RANDOM WALKS FROM STATISTICAL PHYSICS III

What Do We Know about the Schramm-Loewner Evolution?

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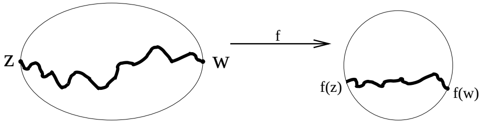
REVIEW OF CHORDAL SLE

Probability measure $\mu_D(z, w)$ on curves connecting boundary points of a domain D.



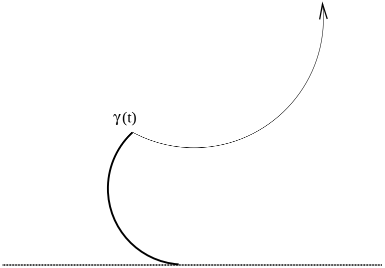
- **Conformal invariance**: If f is a conformal transformation

$$f \circ \mu_D(z, w) = \mu_{f(D)}(f(z), f(w)).$$

- For simply connected D, it suffices to know $\mu_{\mathbb{H}}(0, \infty)$ (Riemann mapping theorem).

- **Domain Markov property** Given $\gamma[0,t]$, the conditional distribution on $\gamma[t,\infty)$ is the same as

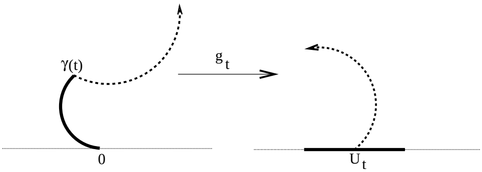
$$m_{\mathbb{H}\setminus\gamma(0,t]}(\gamma(t),\infty).$$



- Satisfied on discrete level by SAW, LERW, percolation exploration, Ising exploration ... (but not by simple random walk)

LOEWNER EQUATION IN UPPER HALF PLANE

- Let $\gamma : (0, \infty) \to \mathbb{H}$ be a simple curve with $\gamma(0+) = 0$ and $\gamma(t) \to \infty$ as $t \to \infty$.
- $g_t : \mathbb{H} \setminus \gamma(0, t] \to \mathbb{H}$



- Can reparametrize if necessary so that

$$g_t(z) = z + \frac{at}{z} + \cdots, \qquad z \to \infty$$

- g_t satisfies

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Moreover, $U_t = g_t(\gamma(t))$ is continuous.

(Schramm) Suppose γ is a random curve satisfying conformal invariance and Domain Markov property. Then U_t must be a random continuous curve satisfying

- For every $s < t$, $U_t - U_s$ is independent of $U_r, 0 \leq r \leq s$ and has the same distribution as U_{t-s}.
- $c^{-1}\, U_{c^2 t}$ has the same distribution as U_t.

Therefore, U_t is a driftless Brownian motion. We can make the variance 1 by choosing a appropriately.

The *(chordal) Schramm-Loewner evolution with parameter* κ (SLE_κ) is the solution obtained by choosing $U_t = B_t$ and $a = 2/\kappa$.

(Rohde-Schramm) Solving the Loewner equation with a Brownian input gives a random curve.

The qualitative behavior of the curves varies greatly with κ

- $0 < \kappa \leq 4$ — simple (non self intersecting) curve
- $4 < \kappa < 8$ — self-intersections (but not crossing); not plane-filling
- $8 \leq \kappa < \infty$ — plane-filling

(Beffara) For $\kappa < 8$, the Hausdorff dimension of the paths is

$$1 + \frac{\kappa}{8}.$$

- $\kappa = 2$ loop-erased random walk [L,Schramm, Werner]

- $\kappa = 8/3$ self-avoiding walk. Strong theoretical reasoning why this is correct limit but scaling limit not proved. [LSW]

- $\kappa = 3, \kappa = 16/3$ Ising model and related cluster model interfaces. [Smirnov]

- $\kappa = 4$ interfaces of Gaussian free field and related harmonic explorer. [Schramm, Sheffield]

- $\kappa = 6$ percolation interfaces. [Smirnov, LSW]

- $\kappa = 8$ Peano curves arising from uniform spanning trees. [LSW]

QUESTIONS

How do we describe the evolution of a random curve $\gamma(t)$ which is fractal and has self-repulsion?

- Fractal dimension d of the path
- Hölder continuity of paths in given parametrization.
- Can we find a parametrization of the path that is Hölder continuous of order $\alpha < 1/d$?
- Can we find an analogue of the "derivative" of the path? In two dimensions, the local behavior of the conformal map is the key.
- Can we describe the exceptional points?

