ALMOST SURE MULTIFRACTAL SPECTRUM FOR THE TIP OF AN SLE CURVE

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Abstract. The tip multifractal spectrum of a two-dimensional curve is one way to describe the behavior of the uniformizing map of the complement near the tip. We give the multifractal spectrum for a Schramm-Loewner evolution (SLE) curve and we prove that the spectrum is valid with probability one.

1. Introduction

The chordal Schramm-Loewner evolution (SLE_κ) is a one parameter family of probability measures on curves $\gamma : [0,\infty) \to \overline{\mathbb{H}}$. It was invented by Oded Schramm [17] as a candidate for the scaling limit of lattice statistical physics models in two-dimensional that satisfy conformal invariance in the limit. See [7, 8, 18] for surveys. In this paper we consider the behavior of the path at a point $\gamma(t)$. Since the curves are fractals, one cannot make sense of derivatives. Instead, one considers the behavior of $|g_t'(z)|$ for z near $\gamma(t)$ where g_t is a uniformizing map from the complement of the curve to the upper half plane.

For technical reasons, it is often easier to consider $|f_t'(\gamma(t)+w)|$ where $f_t = g_t^{-1}$ and w is near the origin. The starting point is estimation of moments of the derivative; this was started in [16] and extended in many places, e.g., [2, 5, 6, 9, 14]. In order to get almost sure results, one needs second moment estimates. The ideas for that appear in [9], and these ideas are also important in understanding the "natural parametrization" of SLE curves, see [10].

1.1. **Tip multifractal spectra.** Suppose $\tilde\gamma : (-\infty,\infty) \to \mathbb{C}$ is a simple curve with $\gamma(t) \to \infty$ as $t \to \pm\infty$. For each t, let $D_t = \mathbb{C} \setminus \tilde\gamma(-\infty, t]$, which is a simply connected domain whose boundary contains $\tilde\gamma(t)$ and ∞. The *(nontangential) tip multifractal spectrum* which we describe in this section is one way to describe the behavior of harmonic measure in D_t near $\tilde\gamma(t)$ for different values of t.

Suppose z is a boundary point of a simply connected domain D. We say that z is *accessible (in D by η)* if $\eta : [0,1] \to \mathbb{C}$ is a simple curve with $\eta(0) = z, \eta(0,1] \subset D$. If z is accessible by η, let h be a conformal

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transformation of D onto $\mathbb{C} \setminus (-\infty, 0]$ with $h(z) = 0$. By $h(z) = 0$, we mean $h(\eta(0+)) = 0$. (In the previous paragraph, $\tilde{\gamma}(t)$ is accessible in D_t by the curve $\eta^{(t)}(s) = \tilde{\gamma}(t+s)$. For endpoints of "slit" domains like $\tilde{\gamma}(t)$ in D_t, there is only one possible meaning for $h(\tilde{\gamma}(t)) = 0$, but for general D a boundary point z might be approached from different directions that correspond to different values of $h(z)$. The curve η specifies a particular direction.) Let $g(z) = i\sqrt{h(z)}$, where the branch of the square root is chosen so that $\sqrt{1} = 1$. Then g is a conformal transformation of D onto the upper half plane $\mathbb{H}$ with $g(0) = 0$. The map g is not unique, but if $\tilde{g}$ is another such map,

$$\tilde{g}(z) = T[g(z)],$$

where T is a Möbius transformation of $\mathbb{H}$ fixing 0. Similarly h is not unique.

Let $\eta^*(s) = g(\eta(s))$. Then $\eta^* : (0,1] \to \mathbb{H}$ with $\eta^*(0+) = 0$. If $\eta_1^* : (0,1] \to \mathbb{H}$ is another curve with $\eta_1^*(0+) = 0$, and $\eta_1(t) = g^{-1}(\eta_1^*(s))$, then $\eta_1(0+) = z$ and z is accessible by η_1. (This uses the fact that the curve η exists. If z is an inaccessible boundary point, then $\lim_{s\to 0+}(\eta_1(s))$ does not exist.) We say that η^* satisfies a *weak cone condition* if there is a subpower function (see Section 2.1) ψ such that for all $s > 0$,

$$|\operatorname{Re}[\eta^*(s)]| \leq \operatorname{Im}[\eta^*(s)]\, \psi\left(\frac{1}{\operatorname{Im}[\eta^*(s)]}\right),$$

and we say that η is weakly nontangential if $g \circ \eta$ satisfies a weak cone condition. It is not difficult to see that this definition is independent of the choice of g. One example of a nontangential curve for D is

$$\eta(s) = g^{-1}(si), \quad 0 < s \leq 1.$$

We will use this curve to define the tip multifractal spectrum but the definition will be the same for any nontangential curve.

We make the following assumptions and comments. Let $f = g^{-1}$.

- f is a conformal transformation of $\mathbb{H}$ onto D.
- Let $\eta(s) = f(is)$. There exists z such that $\eta(0+) = z$.
- Since $\eta(s), s > 0$ is a simple curve, the length of $\eta(0, s]$ is given by

  $$v(f; s) := \int_0^s |f'(iy)|\, dy. \tag{1}$$

  A sufficient condition for the existence of a limiting $z = \eta(0+)$ is that $v(f; 0+) = 0$ which is equivalent to

  $$v(f; t) < \infty, \quad t > 0.$$
- We can write $f(w) = F(-w^2)$ where $F : \mathbb{C} \setminus (-\infty, 0] \to D$ with $F(0) = z$. Then $F^{-1}(\eta(0, s]) = [0, s^2]$. In particular, the length of $F^{-1}(\eta(0, s])$ is s^2, and the length of $f^{-1}(\eta(0, s])$ is s.

We will say that the *(nontangential) scaling exponent* at boundary point z is θ if

$$v(f; s) \approx^* s^{2\theta}, \quad y \to 0+.$$

In particular, if $D = \mathbb{C} \setminus (-\infty, t]$, then the scaling exponent at t equals 1. (See (2) for the definition of $\approx^*$.) More generally, if γ is differentiable at t, then $\theta = 1$ at t. We say that the scaling exponent is the interval $[\theta_-, \theta_+]$ if corresponding statements with liminf and limsup hold. The Beurling estimates (see Lemma 2.6) implies that $\theta_+ \leq 1$. The scaling exponent is closely related to the behavior of $f'(iy)$ as $y \to 0+$. Indeed, if $|f'(iy)| \approx^* y^{-\beta}$ for some $\beta \leq 1$, then (see Proposition 2.7)

$$v(f; y) \approx^* y^{1-\beta}, \qquad \theta = \frac{1-\beta}{2}.$$

Although the definition of $v(f; y)$ depends on the choice of conformal map f, it is not hard to see that the scaling exponent θ is independent of the choice.

Returning to the curve $\tilde{\gamma}$, we can study T_θ, the set of t such that the scaling exponent of D_t at $\tilde{\gamma}(t)$ equals θ. The tip multifractal spectrum can be defined to be either of two functions:

$$\theta \longmapsto \dim_h(T_\theta), \qquad \theta \longmapsto \dim_h(\tilde{\gamma}[T_\theta]),$$

where $\dim_h$ denotes Hausdorff dimension. The first function depends on the choice of parametrization of $\tilde{\gamma}$ and the second is independent of parameterization. We could also define liminf and limsup versions of this. The main goal of this paper is to compute the tip multifractal spectrum for chordal SLE. For technical convenience, we will use an alternative definition in terms of the behavior of $|f'(iy)|$ as $y \to 0+$ and we will use β rather than θ as our variable.

1.1.1. *Harmonic measure spectrum.* The tip multifractal spectrum is closely related to decay of harmonic measure at the tip of the curve. Suppose $\gamma = \gamma(t)$ is a curve in $\overline{\mathbb{H}}$ with $\gamma(0+) \in \mathbb{R}$. Let H_t be the unbounded connected component of $\mathbb{H} \setminus \gamma[0, t]$. One way to define the multifractal spectrum of harmonic measure at the tip is as the function

$$\alpha \longmapsto \dim_h(\gamma[T^{\text{hm}}_\alpha]),$$

where

$$T^{\text{hm}}_\alpha = \{t : \text{hm}(B(\gamma(t), r)) \approx^* r^\alpha, \, r \to 0+\}.$$

Here $\text{hm}(\cdot) = \text{hm}(z_0, \cdot, H_t)$, is the harmonic measure evaluated at a suitable point that could depend on t. We will both this defintion and a slightly different (nonequivalent) definition that is more closely related to the tip spectrum. See Section 2.3 for precise definitions.

Harmonic measure spectra for planar domains have been studied extensively in the physics and mathematics literature. In the case of the paths of Brownian motion, the spectrum is determined by the intersection exponents which in two dimensions where established in [11, 12, 13]. Duplantier derived, using non-rigorous quantum gravity methods, an explicit formula

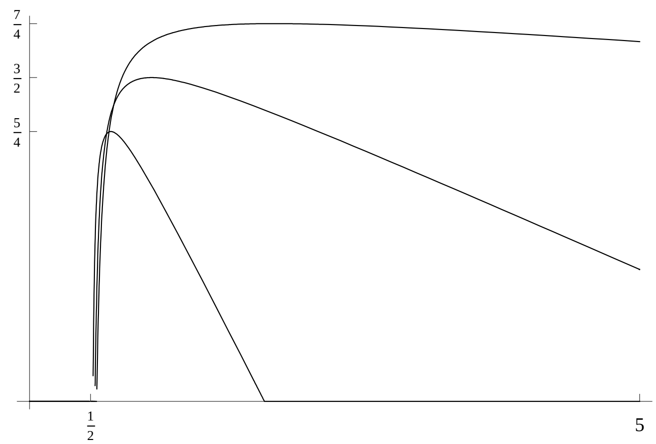

FIGURE 1. Multifractal spectrum of harmonic measure at the tip for $SLE(\kappa)$, $\kappa = 2, 4, 6$. The maximum is the dimension of the curve.

for the harmonic measure spectrum for the *bulk* of $SLE(\kappa)$, that is, for the SLE curve away from the tip [3], [4]. His formula which may be stated as

$$f_{\text{bulk}}(\alpha) = \alpha + \frac{(4+\kappa)^2}{8\kappa} - \left(\frac{(4+\kappa)^2}{8\kappa}\right)\left(\frac{\alpha^2}{2\alpha - 1}\right).$$

Beliaev and Smirnov [2] made a start to proving this result by establishing the average integral means spectrum (measuring the growth of $\int_{\partial\mathbb{D}} \mathbb{E}[|f_t'((1+r)e^{i\theta})|^\lambda]d\theta$ as $r \to 0+$) for the (whole-plane) SLE mapping. To get the multifractal spectrum from the integral means spectrum one can formally apply the so-called multifractal formalism (see, e.g., [15]) and find f_{bulk} by taking a Legendre transform of the average spectrum. This approach is believed to be valid for SLE, athough it has not been established in this case.

In this paper we prove a similar formula for a version the tip spectrum (see Section 2.3):

$$f_{\text{tip}}(\alpha) = \alpha\left(1 - \frac{4}{\kappa}\right) + \frac{(4+\kappa)^2}{8\kappa} - \left(\frac{\kappa}{8}\right)\left(\frac{\alpha^2}{2\alpha - 1}\right).$$

Although we modify the definition somewhat, we prove the stronger version of the theorem which states that for each α, with probability one this result holds. This is a stronger result than a result on the average integral mean spectrum. Indeed the analogue of the average integral mean spectrum result for our problem is the estimate (38) which is the starting point for the more intricate analysis.

