RANDOM WALKS FROM STATISTICAL PHYSICS I

Random Walks: Simple and Self-Avoiding

Wald Lectures
2011 Joint Statistical Meetings

Gregory F. Lawler

Department of Mathematics
Department of Statistics
University of Chicago

August 2, 2011

- Random walks appear in many places in statistical physics.
- Behavior strongly depends on the dimension in which the walk lives. Our walks will live in the integer lattice

  $$\mathbb{Z}^d = \{(z_1, \ldots, z_d) : z_j \in \mathbb{Z}\}.$$

  The choice of lattice is not so important (universality) but the dimension d is.
- There is an upper critical dimension above which the behavior is in some sense easy to understand (mean-field behavior)
- Below the critical dimension, there are nontrivial "critical exponents"
- Still many open questions in these areas.
- Today I will introduce some models and will discuss all dimensions. The second two lectures will focus on work in the last twelve years on two-dimensional systems where conformal invariance becomes important.

SIMPLE RANDOM WALK

- At each step the random walker chooses randomly among its $2d$ nearest neighbors to move to.
- Can write the position of the walker at time n as

  $$S_n = X_1 + \cdots + X_n$$

  where $X_1, \dots, X_n$ are independent, identically distributed, mean zero.
- Classical probability — $\mathbb{E}[|S_n|^2] = n$ and $S_n/\sqrt{n}$ converges in distribution to a standard normal random variable.
- At time n the walker is distance of order $\sqrt{n}$ away from the origin. Assuming $d \geq 2$, we can restate this as "the number of points in the ball of radius R visited by the walker is R^2.
- The "fractal dimension" of the set of points visited by the walker is 2. This holds for all $d \geq 2$.

FRACTAL DIMENSION

- Roughly speaking, a subset of $\mathbb{Z}^d$ has fractal dimension D if the number of points in the disk of radius R grows like R^D. Note that $D \leq d$.
- The rule for variance of sums of independent random walks shows that the fractal dimension of simple random walk path is $D = 2$ (provided that $d \geq 2$).
- There are a number of rigorous notions of fractal dimension such as Hausdorff dimension and box dimension. We will not go into details about this.
- The Hausdorff or box dimension of the path of Brownian motion in $\mathbb{R}^d$ is two.

WARM-UP: USING "DIMENSIONAL" ARGUMENTS

- What is the probability that a random walk is at the origin at time $2n$?
- The walk is about distance $\sqrt{n}$ from the origin.
- There are about $n^{d/2}$ points in $\mathbb{Z}^d$ at that distance.
- The probability of being at a particular one is $n^{-d/2}$. This indicates the result

$$\mathbb{P}\{S(2n) = 0\} \sim c\, n^{-d/2}.$$

- Let e_n denote the expected number of visits up to time n. Then $e_\infty < \infty$ for $d \geq 3$ and

$$e_n \sim c\, n^{1/2}, \quad , d = 1$$

$$e_n \sim c\, (\log n) \quad d = 2.$$

- Two is the critical dimension. This is the critical dimension for intersection of a two-dimensional set (random walk) and a zero-dimensional set (origin).
- Let p_n denote the probability that the walk does not return up to time n. It can be shown that

$$p_n \asymp \frac{1}{e_n} \asymp \begin{cases} n^{-1/2}, & d = 1 \\ (\log n)^{-1}, & d = 2 \\ 1, & d \geq 3 \end{cases}$$

- This argument uses a Markov or renewal argument — if a random walk returns to the origin, then the expected number of visits after that is the same as the expected number (This last argument is easier if we allow a geometric number of steps which is the same as generating function techniques.)
- In particular, we get Polya's theorem that the random walk is recurrent if and only if $d \leq 2$.

SELF-AVOIDING WALK (SAW)

- A self-avoiding walk (SAW) of length n is a simple random walk path that does not visit any vertex more than once.
- Used as a model of polymer chains. (A polymer consists of monomers that form themselves randomly with the constraint that it cannot come back on itself.)
- Analyzing SAWs is much harder than simple random walks. Many questions remain open today.
- Let c_n denote the number of SAWs of length n starting at the origin. (The number of simple walks is $(2d)^n$.) There is a connective constant μ such that

$$c_n \approx \mu^n.$$

  - Numerically we know that $\mu \approx 2.64$ if $d = 2$. There is a good chance we will never know the number exactly.
  - The number μ depends on the lattice. Duminil-Copin and Smirnov recently proved a conjecture by Nienhuis that on the honeycomb lattice $\mu = \sqrt{2 + \sqrt{2}}$.

