A note on the boundary exponent and rate of escape for the Schramm-Loewner evolution

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Abstract

We give an estimate for the probability that a Schramm-Loewner evolution (SLE) curve of parameter $\kappa \in (4, 8)$ hits a disconnected set connected to the boundary. This is used to estimate the rate of escape. The main estimate concerns the relationship between the Hausdorff content of a set and its extremal length.

1 Introduction

The purpose of this note is to prove the following theorem. Let $0 < \kappa < 8$ and let $\alpha = \frac{8}{\kappa} - 1$ be the boundary exponent for the Schramm-Loewner evolution with parameter κ (SLE_κ). Let $\mathbb{D}$ and $\mathbb{H}$ denote the unit disk and upper half plane, respectively.

Theorem 1.1. *For every $4 < \kappa < 8$, there exists $c = c_\kappa < \infty$ such that the following is true for all $0 < r < 1$. Suppose that $\eta : [0, 1] \to \overline{\mathbb{H}}$ is a curve with $\eta(0) = 0, |\eta(1)| = 1$, and $|\eta(t)| < 1$ for $t < 1$. Let D_η denote the unbounded component of $\mathbb{H} \setminus \eta$ and let γ be a chordal SLE_κ path from $\eta(1)$ to ∞ in D_η. Then,*

$$\mathbf{P}\{\gamma \cap \{|z| \leq r\} \neq \emptyset\} \leq c\, r^{\alpha}.$$

Theorem 1.2. *For every $4 < \kappa < 8$, there exists $c = c_\kappa < \infty$ such that the following is true for all $0 < s, r < 1$. Suppose that $\eta : [0, 1] \to \overline{\mathbb{D}}$ is a curve with $\eta(0) = 1, |\eta(1)| = sr$, and $|\eta(t)| > sr$ for $t < 1$. Let D_η denote the component of $\mathbb{D} \setminus \eta$ containing the origin, and let γ be a radial SLE_κ path from $\eta(1)$ to 0 in D_η. Then,*

$$\mathbf{P}\{\gamma \cap \{|z| \geq r\} \neq \emptyset\} \leq c\, s^{\alpha/2}.$$

The important thing is that the constant is uniform over all such curves η. These theorems were proved for $\kappa \leq 4$ in [2, 3], but the argument there does not apply to $4 < \kappa < 8$. For this paper, we will assume that $4 < \kappa < 8$ and hence that $0 < \alpha < 1$. The proofs of the two theorems are similar; we will discuss the chordal case first and the radial case will be handled in the last section.

Although we have stated this as a theorem about SLE, the result in this paper is not about SLE. It is a variant of Pfluger's Theorem [5, Theorem 9.17] that relates the Hausdorff content

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of a set with its capacity. We will start by discussing SLE but we will reduce the theorem to an estimate (4) which is independent of SLE. The work in this paper will be to establish (4) and the argument will use only Brownian motion and potential theory.

The theorems are statements about the probability that an SLE path hits a (perhaps disconnected) subset attached to the boundary. Suppose η is a curve as in Theorem 1.1 and let $0 < r < 1$. Let $g = g_\eta : D_\eta \to \mathbb{H}$ be the unique conformal transformation with $g(\eta(1)) = 0$ and $g(z) \sim z$ as $z \to \infty$. Note that $D_\eta \cap \{|z| = r\}$ is a finite or countably infinite collection of open crosscuts l_j of D_η. We write $\eta_j = g \circ l_j$ for the image of the crosscuts under the map g. There is a subcollection of these crosscuts which we call the *reachable* crosscuts with the property that there is an infinite curve in D_η from the crosscut to infinity that does not enter $\{|z| < r\}$. If γ is an SLE_κ path in D_η from $\eta(1)$ to ∞, then in order to enter $D_\eta \cap \{|z| < r\}$ the path must first intersect the closure of one of the reachable crosscuts. Since there are only countably many crosscuts and $SLE_\kappa, \kappa < 8$ does not hit points (with probability one), the path must, in fact, hit one of the open crosscuts. The upshot is that the probability that SLE_κ enters $\{|z| \leq r\}$ is the probability that SLE intersects one of the reachable crosscuts. Hence, to prove Theorem 1.1 it suffices to show the following.

- There exists c such that for every finite subset of reachable crosscuts $l_1, \ldots, l_n$, the probability that SLE_κ hits $l_1 \cup \cdots \cup l_n$ is bounded above by $c\, r^\alpha$.

The set of crosscut $\{l_j\}$ is naturally partitioned into two sets — those for which the endpoints of η_j are positive and those for which they are negative. We say that l_j is a positive crosscut if the endpoints of the images η_j are positive. By symmetry and conformal invariance, it suffices to prove the following.

- There exists c such that for every finite subset of positive reachable crosscuts $l_1, \ldots, l_n$, the probability that chordal SLE_κ from 0 to ∞ hits $\eta_1 \cup \cdots \cup \eta_n$ is bounded above by $c\, r^\alpha$.

Clearly it suffice to prove this for r sufficiently small. The main tool for proving this lemma is the boundary estimate for SLE that we recall now.