1.2. **Outline of the paper.** We give an outline of the paper. The next section discusses some preliminary facts. After setting some notation about asymptotics in Section 2.1, the deterministic Loewner equation is discussed in Section 2.2. Much in this subsection is standard but we have included this in order to phrase the results appropriately for our purposes. Also, we want to separate estimates that deal only with the Loewner equation itself from those that are particular to SLE. In this subsection, there are three kinds of results: those that hold for all conformal maps of $\mathbb{H}$ for which we use the letter h; those that hold for all solutions of the chordal Loewner equation for which we use g_t and $f_t = g_t^{-1}$; and towards the end fact about solutions of the Loewner equation for driving functions that are weakly Hölder-(1/2). We also define the tip multifractal spectrum in this section.

The main theorem is not stated until Section 3 where the Schramm-Loewner evolution (SLE), i.e., the solution of the Loewner equation with a Brownian motion input, is discussed. From here on a value of the SLE parameter κ is fixed and a large number of κ-dependent parameters are defined. Although we do not discuss it directly, what we are doing is establishing the guess for the value of the multifractal spectrum in terms of the Legendre transform of a logarithmic moment generating function.

The basic proofs of the main theorem can be found in Section 4. This section is relatively short because it relies on estimates on the moments of the derivative some of which were established in [9, 5]; the necessary additions are proved in Section 5. Section 6 uses the *forward* Loewner flow to prove a result on the harmonic measure spectrum stated in Section 3.3. We warn the reader that some of the notation in Section 6 does not agree with that earlier and that the assumption $\kappa < 8$ is made there.

2. Preliminaries

2.1. **Notations.**

In order to avoid writing bulky expressions with ratios of logarithms, we will adopt the following notations.

We call a function $\psi : [0,\infty) \to (0,\infty)$ a *(positive) subpower function* if it is continuous, nondecreasing, and

$$\lim_{x\to\infty} x^{-u}\,\psi(x) = 0,$$

for all $u > 0$.

If f, g are positive we write

$$f(y) \approx^* g(y), \qquad y \to 0+, \tag{2}$$

if there exists a subpower function ψ such that

$$\psi(1/y)^{-1}\, g(y) \le f(y) \le \psi(1/y)\, g(y), \qquad y \to 0+$$

We write

$$f(y) \preccurlyeq g(y), \qquad y \to 0+,$$

if

$$\limsup_{y\to 0+} \frac{\log f(y)}{\log g(y)} \le 1,$$

and

$$f(y) \preccurlyeq_{\text{i.o.}} g(y), \qquad y \to 0+,$$

if

$$\liminf_{y\to 0+} \frac{\log f(y)}{\log g(y)} \leq 1.$$

Here i.o. stands for "infinitely often". Clearly $f(y) \preccurlyeq g(y)$ implies $f(y) \preccurlyeq_{\text{i.o.}} g(y)$ but the converse is not true. Similarly we write $f(y) \succcurlyeq g(y)$ and $f(y) \succcurlyeq_{\text{i.o.}} g(y)$ for

$$\liminf_{y\to 0+} \frac{\log f(y)}{\log g(y)} \geq 1 \quad \text{and} \quad \limsup_{y\to 0+} \frac{\log f(y)}{\log g(y)} \geq 1,$$

respectively. We write $f(y) \approx g(y)$ if $f(y) \preccurlyeq g(y)$ and $f(y) \succcurlyeq g(y)$, i.e., if

$$\lim_{y\to 0+} \frac{\log f(y)}{\log g(y)} = 1.$$

Note that if $\beta \neq 0$, then

$$f(y) \approx y^{\beta} \quad \Longleftrightarrow \quad f(y) \approx^{*} y^{\beta}.$$

but this does not hold for $\beta = 0$ (for example $f(y) = 1, g(y) = \log(1/y)$).

We will also use the notations for asymptotics for functions $f(n), g(n)$ as positive integer $n \to \infty$.

2.2. **Chordal Loewner equation.** In this section, we review some facts about conformal mappings and the chordal Loewner equation. See [7, Chapters 3,4] for proofs of theorems stated without proof here.

Suppose $\tilde{\gamma}$ is a curve as in the introduction. The chordal Loewner equation is an equation that describes the evolution of $\tilde{\gamma}(0,\infty)$ given $\tilde{\gamma}(-\infty,0]$. Let $\tilde{g}$ be a conformal transformation of $\mathbb{C} \setminus \tilde{\gamma}(-\infty,0]$ onto the upper half plane $\mathbb{H}$ with $\tilde{g}(\gamma(0)) = 0$, $\tilde{g}(\infty) = \infty$. In order to describe $\tilde{\gamma}(t), t > 0$ it suffices to describe

$$\gamma(t) := \tilde{g}(\tilde{\gamma}(t)), \quad 0 \leq t < \infty,$$

and this is what the Loewner equation in $\mathbb{H}$ does. For the remainder of the paper, we will consider a curve γ in $\mathbb{H}$ as above. The Riemann mapping theorem implies that there is a unique conformal transformation g_t of $\mathbb{C} \setminus \gamma(0,t]$ onto $\mathbb{H}$ with $g_t(z) = z + o(1)$ as $z \to \infty$. We can expand g_t at infinity,

$$g_t(z) = z + \frac{a(t)}{z} + O(|z|^{-2}),$$

where $a(t)$ denotes the half-plane capacity of $\gamma(0,t]$, which is continuous and strictly increasing. We make the (slightly) stronger assumption that $a(t) \to \infty$ as $t \to \infty$. Then the chordal Loewner integral equation states that

$$g_t(z) = z + \int_0^t \frac{da(s)}{g_s(z) - V_s}, \qquad t \leq T_z,$$

where $V_s = g_s(\gamma(s))$ and $T_z = \inf\{t : \mathrm{Im}[g_t(z)] = 0\} = \inf\{t : g_t(z) - V_t = 0\}$. It can be shown that $s \mapsto V_s$ is a continuous function. It is convenient to choose a parameterization of γ such that $a(t) = at$ for some $a > 0$ in which case we get the Loewner differential equation

$$\partial_t g_t(z) = \frac{a}{g_t(z) - V_t}, \quad g_0(z) = z. \tag{3}$$

Let

$$f_t(z) = g_t^{-1}(z), \quad \hat{f}_t(z) = f_t(z + V_t) = g_t^{-1}(z + V_t).$$

By differentiating both sides of $f_t(g_t(z)) = z$ with respect to t we see that

$$\partial_t f_t(z) = -f_t'(z)\,\frac{a}{z - V_t}, \tag{4}$$

and since $g_t(\gamma(t)) = V_t$, we get

$$\gamma(t) = f_t(V_t) = \lim_{y\to 0+} f_t(V_t + iy) = \lim_{y\to 0+} \hat{f}_t(iy). \tag{5}$$

We let

$$v_t(y) = v(\hat{f}_t; y) = \int_0^y |\hat{f}_t'(iu)|\,du.$$

As mentioned before, if $v_t(y) < \infty$ for some $y > 0$, then $v_t(0+) = 0$ and the limit in (5) exists. More work is needed to determine whether or note γ is a continuous function of t. Note that if g_t satisfies (3) and $g_t^* = g_{t/a}$, then

$$\partial_t g_t^*(z) = \frac{1}{g_t^*(z) - V_t^*}, \quad g_0^*(z) = z,$$

where $V_t^* = V_{t/a}$.

Conversely, we can start with a continuous function $t \mapsto V_t$ and $a > 0$ and define a Loewner chain g_t by (3). We define $\gamma(t)$ by (5) provided that the limit exists. We say that the family of conformal maps g_t *generates a curve* if γ exists and is a continuous function of t. We do *not* assume that the curve is simple. If H_t denotes the unbounded component of $\mathbb{H} \setminus \gamma(0,t]$, then g_t is the unique conformal transformation of H_t onto $\mathbb{H}$ satisfying

$$g_t(z) = z + \frac{at}{z} + O(|z|^{-2}), \quad z \to \infty.$$

Lemma 2.1. *For every t and every $y > 0$ with $v_t(y) < \infty$,*

$$\frac{y\,|\hat{f}_t'(iy)|}{4} \le |\gamma(t) - \hat{f}_t(iy)| \le v_t(y). \tag{6}$$

Proof. The second estimate is immediate from the definition of $v_t(y)$ and the first inequality follows from the Koebe (1/4)-theorem applied to $\hat{f}_t$ on the open disk of radius y about iy. □

Lemma 2.2. *If f_t satisfies (4) and $z = x + iy \in \mathbb{H}$, then for $s \ge 0$*

$$e^{-5as/y^2}\,|f_t'(z)| \le |f_{t+s}'(z)| \le e^{5as/y^2}\,|f_t'(z)|. \tag{7}$$

In particular, if $s \le y^2$,

$$e^{-5a}\,|f_t'(z)| \le |f_{t+s}'(z)| \le e^{5a}\,|f_t'(z)|.$$

Proof. Without loss of generality we may assume that $a = 1$. Differentiating (4) yields

$$\partial_t f_t'(z) = -f_t''(z)\,\frac{1}{z-V_t} + f_t'(z)\,\frac{1}{(z-V_t)^2}.$$

Note that $|z - V_t| \geq y$. Applying Bieberbach's theorem (the $n = 2$ case of the Bieberbach conjecture) to the disk of radius y about z, we can see that

$$|f_t''(z)| \leq 4\,y^{-1}\,|f_t'(z)|.$$

and hence

$$|\partial_t f_t'(z)| \leq 5\,y^{-2}\,|f_t'(z)|,$$

which implies (7). □

The distortion and growth theorems are traditionally stated in terms of univalent functions defined on the unit disk. We will use these theorems for univalent functions on $\mathbb{H}$, and the next proposition gives the appropriate results.

Proposition 2.3. *Suppose $h : \mathbb{H} \to \mathbb{C}$ is a conformal transformation, $x \in \mathbb{R}, y > 0, r \geq 1$. Then*

$$(x^2+4)^{-2}\,|h'(iy)| \leq |h'(y(x+i))| \leq (x^2+4)^2\,|h'(iy)|, \tag{8}$$

$$|h(y(x+i)) - h(iy)| \leq \frac{(x^2+4)^{3/2}\,|x|}{2}\,y\,|h'(iy)|, \tag{9}$$

$$r^{-3}\,|h'(iy)| \leq |h'(iyr)| \leq r\,|h'(iy)|, \tag{10}$$

$$|h(iyr) - h(yi)| \leq \frac{r^2-1}{2}\,y\,|h'(iy)|. \tag{11}$$

Proof. By scaling we may assume that $y = 1$. Let

$$G(z) = \frac{z-i}{z+i}, \qquad G'(z) = \frac{2i}{(z+i)^2},$$

which is a conformal transformation of $\mathbb{H}$ onto the unit disk $\mathbb{D}$ with $G(i) = 0, |G'(i)| = 1/2$. We can write

$$h(z) = f(G(z)), \qquad h'(z) = f'(G(z))\,G'(z),$$

where f is a univalent function on $\mathbb{D}$. The distortion theorem tells us that

$$|f'(w)| \leq \frac{1+|w|}{(1-|w|)^3}\,|f'(0)|, \quad |w| < 1,$$

and the growth theorem states that

$$|f(w) - f(0)| \leq \frac{|w|}{(1-|w|)^2}\,|f'(0)|, \quad |w| < 1.$$

Since $|G'(i)| = 1/2$, we get

$$|h'(z)| \leq \frac{2\,|G'(z)|\,[1+|G(z)|]}{(1-|G(z)|)^3}\,|h'(i)|, \tag{12}$$

and

$$|h(z) - h(i)| \leq \frac{2\,|G(z)|}{(1-|G(z)|)^2}\,|h'(i)|. \tag{13}$$

Since

$$|G(x+i)| = \frac{|x|}{\sqrt{x^2+4}}, \qquad |G'(x+i)| = \frac{2}{x^2+4},$$

we plug into (12) and see that

$$|h'(x+1)| \leq \frac{1}{16}\,[\sqrt{x^2+4}+|x|]^4\,|h'(i)| \leq (x^2+4)^2\,|h'(i)|.$$

This gives the second inequality in (8) and the first follows easily by real translation. Plugging into (13) gives

$$\begin{aligned} |h(x+i) - h(x)| &\leq \frac{2\,|x|\,\sqrt{x^2+4}\,(\sqrt{x^2+4}+x)^2}{16}\,|h'(i)| \\ &\leq \frac{(x^2+4)^{3/2}\,|x|}{2}\,|h'(i)|. \end{aligned}$$

Since $r \geq 1$,

$$|G(ir)| = \frac{r-1}{1+r} = |G(i/r)|, \qquad |G'(ir)| = \frac{2}{(1+r)^2}, \qquad |G'(i/r)| = \frac{2r^2}{(r+1)^2}.$$

Plugging into (12) and (13) gives (10) and (11).