- More important than μ is the correction term:

$$c_n \sim \mu^n \, \phi(n).$$

- There is an interpretation of

$$q(n) = \frac{\phi(2n)}{\phi(n)\,\phi(n)}$$

as the probability that two SAWs starting at the origin of length n do not intersect.

- It is expected that:

$$q(n) \sim c\, n^{-11/32}, \quad d = 2$$

$$q(n) \sim c\, n^{\gamma - 1}, \quad d = 3,$$

$$q(n) \sim c\, (\log n)^{-1/4}, \quad d = 4,$$

$$q(n) \to c(d) > 0, \quad d \geq 5.$$

- Roughly speaking, one expects two two-dimensional sets in $\mathbb{R}^d$ to intersect it $d < 4$ and not to intersect if $d > 4$. This is why $d = 4$ should be critical for this problem.
- The prediction of $11/32$ for $d = 2$ was first done by Nienhuis and has been supported both by numerical simulation and work on the Schramm-Loewner evolution. However, it is still open.
- At the moment, there is no reason to believe that we will ever know the value γ in three dimensions exactly.
- The result for $d \geq 5$ was proved by Hara and Slade.
- The $d = 4$ is still open (but interesting work by Brydges, Imbrie, and Slade may be getting close to proving this).
- The exponent, unlike the constant μ, should not depend on the choice of lattice. Physicists use the word universality to indicate the independence. (The central limit theorem is an example of a universality result.)

- Another critical exponent for the SAW $\nu = \nu_d$ is defined by saying that the typical displacement of a SAW of n steps looks like n^ν.
- For simple random walks $\nu = 1/2$ for all d.
- Flory gave a heuristic argument to suggest that

$$\nu_d = \max\left\{\frac{1}{2}, \frac{3}{d+2}\right\}.$$

- Trivial if $d = 1$. Not at all trivial but proved by Hara and Slade if $d \geq 5$.
- For $d = 4$ a logarithmic correction is expected. Not proved.
- For $d = 2$ it is expected to be true. Although it has not been proved, the exponent can be seen in the fractal dimension of the conjectured scaling limit (Schramm-Loewner evolution)
- For $d = 3$, the prediction is not expected to be exactly true. Numerically $\nu_3 \approx .58\cdots$. Perhaps this number will never be known exactly.

ANALOGOUS QUESTION: INTERSECTIONS OF RANDOM WALK PATHS

- Let S_1 and S_2 be independent simple random walks in $\mathbb{Z}^d$ starting at the origin. Let

$$\omega_n = \{S_1(j) : j = 1, \ldots, n\}, \quad \eta_n = \{S_2(j) : j = 1, \ldots, n\},$$

$$q(n) = \mathbb{P}\{\omega_n \cap \eta_n = \emptyset\}.$$

- This is analogous (but not the same!) as the quantity for SAW.
- The *expected* number of intersections tends to infinity if and only if $d \leq 4$.
- If $d \geq 5$, $q(n) \to q(\infty) > 0$.
- For $d = 1$, one can show that $q(n) \asymp n^{-1}$. (The blue and red stay in different half lines. The gambler's ruin estimate for simple random walk is the key tool.)

EASIER PROBLEM

$$\lambda_n = \{S_3(j) : j = 1, \ldots, n\}.$$

$$p(n) = \mathbb{P}\{\omega_n \cap [\eta_n \cup \lambda_n] = \emptyset\}.$$

- The origin looks like a "middle point" of the random walk path $\eta_n \cup \lambda_n$. This makes the quantity easier to estimate.
- Determining $p(n)$ is not so difficult

$$p(n) \asymp n^{-\alpha}, \quad \alpha = \alpha_d = \frac{4-d}{2} \quad d < 4,$$

$$p(n) \asymp (\log n)^{-1}, \quad d = 4.$$

- $p(n)$ equals

$$\mathbb{P}\{\omega_n \cap \eta_n = \emptyset\} \, \mathbb{P}\{\omega_n \cap \lambda_n = \emptyset \mid \omega_n \cap \eta_n = \emptyset\}.$$

- Cauchy-Schwarz implies $q(n)^2 \leq p(n) \leq q(n)$.