Lemma 1.3. *[1, 4] For every $0 < \kappa < 8$, there exists $c_* < \infty, u > 0$ such that if γ is an SLE_κ path from 0 to ∞, $x > 0$, and $0 < r \leq 1/2$, then*

$$\mathbf{P}\{\text{dist}(\gamma, x) \leq rx\} = \mathbf{P}\{\text{dist}(\gamma, 1) \leq r\} = c_*\, r^\alpha\, [1 + O(r^u)].$$

In particular, there exists $c = c_\kappa$ such that for all $r \leq 1/2$,

$$\mathbf{P}\{\text{dist}(\gamma, x) \leq rx\} \leq c\, r^\alpha. \tag{1}$$

Instead of using the upper half plane, it will be more convenient here to consider a doubly infinite strip. We will set the following notation for this paper:

$$D = \{x + iy : |y| < \pi\}, \quad D_+ = D \cap \mathbb{H} = \{x + iy : 0 < y < \pi\},$$

$$I_\pm = \{x \pm i\pi : x \in \mathbb{R}\}.$$

We will write $\pm\infty$ for the infinite boundary points of D or D_+ corresponding to $x \to \pm\infty$. Let $F(z) = \log z$ which is a conformal transformation of $\mathbb{C} \setminus (-\infty, 0]$ onto D with $F(0) = -\infty, F(\infty) = \infty$, and $F(\mathbb{H}) = D_+$. We write $\mathcal{B}(z, r)$ for the closed disk of radius r centered at z. We can restate the last estimate as follows.

- If γ is an SLE_κ curve from $-\infty$ to ∞ in D_+, and $A = A_1 \cup \cdots \cup A_n$ where $A_1, \ldots, A_n$ are compact, connected sets with $A_j \cap \mathbb{R} \neq \emptyset$, then

$$\mathbf{P}\{\gamma \cap A \neq \emptyset\} \leq c \sum_{j=1}^{n} [\operatorname{diam} A_j]^\alpha.$$

If K is a compact subset of D, we define

$$H^\alpha(K) = \inf \sum_{j=1}^{n} r_j^\alpha,$$

where the infimum is over all finite collections of closed disks $\mathcal{B}(x_1, r_1), \mathcal{B}(x_2, r_2), \ldots \mathcal{B}(x_n, r_n)$ with $x_j \in \mathbb{R}$ and

$$K \subset \bigcup_{j=1}^{n} \mathcal{B}(x_j, r_j).$$

Note that we are requiring the centers of the disks to be on the real line. Using compactness of K we see that we get the same value if we take the infimum over all countable covers. Note that if $\bar{K} = \{z : z \in K \text{ or } \bar{z} \in K\}$, then $H^\alpha(\bar{K}) = H^\alpha(K)$; indeed, any cover of K by balls centered on the real line also covers $\bar{K}$. Lemma 1.3 immediately gives the following.

- If γ is an SLE_κ curve from $-\infty$ to ∞ in D_+, and K is a compact set, then

$$\mathbf{P}\{\gamma \cap K \neq \emptyset\} \leq c\, H^\alpha(K).$$

The quantity H^α is very similar to the α-Hausdorff content of a compact set K defined by

$$\mathcal{H}^\alpha(K) = \inf \sum_{j=1}^{n} r_j^\alpha,$$

where the infimum is over all covers of K by a finite number of disks $\mathcal{B}(z_1, r_1)$, $\mathcal{B}(z_2, r_2)$, $\ldots$, $\mathcal{B}(z_n, r_n)$. Here we do not require z_j to be real. Again this is the same as the infimum over countable covers. For finite unions of compact, connected sets touching the real line, H^α and $\mathcal{H}^\alpha$ are comparable (see Lemma 4.1). We therefore conclude the following version of the boundary estimate.

Lemma 1.4. *For every $\kappa < 8$, there exists $c < \infty$ such that if $A = A_1 \cup \cdots \cup A_n$ where A_j are disjoint connected compact sets intersecting the real line, and γ is a chordal SLE_κ path from $-\infty$ to ∞ in D_+, then*

$$\mathbf{P}\{\gamma \cap A \neq \emptyset\} \leq c\, \mathcal{H}^\alpha(A). \tag{2}$$

The main work in this paper is to relate $\mathcal{H}^\alpha(A)$ to a conformally invariant quantity. We state the result in the next proposition; it is an analogue of Pfluger's theorem [5, Theorem 9.17]. We recall the definition of the module or modulus of curves (see, e.g., [5, Section 9.2] for more details). Suppose U is a domain and ∂_1, ∂_2 are disjoint closed subsets of ∂U. Let $\mathcal{C} = \mathcal{C}_U(\partial_1, \partial_2)$ denote the set of curves γ from ∂_1 to ∂_2 otherwise lying in U. We say that a positive function $\rho : D \to [0,\infty)$ is *admissible* (with respect to $\mathcal{C}$), if for every $\gamma \in \mathcal{C}$,

$$\int_\gamma \rho(z)\,|dz| \geq 1. \tag{3}$$

The *modulus* of $\mathcal{C}$ is defined by

$$\Lambda(\mathcal{C}) = \inf \int_U \rho(z)^2\, dA(z),$$

where the infimum is over all admissible functions ρ. The modulus is conformally invariant. The reciprocal of the modulus is called the *extremal length* or *extremal distance*. Three important cases are:

- If $U = \{x+iy : 0 < x < 1, 0 < y < \theta\}$, and ∂_1, ∂_2 are the vertical boundary segments, then $\Lambda(\mathcal{C}) = \theta$.
- If $U = \{z : r < |z| < 1\}$ and ∂_1, ∂_2 are the inner and outer circles, then $\Lambda(\mathcal{C}) = 2\pi/\log(1/r)$.
- If $U = \{z \in \mathbb{H} : r < |z| < 1\}$ and ∂_1, ∂_2 are the inner and outer half circles, then $\Lambda(\mathcal{C}) = \pi/\log(1/r)$.

An important property of the modulus is monotonicity. We give two examples.

- If U, ∂_1 are the same but $\partial_2' \subset \partial_2$, then $\mathcal{C}' \subset \mathcal{C}$. Hence any ρ that is admissible for $\mathcal{C}$ is admissible for $\mathcal{C}'$ and
$$\Lambda(\mathcal{C}') \leq \Lambda(\mathcal{C}).$$
- If ∂_1, ∂_2 are the same but $U' \subset U$, then any ρ that is admissible for $\mathcal{C}$ is admissible for $\mathcal{C}'$. Moreover,
$$\int_{U'} \rho(x)^2\, dA(x) \leq \int_U \rho(x)^2\, dA(x),$$
and hence,
$$\Lambda(\mathcal{C}') \leq \Lambda(\mathcal{C}).$$

In Section 3 we give an alternative definition of the modulus of such curves in terms of excursion measure for reflecting Brownian motion. This relationship is known by many but the facts that we need do not seem to be written anywhere so we do it here. The alternate definition is more useful in many ways, but the original definition is useful because the monotonicity properties follow easily. Our main result is the following.