∎

Corollary 2.4. *If $h : \mathbb{H} \to \mathbb{C}$ is a conformal transformation, then for every $y > 0$,*

$$v(h;y) \geq \frac{y\,|h'(iy)|}{2}, \tag{14}$$

$$\frac{2}{3}\,v(h;2^{-n}) \leq \sum_{j=n}^{\infty} 2^{-j}\,|h'(i2^{-j})| \leq \frac{8}{3}\,v(h;2^{-n}).$$

Proof. We write

$$v(h;2^{-n}) = \sum_{j=n}^{\infty} \int_{2^{-j-1}}^{2^{-j}} |h'(iy)|\,dy.$$

Using (10) (which holds for $r > 1$)

$$v(h;y) = \int_0^y |h'(is)|\,ds = y \int_0^1 |h'(iry)|\,dr \geq y\,|h'(iy)| \int_0^1 r\,dr = \frac{y\,|h'(iy)|}{2},$$

$$\int_{r/2}^{r} |h'(iy)|\,dy \leq r\,|h_t'(ir)| \int_{1/2}^{1} s^{-3}\,ds = \frac{3r}{2}\,|h'(ir)|,$$

$$\int_{r/2}^{r} |h'(iy)|\,dy \geq r\,|h'(ir)| \int_{1/2}^{1} s\,ds = \frac{3r}{8}\,|h'(ir)|.$$

∎

We define the following measure of the modulus of continuity of U_t:

$$\Delta(t,s) = \sup_{0\le r\le s^2} \sqrt{s^{-2}(U_{t+r}-U_t)^2+4}.$$

Note that $\Delta(t,s)\ge 2$ and it is of order one if

$$\sup_{0\le r\le s^2} |U_{t+r}-U_t| \approx s.$$

The definition of $\Delta(t,s)$ with the 4 has been chosen to make the statement of the next proposition cleaner.

Proposition 2.5. *If $t\ge 0$ and $0\le r\le s^2$ with $v_t(s)+v_{t+r}(s)<\infty$, then*

$$|\gamma(t+r)-\gamma(t)| \le v_t(s)+v_{t+r}(s)+e^{5a}\,|\hat f_t'(si)|\,\Delta(t,s)^4\,s. \tag{15}$$

Proof. By the triangle inequality and (6),

$$\begin{aligned} &|\gamma(t+r)-\gamma(t)| \\ &\quad\le\ |\hat f_t(si)-\gamma(t)|+|\hat f_{t+r}(si)-\gamma(t+r)|+|\hat f_t(si)-\hat f_{t+r}(si)| \\ &\quad\le\ v_t(s)+v_{t+r}(s)+|f_{t+r}(V_{t+r}+si)-f_t(V_t+si)|. \end{aligned}$$

Also,

$$\begin{aligned} &|f_{t+r}(V_{t+r}+si)-f_t(V_t+si)| \le \\ &\qquad |f_{t+r}(V_{t+r}+si)-f_{t+r}(V_t+si)|+|f_{t+r}(V_t+si)-f_t(V_t+si)|. \end{aligned}$$

Using (9) and (7), we see that

$$\begin{aligned} |f_{t+r}(V_{t+r}+si)-f_{t+r}(V_t+si)| &\le \frac12\,|f_{t+r}'(V_t+si)|\,\Delta(t,s)^4\,s \\ &\le \frac12\,e^{5a}\,|f_t'(V_t+si)|\,\Delta(t,s)^4\,s \\ &= \frac12\,e^{5a}\,|\hat f_t'(si)|\,\Delta(t,s)^4\,s \end{aligned}$$

Also (7) and (4) imply that

$$|\partial_r f_{t+r}'(V_t+is)| \le \frac{a}{s}\,e^{5ar/s^2}\,|f_t'(is)|, \tag{16}$$

and hence

$$\begin{aligned} |f_{t+r}(V_t+is)-f_t(V_t+is)| &\le |f_t'(V_t+is)|\int_0^{s^2}\frac{a}{s}\,e^{5au/s^2}\,du \\ &= \frac{s}{5}\,e^{5a}\,|\hat f_t'(is)| \qquad (17)\\ &< \frac{s}{2}\,e^{5a}\,\Delta(t,s)^4\,|\hat f_t'(is)|. \qquad (18) \end{aligned}$$

□

Lemma 2.6. *There exist $c>0$ such that for $t\ge 0$ and $0<y\le 1$,*

$$c\,\frac{y}{\sqrt{2at+1}} \le |\hat f_t'(iy)| \le \frac{\sqrt{2at+1}}{y}.$$

Proof. We may assume $a = 1$ for otherwise we consider $g_t^* = g_{t/a}$. Let $w = \hat{f}_t(iy)$, i.e., $g_t(w) = V_t + iy$, and let $Y_t = \mathrm{Im}[g_t(w)]$. The Loewner equation implies that $\partial_t[Y_t^2] \geq -2$ and hence $\mathrm{Im}[w] \leq \sqrt{2t+1}$. Similarly, $\mathrm{Im}[\gamma(s)] \leq \sqrt{2t} \leq \sqrt{2t+1}$ for $0 \leq s \leq t$. The Loewner equation also implies that $\partial_t[Y_t/|g_t'(w)|] \leq 0$, which implies

$$y\,|\hat{f}_t'(iy)| = \frac{Y_t}{|g_t'(w)|} \leq \frac{Y_0}{|g_0'(w)|} = \mathrm{Im}[w] \leq \sqrt{2t+1}.$$

This gives the second inequality.

For the first inequality, let $d = \mathrm{dist}(w, \gamma[0,t] \cup \mathbb{R})$. The Beurling estimate [7, Theorem 3.76] implies that there is a $c_* < \infty$ such that the probability that a Brownian motion starting at w goes distance $\sqrt{2t+1}$ without hitting $\gamma[0,t] \cup \mathbb{R}$ is bounded above by

$$c_* \left(\frac{d}{\sqrt{2t+1}}\right)^{1/2}.$$

By the gambler's ruin estimate, the probability that a Brownian motion in $\mathbb{H}$ starting at iy reaches $K_t := \{\tilde{w} : \mathrm{Im}[\tilde{w}] = 2\sqrt{2t+1}\}$ before hitting the real line equals $y\,[2\sqrt{2t+1}]^{-1}$. Since the imaginary part decreases in the Loewner flow, it follows from conformal invariance that the probability that a Brownian motion starting at w reaches K_t before hitting $\gamma[0,t] \cup \mathbb{R}$ is at least $y\,[2\sqrt{2t+1}]^{-1}$. Therefore

$$c_* \left(\frac{d}{\sqrt{2t+1}}\right)^{1/2} \geq \frac{y}{2\sqrt{2t+1}}.$$

The Koebe (1/4)-theorem implies that $d \leq 4\,y\,|\hat{f}_t'(iy)|$, and plugging in we get

$$|\hat{f}_t'(iy)| \geq \frac{y}{16\,c_*^2\,\sqrt{2t+1}}.$$

$\square$

Proposition 2.7. *Suppose $h : \mathbb{H} \to \mathbb{C}$ is a conformal transformation and $v(h;y)$ is as defined in* (1). *Then for every $\beta < 1$, as $y \to 0+$*

$$|h'(iy)| \preccurlyeq y^{-\beta} \text{ if and only if } v(h;y) \preccurlyeq y^{1-\beta},$$

$$|h'(iy)| \succcurlyeq_{\text{i.o.}} y^{-\beta} \text{ if and only if } v(h;y) \succcurlyeq_{\text{i.o.}} y^{1-\beta}.$$

$$|h'(iy)| \approx^* y^{-\beta} \text{ if and only if } v(h;y) \approx^* y^{1-\beta}. \tag{19}$$

Proof. Using Corollary 2.4, all of the assertions follow easily except that $v(h;y) \approx^* y^{1-\beta}$ implies $|h'(iy)| \approx^* y^{1-\beta}$ which we will show here. Assume $v(h;y) \approx^* y^{1-\beta}$. By (14), we know that $|h'(iy)| \preccurlyeq y^{1-\beta}$. For $0 < \epsilon < 1-\beta$, let

$$u = u_\epsilon = \frac{3\epsilon}{1-\beta-\epsilon},$$

and note that (10) implies for y sufficiently small

$$\begin{aligned} y^{1-\beta+\epsilon} \leq v(h;y) &= v(h;y^{1+u}) + \int_{y^{1+u}}^{y} |h'(is)|\, ds \\ &\leq y^{1-\beta+2\epsilon} + \int_{y^{1+u}}^{y} |h'(is)|\, ds \\ &\leq y^{1-\beta+2\epsilon} + y^{1-3u}\, |h'(iy)|. \end{aligned}$$

Hence for all y sufficiently small,

$$|h'(iy)| \geq \frac{1}{2}\, y^{-\beta}\, y^{3u_\epsilon+\epsilon}.$$

Since $u_\epsilon \to 0$ as $\epsilon \to 0+$, this gives $|h'(iy)| \succcurlyeq y^{-\beta}$. $\square$

Definition For every $-1 \leq \beta \leq 1$, let

$$\overline{\Theta}_\beta = \{t \in (0,2] : |\hat{f}_t'(iy)| \succcurlyeq_{\text{i.o.}} y^{-\beta}\},$$

$$\Theta_\beta = \{t \in (0,2] : |\hat{f}_t'(iy)| \approx^* y^{-\beta}\},$$

$$\tilde{\Theta}_\beta = \{t \in (0,2] : |\hat{f}_t'(iy)| \preccurlyeq y^{-\beta}\},$$

$$\underline{\Theta}_\beta = \{t \in (0,2] : |\hat{f}_t'(iy)| \preccurlyeq_{\text{i.o.}} y^{-\beta}\},$$

$$\underline{\Theta}_\beta^* = \{t \in (0,2] : v_t(y) \preccurlyeq_{\text{i.o.}} y^{1-\beta}\},$$

where in each case the asymptotics are as $y \to 0+$.

If $\beta \neq 0$, we can write these sets as the set of $t \in (0,2]$ such that

$$\limsup_{y\to 0+} \frac{\log|\hat{f}_t'(iy)|}{\log(1/y)} \geq \beta,$$

$$\lim_{y\to 0+} \frac{\log|\hat{f}_t'(iy)|}{\log(1/y)} = \beta,$$

$$\limsup_{y\to 0+} \frac{\log|\hat{f}_t'(iy)|}{\log(1/y)} \leq \beta,$$

$$\liminf_{y\to 0+} \frac{\log|\hat{f}_t'(iy)|}{\log(1/y)} \leq \beta,$$

$$\liminf_{y\to 0+} \frac{\log v_t(y)}{\log(1/y)} \leq \beta - 1,$$

respectively.

