

RANDOM WALKS FROM STATISTICAL PHYSICS II

Two Dimensions and Conformal Invariance

Wald Lectures
2011 Joint Statistical Meetings

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August 3, 2011

CRITICAL PHENOMENA IN STATISTICAL PHYSICS

- ▶ Study systems at or near parameters at which a phase transition occurs
- ▶ Parameter $\beta = C/T$ where $T =$ temperature
- ▶ Large β (low temperature) — long range correlation.
- ▶ Small β (high temperature) — short range correlation
- ▶ Critical value β_c at which sharp transition occurs
- ▶ Belief: systems at criticality “in the scaling limit” exhibit fractal-like behavior (power-law correlations) with nontrivial critical exponents.
- ▶ The exponents depend on dimension.

TWO DIMENSIONS

- ▶ Belavin, Polyakov, Zamolodchikov (1984) — critical systems in two dimensions in the scaling limit exhibit some kind of “conformal invariance”.
- ▶ A number of theoretical physicists (Nienhuis, Cardy, Duplantier, Saleur, ...) made predictions about critical exponents using nonrigorous methods — conformal field theory and Coulomb gas techniques.
- ▶ Exact rational values for critical exponents — predictions strongly supported by numerical simulations
- ▶ While much of the mathematical framework of conformal field theory was precise and rigorous (or rigorizable), the nature of the limit and the relation of the field theory to the lattice models was not well understood.

SELF-AVOIDING WALK (SAW)

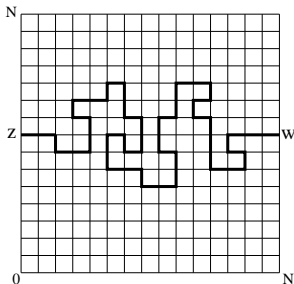
- ▶ Model for polymer chains — polymers are formed by monomers that are attached randomly except for a self-avoidance constraint.

$$\omega = [\omega_0, \dots, \omega_n], \quad \omega_j \in \mathbb{Z}^2, \quad |\omega| = n$$

$$|\omega_j - \omega_{j-1}| = 1, \quad j = 1, \dots, n$$

$$\omega_j \neq \omega_k, \quad 0 \leq j < k \leq n.$$

- ▶ Critical exponent ν : a typical SAW has diameter about $|\omega|^\nu$.
- ▶ If no self-avoidance constraint $\nu = 1/2$; for 2-d SAW Flory predicted $\nu = 3/4$.



Each SAW from z to w gets measure $e^{-\beta|\omega|}$. Partition function

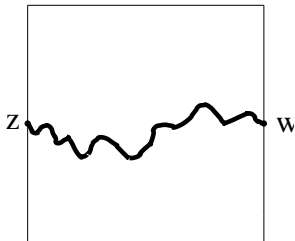
$$Z = Z(N, \beta) = \sum e^{-\beta|\omega|}.$$

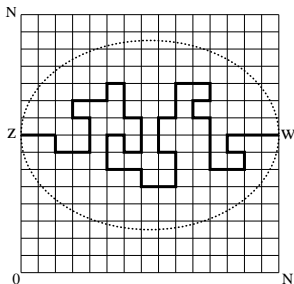
β small — typical path is two-dimensional

β large — typical path is one-dimensional

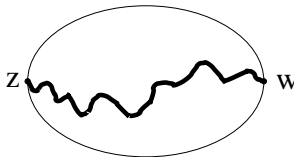
β_c — typical path is $(1/\nu)$ -dimensional

Choose $\beta = \beta_c$; let $N \rightarrow \infty$; divide by $Z(N, \beta)$ and hope to get a probability measure on curves connecting boundary points of the square.



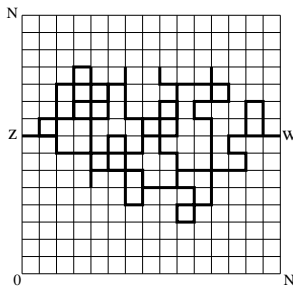


Similarly, if we fix $D \subset \mathbb{C}$, we can consider walks restricted to the domain D



Predict that these probability measures are conformally invariant.

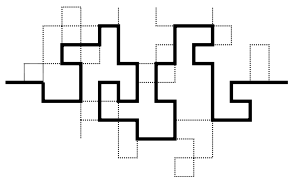
SIMPLE RANDOM WALK



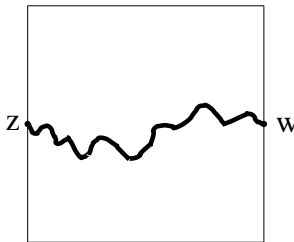
- ▶ Simple random walk — no self-avoidance constraint. Criticality: each walk ω gets weight $(1/4)^{|\omega|}$.
- ▶ Scaling limit is **Brownian motion** which is conformally invariant (Lévy).