Proposition 1.5. *For every $\alpha > 0$, there exist $c < \infty, s > 0$ such that if $A = A_1 \cup \cdots \cup A_n$ where A_j are disjoint connected compact sets intersecting the real line, and $\mathcal{C}$ is the set of curves in D_+ connecting I_+ with ∂A, then for $\Lambda(\mathcal{C}) \leq s$,*

$$\mathcal{H}^\alpha(A) \leq c\, \exp\left\{-\frac{\alpha\pi}{\Lambda(\mathcal{C})}\right\}. \tag{4}$$

Given this proposition and the monotocity of extremal length the proof of the theorem is almost immediate as we now explain.

Proof of Theorem 1.1. Let $\mathcal{C}$ be the set of curves in $\mathbb{H}$ connecting $(-\infty, 0]$ with $\eta_1 \cup \cdots \cup \eta_n$. By conformal invariance, the modulus of $\mathcal{C}$ is the same as that for the curves connecting $g^{-1}(-\infty, 0]$ with $l_1 \cup \cdots \cup l_n$. Since any such curve must go through the circle of radius 1, monotonicity implies that this modulus is bounded above by the modulus of curves from the circle of radius 1 to the circle of radius r in $\mathbb{H}$ which is $\pi / \log(1/r)$. Using (2) and (4) we see that

$$\mathbf{P}\{\gamma \cap (l_1 \cup \cdots l_n) \neq \emptyset\} \leq c\, r^{\alpha}.$$

□

2 Excursion measure and capacity

Let $\mathcal{Q}$ denote the collection of sets of the form

$$K = \mathcal{B}(x_1, r_1) \cup \cdots \cup \mathcal{B}(x_n, r_n),$$

where $x_j \in \mathbb{R}$ and $r_j \leq 1/2$. We do not require the disks to be disjoint. Note that ∂K is a finite union of circular arcs and hence is piecewise analytic. Given $K \in \mathcal{Q}$, let $D_K = D \setminus K$ which is a (possibly multiply connected) domain. Let $\mathcal{Q}_j$ denote the collection of such sets such that all of the centers lie in $[j, j+1)$; each $K \in \mathcal{Q}$ can be written as a union

$$K = \bigcup_{j=-\infty}^{\infty} \tilde{K}_j, \quad \tilde{K}_j \in \mathcal{Q}_j,$$

where all but a finite number of $\tilde{K}_j$ are empty. We also write $K = K^o \cup K^e$ where

$$K^o = \bigcup_{j \text{ odd}} \tilde{K}_j, \quad K^e = \bigcup_{j \text{ even}} \tilde{K}_j.$$

Let $g = g_K$ denote the harmonic function on D_K with boundary value 1 on ∂D and 0 on K, that is, $g(z)$ is the probability that a Brownian motion starting at z leaves D before visiting K. Note that $g(z) = g(\bar{z})$. The excursion measure (between ∂D and ∂K in D_K) is defined by

$$\mathcal{E}_K = \mathcal{E}_{D_K}(\partial D, \partial K) = \int_{\partial K} \partial_{\mathbf{n}} g(z)\, |dz| = -\int_{\partial D} \partial_{\mathbf{n}} g(z)\, |dz|.$$

The term excursion measure often refers not just to the quantity above but also the measure on paths corresponding to Brownian motions going from one boundary to the other. The Brownian interpretation is very useful when estimating $\mathcal{E}_K$, but the number itself is more classical. It is a version of a capacity of the set K as seen from ∂D, or, as probabilists view it, how likely it is for a Brownian motion to hit the set. The goal of this section is to prove the following.

Proposition 2.1. *For every $\alpha > 0$ there exist $c < \infty$ and $s > 0$ such that if $K \in \mathcal{Q}$ with $\mathcal{E}_K \leq s$, then*

$$\mathcal{H}^{\alpha}(K) \leq c\, e^{-2\pi/\mathcal{E}_K}. \tag{5}$$

For the remainder of this section, we fix α and allow constants to depend on α. We note that it suffices to prove (5) for K^e. Indeed, if this is true, by symmetry it also holds for K^o, and since $\mathcal{E}_K \geq \max\{\mathcal{E}_{K^o}, \mathcal{E}_{K^e}\}$,

$$\mathcal{H}^\alpha(K) \leq \mathcal{H}^\alpha(K^e) + \mathcal{H}^\alpha(K^o) \leq 2\, c\, e^{-2\pi/\mathcal{E}_K}.$$

For the remainder of this section, we assume that K has the form

$$K = \bigcup_{j=-M}^{M} K_j, \quad M \in \mathbb{N}, \quad K_j = \tilde{K}_{2j} \in \mathcal{Q}_{2j}.$$

We write $\mathcal{E}_j = \mathcal{E}_{K_j}$. We start by proving (5) for sets in $\mathcal{Q}_j$.

