The probability that planar loop-erased random walk uses a given edge

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Abstract

We give a new proof of a result of Rick Kenyon that the probability that an edge in the middle of an $n \times n$ square is used in a loop-erased walk connecting opposite sides is of order $n^{-3/4}$. We, in fact, improve the result by showing that this estimate is correct up to multiplicative constants.

1 Introduction

Loop-erased random walk is a process obtained by erasing loops from simple random walk. Although the process can be defined for arbitrary Markov chains, we will discuss the process only on the planar integer lattice $\mathbb{Z}^2 = \mathbb{Z} + i\mathbb{Z}$.

Let

$$A_n = \{j + ik \in \mathbb{Z} + i\mathbb{Z} : -n + 1 < j < n, -n < k < n\},$$

$$\partial A_n = \{z \in \mathbb{Z}^2 : \text{dist}(z, A_n) = 1\}.$$ 

Let $\mathcal{K}_n$ denote the set of nearest neighbor paths $\omega = [\omega_0, \ldots, \omega_k]$ with $\text{Im}[\omega_0] = -n, \text{Im}[\omega_k] = n+1$ and $\{\omega_1, \ldots, \omega_{k-1}\} \subset A_n$. We write $|\omega| = k$ for the number of steps, and let $p(\omega) = 4^{-|\omega|}$ be the simple random walk measure. Let

$$f(n) = \sum_{\omega \in \mathcal{K}_n} p(\omega).$$

It is known that $\lim_{n \to \infty} f(n) = c_1 \in (0, \infty)$ (see, e.g., [4, Proposition 8.1.3]), where the constant $c_1$ can be given in terms of the Green’s function of Brownian motion on a domain bounded by a square.

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Let \( R_n \) denote the set of self-avoiding walks (SAWs) \( \eta = [\eta_0, \ldots, \eta_k] \) contained in \( \mathcal{K}_n \). For each \( \omega \in \mathcal{K}_n \) there is a unique path \( L(\omega) \in R_n \) obtained by chronological loop-erasing. The loop-erased measure \( \hat{p}_n(\eta) \) is defined by

\[
\hat{p}_n(\eta) = \sum_{\omega \in \mathcal{K}_n, L(\omega) = \eta} p(\omega).
\]

Note that

\[
\sum_{\eta \in R_n} \hat{p}_n(\eta) = f(n).
\]

Let \( R_n^+ \) denote the set of \( \eta \in R_n \) that contain the directed edge \([0, 1]\) and \( R_n^- \) those that contain \([1, 0]\). Let \( R_n^* = R_n^- \cup R_n^+ \) be the set of \( \eta \in R_n \) that contain the edge \([0, 1]\) in either direction. We write \( a_n \asymp b_n \) to mean than \( a_n/b_n \) and \( b_n/a_n \) are uniformly bounded. The goal of this paper is to prove the following theorem.

**Theorem 1.** As \( n \to \infty \),

\[
\sum_{\eta \in R_n^*} \hat{p}_n(\eta) \asymp n^{-3/4}.
\]

We expect that with a little more work, we could establish the existence of the limit

\[
\lim_{n \to \infty} n^{3/4} \sum_{\eta \in R_n^*} \hat{p}_n(\eta),
\]

but we will not do it here. Our result can be considered a strengthening of a result of Kenyon [1] who proved that

\[
\sum_{\eta \in R_n^*} \hat{p}_n(\eta) \approx n^{-3/4},
\]

where \( \approx \) indicates that the logarithms of both sides are asymptotic. His proof used the relationship between loop-erased walks and two other models, dimers and uniform spanning trees. Another proof of (2) was given by Masson [7] using the relationship between loop-erased walk and the Schramm-Loewner evolution (SLE). We do not need to make reference to any of these models in our proof. There are two main steps.

- A combinatorial identity is proved which writes the left-hand side of (1) in terms of simple random walk quantities.
- The simple random walk quantities are estimated.

Our computation to obtain the exponent \( 3/4 \) uses the Brownian loop measure to estimate the random walk loop measure. This is in the spirit of Kenyon’s calculations [1] since the measure is closely related to the determinant of the Laplacian.