FINDING THE DIMENSION OF PATHS FOR $\kappa < 8$

- The first step in computing the dimension is to find $d, G(z)$ such that

$$\mathbf{P}\{\mathrm{dist}(z, \gamma(0,\infty)) \leq \epsilon\} \asymp G(z)\, \epsilon^{2-d}.$$

(Expected number of disk of radius ϵ needed to cover a domain of area 1 is of order ϵ^{-d}.)

- Using stochastic calculus, Rohde and Schramm (essentially) showed that the only possibility is

$$d = 1 + \frac{\kappa}{8}$$

$$G(re^{i\theta}) = r^{d-2} \sin^u \theta, \quad u = 8\kappa + \frac{1}{8\kappa} - 2 > 0.$$

They showed this gives an upper bound on the dimension.

- Easier to consider

$$\Upsilon_t = \Upsilon_t(z) = \frac{Y_t}{|g_t'(z)|}.$$

This is (1/2 times) the conformal radius of z in H_t, the unbounded component of $\mathbb{H} \setminus \gamma(0,t]$. By the Koebe-(1/4) Theorem or Schwarz lemma,

$$\Upsilon_t \asymp_2 \mathrm{dist}(z, \gamma(0,t] \cup \mathbb{R}).$$

$\Upsilon = \Upsilon_\infty$.

- Using another technique, the Girsanov theorem, one can show as $\epsilon \to 0$,

$$\mathbf{P}\{\Upsilon < \epsilon\} \sim c_*\, G(z)\, \epsilon^{2-d}.$$

- Given the local martingale

$$M_t(z) = |g_t'(z)|^{2-d}\, G(Z_t(z))$$

one can use Girsanov theorem to study paths weighted by the local martingale.
- If $\kappa < 8$, this gives a measure on paths corresponding to chordal SLE_κ conditioned to go through a particular point. This is called two-sided radial SLE_κ.
- Although this is conditioning on an event of probability zero, one can show that it makes sense and corresponds to limits of appropriate measure.
- The term two-sided radial comes from the fact that the path at interior point z has two interacting paths coming out, the past and the future.

- Beffara proved a more difficult two-point estimate

$$\mathbf{P}\{\Upsilon(z) < \epsilon, \Upsilon(w) < \epsilon\} \leq c\, \epsilon^{2-d} \left(\frac{\epsilon}{|z-w|} \right)^{2-d},$$

  at least for z, w restricted to a bounded domain bounded away from $\mathbb{R}$.
- With this estimate one can use standard techniques (Frostman's lemma) to show that Hausdorff dimension of path is d.
- One can also prove the Hausdorff dimension result using the reverse Loewner flow .

- L-Werness proved the existence of a "two-point" Green's function, that is, a function $G(z,w)$ such that

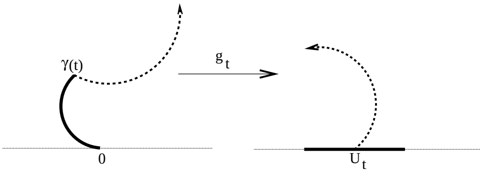
$$\mathbf{P}\{\Upsilon(z) < \epsilon, \Upsilon(w) < \delta\} \sim c_*^2\, G(z,w)\, \epsilon^{2-d}\, \delta^{2-d}, \quad \epsilon, \delta \to 0.$$

- The function G is given implicitly in terms of an expectation with respect to two-sided radial SLE_κ. Unfortunately, the proof does not give an explicit form for $G(z,w)$.
- One can write a PDE that $G(z,w)$ must satisfy.
- This also gives a two-point local martingale

$$G(z,w) = |g_t'(z)|^{2-d}\, |g_t'(w)|^{2-d}\, G(Z_t(z), Z_t(w)).$$

NATURAL PARAMETRIZATION OR LENGTH

- Suppose $\gamma : [0,1] \to \mathbb{R}^2$ is a curve such that for each $a < b$, $\dim_h(\gamma[a,b]) = d$.
- There are a number of properties that can describe a d-dimensional parametrization of a curve:
  - Hölder continuous of order $\alpha < 1/d$. Roughly speaking,
  $$|\gamma(s) - \gamma(t)| \approx |s - t|^{1/d}.$$
  - If $f : D \to f(D)$ is a conformal transformation, then the time needed to traverse $f(\gamma[0,t])$ is
  $$\int_0^t |f'(\gamma(s))|^d \, ds.$$
- Brownian motion in $\mathbb{C}$ has a 2-dimensional parametrization.
- The capacity parametrization of SLE is not a d-dimensional parametrization.