Using Lemma 2.6, we can see that for every $\beta > 1$,

$$\overline{\Theta}_\beta = \Theta_\beta = \Theta_{-\beta} = \tilde{\Theta}_{-\beta} = \underline{\Theta}_{-\beta} = \emptyset.$$

This is why the definition restricts to $|\beta| \leq 1$. Note that (14) implies that $\underline{\Theta}_\beta^* \subset \underline{\Theta}_\beta$. Using Proposition 2.7 we can see that we can also write

$$\Theta_\beta = \{t \in (0,2] : v_t(y) \approx^* y^{1-\beta}, \;\; y \to 0+\},$$

and similarly for $\overline{\Theta}_\beta, \tilde\Theta_\beta$. Also, $\overline{\Theta}_\beta \cup \tilde\Theta_\beta = (0,2]$ and

$$\Theta_\beta \subset \overline{\Theta}_\beta \cap \tilde\Theta_\beta \cap \underline{\Theta}^*_\beta.$$

Definition The driving function V_t is *weakly Hölder-(1/2)* on $[0,2]$ if for each $\alpha < 1/2$, V_t is Hölder continuous of order α on [0,2].

Two equivalent definitions are

- If
$$\delta(s) = \sup\{|V_{t+s} - V_t| : 0 \le t \le t+s \le 2\},$$
then
$$\psi(x) = \sup_{s \ge 1/x} s^{-1/2}\,\delta(s)$$
is a subpower function.
- There is a subpower function ψ such that for all $0 \le t \le 2, 0 \le s \le 1$,
$$\Delta(t,s) \le \psi(1/s).$$

The next proposition shows that for weakly Hölder-(1/2) functions V_t, it suffices to consider dyadic times in the defintion of Θ_β, etc.

Proposition 2.8. *Suppose V_t is weakly Hölder-$(1/2)$ on $[0,2]$. For each $t \in [0,2]$ define $t_n = t_n(t) = (j-1)/2^{2n}$ if*

$$\frac{j-1}{2^{2n}} \le t < \frac{j}{2^{2n}}.$$

Then for $-1 \le \beta \le 1$,

- $$\Theta_\beta = \left\{ t \in (0,2] : |f'_{t_n}(i2^{-n})| \approx^* 2^{n\beta} \right\},$$
$$\overline{\Theta}_\beta = \left\{ t \in (0,2] : |f'_{t_n}(i2^{-n})| \succcurlyeq_{\text{i.o.}} 2^{n\beta} \right\},$$
$$\underline{\Theta}_\beta = \left\{ t \in (0,2] : |f'_{t_n}(i2^{-n})| \preccurlyeq_{\text{i.o}} 2^{n\beta} \right\},$$
Here the asymptotics are as $n \to \infty$ along the integers.
- *If $t \in \overline{\Theta}_\beta$,*
$$v(t,y) \succcurlyeq_{\text{i.o.}} y^{1-\beta}, \quad |\gamma(t) - \hat f_t(iy)| \succcurlyeq_{\text{i.o.}} y^{1-\beta}, \quad y \to 0+.$$
- *If $t \in \tilde\Theta_\beta$,*
$$v(t,y) \preccurlyeq y^{1-\beta}, \quad |\gamma(t) - \hat f_t(iy)| \preccurlyeq y^{1-\beta}, \quad y \to 0+. \tag{20}$$
- *If $t \in \Theta_\beta$,*
$$v(t,y) \approx^* y^{1-\beta}, \quad |\gamma(t) - \hat f_t(iy)| \approx^* y^{1-\beta}, \quad y \to 0+.$$

Proof. Note that

$$\begin{aligned} |\hat f'_t(i2^{-n})| = |f'_t(V_t + i2^{-n})| &\le \Delta(t,2^{-n})^4\,|f'_t(V_{t_n} + i2^{-n})| \\ &\le e^{3a}\,\Delta(t,2^{-n})^4\,|\hat f'_{t_n}(i2^{-n})|, \end{aligned}$$

and similarly

$$|\hat f'_t(i2^{-n})| \ge e^{-3a}\,\Delta(t,2^{-n})^{-4}\,|\hat f'_{t_n}(i2^{-n})|.$$

Hence if V_t is weakly Hölder$-(1/2)$, there is a subpower function ψ such that for all t, n,

$$\psi(2^n)^{-1})\,|\hat{f}'_{t_n}(i2^{-n})| \le |\hat{f}'_t(i2^{-n})| \le \psi(2^n))\,|\hat{f}'_{t_n}(i2^{-n})|.$$

This implies the first assertion and the remaining follow from (6).

□

2.3. **Harmonic measure at the tip.** We will now discuss harmonic measure giving two nonequivalent definitions, one that is standard and one which is more directly related to the multifractal spectrum we have discussed.

In this subsection γ denotes a curve in $\mathbb{H}$ with one endpoint on the real line. We assume that the curve comes from a Loewner chain driven by a continuous function U_t, so it may have double points but it does not cross itself. Let H_t be the unbounded connected component of $\mathbb{H} \setminus \gamma[0,t]$. As above, we write $g_t : H_t \to \mathbb{H}$ for the normalized conformal mapping so that $\lim_{y\to 0+} f_t(U_t + iy) = \gamma(t)$, where $f_t = g_t^{-1}$ and $\hat{f}_t = f_t(z + U_t)$. If the curve has double points, we are interpeting $\gamma(t)$ in terms of prime ends, and we then tacitly understand $\gamma(t)$ as the prime end corresponding to U_t.

If $z \in H_t$, then $\mathrm{hm}_{t,z}$ will denote the usual harmonic measure of $\mathbb{R} \cup \gamma(0,t]$ starting at z, that is to say the hitting measure of Brownian motion starting at z stopped when it reaches ∂H_t. We let

$$\mathrm{hm}_t(V) = \lim_{y\to\infty} y\, \mathrm{hm}_{t,iy}(V),$$

which is the normalized harmonic measure from the boundary point infinity. Note that for each $z \in H_t$, hm_t and $\mathrm{hm}_{t,z}$ are mutually absolutely continuous. Also, conformal invariance and the well known Poisson kernel in $\mathbb{H}$ show that for bounded V,

$$\mathrm{hm}_t(V) = \frac{1}{\pi}\,\mathrm{length}\,[g_t(V)]\,.$$

Let

$$\tilde{\mu}(t,\epsilon) = \mathrm{hm}_t\left[\overline{\mathcal{B}}(\gamma(t),\epsilon)\right],$$

where $\mathcal{B}(z,\epsilon)$ denote the open disk of radius ϵ about z with closure $\overline{\mathcal{B}}(z,\epsilon)$. For $\alpha > 0$, define

$$\tilde{\Theta}^{\mathrm{hm}}_\alpha = \{t \in (0,2] : \tilde{\mu}(t,\epsilon) \approx^* \epsilon^\alpha,\ \epsilon \to 0+\}.$$

We define the multifractal spectrum for harmonic measure by

$$\alpha \mapsto \dim_h\left[\gamma(\tilde{\Theta}^{\mathrm{hm}}_\alpha)\right].$$

This multifractal spectrum can be hard to compute. One of the difficulties is that $\mathcal{B}(\gamma(t),\epsilon) \cap H_t$ can contain many connected components whose images under g_t are far apart. We will give a different definition that is more directly related to the tip multifractal spectrum in this paper.

Fix $t > 0, \epsilon > 0$ and let $\mathcal{B} = \mathcal{B}(\gamma(t),\epsilon)$. Let $O = O_{t,\epsilon}$ denote the connected component of $\mathcal{B} \cap H_t$ that contains $\gamma(t)$ (considered as a prime end) on its boundary. There is a unique connected component (open arc), $\sigma = \sigma_{t,\epsilon}$ of

$\partial\mathcal{B} \cap H_t$ that is in ∂O and such that every curve from $\gamma(t)$ (again viewed as a prime end) to infinity in H_t passes through σ. Let $x_- = x_{-,t,\epsilon} < U_t < x_+ = x_{+,t,\epsilon}$ denote the images of the endpoints of σ under g_t. Let $E = E_{t,\epsilon}$ denote the inverse image of $[x_-, x_+]$ under g_t, and let

$$\mu(t,\epsilon) = \mathrm{hm}_t[E] = \frac{x_+ - x_-}{\pi}.$$

It is not necessarily true that $E \subset \overline{\mathcal{B}}(\gamma(t),\epsilon)$. However a standard estimate using the Beurling projection theorem shows that there is a c such that

$$\mu(t,\epsilon/2) \leq c\,\tilde{\mu}(t,\epsilon). \tag{21}$$

We define

$$\Theta_\alpha^{\mathrm{hm}} = \{t \in (0,2] : \mu(t,\epsilon) \approx^* \epsilon^\alpha,\ \epsilon \to 0+\}$$

Lemma 2.9. *There exist c_1, c_2 such that for all t and all y sufficiently small,*

$$\mu(t, 2v(t,y)) \geq c_1 y, \tag{22}$$

$$\mu(t, y\,|f_t'(iy)|) \leq c_2 y. \tag{23}$$

Proof. Let η_y denote the line segment $(0, iy]$. The harmonic measure from infinity of η_y in $\mathbb{H} \setminus \eta_y$ equals cy, and hence by conformal invariance the harmonic measure from infinity of $\eta_y^* := \hat{f}_t \circ \eta_y$ in $H_t \setminus \eta_y^*$ is also cy. Since η_y^* is a curve of length $v(t,y)$ and one of its endpoints is $\gamma(t)$, the interior of η_y^* is contained in $O_{t,v_t(y)}$. From this and the Beurling estimate as in (21), we get (22).

Let $\sigma_\epsilon = \sigma_{\epsilon,t}$ whose endpoints are mapped to $x_{-,\epsilon} < x_{+,\epsilon}$ as above. As $\epsilon \to 0+$, $x_{+,\epsilon} - x_{-,\epsilon} \to 0$ (using, e.g., the Beurling estimate) and hence for y sufficiently small we can choose ϵ_y with $y = x_{+,\epsilon} - x_{-,\epsilon}$. Note that $\mu(t,\epsilon_y) = y$. To prove (23) it suffices to show that $\epsilon_y \geq c\,y\,|f_t'(iy)|$. Note that there exists c_1 such that with probability one a Brownian motion starting at iy hits $g_t \circ \sigma_{\epsilon_y}$ before leaving $\mathbb{H}$ is at least c_1. By conformal invariance, this is also true for a Brownian motion starting at $\hat{f}_t(iy)$ hitting σ_{ϵ_y} before leaving H_t. The distortion theorem and the Koebe-1/4 theorem show that $\mathrm{dist}[\hat{f}_t(iy), \partial H_t] \asymp y\,|f_t'(iy)|$. The needed estimate then comes from the Beurling estimate which implies in any simply connected domain D, if $V \subset \partial D$,

$$\mathrm{hm}_D(z,V) \leq c\sqrt{\frac{\mathrm{diam}(V)}{\mathrm{dist}(z,\partial D)}}.$$

□

Corollary 2.10. *If $1/2 \leq \alpha < \infty$*

$$\Theta_\alpha^{\mathrm{hm}} = \Theta_{1-1/\alpha}.$$

3. Tip spectrum for SLE

Let $\kappa > 0$ and $a = 2/\kappa$. Then the chordal Schramm-Loewner evolution with parameter κ (SLE_κ) is the solution to (3) with $a = 2/\kappa$ where V_t is a standard Brownian motion. It is well known that with probability one, V_t is weakly Hölder-(1/2). Let

$$d = \min\left\{1 + \frac{\kappa}{8}, 2\right\}.$$

It was proved by Beffara [1] that d is the Hausdorff dimension of the path $\gamma[0,2]$. This will follow as a particular case of our main theorem, so we will not need to assume this result. However, it is convenient to use this notation.