LOOP-ERASED RANDOM WALK

Start with simple random walks and erase loops in chronological order to get a path with no self-intersections.

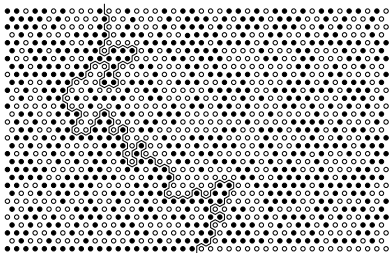


Limit should be a measure on paths with no self-intersections.



CRITICAL PERCOLATION

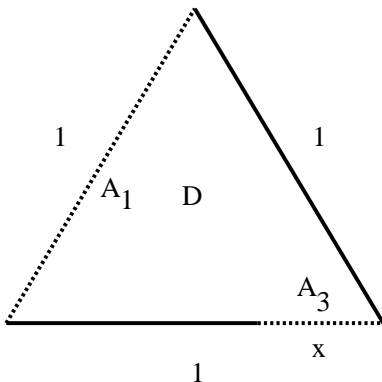
Color vertices of the triangular lattice in the upper half plane black or white independently each with probability $1/2$.



Put a boundary condition of black on negative real axis and white on positive real axis. The [percolation exploration process](#) is the boundary between black and white.

CARDY'S FORMULA

The probability of a black crossing at criticality (in the limit as lattice space goes to zero) was predicted to be a conformal invariant.



Cardy used conformal field theory to predict the value. it is most easily given for an equilateral triangle.

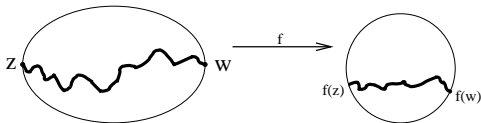
STRATEGY

- ▶ Make precise the conformal invariance assumption and other properties expected of scaling limit.
- ▶ Find all possible limits satisfying these assumptions.
- ▶ For a given discrete process, identify which is the correct limit.
- ▶ Prove the discrete converges to continuous.

Nonrigorous approaches in mathematical physics using conformal field theory had some of the properties of this strategy.

ASSUMPTIONS ON SCALING LIMIT

Probability measure $\mu_D(z, w)$ on curves connecting boundary points of a domain D .



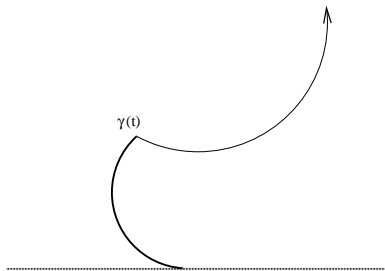
- ▶ **Conformal invariance:** If f is a conformal transformation

$$f \circ \mu_D(z, w) = \mu_{f(D)}(f(z), f(w)).$$

- ▶ For simply connected D , it suffices to know $\mu_{\mathbb{H}}(0, \infty)$ (Riemann mapping theorem).

- ▶ **Domain Markov property** Given $\gamma[0, t]$, the conditional distribution on $\gamma[t, \infty)$ is the same as

$$\mu_{\mathbb{H} \setminus \gamma(0, t]}(\gamma(t), \infty).$$



- ▶ Satisfied on discrete level by SAW, LERW, percolation exploration, Ising exploration ... (but not by simple random walk)

LOEWNER DIFFERENTIAL EQUATION

- ▶ Developed by Loewner in studying Bieberbach conjecture
- ▶ Let D be a simply connected proper domain in \mathbb{C} containing 0. Riemann mapping theorem says there is a unique conformal transformation

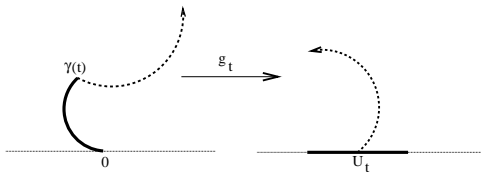
$$f : \mathbb{D} \rightarrow D$$

with $f(0) = 0, f'(0) > 0$.

- ▶ $f(z) = \sum_{j=1}^{\infty} a_j z^j$
- ▶ Bieberbach conjecture: $|a_j| \leq j a_1$
- ▶ Suffices to prove for slit domains $D = \mathbb{C} \setminus \gamma[t, \infty)$ where $\gamma(s) \rightarrow \infty$ as $s \rightarrow \infty$
- ▶ Differential equation studies how coefficients change as function of t
- ▶ Loewner proved $|a_3| \leq 3 a_1$. BC proved by de Branges (proof uses Loewner diff eq)

LOEWNER EQUATION IN UPPER HALF PLANE

- ▶ Let $\gamma : (0, \infty) \rightarrow \mathbb{H}$ be a simple curve with $\gamma(0+) = 0$ and $\gamma(t) \rightarrow \infty$ as $t \rightarrow \infty$.
- ▶ $g_t : \mathbb{H} \setminus \gamma(0, t] \rightarrow \mathbb{H}$



- ▶ Can reparametrize if necessary so that

$$g_t(z) = z + \frac{2t}{z} + \dots, \quad z \rightarrow \infty$$

- ▶ g_t satisfies

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Moreover, $U_t = g_t(\gamma(t))$ is continuous.