Although the proof is self-contained (other than some simple random walk estimates) it does use a key idea from Kenyon’s work as discussed in [2, Section 5.7]. For each random
walk path \( \omega \), we let \( J(\omega) \) be the number of times that the path crosses an edge of the form \([-ki, -ki + 1]\) or \([-ki + 1, -ki]\) where \( k \) is a positive integer. Let \( q(\omega) = (-1)^{J(\omega)} \). Let \( Y_+ (\omega) \) denote the number of times that \( \omega \) uses the directed edge \([0, 1]\), \( Y_- \) the number of times that \( \omega \) uses the directed edge \([1, 0]\), and \( Y(\omega) = Y_+ (\omega) - Y_- (\omega) \). The combinatorial identity is obtained by writing the quantity

\[
\Lambda_n = \sum_{\omega \in \mathcal{R}_n} q(\omega) Y(\omega).
\]

in two different ways.

The paper is written using the perspective of loop-erased walk in terms of the random walk loop measure as in [4, Chapter 9]. We start by reviewing this perspective in Section 2 and then we prove the identity in Section 3. The final section discusses the random walk estimates.

## 2 Random walk loop measure

The random walk loop measure is a measure on unrooted random walk loops. A rooted loop is a nearest neighbor path

\[
l = [l_0, l_1, \ldots, l_{2k}]
\]

with \( k \geq 0 \) and \( l_0 = l_{2k} \). An unrooted loop \( \bar{l} \) is an equivalence class of rooted loops with \( k > 0 \) under the equivalence

\[
[l_j, l_{j+1}, \ldots, l_{2k}, l_1, l_2, \ldots, l_j] \sim [l_0, l_1, \ldots, l_{2k}]
\]

for all \( j \). Note that the orientation of the loops is maintained. The random walk loop measure \( m \) is defined by

\[
m(\bar{l}) = 4\cdot d(\bar{l}) \frac{d(\bar{l})}{|\bar{l}|},
\]

where \( d(\bar{l}) \) is the number of rooted loops in the equivalence class of the unrooted loop \( \bar{l} \). Note that \( d(\bar{l}) \) is always an integer dividing \( |\bar{l}| \).

There is an equivalent way of defining this measure that is sometimes useful. Enumerate \( \mathbb{Z}^2 = \{v_1, v_2, \ldots, \} \) and let \( V_n = \{v_1, \ldots, v_n\} \). We define a different measure on rooted loops by assigning to each (rooted) loop as in (4) with \( k > 0, l \subset V_n \), and \( l_0 = v_n \) measure \( s^{-1} 4^{-2k} \) where \( s = \# \{j; 1 \leq j \leq 2k, l_j = v_n\} \). This induces a measure on unrooted loops by summing over rooted loops that generate an unrooted loop. (The factor \( s^{-1} \) compensates for the fact that several rooted loops give the same unrooted loop.) One can check that the induced measure on unrooted loops is the same as the loop measure above regardless of what enumeration is chosen. For computations it is often convenient to choose an enumeration in which \( |v_j| \) is nondecreasing.
If \( V = \{v_1, \ldots, v_k\} \subset A \subseteq \mathbb{Z}^2 \), we define

\[
F_V(A) = \exp \left\{ \sum_{\bar{l} \subset A, \bar{l} \cap V \neq \emptyset} m(\bar{l}) \right\} = \prod_{j=1}^{k} G_{U_j}(v_j, v_j).
\]

Here \( U_j = A \setminus \{v_1, \ldots, v_{j-1}\} \) and \( G_U \) denotes the usual random walk Green’s function in the set \( U \). The loop-erased measure satisfies [4, Proposition 9.5.1]

\[
\hat{p}_n(\eta) = p(\eta) F_\eta(A_n).
\]

We can also define a loop measure using the signed weight \( q(\omega) = (-1)^{J(\omega)} p(\omega) \). The quantities \( J(l), Y(l) \) as defined in the introduction are functions of the unrooted loop \( \bar{l} \). Let \( J_A \) denote the set of unrooted loops \( \bar{l} \subset A \) with \( J(\bar{l}) \) odd. If \( V \subset A \), let \( J_{A,J} \) denote the set of unrooted loops \( \bar{l} \in J_A \) that intersect \( V \). Let

\[
Q_V(A) = \exp \left\{ \sum_{\bar{l} \subset A, \bar{l} \cap V \neq \emptyset} m(\bar{l}) (-1)^{J(\bar{l})} \right\} = F_V(A) \exp \{-2m(J_{A,J})\}.
\]