3.1. Main theorem.

Before stating the main theorem, we will define some special values of the parameter β. See Section 3.4 for more details. Let

$$\rho(\beta) = \frac{\kappa}{8(\beta+1)} \left[\left(\frac{\kappa+4}{\kappa}\right)(\beta+1) - 1\right]^2, \tag{24}$$

$$\hat{d}_\beta = \frac{2-\rho(\beta)}{2},$$

$$d_\beta = \frac{2\,\hat{d}_\beta}{1-\beta} = \frac{2-\rho(\beta)}{1-\beta}.$$

The maximum value of $\hat{d}_\beta$ equals 1 and is obtained at

$$\beta_\# := \frac{\kappa}{\kappa+4} - 1.$$

The maximum value of d_β equals d and is obtained at

$$\beta_* := \frac{\kappa}{\max\{4, \kappa - 4\}} - 1.$$

We define $\beta_- \leq \beta_\# \leq \beta_* \leq \beta_+$ by $\rho(\beta_-) = \rho(\beta_+) = 2$. A straightforward computation gives

$$\beta_+ = -1 + \frac{\kappa}{12 + \kappa - 4\sqrt{8+\kappa}}, \tag{25}$$

$$\beta_- = -1 + \frac{\kappa}{12 + \kappa + 4\sqrt{8+\kappa}} < 0. \tag{26}$$

Also $-1 < \beta_- < \beta_+ \leq 1$ with equality only for $\kappa = 8$.

Theorem 3.1. *For chordal SLE_κ, if $-1 \leq \beta \leq 1$, the following holds with probability one.*

- *If $\beta_- \leq \beta \leq \beta_+$,*
$$\dim_h(\Theta_\beta) = \hat{d}_\beta, \qquad \dim_h[\gamma(\Theta_\beta)] = d_\beta. \tag{27}$$
- *If $\beta_\# \leq \beta \leq \beta_+$,*
$$\dim_h(\overline{\Theta}_\beta) = \hat{d}_\beta. \tag{28}$$

- *If* $\beta_* \le \beta \le \beta_+$,
$$\dim_h[\gamma(\overline{\Theta}_\beta)] = d_\beta. \tag{29}$$
- *If* $\beta_- \le \beta \le \beta_\#$,
$$\dim_h(\underline{\Theta}_\beta) = \hat{d}_\beta. \tag{30}$$
- *If* $\beta_- \le \beta \le \beta_*$,
$$\dim_h[\gamma(\underline{\Theta}^*_\beta)] = d_\beta. \tag{31}$$
- *If* $\beta > \beta_+$, $\overline{\Theta}_\beta = \emptyset$.
- *If* $\beta < \beta_-$, $\underline{\Theta}_\beta = \emptyset$.

3.2. **Remarks.**

- It follows from the theorem that with probability one the results hold for a dense set of β. This implies that with probability one, (28)–(31) hold for all β. However, we have not shown whether or not for a particular realization, there might be an exceptional β for which (27) does not hold.
- The restriction to $t \in (0, 2]$ is only a convenience. By scaling we get a similar result for $t \in (0, \infty)$.
- The relationship $\dim_h[\gamma(\Theta_\beta)] = 2 \dim_h[\Theta_\beta]/(1 - \beta)$ can be understood as follows. For s small, the image of the interval $[t, t + s^2]$ under $\hat{f}_t$ can be approximated by a set of diameter $s|\hat{f}'_t(is)|$ containing $\hat{f}_t(is)$. If $|\hat{f}'_t(is)| \approx s^{-\beta}$, then this set has diameter $s^{1-\beta}$. That is to say, intervals of length (diameter) s^2 in a covering of Θ_β are sent to sets of diameter $s^{1-\beta}$. Note that this is in contrast to complex Brownian motion where intervals of length s^2 are always sent to sets whose diameter is of order s.
- Since $\Theta_\beta \subset \overline{\Theta}_\beta \cap \tilde{\Theta}_\beta \cap \underline{\Theta}^*_\beta$ and $\underline{\Theta}^*_\beta \subset \underline{\Theta}_\beta$, it suffices to prove the lower bounds for Θ_β in (27) and the upper bounds for $\overline{\Theta}_\beta, \underline{\Theta}_\beta, \underline{\Theta}^*_\beta$ in (28)–(31). The upper bounds will be proved in Section 4.1 and the lower bounds in Section 4.2.
- To prove the upper bound (28) it suffices to show for each $s > 0$,
$$\dim_h(\overline{\Theta}_\beta \cap (s, 2]) \le \hat{d}_\beta,$$
and similarly for (29)–(31). This is what we do in Section 4.1.
- Recall that $\underline{\Theta}^*_\beta \subset \underline{\Theta}_\beta$. It is open whether or not
$$\dim_h[\gamma(\underline{\Theta}_\beta)] \le d_\beta.$$
- Note that $(0, 2] = \overline{\Theta}_{\beta_*} \cup \underline{\Theta}^*_{\beta_*}$. It follows that
$$\dim_h(\gamma(0, 2]) = d_{\beta_*} = d.$$
Hence, Beffara's theorem on the dimension of the path is a particular case of the theorem.
- The statements about the dimension of $\gamma(\Theta_\beta), \gamma(\overline{\Theta}_\beta), \gamma(\underline{\Theta}^*_\beta)$ are independent of the parametrization of the curve.

- Using the Markov property for SLE it is not hard to show that with probability one, either Θ_β is dense in $(0,\infty)$ or it is empty. Also, $\dim_h[\gamma(\Theta_\beta \cap [t_1,t_2])]$ is the same for all $0 < t_1 < t_2 \leq 2$. In particular, in order to prove the lower bound on dimension, it suffices to prove that for all $\alpha < d_\beta$,
$$\mathbb{P}\left\{\dim_h[\gamma(\Theta_\beta \cap [1,2])] \geq \alpha\right\} > 0.$$
This is what we will do in Section 4.2. The proof proves the slightly stronger (for $\kappa > 4$) result
$$\mathbb{P}\left\{\dim_h[\mathbb{H} \cap \gamma(\Theta_\beta \cap [1,2])] \geq \alpha\right\} > 0.$$
- If $\kappa = 8$, $\beta_* = \beta_+ = 1$ and $\dim_h(\gamma(\Theta_1)) = 2$. This is related to the fact that this is the hardest case to establish the existence of the curve. For other values of κ, $\beta_* < \beta_+ < 1$.
- The tip multifractal spectrum for chordal SLE is not the same as the multifractal spectrum for radial SLE. If $\gamma[0,t]$ is the initial segment of a chordal SLE path, then the local behavior near $\gamma(t)$ is not the same as the behavior of a radial SLE path. This is because the chordal path is "conditioned" or "weighted" by the fact that it needs to be continued by an SLE path from $\gamma(t)$ to ∞ in the slit domain.

3.3. **Harmonic measure spectrum.** Let $\Theta_\alpha^{\rm hm}$ be defined as in Section (2.3). Let
$$F(\alpha) = d_{1+\frac{1}{\alpha}} = \alpha\left(1 - \frac{4}{\kappa}\right) + \frac{(4+\kappa)^2}{8\kappa} - \left(\frac{\kappa}{8}\right)\left(\frac{\alpha^2}{2\alpha - 1}\right),$$
and let $\alpha_-, \alpha_*, \alpha_+$ correspond to $\beta_-, \beta_*, \beta_+$ through the relation $\alpha = 1/(1-\beta)$. The following is an immediate corollary of Theorem 3.1 and Lemma 2.9.

Theorem 3.2. *If $\alpha_+ < \alpha < \alpha_*$, then With probability one*
$$\dim_h[\gamma(\Theta_\alpha^{\rm hm})] = F(\alpha).$$

This theorem combined with (21) gives gives some information on $\tilde\Theta_\alpha^{\rm hm}$. In Section 6, we will use the forward Loewner flow to give a proof of the following.

Theorem 3.3. *If $\kappa < 8$ and $\alpha_+ < \alpha \leq \alpha_*$ then with probability one there exists a set V such that $\dim[\gamma(V)] = F(\alpha)$ and for $t \notin V$, $\gamma(t) \in \mathbb{H}$,*
$$\tilde\mu_t(\gamma(t), 2^{-n}) \preccurlyeq 2^{-n\alpha}, \quad n \to \infty. \tag{32}$$
In particular, for these values of α,
$$\dim_h[\mathbb{H} \cap \gamma(\tilde\Theta_\alpha^{\rm hm})] = F(\alpha).$$

We note that the second assertion of the theorem follows from the first, Theorem 3.2, and (21) which imply that there is a set V' with $\dim[\gamma(v')] = F(\alpha)$ on which
$$\tilde\mu_t(\gamma(t), 2^{-n}) \succcurlyeq 2^{-n\alpha}, \quad n \to \infty.$$

It is the first assertion that will be established in Section 6.

It is open whether or not

$$\dim_h[\gamma(\tilde{\Theta}_\alpha^{\rm hm})] = F(\alpha)$$

for $\alpha_* < \alpha < \alpha_-$.

3.4. **Parameters.** In the statement of the main theorem, β and ρ were the parameters used. However, in deriving the result it is useful to consider a number of other parameters. Let

$$r_* = \max\left\{1, \frac{8}{\kappa}\right\}, \quad r_c = \frac{1}{2} + \frac{4}{\kappa},$$

and note that

$$0 < r_* \leq r_c,$$

where the second inequality is strict unless $\kappa = 8$. Let $r < r_c$; we define $\lambda, \zeta, \beta, \rho$ as functions of r.

Let

$$\lambda = \lambda(r) = r\left(1 + \frac{\kappa}{4}\right) - \frac{\kappa r^2}{8}. \tag{33}$$

We write $\lambda_* = \lambda(r_*)$, and similarly for other parameters. As r increases from $-\infty$ to r_c, λ increases from $-\infty$ to

$$\lambda_c = 1 + \frac{3\kappa}{32} + \frac{2}{\kappa}.$$

Since the relationship is injective, we can write either $\lambda(r)$ or $r(\lambda)$. Solving the quadratic equation gives

$$r(\lambda) = \frac{4 + \kappa - \sqrt{(4+\kappa)^2 - 8\lambda\kappa}}{\kappa}.$$

Also,

$$\lambda(0) = 0, \quad \lambda_* = d.$$

Let

$$\zeta = \zeta(r) = r - \frac{\kappa r^2}{8} = \lambda(r) - \frac{\kappa r}{4}, \tag{34}$$

and note that

$$\zeta_* = 2 - d.$$

We can write ζ as a function of λ,

$$\zeta(\lambda) = \lambda + \frac{\sqrt{(4+\kappa)^2 - 8\lambda\kappa} - 4 - \kappa}{4}.$$

We now briefly discuss some results from [9, 5]. The reverse-time Loewner flow h_t (see Section 5.2 for definitions) has the property that for fixed t, the distribution of $|h_t'(z)|$ is the same as that of $|\hat{f}_t'(z)|$. For the reverse-time flow, if $r \in \mathbb{R}$ and λ, ζ are defined as above, then

$$|h_t'(z)|^\lambda \, Y_t(z)^\zeta \, [\sin\arg Z_t(z)]^{-r}$$

is a martingale. Typically one expects $Y_t(i) \asymp \sqrt{t}$ and $\sin\arg Z_t(z) \asymp 1$. If this is true, then the martingale property would imply

$$\mathbb{E}\left[|\hat{f}'_{t^2}(i)|^\lambda\right] = \mathbb{E}\left[|h'_{t^2}(i)|^\lambda\right] \asymp t^{-\zeta}.$$

It turns out that this argument can be carried out if $r < r_c$, and this is the starting point for determining the multifractal spectrum.