Proposition 2.2. *For every $0 < \alpha \leq 1$ there exists $c < \infty$ such that if $K \in \mathcal{Q}_j$ then*

$$\mathcal{H}^\alpha(K) \leq c\, e^{-2\pi/\mathcal{E}_j}.$$

By translation invariance, we may assume $j = 0$ and hence

$$K \subset V := \left\{ x + iy : |x - \frac{1}{2}| \leq 1, |y| \leq \frac{1}{2} \right\}.$$

We write $G = G_D$ for the Green's function of D normalized so that

$$G(z,w) = -\log|z-w| + O(1), \quad z \to w.$$

Standard arguments (e.g., by mapping to a disk) show that there exists β such that for $z, w \in V$,

$$|G(z,w) + \log|z-w|| \leq \beta, \tag{6}$$

Let ν_K denote the measure supported on ∂K, given by

$$\frac{d\nu_K}{|dw|} = \frac{\partial_{\mathbf{n}} g(w)}{2\pi}.$$

This is the equilibrium measure from potential theory. Then,

$$\mathcal{E}_K = 2\pi \int_{\partial K} \nu_K(dz),$$

We claim that

$$1 - g(z) = \int_{\partial K} G(z,w)\, \nu_K(dw). \tag{7}$$

This is a "last-exit" decomposition — for each path starting at z that hits K we focus on the last visit to K. In particular, if $z \in K$,

$$\int_{\partial K} G(z,w)\, \nu_K(dw) = 1.$$

The next proposition is standard from potential theory but we give a quick proof.

Proposition 2.3.
$$\mathcal{E}_K = 2\pi \sup m(K),$$
where the supremum is over all positive measures supported on K such that for all $z \in K$,

$$Gm(z) := \int_{\partial K} G(z,w)\, m(dw) \leq 1. \tag{8}$$

Proof. By (7), we know that ν_K satisfies (8) and we have seen that

$$\mathcal{E}_K = 2\pi \int_{\partial K} \nu_K(dz).$$

Suppose m satisfies (8) and consider the signed measure $\hat{m} = \nu_K - m$. Then, if $z \in K$, $G\hat{m}(z) \geq 0$. Therefore,

$$0 \leq \int_K G\hat{m}(z)\, \nu_K(dz) = \int_K \int_K G(z,w)\, \hat{m}(dw)\, \nu_k(dz) = \int_K d\hat{m}(w) = \hat{m}(K).$$

Therefore $\nu_K(K) = m(K) + \hat{m}(K) \geq m(K)$.

□

We now use this proposition to give a relation between $\mathcal{E}_K$ and the capacity of K in the whole plane. Let $h = h_K$ be the unique continuous function on $\mathbb{C}$ that is harmonic on $\mathbb{C} \setminus K$, vanishes on K, and is asymptotic to $\log|z|$ as $z \to \infty$. It is given explicitly by

$$h(z) = \log|z-x| - \mathbf{E}^z \left[\log|B_\tau - x|\right].$$

where $x \in K$. Here B_t is a complex Brownian motion and $\tau = \tau_K$ is the first time that it visits K. We can expand h at infinity,

$$h(z) = \log|z| - \mathrm{cap}(K) + O(|z|^{-1}), \quad z \to \infty \tag{9}$$

where $\mathrm{cap}(K)$ denotes the capacity given by

$$\mathrm{cap}(K) = \int_K \log|w|\, \mu_K(dw).$$

(Many authors, in particular [5], define the capacity to be $e^{\mathrm{cap}(K)}$, so some care must be used in reading results.) Here μ_K denote the harmonic measure (that is, the hitting measure of Brownian motion) of K as seen from infinity. It is supported on ∂K and can be written as

$$\mu_K(dz) = \partial_{\mathbf{n}} h(z)\, \frac{|dz|}{2\pi}.$$

To check this is a probability measure (and that the 2π is needed) note that

$$\int_{\partial K} \mu_K(dz) := \int_{\partial K} \partial_{\mathbf{n}} h(z)\, \frac{|dz|}{2\pi} = 1, \tag{10}$$

where $\mathbf{n}$ denotes the outward normal (that is, pointing inward to $\mathbb{C} \setminus K$). This can be deduced from (9) by letting

$$U_R = \{|z| < R\} \setminus K,$$

and using

$$\int_{\partial U_R} \partial_{\mathbf{n}} h(z)\,|dz| = 0.$$

It is known (see [5, Theorem 9.7]) that for every $z \in K$,

$$\mathrm{cap}(K) = \int_K \log|w-z|\,\mu_K(dw). \tag{11}$$

We will use one other fact.

Lemma 2.4. *[5, Theorem 10.3] For every $0 < \alpha \le 1$, there exists $c_\alpha < \infty$ such that if K is a compact set, then $\mathcal{H}^\alpha(K) \le c_\alpha\, e^{\alpha\,\mathrm{cap}(K)}$.*

Proof of Proposition 2.2. Given what we have done so far, it suffices to show that if $K \in \mathcal{Q}_0$,

$$\mathrm{cap}(K) \le -\frac{2\pi}{\mathcal{E}_K} + \beta.$$

To see this, we use (11), (10), and (6) to see that if $w \in K$,

$$\begin{aligned}\int_{\partial K} G(z,w)\,\partial_n h(z)\,\frac{|dz|}{2\pi} &\le \int_{\partial K} [-\log|z-w| + \beta]\;\partial_n h(z)\,\frac{|dz|}{2\pi}\\ &= -\mathrm{cap}(K) + \beta.\end{aligned}$$

In other words, if m is the measure supported on ∂K given by

$$m(dz) = \frac{\partial_n h(z)}{-\mathrm{cap}(K)+\beta}\,\frac{|dz|}{2\pi},$$

then $Gm \le 1$ on ∂K, and hence,

$$\mathcal{E}_{D_K}(\partial D, \partial K) \ge 2\pi \int_{\partial K} m(dz) = \frac{2\pi}{-\mathrm{cap}(K)+\beta}.$$

□

Proof of Proposition 2.1. let

$$\mathcal{E}_j = \mathcal{E}_{D_{K_j}}(\partial D, K_j), \quad \hat{\mathcal{E}}_j = \mathcal{E}_{D_K}(\partial D, K_j).$$