As in the case for \( F \), if \( V = \{v_1, \ldots, v_k\} \subset A \), then

\[
Q_V(A) = \prod_{j=1}^{k} g_{U_j}(v_j, v_j),
\]

where \( U_j = A \setminus \{v_1, \ldots, v_{j-1}\} \) and

\[
g_U(v_j, v_j) = \sum_\omega q(\omega)
\]

where the sum is over all (rooted) loops from \( v_j \) to \( v_j \) staying in \( U \). In particular, if \( \eta \in \mathcal{R}_n \), then

\[
\sum_{\omega \in \mathcal{K}_n, \lambda(\omega) = \eta} (-1)^{J(\omega) - J(\eta)} p(\omega) = Q_\eta(A_n).
\]

If \( V = \{0\} \) is a singleton set, then

\[
\lim_{n \to \infty} Q_V(A_n) = Q_V(\mathbb{Z}^2) = (1 - s)^{-1} > 0.
\]

Here \( s \) is defined as the expectation of \((-1)^{J(S[0,T])}\) where \( S \) is a simple random walk starting at the origin and \( T_0 = \min\{n \geq 1 : S_n > 0\} \). It is easy to see that \(|s| < 1\) and hence the limit exists and is positive. Given this, it is straightforward to show that if \( V \) is finite, then

\[
Q_V = Q_V(\mathbb{Z}^2) = \lim_{n \to \infty} Q_V(A_n) \quad (5)
\]
exists and is strictly positive. We will write $Q_{01}(A_n)$ for $Q_{\{0,1\}}(A_n)$.

For the important computation of the random walk loop measure, we will use the Brownian loop measure as introduced in [6]. There are several equivalent definitions. We give one here that is convenient for computational purposes and is analogous to the second definition we gave for the random walk loop measure. We start with the Brownian (boundary) bubble measure in the upper half plane $\mathbb{H}$ started at the origin. It is the limit as $\epsilon \downarrow 0$ of a measure on paths from $i\epsilon$ to 0 in $\mathbb{H}$. For each $\epsilon$ consider the measure on paths of total mass $\epsilon^{-1}$ whose normalized probability measure is that of an $h$-process to 0. An $h$-process can be viewed roughly as a Brownian motion conditioned to leave $\mathbb{H}$ at 0. As $\epsilon \downarrow 0$, the limit measure is a $\sigma$-finite measure $\nu_\mathbb{H}(0)$ on loops from 0 to 0 otherwise in $\mathbb{H}$. The normalization is such that the measure of bubbles that hit the unit circle equals one. This measure on simply connected domains with smooth boundaries either by the analogous definition or by the following conformal covariance rule: if $f : \mathbb{H} \to D$ is a conformal transformation, then

$$f \circ \nu_\mathbb{H}(0) = |f'(0)|^2 \nu_D(f(0)).$$

(In the definition of $f \circ \nu_\mathbb{H}(0)$, we need to modify the parametrization using Brownian scaling, but the parametrization is not important in this paper.)

Given the Brownian bubble measure, the Brownian loop measure restricted to curves in the unit disk $D$ can be written as

$$\frac{1}{\pi} \int_0^1 \int_0^{2\pi} \nu_D(re^{i\theta}) r \, d\theta \, dr,$$

To be more precise, the loop measure is the measure on unrooted loops induced by the above measure on rooted loops. (This representation of the measure on unrooted loops focuses on the rooted representative with root as far from the origin as possible.) The Brownian loop measure is the scaling limit of the random walk loop measure in a sense made precise in [5]. We discuss this more in the last section.

3 A combinatorial identity

Let $\mathcal{K}_n'$ denote the set of nearest neighbor paths $\omega = [\omega_0, \omega_1, \ldots, \omega_k]$ with $\text{Re}[\omega_0] = -n, \omega_k = 0$ and $\{\omega_1, \ldots, \omega_{k-1}\} \in A_n \setminus [1, \infty)$. Let $\mathcal{K}_n''$ denote the set of nearest neighbor paths $\omega = [\omega_0, \omega_1, \ldots, \omega_k]$ with $\text{Re}[\omega_0] = n+1, \omega_k = 1$ and $\{\omega_1, \ldots, \omega_{k-1}\} \in A_n \setminus (-\infty, 0]$. There is a natural bijection between $\mathcal{K}_n'$ and $\mathcal{K}_n''$ obtained by reflection about the line $\{\text{Re}(z) = 1/2\}$. Let

$$R_n = \sum_{\omega \in \mathcal{K}_n'} p(\omega) = \sum_{\omega \in \mathcal{K}_n''} p(\omega).$$

Note that $R_n$ equals $\mathbb{P}\{\text{Re}(S_\tau) = -n\}$ where $S$ is a simple random walk starting at the origin and $\tau = \min\{j > 0 : S_j \in \partial A_n \cup [0, \infty)\}$. It is known (see e.g., [4, Proposition 5.3.2]) that

$$R_n \asymp n^{-1/2}, \ n \to \infty.$$  

(7)
The goal of this section is to prove the following combinatorial identity which relates the probability that loop-erased walk uses the undirected edge \{0, 1\} to some simple random walk quantities.