- Mean-field behavior would be

$$\mathbb{P}\{\omega_n \cap \lambda_n = \emptyset \mid \omega_n \cap \eta_n = \emptyset\} \asymp \mathbb{P}\{\omega_n \cap \lambda_n = \emptyset\}.$$

- If mean-field, then $p(n) \asymp q(n)^2$.
- Systems at the critical dimension tend to exhibit mean-field behavior. This is true here [L]: if $d = 4$,

$$q(n) \asymp p(n)^{1/2} \asymp (\log n)^{-1/2}.$$

- If $d = 1$, $q(n) \asymp n^{-1}, \quad p(n) \asymp n^{-3/2}$. Not mean-field.
- If $d = 2, 3$, it is not mean-field. In fact [Burdzy-L], there exists an intersection exponent $\zeta = \zeta_d$ such that

$$q(n) \asymp n^{-\zeta}.$$

  This is the same as an exponent for Brownian motion.
- The argument shows existence of the exponent, but does not compute its value.

- Estimates were given for the exponent ζ. In particular,

$$\frac{4-d}{4} < \zeta_d < \frac{4-d}{2}.$$

- Duplantier and Kwon gave a nonrigorous, physical argument using conformal field theory to predict $\zeta_2 = 5/8$. This was consistent with simulations.
- This prediction was proved by L-Schramm-Werner using the Schramm Loewner evolution (SLE).
- Numerical simulations suggest $.28 < \zeta_3 < .29$. It is possible that this value will never be known exactly!
- Relates to fractal nature of Brownian motion paths. Suppose $B_t, 0 \leq t \leq 1$ is a d-dimensional Brownian motion ($d = 2, 3$). A cut time is a time that satisfies

$$B[0,t) \cap B(t,1] = \emptyset.$$

  Then (w.p.1), almost all times t will *not* be cut times. In fact, the Hausdorff dimension of the set of cut times is $1 - \zeta_d$.

A PROBLEM OF MANDELBROT

- In his *Fractal Geometry of Nature*, Benoit Mandelbrot looked at Brownian islands obtained by taking a planar Brownian motion or random loop and filling in the bounded areas. The coastline is also called the "Brownian frontier".
- The frontier looked to him to have the same fractal characteristics as self-avoiding walk. In particular, the fractal dimension was $4/3$.
- Studying this leads to the disconnection exponent λ defined by saying that the probability that a two-sided n-step random walk does not disconnect the origin decays like $n^{-\lambda}$. (Such a point looks like a typical point of the frontier.)
- Similar methods show that this exists and is equivalent to a Brownian exponent. The Hausdorff dimension of the boundary is $2(1 - \lambda)$. This method did not compute λ.

- The nonrigorous prediction $\lambda = 1/3$ was made using conformal field theory.
- The disconnection exponent is a type of intersection exponent and LSW proved Mandelbrot's conjecture by showing $\lambda = 1/3$.
- Further work has shown that Mandelbrot was even "more right". The statistics of the outer boundary of the Brownian motion are very similar to the conjectured scaling limit of self-avoiding walk.
- In three dimensions, the outer boundary is the same as the entire path. However, one can consider the "dimension of harmonic measure" which is the effective dimension of the set which are the first visit of a Brownian motion. One can prove that this dimension is strictly less than two (multifractal behavior).
- In contrast, the effective dimension in two dimensions is one. (This is true for all connected sets by a theorem of Makarov.)

LOOP-ERASED RANDOM WALK

- Take a simple random walk path and erase loops chronologically.
- Easy to define for $d \geq 3$ (random walk is transient). Need a limiting argument (not too difficult) for $d = 2$. ($d = 1$ is not interesting)
- First considered in order to hope to understand self-avoiding walk. However, it turns out that the paths look qualitatively different. (Physicists would say that it is in a different universality class.)
- The problem turns out to be related to a number of other problems: electrical networks, uniform spanning trees, determinant of the Laplacian. We will not consider these today.
- We ask: how far does a loop-erased walk go in n steps?

- Let Y_n be the number of steps on an n step walk that remain after loop erasure. Define $\Phi(n)$ by

$$\Phi(n) = \mathbb{E}[Y_n]$$

  In $\Phi(n)$ steps one expects to be distance $n^{1/2}$.
- In n steps expect distance $[\Phi^{-1}(n)]^{1/2}$.
- Critical dimension is $d = 4$.
- If $d > 4$, with probability one $Y_n \sim c\, n$ and the loop-erased random walk has the scaling limit as random walk (Brownian motion).