Note that

$$\mathcal{E}_K = \sum_{j=-\infty}^{\infty} \hat{\mathcal{E}}_j.$$

Using the strong Markov property,

$$\hat{\mathcal{E}}_j \le \mathcal{E}_j \le \sum_{k=-\infty}^{\infty} \hat{\mathcal{E}}_k\, q(k,j)$$

where $q(k,j) = q(k,j,K)$ is the maximum over z in K_k of the probability that a Brownian motion starting at k hits K_j before leaving D. Using standard estimates and the Harnack inequality, we see that there exist c, u such that if $k \neq j$,

$$q(k,j) \leq c\, e^{-u|k-j|}\, \hat{\mathcal{E}}_j.$$

The term $e^{-u|k-j|}$ gives the probability of getting within distance 2 of K_j before leaving D and given this the probability of hitting K_j is $O(\mathcal{E}^j)$. Using inclusion-exclusion, we see that

$$\begin{aligned} \mathcal{E}_K &\geq \sum_{j=-\infty}^{\infty} \mathcal{E}_j - \sum_{j=-\infty}^{\infty} \sum_{k \neq j} \hat{\mathcal{E}}_k\, q(k,j) \\ &\geq \sum_{j=-\infty}^{\infty} \mathcal{E}_j - c \sum_{j=-\infty}^{\infty} \sum_{k \neq j} \hat{\mathcal{E}}_k\, \hat{\mathcal{E}}_j\, e^{-u|k-j|} \\ &\geq \sum_{j=-\infty}^{\infty} \mathcal{E}_j - c\, \mathcal{E}_K^2. \end{aligned}$$

That is, there exists $c < \infty$ such that

$$\sum_{j=-\infty}^{\infty} \mathcal{E}_j \leq \mathcal{E}_K\, [1 + c\, \mathcal{E}_K], \tag{12}$$

$$\begin{aligned} \left[\sum_{j=-\infty}^{\infty} \mathcal{E}_j \right]^{-1} &\geq \frac{1}{\mathcal{E}_K\, [1 + c\, \mathcal{E}_K]} \\ &= \frac{1}{\mathcal{E}_K} - \frac{c\, \mathcal{E}_K}{1 + c\, \mathcal{E}_K} \\ &\geq \frac{1}{\mathcal{E}_K} - 1. \end{aligned}$$

Combining the previous two lemmas, we have for each j, $\mathcal{H}^\alpha(K_j) \leq c\, e^{-2\pi/\mathcal{E}_j}$, and hence

$$\mathcal{H}^\alpha(K) \leq c \sum_{j=-\infty}^{\infty} e^{-2\pi/\mathcal{E}_j} \leq c \exp\left\{ -\frac{2\pi}{\sum \mathcal{E}_j} \right\} \leq c\, e\, \exp\left\{ -\frac{2\pi}{\mathcal{E}_K} \right\}.$$

The second inequality holds provided that $\sum \mathcal{E}_j \leq 2\pi$ which will be true for $\mathcal{E}_K$ sufficiently small by (12).

□

In applications, we will not have a union of disk, so it will be useful to give a slight generalization of this.

Corollary 2.5. *For every $\alpha \leq 1$, there exists $c < \infty, s > 0$, such that the following holds. Suppose $A_1, A_2, \ldots, A_n$ are compact subsets with $A_j \cap \mathbb{R} \neq \emptyset$ and $r_j := \mathrm{diam}(A_j) \leq s$. Let x_j be any point in $A_j \cap \mathbb{R}$, let $K_j = \mathcal{B}(x_j, r_j)$ and*

$$A = A_1 \cup \cdots \cup A_n, \quad K = K_1 \cup \cdots \cup K_n.$$

Then,

$$\mathcal{H}^\alpha(A) \leq e^{-2\pi/\mathcal{E}_A}.$$

Proof. Since $A \subset K$, we have

$$\mathcal{H}^\alpha(A) \leq \mathcal{H}^\alpha(K) \leq c\, e^{-2\pi/\mathcal{E}_K}.$$

Hence we only need to show that

$$\frac{1}{\mathcal{E}_K} \geq \frac{1}{\mathcal{E}_A} - O(1). \tag{13}$$

For every $z \in K_j$, the probability that a Brownian motion starting at z leaves D without hitting A_j is $O(-1/\log d_j) \asymp \mathcal{E}_{A_j} \leq \mathcal{E}_A$. Therefore, by the strong Markov property,

$$\mathcal{E}_A \geq \mathcal{E}_K\,[1 - O(\mathcal{E}_A)],$$

which gives (13).

□

3 Modulus and reflected excursion measure

We will relate some the modulus of a set of curves and the excursion measure given by (partially) reflected Brownian motion. Although it can be done in more generality, we will restrict our consideration to conformal rectangles, that is, simply connected domains U with four marked boundary points, z_1, z_2, z_3, z_4 in counterclockwise order. Any such domain is conformally equivalent to a rectangle

$$\mathcal{R}_\theta = \{x + iy : 0 < x < 1, 0 < y < \theta\},$$

with $z_1 = \theta i, z_2 = 0, z_3 = 1, z_4 = 1 + \theta i$. Let ∂_- denote the boundary arc connecting z_1, z_2 that does not include z_3, z_4, and ∂_+ the arc connecting z_3, z_4 that does not contain z_1, z_2. We will also divide the closed arc ∂_+ into

$$\partial_+ = A_1 \cup l_1 \cup A_2 \cup l_2 \cup \cdots \cup l_k \cup A_{k+1},$$

where $l_1, l_2, \ldots, l_k$ are disjoint open subarcs in ∂_+, listed in counterclockwise order, and $A_1, A_2, \ldots, A_{k+1}$ are nontrivial closed subarcs. We write $\hat{\partial} = A_1 \cup \cdots \cup A_{k+1}$. If $U = \mathcal{R}_\theta$ as above, then $A_1, A_2, \ldots, A_{k+1}$ are nontrival, closed, disjoint subintervals of $[1, 1+\theta i]$ with $1 \in A_1, 1+\theta i \in A_{k+1}$, listed in increasing order of their imaginary parts. We let $\mathcal{C}$ denote the set of curves in U from ∂_- to ∂_+ and $\hat{\mathcal{C}}$ the set of curves in U from ∂_- to $\hat{\partial}$. Note that $\hat{\mathcal{C}} \subset \mathcal{C}$, and hence $\Lambda(\hat{\mathcal{C}}) \leq \Lambda(\mathcal{C})$ since the set of admissible functions for $\hat{\mathcal{C}}$ is larger than that for $\mathcal{C}$.

