}_{C_n}) - \frac{1}{8} \right| < \infty. \tag{17}$$

Let μ denote the Brownian loop measure, and let $\tilde{\mathcal{J}}$ denote the set of unrooted Brownian loops γ in the unit disk that intersect $\{|z| \geq e^{-1}\}$ and such that the winding number of γ about the origin is odd. We will establish (17) by showing that $\mu(\tilde{\mathcal{J}}) = 1/8$ and

$$\left| m(\mathcal{J}_{C_{n+1}} \setminus \mathcal{J}_{C_n}) - \mu(\tilde{\mathcal{J}}) \right| = O(n^{-2}). \tag{18}$$

For the Brownian loop measure, we do a computation similar to that in [4, Proposition 3.9]. Using (8), we write

$$\mu(\tilde{\mathcal{J}}) = \frac{1}{\pi} \int_{e^{-1}}^{1} \int_{0}^{2\pi} \phi(r, \theta) \, d\theta \, r \, dr,$$

where $\phi(r, \theta)$ denotes the Brownian bubble measure of loops in $r\mathbb{D}$ rooted at $re^{i\theta}$ with odd winding number about the origin. Rotational symmetry implies that $\phi(r, \theta) = \phi(r, 0)$ and conformal covariance implies that $\phi(r, 0) = r^{-2} \phi$ where $\phi = \phi(1, 0)$. Hence,

$$\mu(\tilde{\mathcal{J}}) = \frac{\phi}{\pi} \int_{e^{-1}}^{1} \int_{0}^{2\pi} r^{-2} \, d\theta \, r \, dr = 2\phi. \tag{19}$$

By considering the (multi-valued) covering map $f(z) = i \log z$ which satisfies $|f'(1)| = 1$, we see that

$$\phi = \sum_{k \text{ odd}} H_{\partial \mathbb{H}}(0, 2\pi k),$$

where $H_{\partial\mathbb{H}}$ denotes the boundary Poisson kernel (normal derivative of the Poisson kernel) in the upper half-plane $\mathbb{H}$ normalized as before so that $H_{\partial\mathbb{H}}(0,x) = x^{-2}$. Therefore,

$$2\phi = 2\sum_{k=-\infty}^{\infty} \frac{1}{[2\pi(2k+1)]^2} = \frac{1}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{1}{8}.$$

If $s > 2$, let $\tilde{\mathcal{J}}_s^*$ denote the set of Brownian loops in $\mathbb{D}$ that intersect both $\{|z| \geq e^{-1}\}$ and $\{|z| \leq e^{-s}\}$. We claim that as $s \to \infty$,

$$\mu(\tilde{\mathcal{J}}_s^*) = s^{-1} + O(s^{-2}), \tag{20}$$

$$\mu[\tilde{\mathcal{J}} \cap \tilde{\mathcal{J}}_s^*] = (2s)^{-1} + O(s^{-2}). \tag{21}$$

To see this, we first consider the boundary bubble measure λ_s of loops in $\mathbb{D}$ rooted at 1 that enter $\{|z| \leq e^{-s}\}$. An exact expression is given as follows. Let B_t be a Brownian motion and $\sigma_s = \inf\{t : |B_t| = e^{-s}\}$. Then,

$$\lambda_s = \lim_{\epsilon \downarrow 0} \mathbb{E}^{\epsilon}\left[H_{\mathbb{D}}(B_{\sigma_s}, 1); \sigma_s < \sigma_0\right].$$

The Poisson kernel in the disk is well known; for our purpose we need only know that

$$H_{\mathbb{D}}(e^{-s+xi}, 1) = \sum_{k\in\mathbb{Z}} H_{\mathbb{H}}(x + 2\pi k + is, 0) = \sum_{k\in\mathbb{Z}} \frac{s}{(x+2\pi k)^2 + s^2} = \frac{1}{2} + O(e^{-s}),$$

(recall that $H_{\mathbb{D}}(z,1)$ is π times the hitting density which is uniform on the circle), and a standard estimate for Brownian motion gives

$$\mathbb{P}^{1-\epsilon}\{\sigma_s < \sigma_0\} = \frac{\log(1-\epsilon)}{-s} \sim \frac{\epsilon}{s}.$$

Therefore, $\lambda_s = (2s)^{-1} + O(e^{-s})$. Using rotational invariance, and conformal covariance, if $r \geq e^{-1}$ and $\lambda(r,\theta,s)$ denotes the bubble measure of bubbles in $r\,\mathbb{D}$ rooted at $re^{i\theta}$ that enter $\{|z| \leq e^{-s}\}$, then

$$\lambda(r,\theta,s) = r^{-2}\,(2s)^{-1}\,[1 + O(s^{-1})].$$

If we compute as in (19), we get (20). The relation (21) is done similarly except that we have to worry about the winding number of the loop. Here we use

$$\sum_{k \text{ even}} H_{\mathbb{H}}(x + 2\pi k + is, 0) = \sum_{k \text{ even}} \frac{s}{(x+2\pi k)^2 + s^2} = \frac{1}{4} + O(e^{-s}),$$

to see that

$$\mu[\tilde{\mathcal{J}} \cap \tilde{\mathcal{J}}_s^*] = \frac{1}{2}\,\mu[\tilde{\mathcal{J}}_s^*]\,[1 + O(e^{-s})]. \tag{22}$$

