A. (18) For each of the following functions, state whether or not the function is holomorphic on \( D = \{ z \in \mathbb{C} : |z| < 1 \} \). Reasons must be given.

1. 
   \( f_1(x + iy) = x^2 - iy^3 \)
   
   \( u(x, y) = x^2, v(x, y) = -y^3 \). Since \( \partial_x u = 2x \neq -3y^2 = \partial_y v \), it does not satisfy the Cauchy-Riemann equations and hence is not holomorphic.

2. 
   \( f_2(z) = \sum_{n=0}^{\infty} n^5 z^n. \)
   
   Using the definition or using the ratio test, we can see that the radius of convergence of this power series about 0 is 1. Hence it is holomorphic.

3. 
   \( f_3(z) = \sum_{n=0}^{\infty} \frac{e^{z^2}}{n! + n^5}. \)
   
   This actually was a typo. I had meant to have \( e^{z^n} \) instead of \( e^{z^2} \). A proof that works for either \( e^{z^2} \) or \( e^{z^n} \) is as follows. Note that
   
   \[ \left| \frac{e^{z^2}}{n! + n^5} \right| \leq \frac{e}{n! + n^5}, \]
   
   and hence the sequence of holomorphic functions \( \sum_{n=1}^{N} e^{z^2}/(n! + n^5) \) converges uniformly. For the problem as written, it was sufficient to note that
   
   \[ \sum_{n=0}^{\infty} \frac{1}{n! + n^5} < \infty \]
   
   and hence the function is \( Ce^{z^2} \) for some \( C \).

B. (12) Find the radius of convergence of the following power series.
1. \[
\sum_{n=1}^{\infty} \left(2i + \frac{1}{n}\right)^n (z - 1)^n.
\]

\[\limsup_{n \to \infty} \left|\left(2i + \frac{1}{n}\right)^n\right|^{1/n} = \lim_{n \to \infty} \left|2i + \frac{1}{n}\right| = 2.\]

Hence the radius of convergence is 1/2.

2. The power series of the function

\[f(z) = \frac{z^5 - z - 1}{z^2 + 1}\]

about \(z_0 = 4 + i\).

- The denominator equals \((z + i)(z - i)\). The function is holomorphic every except those two points. Hence the radius of convergence is the distance from 4 + i to the closer of those two points which is 4.

C. (24) Give the value of \(\int_\gamma f\) for the following.

1. \(f(z) = z e^{-z^2/2}\) and \(\gamma(t) = \sin(\pi(t^3 - 3t^2 + 3t)/2), \quad 0 \leq t \leq 1\).

- \(f\) has antiderivative (primitive) \(g(z) = -e^{-z^2/2}\) everywhere on \(\mathbb{C}\). Therefore the integral equals \(g(\gamma(1)) - g(\gamma(0)) = g(1) - g(0) = 1 - e^{-1/4}\).

2. \(f(z) = \frac{ze^z}{z - 1}\), and \(\gamma(t) = 2e^{it}, 0 \leq t \leq 2\pi\).

- The Cauchy integral formula tells us that this integral equals \(2\pi i f(1) = 2\pi i e\).

3. \(f = f_3\) from the first problem and \(\gamma\) is the circle of radius 1/4 about \(i/2\).

- \(f_3\) is holomorphic in the unit disk and hence is holomorphic on and inside the circle of 1/4 about \(i/2\). Therefore, the integral equals 0.