Some of the arguments here are similar to those for a well known theorem of Pfluger's theorem [5, Theorem 9.17]. The comb domains described below are the chordal analogue of "starlike" domains in [5].

- Suppose B_t is a Brownian motion in U that is reflected on $\partial U \setminus (\partial_- \cup \hat{\partial})$ and is stopped at time $\tau = \inf\{t : B_t \in \partial_- \cup \hat{\partial}\}$. let

$$h(z) = \mathbf{P}^z\{B_\tau \in \hat{\partial}\}. \tag{14}$$

Then $h(z)$ is the unique harmonic function on U satisfying the boundary conditions

$$h(z) = 0, \quad z \in \partial_-, \qquad h(z) = 1, \quad z \in \hat{\partial},$$

$$\partial_{\mathbf{n}} h(z) = 0, \quad z \in \partial U \setminus (\partial_- \cup \hat{\partial}).$$

Here $\partial_{\mathbf{n}}$ denotes the normal derivative (which can be interpreted for nonsmooth boundaries by first mapping to a domain with smooth boundaries). Then, the excursion measure between ∂_- and $\hat{\partial}$ in U with reflecting boundary on $\partial U \setminus (\partial_- \cup \hat{\partial})$ is given by

$$\mathcal{E}^*_U(\partial_-, \hat{\partial}) = \int_{\partial_-} \partial_{\mathbf{n}} h(z)\, |dz| = -\int_{\hat{\partial}} \partial_{\mathbf{n}} h(z)\, |dz|.$$

The * in the notation indicates that were are using reflecting boundary conditions; we use $\mathcal{E}_U(\partial_-, \hat{\partial})$ for the excursion measure for paths killed upon reaching $\partial U \setminus (\partial_- \cup \hat{\partial})$. An argument similar to that for usual excursion measure, using the fact that reflected Brownian motion is a conformal invariant, shows that reflected excursion measure is a conformal invariant. Also, $\mathcal{E}^*_{\mathcal{R}_\theta}(\partial_-, \partial_+) = \theta$, and hence $\mathcal{E}^*_{\mathcal{R}_\theta}(\partial_-, \partial_+) = \Lambda(\mathcal{C})$.

- We will call $\mathcal{D}$ a "comb" domain if it is of the form

$$\mathcal{D} = \mathcal{R}_{\theta'} \setminus (l'_1 \cup \cdots \cup l'_k)$$

where $l'_1, \ldots, l'_k$ are disjoint intervals of the form

$$l'_j = \{x + iy_j : x_j \le x \le 1\},$$

where $0 < x_j < 1$, $0 < y_j < \theta'$. In this case, $h(x + iy) = x$, the same as for $\mathcal{R}_{\theta'}$, since the reflection is always in the y-direction and is independent of the real part. Using the same argument as for $\mathcal{R}_{\theta'}$, we see that if $\mathcal{C}$ denotes the set of curves connecting the vertical boundaries in $\mathcal{D}$, then $\Lambda(\mathcal{C}) = \theta'$. Similarly, $\mathcal{E}^*_{\mathcal{D}}(\partial_-, \hat{\partial}) = \theta'$, where ∂_-, ∂_+ denote the vertical boundaries.

- Given $\mathcal{R}_\theta, A_1, l_1, A_2, \ldots, l_k, A_{k+1}, h$ as above, there is a unique comb domain $\mathcal{D}$ and conformal transformation $g : \mathcal{R}_\theta \to \mathcal{D}$ such that $g(\partial_-) = \partial_-$ and $g(\hat{\partial}) = \partial_+$. The intervals $l_1, \ldots, l_k$ are mapped to $l'_1, \ldots, l'_k$. Note that $\theta' \le \theta$ since $\Lambda(\hat{\mathcal{C}}) \le \Lambda(\mathcal{C})$. The domain $\mathcal{D}$ can be determined from $\mathcal{R}_\theta, A_1, l_1, A_2, \ldots, l_k, A_{k+1}, h$ by the relations,

$$\theta' = \int_0^\theta \partial_x h(iy)\, dy,$$

$$y_j - y_{j-1} = \int_{A_j} -\partial_x h(1 + iy)\, dy \quad (\text{where } y_0 = 0),$$

$$x_j = \min\{h(z) : z \in l_j\}.$$

The conformal map g is the unique holomorphic function on $\mathcal{R}_\theta$ with $g(0) = 0$ and $\mathrm{Re}[g(z)] = h(z)$.

- Suppose $D = \{|y| < \pi\}$ as before and $D_+ = D \cap \mathbb{H} = \{0 < |y| < \pi\}$. Suppose $K = K_1 \cup \cdots \cup K_n$, where K_j are disjoint connected compact sets invariant under $z \mapsto \bar{z}$. As before, let $D_K = D \setminus K$ and let $D_K^+ = D_K \cap \mathbb{H}$. We let $\partial_- = I_+ := \{x + i\pi\}$ and $A_j = \partial U \cap K_j$. Then,
$$\mathcal{E}^*_{D_K^+}(\partial_-, \hat{\partial}) = \mathcal{E}_{D_K}(I_+, K) = \frac{1}{2}\,\mathcal{E}_K.$$
In particular, $\mathcal{E}_K = 2\Lambda(\mathcal{C})$ where $\mathcal{C}$ denotes the set of curves in D_K^+ that connect I_+ with K.