**Theorem 2.**

\[
4 \sum_{\eta \in \mathcal{R}_n^*} p_n(\eta) = Q_{01}(A_n) R_n^2 \exp \{2m(J_{A_n})\}.
\]

**Proof.** We claim the following:

\[
(-1)^{J(\eta)} Y(\eta) = 1 \quad \text{if} \quad \eta \in \mathcal{R}_n^*.
\]

To see this, suppose \(\eta \in \mathcal{R}_n^+\) and write

\[
\eta = \eta^1 \oplus [0, 1] \oplus \eta^2,
\]

where \(\eta^1\) is a SAW from \(-n\) to 0, \(\eta^2\) is a SAW from 1 to \(n + 1\) and \(\eta^1 \cap \eta^2 = \emptyset\). Consider the simple curve

\[
\gamma = \eta^1 \oplus [0, 1/4] \oplus \gamma^1 \oplus [3/4, 1] \oplus \eta^2,
\]

where \(\gamma^1\) is a half-circle of radius 1/4 about 1/2 traversed clockwise. Let us view the continuous argument of the path where the argument is computed with respect to the center point 1/2 and is started at argument \(\pi\). At the end the argument is 0. The change in the argument during the half-circle is \(-\pi\). Therefore the change in argument in \(\eta^1\) must be the negative of the change of argument in \(\eta^2\). However, \(J(\eta^1)\) is odd if and only the change of argument is \(2\pi k\) for an odd integer \(k\). Therefore \(J(\eta^1) + J(\eta^2)\) is even and \(J(\eta)\) is even. Since \(Y(\eta) = 1\) for such a walk, \(Y(\eta) (-1)^{J(\eta)} = 1\). A similar argument shows that if \(\eta \in \mathcal{R}_n^-\), then \(J(\eta)\) is odd. Since \(Y(\eta) = -1\), we see that \(Y(\eta) (-1)^{J(\eta)} = 1\).

Let \(\Lambda_n\) be defined as in (3). We claim that

\[
\Lambda_n = \sum_{\omega \in \mathcal{K}_n} q(\omega) Y(L(\omega)) = \sum_{\eta \in \mathcal{R}_n^+} \sum_{L(\omega) = \eta} p(\omega) (-1)^{J_{L}(\omega)}.
\]

To see this, suppose that \(L(\omega) = \eta = [\eta_0, \ldots, \eta_k]\). We can write

\[
\omega = [\eta_0, \eta_1] \oplus l_1 \oplus [\eta_1, \eta_2] \oplus l_2 \oplus \cdots \oplus [\eta_{k-2}, \eta_{k-1}] \oplus l_{k-1} \oplus [\eta_{k-1}, \eta_k],
\]

where \(l_j\) is a loop rooted at \(\eta_j\) that does not enter \(\{\eta_1, \ldots, \eta_{j-1}\}\). We write

\[
J(\omega) = J(\eta) + J_{L}(\omega), \quad Y(\omega) = Y(\eta) + Y_{L}(\omega),
\]

where \(J_{L}, Y_{L}\) denote the contributions from the loops. Then

\[
Y(\omega) = Y(\eta) + \sum_{j=1}^{k-1} Y(l_j).
\]
For each loop \( l_j \) there is the corresponding reversed loop \( l_j^R \) for which \( Y(l_j^R) = -Y(l_j) \). Since 
\[ J(l_j^R) = J(l_j) \text{ and } Y(l_j^R) = -Y(l_j), \]
we get cancellation. Doing this for all the loops, we see that
\[ \sum_{\omega \in \mathcal{K}_n, L(\omega) = \eta} q(\omega) [Y(\omega) - Y(\eta)] = 0. \]

This gives the first equality in (9). The second equality uses (8) and the fact that \( Y(\eta) = 0 \) if \( \eta \not\in \mathcal{R}_n^* \).