We define $\beta = \beta(r)$ by the relation

$$\frac{d\zeta}{d\lambda} = -\beta.$$

A straightforward calculation gives

$$\beta(r) = -1 + \frac{\kappa}{4+\kappa-\kappa r}, \qquad \kappa\, r(\beta) = 4 + \kappa - \frac{\kappa}{\beta+1}.$$

Note that β increases with r with

$$\beta(-\infty) = -1, \quad \beta(0) = -\frac{4}{4+\kappa} = \beta_{\#}, \quad \beta(r_*) = \beta_*, \quad \beta_c = 1,$$

where $\beta_{\#}, \beta_*$ are as defined in the previous section. Roughly speaking, $\mathbb{E}[|\hat{f}'_{t^2}(i)|^\lambda]$ is carried on an event on which $|\hat{f}'_{t^2}(i)| \approx t^\beta$ and

$$\mathbb{P}\left\{|\hat{f}'_{t^2}(i)| \approx t^\beta\right\} \approx t^{-(\zeta+\lambda\beta)}. \tag{35}$$

We emphasize that the relation between r, λ, β for $-\infty < r < r_c$ is bijective and in order to specify the values of the parameters it suffices to give the value of any one of these. For example, we could choose β as the independent variable and write $r(\beta), \lambda(\beta)$. This is the natural approach when proving Theorem 3.1, but the formulas tend to be somewhat simpler if we choose r to be the independent variable.

From (35), it is natural to define

$$\rho = \rho(r) = \zeta(r) + \lambda(r)\,\beta(r) = \frac{\kappa^2\, r^2}{8(4+\kappa-\kappa r)}.$$

We can also write ρ as a function of β and a computation gives (24). Note that

$$\frac{d\rho}{d\beta} = \frac{d\zeta}{d\lambda}\,\frac{d\lambda}{d\beta} + \lambda + \beta\,\frac{d\lambda}{d\beta} = \lambda.$$

Let r_+, r_- denote the two values of r for which $\zeta(r) + \lambda(r)\,\beta(r) = 2$ with corresponding values $\beta_+ = \beta(r_+), \beta_- = \beta(r_-)$. Then

$$r_\pm = \frac{4}{\kappa}\,\left[-2 \pm \sqrt{8+\kappa}\right],$$

and β_+, β_- are given as in (25)–(26). Note that if $\kappa \neq 8$, then $r_+ < r_c$.

Define

$$d_\beta = \frac{2-\rho(\beta)}{1-\beta} = \frac{2-(\zeta+\beta\lambda)}{1-\beta}.$$

Note that d_β is maximized at $\beta = \beta_*$ (interpreted as a limit for $\kappa = 8$) with $d_* = d$. We can also define d as a function of r,

$$d(r) = 1 + \frac{\kappa - \frac{\kappa^2 r^2}{8}}{8 + \kappa - 2\kappa r}.$$

Straightforward differentiation shows that $d'(r) = 0$ implies $r = 1$ or $r = 8/\kappa$. Note that $1 = 8/\kappa = r_+$ if $\kappa = 8$ and

$$1 < r_+ < \frac{8}{\kappa}, \quad \kappa < 8,$$

$$\frac{8}{\kappa} < r_+ < 1, \quad \kappa > 8.$$

From this we can see that $d(r)$ achieves its maximum on $(-\infty, r_+)$ at $r = r_*$; in fact, $d(\beta)$ increases for $\beta < \beta_*$ and decreases for $\beta_* < \beta < \beta_+$.

In order to match the notation of [9], let

$$q = r_c - r = \frac{1}{2} + \frac{4}{\kappa} - r. \tag{36}$$

Obviously, $q > 0$ if and only if $r < r_c$. For future reference we note that

$$\frac{1-2q}{1+2q} = \beta. \tag{37}$$

4. Proof of the main theorem

In this section we will present the proof of the theorem relying on estimates about moments of derivatives of the map $\hat{f}$. The upper bounds are proved in Section 4.1, and the lower bound is proved in Section 4.2.

4.1. Upper bounds.

In this subsection (and this subsection only) we write

$$\hat{f}_{j,n} = \hat{f}_{(j-1)2^{-2n}}.$$

For each $t \in [0, \infty)$, we associate a dyadic time by defining

$$t_n = t_n(t) = \frac{j-1}{2^{2n}} \quad \text{if} \quad \frac{j-1}{2^{2n}} \le t < \frac{j}{2^{2n}}.$$

We a fix s with $0 < s \le 2$ and allow constants to depend on s.

The next theorem states the derivative estimates that we will use for the upper bounds; a proof can be found in [5].

Theorem 4.1. [5] *If $r < r_c$, there exists $c < \infty$ such that for all $t \ge 1$,*

$$\mathbb{E}\left[|\hat{f}'_{t^2}(i)|^\lambda \right] \le c\, t^{-\zeta}. \tag{38}$$

Corollary 4.2. *If $\beta \ge \beta_\#$, there is a $c < \infty$ such that if*

$$N_{n,\beta} = \sum_{s2^{2n} \le j \le 2^{2n+1}} 1\left\{ |\hat{f}'_{j,n}(i2^{-n})| \ge 2^{n\beta} \right\},$$

then

$$\mathbb{E}[N_{n,\beta}] \le c\, 2^{n(2-\rho)}. \tag{39}$$

Proof. The range $\beta \geq \beta_\#$ corresponds to $\lambda \geq 0$. Hence, by Chebyshev's inequality,

$$\begin{aligned}\mathbb{P}\{|\hat{f}'_{j,n}(i2^{-n})| \geq 2^{n\beta}\} &\leq 2^{-n\beta\lambda}\mathbb{E}\left[|\hat{f}'_{j,n}(i2^{-n})|^{\lambda}\right] \\ &= 2^{-n\beta\lambda}\,\mathbb{E}\left[|\hat{f}'_{j}(i)|^{\lambda}\right] \leq c\,2^{-n\beta\lambda}\,j^{-\zeta/2},\end{aligned}$$

and hence

$$\begin{aligned}\mathbb{E}\left[N_{n,\beta}\right] &= \sum_{s2^{2n}\leq j\leq 2^{2n+1}} \mathbb{P}\{|\hat{f}'_{j,n}(2^{-n})| \geq 2^{n\beta}\} \\ &\leq 2^{-n\beta\lambda} \sum_{s2^{2n}\leq j\leq 2^{2n+1}} j^{-\zeta/2} \\ &\leq c\,2^{n(2-\beta\lambda-\zeta)} = c\,2^{n(2-\rho)}.\end{aligned}$$

□

Corollary 4.3. *If $\beta < \beta_\#$, there is a $c < \infty$ such that if*

$$N^*_{n,\beta} = \sum_{s2^{2n}\leq j\leq 2^{2n+1}} 1\left\{|\hat{f}'_{j,n}(i2^{-n})| \leq 2^{n\beta}\right\},$$

then

$$\mathbb{E}[N^*_{n,\beta}] \leq c\,2^{n(2-\rho)}. \tag{40}$$

Proof. This is proved in the same way using $\lambda < 0$. □

The standard technique to find upper bounds for Hausdorff dimension uses an appropriate sequences of covers for a set. We will now describe the covers that we will use. Let

$$I(j,n) = \left[\frac{j-1}{2^{2n}}, \frac{j}{2^{2n}}\right].$$

If $b, \overline{b} \in \mathbb{R}$ with $-1 < b < \overline{b} < 1$, let $\overline{\mathcal{B}}(j,n,\overline{b})$ be the closed disk in $\mathbb{C}$ of radius $2^{n(\bar{b}-1)}$ centered at $\hat{f}_{j,n}(i2^{-n})$, and let

$$I_n(s,b) = \bigcup I(j,n) \quad \mathcal{B}_n(s,b,\overline{b}) = \bigcup \overline{\mathcal{B}}(j,n,\overline{b}),$$

where in each case the union is over $s2^{2n} \leq j \leq 2^{2n+1}$ with $|\hat{f}'_{j,n}(i2^{-n})| \geq 2^{nb}$. Let

$$I^m(s,b) = \bigcup_{n=m}^{\infty} I_n(s,b), \quad \mathcal{B}^m(s,b,\overline{b}) = \bigcup_{n=m}^{\infty} \mathcal{B}_n(s,b,\overline{b}).$$

Lemma 4.4. *If $-1 < b < \beta < b_1 < \overline{b} < 1$, then for each m,*

$$\overline{\Theta}_\beta \cap (s,2] \subset I^m(s,b),$$

$$\gamma\left(\overline{\Theta}_\beta \cap \tilde{\Theta}_{b_1} \cap (s,2]\right) \subset \mathcal{B}^m(s,b,\overline{b}).$$

Proof. Suppose $t \in \overline{\Theta}_\beta \cap (s,2]$. By Proposition 2.8, there exists a subsequence $n_j \to \infty$ such that

$$|f'_{t_{n_j}}(i2^{-n_j})| \geq 2^{n_j b}.$$

In other words, there is a sequence n_j such that $I(t_{n_j}2^{2n_j}, n_j) \in I_{n_j}(s,b)$. This proves the first assertion.

If $t \in \overline{\Theta}_\beta \cap \tilde{\Theta}_{b_1} \cap (s,2]$ and $b_1 < u < \overline{b}$, then (20) shows that for all n sufficiently large,

$$|\gamma(t) - \hat{f}_t(i2^{-n})| \leq 2^{(u-1)n}.$$

The triangle inequality gives

$$|\gamma(t) - \hat{f}_{t_n}(i2^{-n})| \leq |\gamma(t) - \hat{f}_t(i2^{-n})| + |\hat{f}_t(i2^{-n}) - \hat{f}_{t_n}(i2^{-n})|,$$

and estimating as in (18), we have for n sufficiently large

$$|\hat{f}_t(i2^{-n}) - \hat{f}_{t_n}(i2^{-n})| \leq 2^{(u-1)n}.$$

Hence, for n sufficiently large,

$$|\gamma(t) - \hat{f}_{t_n}(i2^{-n})| \leq 2^{n_j(\bar{b}-1)}.$$

This implies for all j sufficiently large,

$$\gamma(t) \in \mathcal{B}(t_{n_j}2^{2n_j}, b, \overline{b}).$$

□

Proposition 4.5. *If $\beta \geq \beta_\#$, then with probability one,*

$$\dim_h(\overline{\Theta}_\beta \cap (s,2]) \leq \hat{d}_\beta, \tag{41}$$

Moreover, if $\beta > \beta_+$, then with probability one

$$\overline{\Theta}_\beta \cap (s,2] = \emptyset.$$

Proof. It suffices to consider $\beta_\# < \beta < 1$. Suppose $\beta_\# < b < \beta < 1$. Using the cover from Lemma 4.4, we get

$$\mathcal{H}^\alpha\left[\overline{\Theta}_\beta \cap (s,2]\right] \leq \sum_{n=m}^{\infty} N_{n,b}\, 2^{-2\alpha n},$$

and hence (39) implies

$$\mathbb{E}\left(\mathcal{H}^\alpha\left[\overline{\Theta}_\beta \cap (s,2]\right]\right) \leq c \sum_{n=m}^{\infty} 2^{n(2-\rho(b))}\, 2^{-2\alpha n}.$$

The sum goes to zero, provided that $2\alpha > 2 - \rho(b)$, and hence with probability one

$$\mathcal{H}^\alpha(\overline{\Theta}_\beta \cap (s,2]) = 0, \qquad \alpha > \frac{2-\rho(b)}{2}.$$

Letting $b \to \beta$ gives (41).