4. \(f(z) = 4/z\) and \(\gamma(t), 0 \leq t \leq 1\) is a curve with \(\gamma(0) = 2i, \gamma(1) = -2i\) and such that for all \(0 \leq t \leq 1\),

\[\gamma(t) \subset \{re^{i\theta}: r > 0, -\pi < \theta < \pi\}.\]
Since the region \( U = \{ re^{i\theta} : r > 0, -\pi < \theta < \pi \} \) is simply connected and \( 4/z \) is holomorphic in this region, the value of the integral does not depend on the path. Hence, we can choose a convenient path such as \( \gamma(t) = 2e^{-(\pi/2)(1-t)}, 0 \leq t \leq 1 \). For this choice \( \gamma'(t) = i\pi e^{-(\pi/2)(1-t)} \) and
\[
\int_{\gamma} f = \int_{0}^{1} \frac{4}{2e^{-(\pi/2)(1-t)}} i\pi e^{-(\pi/2)(1-t)} dt = -4i\pi.
\]
Alternatively, one can note that \( g(re^{i\theta}) = 4[\log r + i\theta], -\pi < \theta < \pi \) is a primitive of \( 4/z \); then the integral equals
\[
g(2e^{-i\pi/2}) - g(2e^{i\pi/2}) = -4\pi i.
\]

D. (12)

1. Find the Laurent series for
\[
f(z) = \frac{1}{1 - z^2}
\]
about \( z_0 = 1 \).

   \[
   \frac{1}{1 - z^2} = \frac{1}{(1 - z)(1 + z)} = \frac{1}{z - 1} \frac{1}{2 + (z - 1)} = \frac{1}{2(z - 1)} \frac{1}{1 + \frac{z - 1}{2}} = \frac{1}{2(z - 1)} \left[ 1 + \frac{z - 1}{2} + \left( \frac{z - 1}{2} \right)^2 + \ldots \right].
   \]

2. Find the smallest value of \( r \) and largest value of \( R \) with \( 0 \leq r < R \leq \infty \) such that the Laurent series converges absolutely for \( \{ r < |z - 1| < R \} \).

   \bullet \quad \text{The power series above requires } |z - 1|/2 < 1, \text{ i.e., } |z - 1| < 2. \text{ This is the only condition needed, so } r = 0, R = 2.

E. (10) Suppose \( f \) is an entire function such that \( f(z) \neq 0 \) for all \( z \); \( f(1) = 1; f(1+i) = 2 \). Show that for all \( R < \infty \)
\[
\inf \{|f(z)| : |z| \geq R \} = 0.
\]
Suppose that there is an $R < \infty$ and $\delta > 0$ such that $|f(z)| \geq \delta$ for $|z| \geq R$. Let $g(z) = 1/f(z)$. Then $g$ is an entire function and $|g(z)| \leq 1/\delta$ for $|z| \geq R$. Since \{z : |z| \leq R\} is compact, $g$ is also bounded on this set, and hence $g$ is bounded. By Liouville’s Theorem, $g$ must be a constant function. But we know that $g(1) \neq g(1 + i)$. Contradiction.

F. (15) Suppose $D = \{|z| < 1\}$ is the open unit disk and $f(z) = 1/(1 - z)$. Let

$$f_n(z) = f\left(\frac{n}{n+1}z\right).$$

True or false (give justification):

1. $f_n$ converges to $f$ uniformly on $D$.
   - $|f_n(z)| \leq n + 1$ for each $|z| < 1$. However $|f(z)|$ is unbounded on $\{|z| < 1\}$. Hence for each $n$, $|f(z) - f_n(z)|$ is unbounded and hence uniform convergence is impossible.

2. $f_n$ converges to $f$ uniformly on $\{z : |z| \leq 1/2\}$.
   - 
     $$|f_n(z) - f(z)| = \left|\frac{1}{1 - \frac{n}{n+1}z} - \frac{1}{1 - z}\right|$$

     $$= \frac{1}{n + 1} \left|\frac{z}{(1 - \frac{n}{n+1}z)(1 - z)}\right|.$$

     If $|z| \leq 1/2$, the absolute value of the last denominator is greater than or equal to $1/4$. Therefore, for $|z| \leq 1/2$,

     $$|f_n(z) - f(z)| \leq \frac{2}{n + 1}.$$

     Given this, one can check from the definition that $f_n$ converges uniformly to $f$. 