If \( \eta \in \mathcal{R}_n^* \), then
\[
\sum_{L(\omega) = \eta} \frac{p(\omega)}{p(\eta)} (-1)^{J(\omega)} = Q_\eta(A_n) = \exp \left\{ \sum_{\bar{l} \subseteq A_n, \bar{l} \cap \eta \not= \emptyset} (-1)^{\bar{J}(\bar{l})} m(\bar{l}) \right\}
\]
\[
= F_\eta(A_n) \exp \left\{ -2 \sum_{\bar{l} \subseteq A_n, \bar{l} \cap \eta \not= \emptyset, J(\bar{l}) \text{ odd}} m(\bar{l}) \right\}. \]

Topological considerations imply that if \( \eta \in \mathcal{R}_n^* \) and \( J(\bar{l}) \) is odd, then \( \eta \cap \bar{l} \not= \emptyset \). Hence
\[
\sum_{\bar{l} \subseteq A_n, \bar{l} \cap \eta \not= \emptyset, J(\bar{l}) \text{ odd}} m(\bar{l}) = \sum_{\bar{l} \subseteq A_n, J(\bar{l}) \text{ odd}} m(\bar{l}) = m(\mathcal{J}_{A_n}). \]

Combining this with (9), we see that
\[
A_n = \sum_{\eta \in \mathcal{R}_n^*} p(\eta) Q_\eta(A_n) = e^{-2m(\mathcal{J}_{A_n})} \sum_{\eta \in \mathcal{R}_n^*} p(\eta) F_\eta(A_n) = e^{-2m(\mathcal{J}_{A_n})} \sum_{\eta \in \mathcal{R}_n^*} \hat{p}(\eta). \quad (10)
\]

We will now compute \( A_n \) as defined in (3) a different way. Let \( \omega = [\omega_0, \ldots, \omega_r] \in \mathcal{K}_n \). If \( \omega \) does not visit 0 or it does not visit 1, then \( Y(\omega) = 0 \). Otherwise, let \( T_0 = \min\{j : \omega_j = 0\}, T_0' = \max\{j < \tau : \omega_j = 0\} \) and define \( T_1, T_1' \) similarly.

Suppose that \( T_0 < T_1, T_0' > T_1' \). In this case we write
\[ \omega = \omega^- \oplus l \oplus \omega^+, \]
where \( l \) is the loop \([\omega_{T_0}, \ldots, \omega_{T_0'}] \). For any such loop, there is the corresponding reversed loop \( l^R = [\omega_{T_0'}, \omega_{T_0'-1}, \ldots, \omega_{T_0}] \) for which \( Y(l^R) = -Y(l) \). These terms cancel and hence the sum in (3) over \( \omega \) with \( T_0 < T_1, T_0' > T_1' \) is zero. Similarly, the sum over \( \omega \) with \( T_1 < T_0 \leq T_0' < T_1' \) is zero.

Suppose that \( T_0 > T_1, T_0' > T_1' \). Then we can write \( \omega \) as
\[ \omega^- \oplus l_1 \oplus \omega' \oplus l_0 \oplus \omega^+. \]
Here \( l_0 \) is a loop rooted at 0, \( l_1 \) is a loop rooted at 1, \( \omega' \) is a path from 1 to 0, \( \omega^- \) is a path from \( \{\text{Re}(z) = -n\} \) to 1 avoiding 0, \( \omega^+ \) is a path from 0 to \( \{\text{Re}(z) = n + 1\} \) avoiding 1. Let
Let $\bar{\omega}^-$ be the reflection of $\omega^-$ about the real axis. Then $J(\omega^-) + J(\bar{\omega}^-)$ is odd and these terms will cancel. Hence the sum over all $\omega$ with $T_0 > T_1, T_0' > T_1'$ is zero.

Let $\mathcal{K}^1_n$ be the set of paths in $\mathcal{K}_n$ that visit both 0 and 1 and satisfy $T_0 < T_1, T_0' < T_1'$. We have shown that

$$\Lambda_n = \sum_{\omega \in \mathcal{K}^1_n} q(\omega) Y(\omega).$$

If $\omega \in \mathcal{K}^1_n$, let $\rho = \min\{j > T_0' : \omega_j = 1\}$. Then we can write $\omega$ as

$$\omega^- \oplus l_0 \oplus \omega' \oplus l_1 \oplus \omega^+.$$  \hspace{1cm} (11)