Let

$$\tilde{\gamma} = g_t\left(\gamma[t, t + \Delta t]\right).$$

- The cap param is defined so that the time to traverse $\tilde{\gamma}$ is Δt, the same as the time to traverse $\gamma[t, t + \Delta t]$
- In natural param, the time to traverse $\tilde{\gamma}$ should be

$$\int_0^{\Delta t} \left|g_s'(\gamma(s))\right|^d \, ds.$$

- Important to understand $|g_t'|$ or $|(g_t^{-1})'|$.

SOME POSSIBLE (UNPROVED) DEFINITIONS

$$\Theta_t(D) = \text{time to traverse } \gamma[0,t] \cap D, \quad \Theta_t = \Theta_t(\mathbb{H}).$$

- $$\Theta_t = \lim_{n\to\infty} \sum_{j=1}^{n} \left| \gamma(\frac{j}{n}t) - \gamma(\frac{j-1}{n}t) \right|^d.$$
- Minkowski measure of Θ_t
- Hausdorff measure with appropriate gauge of Θ_t

RIGOROUS DEFINITION (L - Sheffield)

- Expect (up to multiplicative constant)

$$\mathbf{E}\left[\Theta_\infty(D)\right] = \int_D G(z)\, dA(z).$$

- $$\mathbf{E}[\Theta_\infty(D) \mid \mathcal{F}_t] = \Theta_t(D) + \mathbf{E}[\Theta_\infty(D) - \Theta_t(D) \mid \mathcal{F}_t]$$

- Using conformal invariance, one can show that the last term equals

$$\Psi_t(D) = \int_D M_t(z) = \int_D |g_t'(z)|^{2-d}\, G(Z_t(z)).$$

- $N_t = \mathbf{E}[\Theta_\infty(D) \mid \mathcal{F}_t]$ is a martingale. We have written

$$\Psi_t(D) = N_t - \Theta_t(D),$$

and $\Theta_t(D)$ is increasing.

- Doob-Meyer decomposition. $\Theta_t(D)$ is the unique increasing process that makes

$$\Psi_t(D) + \Theta_t(D)$$

a martingale.
- Technical difficulty:

$$\Psi_t(D) = \int_D M_t(z)\, dA(z)$$

The $M_t(z)$ are local martingales (but not martingales). This makes $\Psi_t(D)$ a supermartingale, but could it possibly be a local martingale (in which case $\Theta_t(D)$ is trivial)?
- Moment estimates needed to show this $\Psi_t(D)$ is a "strict supermartingale" (not a local martingale).

- The proof of the Doob-Meyer decomposition gives formula

$$\Theta_1 = \lim_{n\to\infty} \sum_{j=1}^{n} |f'_{\frac{j-1}{n}}(z)|^d \, \phi(z\sqrt{n})\, G(z)\, dA(z).$$

where ϕ is a bounded function (almost of compact support) and $f_t(z) = g_t^{-1}(z + U_t)$.

Theorem

(L) Choosing

$$\phi(z/\sqrt{n})\, dA(z) = n^{-1/2}\, \delta_{i/\sqrt{n}}$$

there is tightness. This gives subsequential limits but does not prove existence of the limit.

- This gave an alternative proof of Beffara's theorem on Hausdorff dimension of paths which did not rely on the difficult estimate. (However, it had other difficult estimates!)

Theorem

(L, Sheffield) At least for $\kappa < 4(7-\sqrt{33}) = 5.021\cdots$*, the natural parametrization defined as above exists.*

- Proof does not need Beffara's estimate. Also, bounds on Hölder continuity (of nat param with respect to cap param) are given by giving a second moment estimate. The argument should work for all $\kappa < 8$ with corresponding bounds, but this has not been done.

Theorem

(L, Wang Zhou) The natural parametrization exists for all $\kappa < 8$*.*

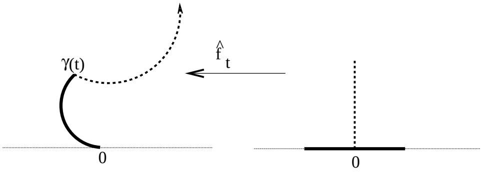
- Proof uses Beffara's estimate and does not give the Hölder continuity bounds. It uses a two-point martingale given by the two-point Green's function in L-Werness.

TIP MULTIFRACTAL SPECTRUM

- Consider the SLE_κ path $\gamma(t)$ with conformal maps g_t, driving function U_t and

$$f_t(z) = g_t^{-1}(z + U_t).$$

- The analogue of the "derivative" at g is the behavior of the derivative of $g_t'(z)$ for z near $\gamma(t)$.
- More convenient to consider behavior of $f_t'(z)$ for z near the origin.