For the second assertion, note that

$$\mathbb{P}\left\{\overline{\Theta}_\beta \cap (s,2] \neq \emptyset\right\} \leq \sum_{n=m}^{\infty} \mathbb{E}[N_{n,b}] \leq c \sum_{n=m}^{\infty} 2^{n(2-\rho(b))}.$$

If $\beta > \beta_+$, then $\rho(\beta) > 2$ and we can find $b < \beta$ with $\rho(b) > 2$. $\square$

Lemma 4.6. *If $\beta_\# \leq \beta < b_1 < 1$, then with probability one,*

$$\dim_h \left[\gamma(\overline{\Theta}_\beta \cap \tilde{\Theta}_{b_1} \cap (s,2])\right] \leq \frac{2-\rho(\beta)}{1-b_1}. \tag{42}$$

Proof. Choose $b, \overline{b}$ with $\beta_\# < b < \beta < b_1 < \overline{b} < 1$. Using the cover from Lemma 4.4, we get

$$\mathcal{H}^\alpha\left[\gamma(\overline{\Theta}_\beta \cap \tilde{\Theta}_{b_1} \cap (s,2])\right] \leq \sum_{n=m}^{\infty} N_{n,b}\, 2^{\alpha(\overline{b}-1)n},$$

and hence (39) implies

$$\mathbb{E}\left(\mathcal{H}^\alpha\left[\gamma(\overline{\Theta}_\beta \cap \tilde{\Theta}_{b_1} \cap (s,2])\right]\right) \leq c \sum_{n=m}^{\infty} 2^{n(2-\rho(b))}\, 2^{\alpha(\overline{b}-1)n}.$$

The sums on the right go to zero provided that

$$\alpha > \frac{2-\rho(b)}{1-\overline{b}},$$

respectively. We now choose a sequence of values for $b, \overline{b}$ that converge to β, b_1 to conclude (42). $\square$

Proposition 4.7. *If $\beta_* \leq \beta \leq 1$, then with probability one,*

$$\dim_h \left[\gamma(\overline{\Theta}_\beta \cap (s,2])\right] \leq d_\beta. \tag{43}$$

Proof. If $\beta = b_0 < b_1 < b_2 < \ldots < b_k < 1$ with $b_k > b_+$, then

$$\overline{\Theta}_\beta = \bigcup_{j=1}^{k} (\overline{\Theta}_{b_{j-1}} \cap \tilde{\Theta}_{b_j}).$$

Therefore, (42) implies

$$\dim_h \left[\gamma(\overline{\Theta}_\beta \cap (s,2])\right] \leq \max\left\{\frac{2-\rho(b_{j-1})}{1-b_j} : j = 1,\ldots,k\right\}.$$

By taking finer and finer partitions and using the continuity of ρ, we see that

$$\dim_h \left[\gamma(\overline{\Theta}_\beta \cap (s,2])\right] \leq \sup_{b \geq \beta} \frac{2-\rho(b)}{1-b} = d_\beta.$$

The last equality uses $\beta \geq \beta_*$ and the fact which can easily be verified (see Section 3.4) that the function

$$F(b) = \frac{2-\rho(b)}{1-b},$$

is decreasing on the interval $[\beta_*, \beta_+]$. $\square$

For $\beta < \beta_{\#}$ we use a slightly different cover. Let $I(j,n)$ be as above and

$$I_n^*(s,b) = \bigcup I(j,n) \quad \mathcal{B}_n^*(s,b,\overline{b}) = \bigcup \mathcal{B}(j,n,\overline{b}),$$

where in each case the union is over $s2^{2n} \le j \le 2^{2n+1}$ with $|\hat{f}'_{j,n}(i2^{-n})| \le 2^{nb}$. Let

$$I_*^m(s,b) = \bigcup_{n=m}^{\infty} I_n^*(s,b), \quad \mathcal{B}_*^m(s,b,\overline{b}) = \bigcup_{n=m}^{\infty} \mathcal{B}_n^*(s,b,\overline{b}).$$

Lemma 4.8. *If $-1 < \beta < b < \overline{b}$,*

$$\underline{\Theta}_\beta \cap (s,2] \subset I_*^m(s,b),$$

$$\gamma\left(\underline{\Theta}_\beta^* \cap (s,2]\right) \subset \mathcal{B}_*^m(s,b,\overline{b}).$$

Proof. Suppose $t \in \underline{\Theta}_\beta \cap (s,2]$. By Proposition 2.8, there exists a subsequence $n_j \to \infty$ such that

$$|f'_{t_{n_j}}(i2^{-n_j})| \le 2^{n_j b}. \tag{44}$$

In other words, there is a sequence n_j such that $I(t_{n_j}2^{2n_j}, n_j) \in I_{n_j}^*(s,b)$. This proves the first assertion.

Now suppose $t \in \underline{\Theta}_\beta^* \cap (s,2] \subset \underline{\Theta}_\beta \cap (s,2]$. Then there exists a sequence n_j such that both (44) holds and

$$v_t(2^{-n_j}) \le 2^{n_j(b-1)}.$$

Using the triangle inequality as in Lemma 4.4, we see that

$$|\gamma(t) - \hat{f}_{t_{n_j}}(i2^{-n_j})| \le v_t(2^{-n_j}) + \left|\hat{f}_t(i2^{-n_j}) - \hat{f}_{t_{n_j}}(i2^{-n_j})\right|,$$

and arguing as before we see that for j sufficiently large

$$|\gamma(t) - \hat{f}_{t_{n_j}}(i2^{-n_j})| \le 2^{n_j(\overline{b}-1)},$$

and

$$\gamma(t) \in \mathcal{B}^*(t_{n_j}2^{2n}, b, \overline{b}).$$

$\square$

Proposition 4.9. *If $\beta < \beta_{\#}$, then with probability one,*

$$\dim_h(\underline{\Theta}_\beta \cap (s,2]) \le \hat{d}_\beta,$$

$$\dim_h(\underline{\Theta}_\beta^* \cap (s,2]) \le d_\beta.$$

Moreover, if $\beta < \beta_-$, with probability one,

$$\underline{\Theta}_\beta \cap (s,2] = \emptyset.$$

Proof. This is proved in the same way as Proposition 4.5 and Lemma 4.6 using (40). $\square$

4.2. **Lower bound.** In this subsection we prove the lower bound for the dimension in (27). We fix r such that $\rho = \lambda\beta + \zeta < 2$ and recall that $r < r_c$. As has been pointed out, it suffices to show that with positive probability

$$\dim_h(\Theta_\beta \cap [1,2]) \geq \frac{2-\rho}{2}, \qquad \dim_h[\gamma(\Theta_\beta \cap [1,2])] \geq \frac{2-\rho}{1-\beta}. \tag{45}$$

We will use a standard technique of Frostman to show that with positive probability there exist nontrivial positive measures μ, ν whose supports are contained in $\Theta_\beta \cap [1,2]$ and $\gamma(\Theta_\beta \cap [1,2])$, respectively, such that

$$\mathcal{E}_\alpha(\mu) < \infty, \ \ \alpha < \frac{2-\rho}{2}; \qquad \mathcal{E}_\alpha(\nu) < \infty, \ \ \alpha < \frac{2-\rho}{1-\beta},$$

where

$$\mathcal{E}_\alpha(\mu) = \int\int \frac{\mu(dx)\,\mu(dy)}{|x-y|^\alpha}$$

is the energy integral. It is well known that this implies (45). For this subsection, we will adopt a different notation than in the previous subsection. We let

$$\hat{f}_{j,n} = 1 + \frac{(j-1)}{n^2}, \qquad j = 1, 2, \ldots, n^2.$$

We will be studying $|\hat{f}'_{j,n}(i/n)|$. The proof considers a subset of times in $\Theta_\beta \cap [1,2]$ that behave in some sense nicely. The hard work is Theorem 4.10 which will be proved in Section 5. This theorem discusses the existence of some events $E_{j,n}$ on which

$$|\hat{f}'_{j,n}(iy)| \approx y^{-\beta}, \quad n^{-1} \leq y \leq 1.$$

The definition of the events ("good times") will be left for Section 5.

Theorem 4.10. *Suppose $\rho = \lambda\beta + \zeta < 2$. There exist $c < \infty$, a subpower function ψ, and events*

$$E_{j,n}, \quad n = 1, 2, \ldots, \quad j = 1, \ldots, n^2,$$

such that the following hold. Let $E(j,n) = 1_{E_{j,n}}$ and

$$F(j,n) = n^{\zeta-2}\,|\hat{f}'_{j,n}(i/n)|^\lambda\, E(j,n).$$

- *If $1 \leq j \leq n^2$, then on the event $E_{j,n}$,*
$$\psi(1/y)^{-1}\, y^{-\beta} \leq |\hat{f}'_{j,n}(iy)| \leq \psi(1/y)\, y^{-\beta}, \quad n^{-1} \leq y \leq 1. \tag{46}$$
- *If $1 \leq j \leq k \leq n^2$,*
$$c_1\, n^{-2} \leq \mathbb{E}\,[F(j,n)] \leq n^{\zeta-2}\,\mathbb{E}\left[|\hat{f}'_{j,n}(i/n)|^\lambda\right] \leq c_2\, n^{-2}, \tag{47}$$
- *If $1 \leq j \leq k \leq n^2$,*
$$\mathbb{E}[F(j,n)\,F(k,n)] \leq n^{-4}\left(\frac{n^2}{k-j+1}\right)^{\frac{\rho}{2}} \psi\left(\frac{n^2}{k-j+1}\right). \tag{48}$$

- *If $1 \leq j < k \leq n^2$ and $E(j,n)\, E(k,n) = 1$,*

$$\left|\hat{f}_{j,n}(i/n) - \hat{f}_{k,n}(i/n)\right|^{-\frac{2-\rho}{1-\beta}} \leq \left(\frac{n^2}{k-j+1}\right)^{\frac{2-\rho}{2}} \psi\left(\frac{n^2}{k-j+1}\right)^{d_\beta}. \tag{49}$$

Proof. This theorem combines Propositions 5.8 and 5.9 proved in Section 5. □

Proposition 4.11. *Under the assumptions of Theorem 4.10, with positive probability there exists $A \subset [1,2]$ such that for $t \in A$,*

$$\frac{1}{4}\, y^{-\beta}\, \psi(1/y)^{-1} \leq |f_t'(iy)| \leq 4\, y^{-\beta}\, \psi(1/y), \quad 0 < y \leq 1. \tag{50}$$

and such that

$$\dim_h(A) \geq \frac{2-\rho}{2}, \qquad \dim_h[\gamma(A)] \geq \frac{2-\rho}{1-\beta}.$$

Proof assuming Theorem 4.10. We use a now standard argument to show that with positive probability a "Frostman measure" of appropriate dimension can be put on the set of t satisfying (50). The proof is very similar to that of [9, Lemma10.3] so we omit some of the details.