For each unrooted random walk loop $\bar{l} \in \mathcal{J}_{C_n} \setminus \mathcal{J}_{C_{n-1}}$, there is a corresponding continuous unrooted loop $\bar{l}^{(n)}$ in $\mathbb{D}$ obtained from linear interpolation and Brownian scaling. We will write $d(\bar{l},\gamma) \leq \delta$, if we can parametrize and root the loops $\bar{l}^{(n)}$ and γ such that the loops are within δ in the supremum norm. In [9] it was shown that there exists $\alpha > 0$ and a coupling of the random walk and Brownian loop measures in D, restricted to loops of diameter at least $1/e$, so that the total masses agree up to $O(e^{-n\alpha})$ and such that in the coupling, except for a set of paths of size $O(e^{-n\alpha})$, we have $d(\bar{l},\gamma) < e^{-n\alpha}$. (Actually, a more precise estimate is given in [9], but this is all we need for this paper.) We would like to say that in the coupling, the Brownian loop has odd winding number if and only $J(\bar{l})$ is odd. If the loops stay away from the origin, this holds. However, if the loops are near the origin, it is possible for the winding numbers of the continuous and the discrete walks to be different. However, and this is why we can prove what we need, it is also true that if a macroscopic loop (either continuous or discrete) gets close to the origin, then it is just about equally likely to have an odd as an even winding number. Let us be more precise.

Let $\beta < \alpha$ and let $\mathcal{J}^n$ denote the set of loops in $\mathcal{J}_{C_{n+1}} \backslash \mathcal{J}_{C_n}$ that intersect $\{|z| \leq e^{-\beta n}\, e^{n+1}\}$ Using the coupling and (21), we see that

$$m(\mathcal{J}^n) = \mu(\tilde{\mathcal{J}}'_{\beta n}) + O(n^{-2}) = (\beta n)^{-1} + O(n^{-2}).$$

Let us split these paths into two sets: those for which $\mathrm{dist}(0,\gamma) \leq 2e^{-n\alpha}$ and those for which $\mathrm{dist}(0,\gamma) > 2e^{-n\alpha}$. If $\mathrm{dist}(0,\gamma) > 2e^{-n\alpha}$ and $d(\bar{l},\gamma) \leq e^{-n\alpha}$, then $J(\bar{l})$ is odd if and only if the winding number of γ is odd. Therefore

$$m((\mathcal{J}_{C_{n+1}} \setminus \mathcal{J}_{C_n}) \setminus \mathcal{J}^n) = \mu(\tilde{\mathcal{J}} \setminus \tilde{\mathcal{J}}'_{\beta n}) + O(n^{-2}).$$

(The error term $O(n^{-2})$ is comparable to the measure of loops γ such that $e^{-n\beta} \leq \mathrm{dist}(0,\gamma) \leq 2e^{-n\beta}$.)

A coupling argument can be used to give a random walk analogue of (22),

$$m\left[\mathcal{J}^n \cap (\mathcal{J}_{C_{n+1}} \setminus \mathcal{J}_{C_n})\right] = \frac{1}{2}\, m(\mathcal{J}^n)\, [1 + o(n^{-1})].$$

We sketch the proof which, in fact, gives an error of $O(e^{-un})$ for some u. We use the definition of the loop measure using an enumeration of $\mathbb{Z}^2 = \{z_1, z_2, \ldots\}$ such that $|z_j|$ increases. Then an unrooted loop in $\mathcal{J}^n$ is obtained from a loop rooted in $C_{n+1} \setminus C_n$. Let us call the root z_k and so the loops lies in $V_k = \{z_1, \ldots, z_k\}$. Let us stop the walk at the first time it reaches a point, say z', in $\{|z| \leq e^{-\beta n}\, e^{n+1}\}$. The remainder of the loop acts like a random walk started at z' conditioned to reach z_k without before leaving V_k. Let J' denote the number of times such a walk crosses the half line $\{(1/2) + iy : y < 0\}$. We claim that the probability that J' is odd equals $\frac{1}{2} + O(e^{-un})$ for some $u > 0$. Indeed, we can couple two walks starting at the point so that each walk has the distribution of random walk conditioned to reach z_k before leaving V_k and that, except for an event of probability $O(e^{-\delta})$, the parity of J' is different for the two walks. This uses a standard technique. The key estimate is the following. There exists $c > 0$ such that if S is a simple random walk starting at $z \in C_{j-1}$ and $T = \min\{j : S_y \in C_j\}$, then for all $w \in \partial C_j$ with $\mathrm{Im}(w) > 0$,

$$\mathbb{P}\{S(T) = w,\ J' \text{ odd}\} \geq c\, e^{-j},$$

$$\mathbb{P}\{S(T) = w,\ J' \text{ even}\} \geq c\, e^{-j}.$$

Without the restriction of the parity of J', see, for example, [5, Lemma 6.3.7]. To get the result with the restriction, we just note that there is a positive probability of making a loop in the annulus $C_j \setminus C_{j-1}$, and this increases J' by one. Hence, we can find a coupling and a $\rho > 0$ such that at each annulus there is a probability ρ of a successful coupling given that the walks have not yet been coupled. Since there are of order $\beta\, n$ annuli, we can couple the processes so that the probability of not being coupled is $(1-\rho)^{\beta n} = O(e^{-un})$ for some u.

From the last two estimates and (21), we see that

$$\left| \mu(\tilde{\mathcal{J}}) - m(\mathcal{J}_{C_{n+1}} \setminus \mathcal{J}_{C_n}) \right| \leq c\, n^{-2}.$$

This gives (18).

□

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