We finish this section by restating Corollary 2.5 using the relation above.

Proposition 3.1. *For every $\alpha \leq 1$, there exist $c < \infty, s > 0$, such that the following holds. Suppose $A_1, A_2, \ldots, A_n$ are connected compact subsets of $\overline{\mathbb{H}}$ with $A_j \cap \mathbb{R} \neq \emptyset$ and $r_j := \operatorname{diam}(A_j) \leq s$. Then,*
$$\mathcal{H}^\alpha(A) \leq c\, e^{-\alpha\pi/\Lambda(\mathcal{C})},$$
where $\mathcal{C}$ denotes the collection of curves in D_K^+ that connect I_+ to A. In particular, if γ is an SLE_κ path from 0 to ∞ and $\alpha = \frac{8}{\kappa} - 1$. then
$$\mathbf{P}\{\gamma \cap A \neq \emptyset\} \leq c\, e^{-\alpha\pi/\Lambda(\mathcal{C})}.$$

4 Lemma

Here we prove the simple lemma comparing H^α and $\mathcal{H}^\alpha$.

Lemma 4.1. *If $0 < \alpha \leq 1$, and $K = K_1 \cup \cdots \cup K_n$ where K_j are connected, compact sets with $K_j \cap \mathbb{R} \neq \emptyset$, then*
$$H^\alpha(K) \leq 6^\alpha\, \mathcal{H}^\alpha(K).$$

Proof. Let $d_j = \operatorname{diam}(K_j)$. Without loss of generality assume that $d_1 \geq d_2 \geq \cdots \geq d_n$. Let x_j denote some point in $K_j \cap \mathbb{R}$.

Suppose we have a covering of K with disks $\mathcal{B}(z_1, r_1), \ldots, \mathcal{B}(z_m, r_m)$. We will find a collection of disks $\mathcal{B}(x_1, \tilde{r}_1), \ldots, \mathcal{B}(x_n, \tilde{r}_n)$ with
$$K \subset \bigcup_{k=1}^n \mathcal{B}(x_j, \tilde{r}_j),$$
and such that
$$\sum_{j=1}^n \tilde{r}_j^\alpha \leq 6^\alpha \sum_{j=1}^m r_j^\alpha.$$
This will establish the lemma. Without loss of generality, we assume that each of the disks $\mathcal{B}(z_j, r_j)$ intersects K.

Let $\mathcal{A}_j$ denote the set of disks in the covering that intersect K_j but do not intersect $K_1 \cup \cdots \cup K_{j-1}$. Note that
$$K_1 \subset \bigcup_{\mathcal{B}(z,r) \in \mathcal{A}_1} \mathcal{B}(z, r),$$
and for $j > 1$, either there exists $\mathcal{B} \in \mathcal{A}_i \cup \cdots \cup \mathcal{A}_{j-1}$ that intersects K_j or
$$K_j \subset \bigcup_{\mathcal{B}(z,r) \in \mathcal{A}_j} \mathcal{B}(z, r).$$

- Let
$$\delta = \max\{r : \mathcal{B}(z,r) \in \mathcal{A}_1\}.$$
If $\delta \geq d_1$, then $\mathcal{B}(z,r) \subset \mathcal{B}(x_1, 2\delta)$ for every $\mathcal{B}(z,r) \in \mathcal{A}_1$ and obviously
$$(2\delta)^\alpha \leq 2^\alpha \sum_{(z,r)\in\mathcal{A}_1} r^\alpha.$$
If $\delta \leq d_1$, then since K_1 is connected and it is a cover we must have
$$\sum_{\mathcal{B}(z,r)\in\mathcal{A}_1} \frac{r}{d_1} \geq \frac{1}{2}.$$
Since $\alpha \leq 1$, this implies that
$$\sum_{\mathcal{B}(z,r)\in\mathcal{A}_1} \left(\frac{r}{d_1}\right)^\alpha \geq \frac{1}{2^\alpha}.$$
Let $\delta_1 = 2(\delta \vee d_1)$ and note that $\mathcal{B}(z,r) \subset \mathcal{B}(x_1, \delta_1)$ for each $\mathcal{B}(z,r) \in \mathcal{A}_1$, and
$$\delta_1^\alpha \leq 4^\alpha \sum_{\mathcal{B}(z,r)\in\mathcal{A}_1} r^\alpha.$$
- For $j > 1$ we proceed recursively as follows. Set $\tilde{r}_j = 3\delta_j/2$.
  - If there exists $i < j$ and a disk $\mathcal{B} \in \mathcal{A}_i$ with $\mathcal{B} \cap K_j = \emptyset$, we let $\delta_j = 0$. Note that since $\delta_i \geq 2d_j$, and $\mathcal{B} \subset \mathcal{B}(x_i, \delta_i)$, $K_j \subset \mathcal{B}(x_i, \tilde{r}_i)$.
  - Otherwise, the balls in $\mathcal{A}_j$ must cover K_j. As before, we let
  $$\delta_j = 2\left[d_j \wedge \max\{r : (z,r) \in \mathcal{A}_j\}\right].$$
  Then $\mathcal{B}(z,r) \subset \mathcal{B}(x_j, \delta_j)$ for every $\mathcal{B}(z,r) \in \mathcal{A}_j$ and
  $$\sum_{\mathcal{B}(z,r)\in\mathcal{A}_j} \left(\frac{r}{d_j}\right)^\alpha \geq \frac{1}{2^\alpha}.$$

□

5 Radial result

We now show that Theorem 1.2 can be done similarly. Again we reduce the *SLE* statement to a fact about capacities and Hausdorff content. Under the assumption of Theorem 1.2, there are a finite or countably infinite number open arcs on the circle of radius r that can be reached as the initial visit of the *SLE*. It suffces to give the estimate for any finite union of these crosscuts (provided that c is independent of the the number of crosscuts). Let us write $l_1, \ldots, l_n$ for the crosscuts. $g : \mathbb{D} \setminus \eta \to \mathbb{D}$ be the unique conformal transformation with $g(\eta(1)) = 1$ and $g(0) = 0$. Let $A_j = g \circ l_j$, $A = A_1 \cup \cdots \cup A_n$, and note that A_j are crosscuts of $\mathbb{D}$ that do not separate 0 from 1.