(Schramm) Suppose γ is a random curve satisfying conformal invariance and Domain Markov property. Then U_t must be a random continuous curve satisfying

- ▶ For every $s < t$, $U_t - U_s$ is independent of $U_r, 0 \leq r \leq s$ and has the same distribution as U_{t-s} .
- ▶ $c^{-1} U_{c^2 t}$ has the same distribution as U_t .

Therefore, $U_t = \sqrt{\kappa} B_t$ where B_t is a standard (one-dimensional) Brownian motion.

The (chordal) Schramm-Loewner evolution with parameter κ (SLE_κ) is the solution obtained by choosing $U_t = \sqrt{\kappa} B_t$.

(Rohde-Schramm) Solving the Loewner equation with a Brownian input gives a random curve.

The qualitative behavior of the curves varies greatly with κ

- ▶ $0 < \kappa \leq 4$ — simple (non self intersecting) curve
- ▶ $4 < \kappa < 8$ — self-intersections (but not crossing); not plane-filling
- ▶ $8 \leq \kappa < \infty$ — plane-filling

(Beffara) For $\kappa < 8$, the Hausdorff dimension of the paths is

$$1 + \frac{\kappa}{8}.$$

The fundamental tools for studying *SLE* are those of **stochastic calculus** (Itô integral and formula, martingales, Girsanov transformation)

For which κ does *SLE* have double points?

Equivalent to ask, for which κ does *SLE* hit the real line?

Let $x > 0$ and $X_t = X_t(x) = g_t(x) - U_t$. Then *SLE* hits $[x, \infty)$ if and only if X_t reaches zero in finite time. X_t satisfies

$$dX_t = \frac{2}{X_t} dt + \sqrt{\kappa} dB_t.$$

Bessel equation. Well known that X_t reaches zero if and only if $\kappa > 4$.

What is the probability that $z = re^{i\theta}$ lies to the left of the SLE_κ curve?

- ▶ Function only of θ , $f(\theta)$ with $f(0) = 0, f(\pi) = 1$.
- ▶ Let M_t be the probability given $\gamma(0, t]$. Then $M_t = f(\arg(g_t(z) - U_t))$ is a martingale, i.e., if $s < t$,

$$\mathbb{E}[M_t \mid \gamma(0, s)] = M_s.$$

- ▶ Itô's formula gives a differential equation for f

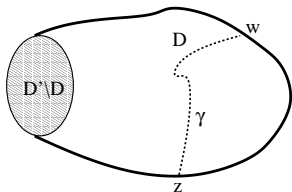
$$f''(\theta) + 2(1 - 2a) f'(\theta) \cot \theta = 0, \quad a = 2/\kappa.$$

▶

$$f(\theta) = \int_0^\theta \frac{c \, dx}{\sin^{2-4a} x}.$$

WHICH κ FOR WHICH MODEL?

How does the measure of a path change when we perturb the boundary? ($\kappa \leq 4$)



$$1\{\gamma \subset D\} \exp \left\{ \frac{c}{2} \Lambda(D'; \gamma, D' \setminus D) \right\}$$

$J(D'; \gamma, D' \setminus \gamma)$ is a conformal invariant given by the measure (using a certain [Brownian loop measure](#)) of loops in D' that intersect both $D' \setminus D$ and γ .

- ▶ For SAW, perturbing the domain does not change the measure. Expect $c = 0$.
- ▶ For LERW, shrinking the domain loses some simple random walks whose loop-erasure is γ . Expect $c < 0$.
- ▶ c is the **central charge** which is the parameter used in conformal field theory to distinguish models.

$$c = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}, \quad \kappa = \frac{(13 - c) \pm \sqrt{(13 - c)^2 - 144}}{3}.$$

Each $c < 1$ corresponds to two values κ, κ' with $\kappa\kappa' = 16$.
 $c = 1$ corresponds to the double root $\kappa = \kappa' = 4$.

BROWNIAN PATHS

- ▶ The first major problem solved with SLE was the Brownian intersection exponents. One example goes back to a conjecture of Mandelbrot. Consider a “Brownian island” formed by taking a Brownian motion (random walk), conditioning to end at the same place it began, and filling in the bounded holds. Mandelbrot noted that simulations of this coastline indicated that it should have dimension $4/3$.
- ▶ L. showed that the dimension could be calculated in terms of a particular value of the intersection exponents.
- ▶ L, Schramm, and Werner showed how the **locality** property of SLE_6 could be used to calculate this exponent and verified Mandelbrot’s conjecture.