Here $l_0$ is a loop rooted at 0, $l_1$ is a loop rooted at 1, $\omega' = [\omega_\rho, \ldots, \omega_1]$ is a path from 0 to 1, $\omega^-$ is a path from $\{\text{Re}(z) = -n\}$ to 0, $\omega^+$ is a path from 1 to $\{\text{Re}(z) = n + 1\}$. The paths $\omega', \omega^-, \omega^+$ do not enter $\{0, 1\}$ except for their endpoints. The loop $l_1$ does not visit 0. All points other than the endpoints must lie in $A_n$. These are all the restrictions on the paths. Note that $Y(\omega) = Y(l_0) + Y(\omega')$. As in the previous arguments, we can replace $l_0$ with the reversed path $l_0^R$, to see that

$$\sum_{\omega \in \mathcal{K}^1_n} (-1)^{J(\omega)} Y(l_0) p(\omega) = 0.$$

Also $Y(\omega') \in \{0, 1\}$ with $Y(\omega') = 1$ if and only if $T_0' + 1 = \rho$, that is, if $\omega' = [0, 1]$. Therefore, if $\mathcal{K}^2_n$ denotes the set of paths in $\mathcal{K}^1_n$ with $\omega' = [0, 1]$, then

$$\Lambda_n = \sum_{\omega \in \mathcal{K}^2_n} (-1)^{J(\omega)} p(\omega) = \sum_{\omega \in \mathcal{K}^2_n} (-1)^{J(\omega^-) + J(l_0) + J(l_1) + J(\omega^+)} p(\omega).$$  \hspace{1cm} (12)

If $\omega \in \mathcal{K}^2_n$, let $\xi$ be the smallest $j$ such that $\omega_j$ is on the positive real axis. Suppose for the moment that $\xi < T_0$. Then we can write

$$\omega^- = \omega^{-1} \oplus \omega^{-2},$$

by splitting the path at time $\xi$. The path $\omega^{-2}$ is a path from the positive real axis to 0 that does not go through the point 1. Hence, $J(\omega^{-2}) + J(\bar{\omega}^{-2})$ is odd, where $\bar{\omega}^{-2}$ is the reflection of $\omega^{-2}$ about the real axis. These terms will cancel in the sum (12), and hence it suffices to sum over $\omega^-$ such that $\omega^- \cap [1, \infty) = \emptyset$. For these $\omega^-$, we have $(-1)^{J(\omega^-)} = 1$. By a similar argument, it suffices to sum over $\omega^+$ satisfying $\omega^+ \cap (-\infty, 0] = \emptyset$, and for these walks $(-1)^{J(\omega^+)} = 1$. Therefore, if $\mathcal{K}^3_n$ denote the sets of paths in $\mathcal{K}^2_n$ satisfying

$$\omega^- \cap [1, \infty) = \emptyset, \quad \omega^+ \cap (-\infty, 0] = \emptyset,$$

we see that

$$\Lambda_n = \sum_{\omega \in \mathcal{K}^3_n} (-1)^{J(l_0) + J(l_1)} p(\omega).$$
Let us write any \( \omega \in \mathcal{K}_n^3 \) as in (11). We must choose \( \omega^- \in \mathcal{K}_n', \omega^+ \in \mathcal{K}_n'' \) and \( \omega' = [0, 1] \). Summing over all of these possibilities, gives a factor of \( R_n^2/4 \). The choices of \( l_0, l_1 \) are independent of the choices of \( \omega^- \) and \( \omega^+ \). The only restriction is that the loops lie in \( A_n \) and \( l_1 \) does not contain zero. By our definition,

\[
\sum_{l_0, l_1} (-1)^{J(l_0)+J(l_1)} = g_{A_n}(0,0) g_{A_n \setminus \{0\}}(1,1) = Q_{01}(A_n).
\]

Therefore,

\[
\Lambda_n = \sum_{\omega \in \mathcal{K}_n^3} (-1)^{J(l_0)+J(l_1)} p(\omega) = \frac{1}{4} R_n^2 Q_{01}(A_n).
\]

Comparing this with (10) gives the theorem. \( \square \)

4 Estimate on random walk loop measure

Using Theorem 2 and the estimates (5) and (7), we see that

\[
\sum_{\eta \in \mathcal{R}_n} \hat{p}_n(\eta) \asymp n^{-1} \exp \{ 2m(J_{A_n}) \}.
\]

The proof of (1) is finished with the following proposition.