MOMENTS FOR DERIVATIVES OF f

- To understand the "multifractal behavior" of $f_1'(iy)$ for small y, one studies the moment

$$\mathbf{E}[|f_1'(iy)|^\lambda] = \mathbf{E}\left[|f_{1/y^2}'(i)|^\lambda\right].$$

- The tool is stochastic calculus and martingales. To state the results we introduce some notation. For a range of λ,

$$r(\lambda) = 2a + 1 - \sqrt{(2a+1)^2 - 4a\lambda},$$

$$\zeta(\lambda) = \lambda - \frac{r}{2a}, \quad \beta(\lambda) = -\zeta'(\lambda).$$

$$\zeta'(\lambda) = 1 - \frac{1}{\sqrt{(2a+1)^2 - 4a\lambda}}, \quad \zeta'(\lambda_c) = -1,$$

$$\rho(\lambda) = -\lambda\beta + \zeta.$$

- For

$$r = 1, \ \ \lambda = d, \ \ \zeta = 2 - d, \ \ \frac{\beta}{2} = d - \frac{3}{2} = \frac{1}{4a} - \frac{1}{2}$$

$$\mathbf{E}\left[|f_1'(i/\sqrt{n})|^d\right] = \mathbf{E}\left[|f_n'(i)|^d\right] \asymp n^{\frac{d}{2}-1}$$

$$\mathbf{P}\{|f_1'(i/\sqrt{n})| \approx n^{d-\frac{3}{2}}\} \approx n^{-(d^2-2d+1)}$$

- The "dimension" of set of *times* $t \in [0,1]$ with $|\hat{f}_t'(i/\sqrt{n})| \approx n^{d-\frac{3}{2}}$ equals

$$1 - (d^2 - 2d + 1) = d(2-d).$$

- This is a set of dimension strictly less than one. However, these are the points on which the natural parameterization lives.
- From this we see that the natural param is singular with respect to the capacity param.
- The dimension of *points* $\gamma(t)$ satisfying the relation is above is d, the full dimension of the path.

Theorem

(L-Johansson Viklund)
Let Λ_β denote the set of t such that

$$|\hat{f}_t'(iy)| \approx y^{-\beta}, \quad y \to 0+ .$$

Then if $\rho \leq 2$,

$$\dim_h(\Lambda_\beta) = \frac{2-\rho}{2},$$
$$\dim_h[\gamma(\Lambda_\beta)] = \frac{2\dim_h(\Lambda_\beta)}{1-\beta} = \frac{2-\rho}{1-\beta}.$$

- This is a statement about the almost sure Hausdorff dimension and not just a statement about expectations. This requires second moment estimates.

- If B_t is a two-dimensional Brownian motion and A is any (perhaps random) Borel subset of $[0,1]$, then

$$\dim_h [B(A)] = 2 \dim_h(A).$$

  For exceptional sets of SLE we don't get a simple rule like this.

- Instead we get the rule

$$\dim_h[\gamma(\Lambda_\beta)] = \frac{2}{\beta - 1} \dim_h[\Theta_\beta].$$

  This follows (roughly) from the fact that on the set Λ_β,

$$|\hat{f}_t'(\epsilon i)| \approx \epsilon^{-\beta}.$$

- Similar ideas allow one to prove the following.

Theorem

(L-Johansson Viklund) If γ is SLE$_\kappa$ (in capacity parametrization) with probability one $\gamma(t), \epsilon \le t \le 1$ is Hölder continuous of order $\alpha < \alpha_*$ but not $\alpha > \alpha_*$ where

$$\alpha_* = 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{8+\kappa}}.$$

- $\alpha_* > 0$ except if $\kappa = 8$.
- This can be conjectured from moment estimates of Rohde and Schramm and one direction was proved by Joan Lind. The new direction requires second moment estimates.

SOME CLOSING COMMENTS

- The theory of SLE is a very powerful tool which allows for detailed analysis of conformally invariant fractals. It is a very active area of research
- The two-dimensional world of "critical phenomena" is particularly nice because of conformal invariance.
- There are also very interesting models of random curves in three dimensions, such as the self-avoiding walk and the loop-erased walk, but we do not have tools to analyze them.
- Interfaces such as in percolation are also interesting but these are now random surfaces.
- It would be good to construct continuous models for three-dimensional random fractals with self-repulsion even if we cannot show that they are the limits of known processes.