MATH 270 MIDTERM
SPRING 2008

In solutions you may quote any theorem proved in class or any theorem proved in the sections of Lang that were required reading for the midterm. You may not quote any other theorems of complex variables that you might know (without proving them!)

A. (18) For each of the following functions, state whether or not the function is holomorphic on $D = \{z \in \mathbb{C} : |z| < 1\}$. Reasons must be given.

1.
$$f_1(x+iy) = x^2 - i\,y^3$$

- $u(x,y) = x^2, v(x,y) = -y^3$. Since $\partial_x u = 2x \neq -3y^2 = \partial_y v$, it does not satisfy the Cauchy-Riemann equations and hence is not holomorphic.

2.
$$f_2(z) = \sum_{n=0}^{\infty} n^5\, z^n.$$

- Using the definition or using the ratio test, we can see that the radius of convergence of this power series about 0 is 1. Hence it is holomorphic.

3.
$$f_3(z) = \sum_{n=0}^{\infty} \frac{e^{z^2}}{n! + n^5}.$$

- This actually was a typo. I had meant to have e^{z^n} instead of e^{z^2}. A proof that works for either e^{z^2} or e^{z^n} is as follows. Note that
$$\left| \frac{e^{z^2}}{n! + n^5} \right| \leq \frac{e}{n! + n^5},$$
and hence the sequence of holomorphic functions $\sum_{n=1}^{N} e^{z^2}/(n! + n^5)$ converges uniformly. For the problem as written, it was sufficient to note that
$$\sum_{n=0}^{\infty} \frac{1}{n! + n^5} < \infty$$
and hence the function is Ce^{z^2} for some C.

B. (12) Find the radius of convergence of the following power series.

1.
$$\sum_{n=1}^{\infty} \left(2i + \frac{1}{n}\right)^n (z-1)^n.$$

- $$\limsup_{n\to\infty} |(2i + \frac{1}{n})^n|^{1/n} = \lim_{n\to\infty} |2i + \frac{1}{n}| = 2.$$

  Hence the radius of convergence is 1/2.

2. The power series of the function
$$f(z) = \frac{z^5 - z - 1}{z^2 + 1}$$
about $z_0 = 4 + i$.

   - The denominator equals $(z+i)(z-i)$. The function is holomorphic every except those two points. Hence the radius of convergence is the distance from $4+i$ to the closer of those two points which is 4.

C. (24) Give the value of $\int_\gamma f$ for the following.

1. $f(z) = z\, e^{-z^2/2}$ and $\gamma(t) = \sin(\pi(t^3 - 3t^2 + 3t)/2), \quad 0 \le t \le 1.$

   - f has antiderivative (primitive) $g(z) = -e^{-z^2/2}$ everywhere on $\mathbb{C}$. Therefore the integral equals
$$g(\gamma(1)) - g(\gamma(0)) = g(1) - g(0) = 1 - e^{-1/1}.$$

2.
$$f(z) = \frac{ze^z}{z-1},$$
and $\gamma(t) = 2e^{it}, 0 \le t \le 2\pi$.

   - The Cauchy integral formula tells us that this integral equals $2\pi i f(1) = 2\pi i e$.

3. $f = f_3$ from the first problem and γ is the circle of radius 1/4 about $i/2$.

   - f_3 is holomorphic in the unit disk and hence is holomorphic on and inside the circle of 1/4 about $i/2$. Therefore, the integral equals 0.

4. $f(z) = 4/z$ and $\gamma(t), 0 \le t \le 1$ is a curve with $\gamma(0) = 2i$, $\gamma(1) = -2i$ and such that for all $0 \le t \le 1$,
$$\gamma(t) \subset \{re^{i\theta} : r > 0, -\pi < \theta < \pi\}.$$

- Since the region $U = \{re^{i\theta} : r > 0, -\pi < \theta < \pi\}$ is simply connected and $4/z$ is holomorphic in this region, the value of the integral does not depend on the path. Hence, we can choose a convenient path such as $\gamma(t) = 2e^{-(\pi/2)(1-t)}, 0 \le t \le 1$. For this choice $\gamma'(t) = i\pi\, e^{-i(\pi/2)(1-t)}$ and

$$\int_\gamma f = \int_0^1 \frac{4}{2e^{-(\pi/2)(1-t)}}\, i\pi\, e^{-(\pi/2)(1-t)} dt = -4i\pi.$$

  Alternatively, one can note that $g(re^{i\theta}) = 4[\log r + i\theta], -\pi < \theta < \pi$ is a primitive of $4/z$; then the integral equals

$$g(2e^{-i\pi/2}) - g(2e^{i\pi/2}) = -4\pi i.$$

D. (12)

1. Find the Laurent series for

$$f(z) = \frac{1}{1-z^2}$$

   about $z_0 = 1$.

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$$\begin{aligned} \frac{1}{1-z^2} &= \frac{1}{(1-z)(1+z)} \\ &= -\frac{1}{z-1}\,\frac{1}{2+(z-1)} \\ &= -\frac{1}{2(z-1)}\,\frac{1}{1+\frac{z-1}{2}} \\ &= -\frac{1}{2(z-1)}\left[1 + \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 + \cdots\right]. \end{aligned}$$

2. Find the smallest value of r and largest value of R with $0 \le r < R \le \infty$ uch that the Laurent series converges absolutely for $\{r < |z-1| < R\}$.

   - The power series above requires $|z-1|/2 < 1$, i.e., $|z-1| < 2$. This is the only condition needed, so $r = 0, R = 2$.

E. (10) Suppose f is an entire function such that $f(z) \neq 0$ for all z; $f(1) = 1$; $f(1+i) = 2$. Show that for all $R < \infty$

$$\inf\{|f(z)| : |z| \ge R\} = 0.$$

- Suppose that there is an $R < \infty$ and $\delta > 0$ such that $|f(z)| \geq \delta$ for $|z| \geq R$. Let $g(z) = 1/f(z)$. Then g is an entire function and $|g(z)| \leq 1/\delta$ for $|z| \geq R$. Since $\{z : |z| \leq R\}$ is compact, g is also bounded on this set, and hence g is bounded. By Liouville's Theorem, g must be a constant function. But we know that $g(1) \neq g(1+i)$. Contradiction.

F. (15) Suppose $D = \{|z| < 1\}$ is the open unit disk and $f(z) = 1/(1-z)$. Let

$$f_n(z) = f\left(\frac{n}{n+1}\,z\right).$$

True or false (give justification):

1. f_n converges to f uniformly on D.

   - $|f_n(z)| \leq n+1$ for each $|z| < 1$. However $|f(z)|$ is unbounded on $\{|z| < 1\}$. Hence for each n, $|f(z) - f_n(z)|$ is unbounded and hence uniform convergence is impossible.

2. f_n converges to f uniformly on $\{z : |z| \leq 1/2\}$.

   - $$\begin{aligned} |f_n(z) - f(z)| &= \left|\frac{1}{1-\frac{n}{n+1}z} - \frac{1}{1-z}\right| \\ &= \frac{1}{n+1}\left|\frac{z}{(1-\frac{n}{n+1}z)(1-z)}\right|. \end{aligned}$$

   If $|z| \leq 1/2$, the absolute value of the last denominator is greater than or equal to $1/4$. Therefore, for $|z| \leq 1/2$,

   $$|f_n(z) - f(z)| \leq \frac{2}{n+1}.$$

   Given this, one can check from the definition that f_n converges uniformly to f.