**Proposition 4.1.** There exists \( c < \infty \) such that for all \( n \),

\[
\left| m(J_{A_n}) - \frac{1}{8} \log n \right| \leq c.
\]

**Proof.** Let \( C_n = \{ z \in \mathbb{Z}^2 : |z| < e^n \} \). We will prove the stronger fact that the limit

\[
\lim_{n \to \infty} \left[ m(J_{C_n}) - \frac{n}{8} \right] = 0 \tag{13}
\]

exists by showing that

\[
\sum_{n=1}^{\infty} \left| m(J_{C_{n+1} \setminus C_n}) - \frac{1}{8} \right| < \infty. \tag{14}
\]

Let \( \mu \) denote the Brownian loop measure, and let \( \tilde{J} \) denote the set of unrooted loops \( \gamma \) in the unit disk that intersect \( \{|z| \geq e^{-1}\} \) and such that the winding number of \( \gamma \) about the origin is odd. We will establish (14) by showing that \( \mu(\tilde{J}) = 1/8 \) and

\[
\left| m(J_{C_{n+1} \setminus C_n}) - \mu(\tilde{J}) \right| = O(n^{-2}). \tag{15}
\]

For the Brownian loop measure, we do a computation similar to that in [3, Proposition 3.9]. Using (6), we write

\[
\mu(\tilde{J}) = \frac{1}{\pi} \int_{e^{-1}}^{1} \int_{0}^{2\pi} \phi(r, \theta) \, d\theta \, dr.
\]
where $\phi(r, \theta)$ denotes the Brownian bubble measure of loops in $r\mathbb{D}$ rooted at $re^{i\theta}$ with odd winding number about the origin. Rotational symmetry implies that $\phi(r, \theta) = \phi(r, 0)$ and conformal covariance implies that $\phi(r, 0) = r^{-2} \phi$ where $\phi = \phi(1, 0)$. Hence,

$$
\mu(\tilde{\mathcal{J}}) = \frac{\phi}{\pi} \int_{e^{-1}}^{1} \int_{0}^{2\pi} r^{-2} d\theta \, dr = 2 \phi.
$$

By considering the covering map $f(z) = i \log z$ which satisfies $|f'(1)| = 1$, we see that

$$
\phi = \sum_{k \text{ odd}} H_{\partial \mathbb{H}}(0, 2\pi k),
$$

where $H_{\partial \mathbb{H}}$ denotes the boundary Poisson kernel normalized so that $H_{\partial \mathbb{H}}(0, x) = x^{-2}$. Therefore,

$$
2 \phi = 2 \sum_{k=-\infty}^{\infty} \frac{1}{[2\pi(2k+1)]^2} = \frac{1}{\pi^2} \left[ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right] = \frac{1}{8}.
$$

If $s > 2$, let $\tilde{\mathcal{J}}^s$ denote the set of loops in $\mathbb{D}$ that intersect both $\{|z| \geq e^{-1}\}$ and $\{|z| \leq e^{-s}\}$. A similar computation shows that the Brownian bubble measure of bubbles at 1 that intersect $\{|z| \leq e^{-s}\}$ is $(2s)^{-1}$ and hence

$$
\mu(\tilde{\mathcal{J}}^s) = s^{-1} + O(s^{-2}), \quad s \to \infty.
$$

If we think of such loops as bubbles starting at a point in $\{|z| \geq e^{-1}\}$, then after the first visit to $\{|z| \leq e^{-s}\}$, the conditional probability that the winding number will be odd is $\frac{1}{2} + O(e^{-us})$ for some $u > 0$. One way to see this is to lift to the half plane and consider a Brownian motion starting at $x + is$ and estimating

$$
\frac{\sum_{k \text{ odd}} H_{\mathbb{H}}(x + is, 2\pi k)}{\sum_k H_{\mathbb{H}}(x + is, 2\pi k)}.
$$

Another way is to use a coupling argument as we describe below for random walk. Therefore,

$$
\mu[\tilde{\mathcal{J}} \cap \tilde{\mathcal{J}}^s] = \frac{1}{2} \mu[\tilde{\mathcal{J}}^s] [1 + o(s^{-1})] = \frac{1}{2s} + O(s^{-2}). \quad (16)
$$

For each unrooted random walk loop $\tilde{l} \in \mathcal{J}_{C_{n+1}} \setminus \mathcal{J}_{C_{n-1}}$, there is a corresponding continuous unrooted loop $\tilde{l}(n)$ in $\mathbb{D}$ obtained from linear interpolation and Brownian scaling. We will write $d(\tilde{l}, \gamma) \leq \delta$, if we can parametrize and root the loops $\tilde{l}(n)$ and $\gamma$ such that the loops are within $\delta$ in the supremum norm. In [5] it was shown that there exists $\alpha > 0$ and a coupling of the random walk and Brownian loop measures in $D$, restricted to loops of diameter at least $1/e$, so that the total masses agree up to $O(e^{-n\alpha})$ and such that in the coupling, except for a set of paths of size $O(e^{-n\alpha})$, we have $d(\tilde{l}, \gamma) < e^{-n\alpha}$.