The analogue of the radial boundary estimate for one crosscut was established in [3, Proposition 4.3]. Suppose l is a crosscut of $\mathbb{D}$. Then the probability that an SLE_κ path from 1 to 0 intersects l is bounded above by $c\,[\text{diam}(l)/\text{dist}(1,l)]^{4a-1}$. This is essentially the same as the chordal estimate, and, indeed, radial and chordal SLE are locally absolutely continuous. In the same paper it was shown that for a fixed crosscut as in Theorem 1.2, we have $\text{diam}(l) \leq c\,r^{1/2}\,\text{dist}(1,l)$ and hence

$$\mathbf{P}\{\gamma \cap A_j \neq \emptyset\} \leq c\,r^{(4a-1)/2}.$$

Therefore, to establish the result it suffices to show that we can cover A by a finite union of disks $\mathcal{B}(z_1,r_1),\ldots,\mathcal{B}(z_m,r_m)$ centered on $\mathbb{D}$ such that

$$\sum_{j=1}^{m}\left[\frac{r_j}{|z_j-1|}\right]^{(4a-1)/2} \leq c\,r^{(4a-1)/2}.$$

To do this, we consider ℓ, the image of the circle of radius rs under g. By monotonicty and the calculation for an annulus, we know that

$$\mathcal{E}^*_{\mathbb{D}}(\ell,A) \leq \frac{2\pi}{\log(1/s)}, \tag{15}$$

where we write $\mathcal{E}^*_{\mathbb{D}}(\ell,A)$ for $\mathcal{E}^*_{D'}(\ell,A)$. and D' is the component of $\mathbb{D}\setminus(\ell\cup A)$ whose boundary intersects A. To use the results in this paper, it is convenient to map $\mathbb{D}$ to D_+. We choose the maps that send 1 to $-\infty$ and 0 either to $i\pi/2$ for $i\pi/4$. It is useful to have two maps for then different parts of the unit circle are sent to $+\infty$ under the map. We then are in a position to use our work once we make the observation in the next proposition.

Proposition 5.1. *For every $M<\infty$, there exists $c<\infty$ such that the following hold. Suppose $A,\mathcal{C}$ are as in Proposition 1.5 with $A\subset\{\text{Re}(z)\leq M\}$. Suppose η is an infinite, connected, closed, subset of $D\cap\{\text{Re}(z)\leq 0\}$ that intersects the imaginary axis, and let $\mathcal{C}'$ be the set of curves in D_+ connecting η with ∂A. Then for $\Lambda(\mathcal{C})$ sufficiently small,*

$$\frac{1}{\Lambda(\mathcal{C}')} \leq \frac{1}{\Lambda(\mathcal{C})} + c.$$

In particular, For every $\alpha>0$, there exist $c'<\infty, s>0$ such that if $A=A_1\cup\cdots\cup A_n$ where A_j are disjoint connected compact sets intersecting the real line, and $\mathcal{C}$ is the set of curves in D_+ connecting I_+ with ∂A, then for $\Lambda(\mathcal{C})\leq s$,

$$\mathcal{H}^\alpha(A) \leq c'\,\exp\left\{-\frac{\alpha\pi}{\Lambda(\mathcal{C}')}\right\}. \tag{16}$$

Proof. We assume that $A\subset\{\text{Im}(z)\leq\pi/4\}$; this will be true for sufficiently small $\Lambda(\mathcal{C})$.

Let $D_A = D_+\setminus A$ and let $h(z)$ be the harmonic function on D with boundary value 0 on I_+, 1 on ∂A, and with zero normal derivative on the rest of the boundary. That is, q is the probability that a reflected Brownian motion in D_A started at z reaches A before reaching I_+. Note that

$$\mathcal{E}^*_{D_A}(I_+,A) = \frac{2}{\pi}\int_{-\infty}^{\infty} h(x+i(\pi/2))\,dx.$$

Using the Harnack principle in a neighborhood about x, we see that this implies that there exists c_1 such that for all x,

$$h(x + i(\pi/2)) \leq c_1 \, \mathcal{E}^*_{D_A}(I_+, A).$$

Let $q(z)$ denote the probability that the reflected Brownian motion hits η before reaching A or I_+. Using planarity and connectivity of η it is not hard to see that there exists c_2 such that

$$q(x + i(\pi/2)) \geq c_2, \quad |x| \leq M.$$

Let $p(z)$ denote the probability that the reflected Brownian motion hits η before reaching A. Using the estimates above we find c_2 such that

$$p(z) \geq 1 - c_2 \, \mathcal{E}^*_{D_A}(I_+, A), \quad z \in I_+,$$

from which we get

$$\mathcal{E}^*_{D_A}(A, \eta) \geq \mathcal{E}^*_{D_A}(A, I_+) \, [1 - c_2 \, \mathcal{E}^*_{D_A}(I_+, A)],$$

and hence

$$\frac{1}{\mathcal{E}^*_{D_A}(A, \eta)} \leq \frac{1}{\mathcal{E}^*_{D_A}(I_+, A)} + c.$$

□

The proposition obviously also holds if we interchange the roles of I_+ and the real line. Theorem 1.2 now follows from the boundary estimate, (15), and (16).

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