CRITICAL PERCOLATION

- ▶ Smirnov proved that the scaling limit of critical percolation on the triangular lattice satisfies Cardy's formula. Using this and the work of LSW he established that the scaling limit of percolation is SLE_6 .
- ▶ $\kappa = 6$ is the only value of κ for which SLE satisfies the locality property — something that would be expected of the scaling limit of percolation. (Schramm had already identified $\kappa = 6$ as the correct candidate for the scaling limit.)
- ▶ Smirnov's proof is particular to the triangular lattice. It is an open problem to establish this limit for other lattices, e.g., critical bond or site percolation in \mathbb{Z}^2

LOOP-ERASED RANDOM WALK

- ▶ LSW proved that the scaling limit of LERW is SLE_2 (Schramm had already identified $\kappa = 2$ as the appropriate candidate and the physics literature had $c = -2$.)
- ▶ In particular, the paths has dimension $5/4$. Rick Kenyon had previously used a relationship with domino tilings to prove a discrete analogue of this statement.
- ▶ The LERW is closely related to the [uniform spanning tree](#). LSW shows that the scaling limit of the uniform spanning tree is SLE_8 .
- ▶ One can obtain information about the loop-erased random walk directly from the SLE_2 result (Masson).

GAUSSIAN FREE FIELD

- ▶ Schramm and Sheffield have shown that the level lines of the blue Gaussian free field correspond to $\kappa = 4$ (as does a similar model called the [harmonic explorer](#).)
- ▶ To see a realization of the Gaussian free field and some level lines, see the picture on the front page of the notes for today's talks.

ISING MODEL

- ▶ The Ising model is a model for ferromagnets.
- ▶ It is one of a large class of models called Potts models which are related to random cluster models.
- ▶ The scaling limit of the interfaces for the Ising model should satisfy conformal invariance and the domain Markov property — hence SLE_κ for some κ . In fact $\kappa = 3, c = 1/2$.
- ▶ Smirnov has established the scaling limit in the case of Ising model on the square lattice. There are also results for the related random cluster model.

SELF-AVOIDING WALK

- ▶ The scaling limit for SAW should have $c = 0$ (restriction property). Assuming that the limit is on simple curves ($\kappa \leq 4$), this gives $\kappa = 8/3$.
- ▶ $SLE_{8/3}$ curves have dimension $4/3$ which gives the prediction $\nu = 3/4$.
- ▶ Simulations (Tom Kennedy) strongly support the conjecture that the limit is $SLE_{8/3}$.
- ▶ It is still an open question to prove that the scaling limit of SAW is $SLE_{8/3}$. (In fact, almost nothing is known rigorously about SAWs in two dimensions!)

COMMENTS AND SOME CURRENT RESEARCH AREAS

- ▶ Overall *SLE* has been well received by the theoretical physics community even though most of the exponents predicted by *SLE* had already been predicted (or at least conjectured) in the physics community. There has already been some progress and more will occur using *SLE*, the Brownian loop measure, and other similar conformally invariant objects to construct fields.

- ▶ *SLE* in simply connected domains is well understood because conformal invariance and the Markov property determine the process (up to one parameter). Simply connected domains have the property that if one slits them from a boundary point, the slit domain is conformally equivalent to the original domain. This is not true for non-simply connected domains and conformal invariance and domain Markov property do not determine the measures.

- ▶ *SLE* describes a path or interface by giving it a random dynamics. However, the path is not formed according to these dynamics — rather the description of the path using the Loewner equation is only a way of collecting information about conditional probabilities as we explore parts of the path/domain.
- ▶ This is why some “obvious” results are difficult to prove. Zhan proved that $SLE_{\kappa}(\kappa \leq 4)$ from z to w in a domain D is the same as the path from w to z . This is immediate on the lattice level for most of the models we are considering but is not easy to prove for *SLE* directly.
- ▶ This talk has focused on *SLE*. There are a lot of other exciting aspects to studying conformally invariant systems in two dimensions.

WHAT ABOUT THREE DIMENSIONS?

- ▶ The use of conformal invariance to study these systems is essentially a two-dimensional phenomenon (although there are some applications to four-dimensional questions). It seems much harder to analyze the very important case of three dimensions.
- ▶ SAW and LERW are expected to give interesting, nontrivial, random fractal paths in three dimensions.
- ▶ Interfaces as in percolation or Ising model become random surfaces rather than random curves.
- ▶ In two dimensions, critical exponents tend to take on rational values. There is no reason to believe that this is true in three dimensions.
- ▶ For example, for SAW, Flory conjectured that a typical SAW of n steps in three dimensions would have diameter n^ν where $\nu = 3/5$. This is no longer believed to be the exact value. Numerical simulations suggest $\nu = .588 \dots$