Let $\beta < \alpha$ and let $\mathcal{J}^n$ denote the set of loops in $\mathcal{J}_{C_{n+1}} \setminus \mathcal{J}_{C_{n}}$ that intersect $\{|z| \leq e^{-\beta n} e^{n+1}\}$. Using the coupling and (16), we see that

$$
m(\mathcal{J}^n) = \mu(\tilde{\mathcal{J}}^n) + O(n^{-2}) = (\beta n)^{-1} + O(n^{-2}).
$$
Let us split these paths into two sets: those for which \( \text{dist}(0, \gamma) \leq 2e^{-n\alpha} \) and those for which \( \text{dist}(0, \gamma) > 2e^{-n\alpha} \). If \( \text{dist}(0, \gamma) > 2e^{-n\alpha} \) and \( d(\bar{l}, \gamma) \leq e^{-n\alpha} \), then \( J(\bar{l}) \) is odd if and only if the winding number of \( \gamma \) is odd. Therefore

\[
m((\mathcal{C}_{n+1} \setminus \mathcal{C}_n) \setminus \mathcal{J}^n) = \mu(\tilde{\mathcal{J}} \setminus \mathcal{J}_{\beta n}) + O(n^{-2}).
\]

(The error term \( O(n^{-2}) \) is comparable to the measure of loops \( \gamma \) such that \( e^{-n\beta} \leq \text{dist}(0, \gamma) \leq 2e^{-n\beta} \).

A coupling argument can be used to see that

\[
m[\mathcal{J}^n \cap (\mathcal{C}_{n+1} \setminus \mathcal{C}_n)] = \frac{1}{2} m(\mathcal{J}^n) [1 + o(n^{-1})].
\]

We sketch the proof. We use the definition of the loop measure using an enumeration of \( \mathbb{Z}^2 = \{z_1, z_2, \ldots\} \) such that \(|z_j|\) increases. Then an unrooted loop in \( \mathcal{J}^n \) is obtained from a loop rooted in \( \mathcal{C}_{n+1} \setminus \mathcal{C}_n \). Let us call the root \( z_k \) and so the loops lies in \( V_k = \{z_1, \ldots, z_k\} \). Let us stop the walk at the first time it reaches a point, say \( z' \), in \( \{|z| \leq e^{-\beta n} e^{n+1}\} \). The remainder of the loop acts like a random walk started at \( z' \) conditioned to reach \( z_k \) without before leaving \( V_k \). Let \( J' \) denote the number of times such a walk crosses the half line \( \{(1/2) + iy : y < 0\} \). We claim that the probability that \( J' \) is odd equals \( \frac{1}{2} + O(e^{-un}) \) for some \( u > 0 \). Indeed, we can couple two walks starting at the point so that each walk has the distribution of random walk conditioned to reach \( z_k \) before leaving \( V_k \) and that, except for an event of probability \( O(e^{-\delta}) \), the parity of \( J' \) is different for the two walks. This uses a standard technique. The key estimate is the following. There exists \( c > 0 \) such that if \( S \) is a simple random walk starting at \( z \in \mathcal{C}_{j-1} \) and \( T = \min\{j : S_y \in \mathcal{C}_j\} \), then for all \( w \in \partial \mathcal{C}_j \) with \( \text{Im}(w) > 0 \),

\[
\mathbb{P}\{S(T) = w, J' \text{ odd}\} \geq c e^{-j},
\]

\[
\mathbb{P}\{S(T) = w, J' \text{ even}\} \geq c e^{-j}.
\]

Without the restriction of the parity of \( J' \), see, for example, [4, Lemma 6.3.7]. To get the result with the restriction, we just note that there is a positive probability of making a loop in the annulus \( \mathcal{C}_j \setminus \mathcal{C}_{j-1} \), and this increases \( J' \) by one. Hence, we can find a coupling and a \( \rho > 0 \) such that at each annulus there is a probability \( \rho \) of a successful coupling given that the walks have not yet been coupled. Since there are of order \( \beta n \) annuli, we can couple the processes so that the probability of not being coupled is \( (1 - \rho)^{\beta n} = O(e^{-un}) \) for some \( u \).

From the last two estimates and (16), we see that

\[
\left| \mu(\tilde{\mathcal{J}}) - m(\mathcal{C}_{n+1} \setminus \mathcal{C}_n) \right| \leq c n^{-2}.
\]

This gives (15).
References