- Let J_n be the conditional probability given $[S(0), \ldots, S(n)]$ of the event that the nth point of the simple random walk is not erased. The probability that the nth point is not erased is $\mathbb{E}[J_n]$.
- The "easy" quantity to compute turns out to be the third moment (recall: returns to origin — first moment; intersections of random walks — second moment)

$$\mathbb{E}\left[J_n^3\right] \asymp \begin{cases} n^{(d-4)/2} & d = 2, 3 \\ (\log n)^{-1} & d = 4. \end{cases}$$

- "Mean-field" corresponds to $\mathbb{E}\left[J_n^3\right] \approx \mathbb{E}[J_n]^3$. Expect to be true if $d = 4$ but not if $d = 2, 3$.
- If $d = 4$, $\mathbb{E}[J_n] \asymp (\log n)^{-1/3}$ and the typical path has $n/(\log n)^{1/3}$ points. The typical distance of an n-step walk is $n^{1/2} (\log n)^{1/6}$. Moreover the scaled process approaches a Brownian motion.

- For $d = 2, 3$ define $\alpha = \alpha_d$ by $\mathbb{E}[J_n] \asymp n^{-\alpha}$. In other words the loop-erasure of an n-step path has $n^{1-\alpha}$ points.
- Let $\nu = \nu_d$ denote the exponent given by the typical distance in n steps is n^ν.Since $[n^{1-\alpha}]^\nu \approx n^{1/2}$,

$$\nu = \frac{1}{2(1-\alpha)}.$$

- Since $\mathbb{E}[J_n^3] \approx n^{(d-4)/2}$, we see that $\alpha \geq (4-d)/6$. This gives the inequality

$$\nu \geq \frac{3}{2+d}.$$

  The right-hand side is the Flory guess for the exponent for SAW. Since we expect not mean-field behavior, would guess that equality does *not* hold.
- This indicates that loop-erased walks are "thinner" than self-avoiding walks. (There are other heuristic arguments for this.)

- For $d = 2$, Kenyon used an analogy with dimers and a calculation using conformal invariance to show that

$$\nu_2 = 4/5.$$

- LSW showed later that the scaling limit of loop-erased walk for $d = 2$ is SLE with $\kappa = 2$. This is the value such that the paths have fractal dimension $5/4$.
- For $d = 3$ the exponent is not known exactly and it is not known if it ever will. Wilson did numerics recently suggesting $\nu_3 = .6157 \cdots$.

LAPLACIAN RANDOM WALK

- The loop-erased random walk (LERW) is equivalent to a model called the Laplacian random walk
- We will give the definition in the transient ($d \geq 3$) case. There is a similar definition for $d = 2$.
- Let $\hat{S}_0, \hat{S}_1, \ldots$ denote the loop-erased walk and let $\omega_n = [\hat{S}_0, \ldots, \hat{S}_n]$.
- The LERW is a non-Markovian process, but we can still give transition probabilities
- Let $p(z)$ be the probability that a simple random walk starting at z never visits ω_n. By definition $p(z) = 0$ if $z \in \omega_n$. Then,

$$\mathbb{P}\{\hat{S}_{n+1} = z \mid \omega_n\} \propto p(z).$$

- $p(z)$ is the unique discrete harmonic function (satisfying discrete Laplace equation) with boundary value 0 on ω_n and 1 at infinity.
- If $d = 2$, the boundary condition at infinity becomes $p(z) \sim \log |z|$.
- At least heuristically, we can write a similar expression for self-avoiding walks, except that $p(z)$ is replaced with the probability that a self-avoiding walk avoids ω_n. Since it should be easier for a SAW to avoid the set, we would guess that the SAW paths are not pushed to infinity as quickly. This is why they should be thicker than LERW paths.
- One can create a one-parameter family of processes by saying that

  $$\mathbb{P}\{\hat{S}_{n+1} = z \mid \omega_n\} \propto p(z)^b.$$

  This is most interesting in two and three dimensions where the fractal dimension is expected to vary with b.
- For $d = 2$, there is a one parameter family of continuous process, SLE.

SUMMARY

We have looked at three problems: SAW, intersection of random walks, loop-erased walk. There are many other models in "critical phenomenon" with these properties:

- An (upper) critical dimension above which behavior is trivial.
- Mean-field behavior at the critical value with logarithmic correction.
- Non mean-field behavior below the critical value with nontrivial "critical exponents"
- In dimensions strictly between 2 and the critical dimension, it may not be possible to give value exactly.
- In two dimensions, conformal invariance plays a major role. Here critical exponents can be determined exactly.
- Much progress has been made in the last twelve years in the two dimensional case. The second two lectures in this series will discuss this.