Let $\mu_{j,n}$ denote the random measure on $\mathbb{R}$ that is a multiple of Lebesgue measure on $I(j,n) := [1+(j-1)n^{-2}, 1+jn^{-2}]$ where the multiple is chosen to that $\|\mu_{j,n}\| = F(j,n)$. Let $\nu_{j,n}$ denote the random measure on $\mathbb{C}$ that is a multiple of Lebesgue measure on the disk of radius $n^{\beta-1}\,\psi(n^2)^{-1}/4$ centered at $\hat{f}_{j,n}(i/n)$ where the constant is chosen so that $\|\nu_{j,n}\| = F(j,n)$. Let $\mu_n = \sum_{j=1}^{n^2} \mu_{j,n}, \nu_n = \sum_{j=1}^{n^2} \nu_{j,n}$. Note that

$$\|\mu_n\| = \|\nu_n\| = \sum_{j=1}^{n^2} F(j,n).$$

From (47) and (48), we see that

$$\mathbb{E}\left[\|\mu_n\|\right] \geq c_1, \quad \mathbb{E}\left[\|\mu_n\|^2\right] \leq c_2.$$

Hence

$$\mathbb{P}\left\{\|\mu_n\| > 0\right\} \geq c > 0,$$

uniformly in n. From (48) and (49), we can show that there is a C_α such that

$$\mathbb{E}[\mathcal{E}_\alpha(\mu_n)] \leq C_\alpha, \qquad \alpha < \frac{2-\rho}{2},$$

$$\mathbb{E}[\mathcal{E}_\alpha(\nu_n)] \leq C_\alpha, \qquad \alpha < \frac{2-\rho}{1-\beta}.$$

We let μ denote a subsequential limit of the μ_n which with positive probability we know is nontrivial and satisfies $\mathcal{E}_\alpha(\mu) < \infty$ for all $\alpha < (2-\rho)/2$. Hence,

$$\dim_h[\operatorname{supp}\mu] \geq \frac{2-\rho}{2}.$$

Similarly, let ν denote a subsequential limit of the ν_n which is nonzero with positive probability and satisfies

$$\dim_h[\operatorname{supp}\nu] \geq \frac{2-\rho}{1-\beta}.$$

We claim that every $t \in \operatorname{supp}\mu$ satisfies (50). Indeed, the construction shows that if $t \in \operatorname{supp}\mu$, then there is a subsequence $n_k \to \infty$ and $j_k \in \{1, \ldots, n_k^2\}$ such that $E(j_k, n_k) = 1$ and

$$\lim_{k\to\infty} \frac{j_k - 1}{n_k^2} = t. \tag{51}$$

Suppose for some $t \in [1,2]$ and $0 < y \leq 1$, we had

$$|\hat{f}_t'(iy)| \geq 4\, y^{-\beta}\, \psi(1/y).$$

Continuity would imply that for all s in a neighborhood of t,

$$|\hat{f}_s'(iy)| \geq 2\, y^{-\beta}\, \psi(1/y).$$

This implies that there is no sequence (j_k, n_k) as above with $E(j_k, n_k)$ satisfying (51). A similar argument shows that there cannot exist $t \in \operatorname{supp}\mu$ and y with $|\hat{f}_t'(iy)| \leq (1/4)\, y^{-\beta}\, \psi(1/y)^{-1}$, and this gives (50). Similarly, $\operatorname{supp}\nu$ is contained in $\gamma(A')$, where A' denotes the set of $t \in [1,2]$ satisfying (50). □

5. Estimating the moments

In this section $r < r_c$ with corresponding values of $\zeta, \beta, \rho, \lambda$. All constants may depend on r.

5.1. **Reverse Loewner flow.** Here we state the basic lemma that relates the reverse Loewner flow to the forward flow for SLE. We will estimate the moments for $h, \tilde{h}$ rather than for $\hat{f}$.

If V_t is a continuous function, define g_t to be the solution to the forward-time (chordal) Loewner equation

$$\partial_t g_t(z) = \frac{a}{g_t(z) - V_t}, \quad g_0(z) = z. \tag{52}$$

Let $f_t(z) = g_t^{-1}(z), \hat{f}_t(z) = f_t(z + V_t)$. If U_t is another continuous function, let h_t be the solution to the reverse-time (chordal) Loewner equation

$$\partial_t h_t(z) = \frac{a}{U_t - h_t(z)}, \quad h_0(z) = z. \tag{53}$$

The next lemma relates the forward-time and reverse-time equations; although versions of this have appeared before we give a short proof. We point out that this is a fact about the Loewner equation itself; no assumptions are made about the function V other than continuity.

Lemma 5.1. *Assume $V_t, -\infty < t < \infty$ is a continuous function with $V_0 = 0$. For each $T \geq 0$, let*

$$U_{t,T} = V_{T-t} - V_T.$$

Let $g_t, 0 \leq t < \infty$ be the solution to (52), and let $f_t, \hat{f}_t$ be as above. Let $h_{t,T}, 0 \leq t < \infty$ be the solution to (53) with $U_t = U_{t,T}$. Let

$$Z_{t,T}(z) = h_{t,T}(z) - U_{t,T}.$$

Then

$$h_{T,T}(z) = \hat{f}_T(z) - V_T = \hat{f}_T(z) + U_T,$$

and if $0 \leq S \leq T$, $z, w \in \mathbb{H}$,

$$h_{T,T}(z) = h_{S,S}\left(Z_{T-S,T}(z)\right) + U_{T-S,T}, \tag{54}$$

$$\hat{f}_T(z) - \hat{f}_S(w) = h_{S,S}\left(Z_{T-S,T}(z)\right) - h_{S,S}(w). \tag{55}$$

In particular,

$$h'_{T,T}(z) = h'_{S,S}\left(Z_{T-S,T}(z)\right)\ h'_{T-S,T}(z).$$

Proof. Fix T and let $U_t = U_{t,T}, h_t = h_{t,T}$. For $0 \leq s \leq T$, we have

$$U_{t,s} = V_{s-t} - V_s = U_{T-s+t} - U_{T-s}.$$

Let $u_t(z) = h_{T-t}(z) + V_T$. Then u_t satisfies

$$\partial_t u_t(z) = -\partial_t h_{T-t}(z) = \frac{a}{h_{T-t}(z) - [V_t - V_T]} = \frac{a}{u_t(z) - V_t}.$$

Also, $u_T(z) = h_0(z) + V_T = z + V_T$. By comparison, with (52),

$$u_0(z) = g_t^{-1}(z + V_T) = \hat{f}_t(z),$$

and hence

$$h_T(z) = u_0(z) - V_T = \hat{f}_T(z) - V_T = \hat{f}_T(z) + U_T.$$

For $0 \leq s \leq T$, define $h_t^{(s)}$ by

$$h_{s+t} = h_t^{(s)} \circ h_s.$$

By (53) we see that

$$\partial_t[h_t^{(s)}(z) - U_s] = \frac{a}{U_{t+s} - h_t^{(s)}(z)} = \frac{a}{U_{t+s} - U_s - [h_t^{(s)}(z) - U_s]},$$

$$h_0^{(s)}(z) - U_s = z - U_s.$$

Therefore,

$$h_t^{(s)}(z) - U_s = h_{t,T-s}(z - U_s), \tag{56}$$

which implies

$$h_{s+t}(z) = h_{t,T-s}(z - U_s) + U_s. \tag{57}$$

Setting $s = T - S$ and $t = S$ gives (54) and setting $s = S, t = T - S$ gives

$$\hat{f}_{T-S}(w) = h_{T-S}(w) - U_{T-S,T-S} = h_{T-S,T-S}(w) - [U_T - U_S],$$

$$\hat{f}_T(w) = h_T(z) - U_T = h_{T-S,T-S}(Z_S(z)) + U_S - U_T.$$

Subtracting these equations gives (55). The final assertion follows from (54) and the chain rule. $\square$

The preceding lemma holds for all continuous V_t. If V_t is a standard Brownian motion, then so is $U_{t,T}$ for each T. We get the following corollary.

Lemma 5.2. *Suppose $0 < S < T$ and g_t, $0 \le t \le T$, is the solution to (52) where V_t is a standard Brownian motion. Suppose U_t is a standard Brownian motion and $h, \tilde{h}$ are the solutions to*

$$\partial_t h_t(z) = \frac{a}{U_t - h_t(z)}, \quad h_0(z) = z, \tag{58}$$

$$\partial_t \tilde{h}_t(z) = \frac{a}{\tilde{U}_t - \tilde{h}_t(z)}, \quad \tilde{h}_0(z) = z,$$

where $\tilde{U}_t = U_{T-S+t} - U_{T-S}$. Then

$$h_{T-S+t}(z) = \tilde{h}_t(h_{T-S}(z) - U_{T-S}) + U_{T-S}.$$

Moreover, the joint distribution of the functions

$$\Big(\ \hat{f}_S'(w) \ , \ \mathrm{Im}\hat{f}_S(w), \ \hat{f}_T'(z) \ , \ \mathrm{Im}\hat{f}_T(w) \ , \ \hat{f}_T(z) - \hat{f}_S(w) \ \Big)$$

is the same as the joint distribution of

$$\Big(\ \tilde{h}_S'(w) \ , \ \mathrm{Im}\,\tilde{h}_S(w) \ , \ \tilde{h}_S'(Z)\, h_{T-S}'(z) \ , \ \mathrm{Im}\,\tilde{h}_S(Z) \ , \ \tilde{h}_S(Z) - \tilde{h}_S(w) \ \Big),$$

where $Z = Z_{T-S}(z) = h_{T-S}(z) - U_{T-S}$.

5.2. **Good times.** Suppose that $T > 0$ and $h_t = h_{t,T}$ is defined as in the proof of Lemma 5.1. More specifically, g_t is the solution to (52) with Brownian motion V_t, and h_t is the solution to (53) with $U_t = V_{T-t} - V_T$. Let

$$Z_t(z) = X_t(z) + Y_t(z) = h_t(z) - U_t.$$

Recalling (56) and (57), we define $\tilde{h}_{t,s}$ by $h_{t+s}(z) = \tilde{h}_{t,s}(Z_s(z)) + U_s$; in the notation of Lemma 5.1, $\tilde{h}_{t,s} = h_{t,T-s}$. Note that

$$h_{t+s}'(z) = \tilde{h}_{t,s}'(Z_s(z))\, h_s'(z). \tag{59}$$

If ψ is a subpower function and $0 < \delta \le 1$, we let

$$\hat{\psi}_\delta(t) = \min\{\psi(t/\delta), \psi(1/t)\} = \begin{cases} \psi(t/\delta) & \text{if } t \le \sqrt{\delta} \\ \psi(1/t) & \text{if } t > \sqrt{\delta} \end{cases} .$$

Note that for every subpower function ψ and every $c < \infty$, there is an $M < \infty$ such that for all $\delta > 0$,

$$\hat{\psi}_\delta(t) \le M \qquad \text{if } \delta \le t \le c\delta \text{ or } t \ge 1/c. \tag{60}$$

Roughly speaking, $\hat{\psi}_\delta(t)$ is $O(1)$ for t comparable to δ or comparable to 1 but can be larger for other $\delta < t < 1$.

Definition We call a time T *ψ-good at δ* if the following five conditions hold with $\hat{\psi} = \hat{\psi}_\delta$ and $Z_t = X_t + iY_t = Z_t(\delta i)$.

$$Y_{t^2} \geq t\,\hat{\psi}(t)^{-1}, \quad \delta \leq t \leq 2, \tag{61}$$

$$|X_{t^2}| \leq [t+\delta]\,\hat{\psi}(t), \quad 0 \leq t \leq 2, \tag{62}$$

$$(t/\delta)^\beta\,\hat{\psi}(t)^{-1} \leq |h'_{t^2}(i\delta)| \leq (t/\delta)^\beta\,\hat{\psi}(t), \quad \delta \leq t \leq 2, \tag{63}$$

$$t^{-\beta}\,\hat{\psi}(t)^{-1} \leq \frac{|h'_4(i\delta)|}{|h'_{t^2}(i\delta)|} = |\tilde{h}'_{4-t^2,t^2}(Z_{t^2})| \leq t^{-\beta}\hat{\psi}(t), \quad \delta \leq t \leq 2, \tag{64}$$

This definition depends on ψ and δ. Note that if T is ψ-good at δ and ϕ is a subpower function with $\psi \leq \phi$, then $\psi_\delta \leq \phi_\delta$ and T is ϕ-good at δ. In the remainder of this subsection, we derive some properties of ψ-good times. These will be used in the later subsections to estimate first and second moments for $|h'_{t^2}(\delta i)|^\lambda$ on the event that T is ψ-good at δ.

Proposition 5.3. *For every subpower function ψ there is a subpower function ϕ such that for all $\delta > 0$, if T is ψ-good at δ, then*

$$|U_{t^2}| \leq t\,\phi(1/t), \quad \delta \leq t \leq 2. \tag{65}$$

Proof. Let $X_t = X_t(i\delta), Y_t = Y_t(i\delta)$. We let ϕ denote a subpower function whose value may change from line to line. From the Loewner equation, we know that

$$dX_t = -\frac{aX_t}{X_t^2 + Y_t^2}\,dt - dU_t.$$

Hence,

$$|U_{t^2}| \leq |X_{t^2}| + a\int_0^{t^2} \frac{|X_s|\,ds}{X_s^2 + Y_s^2}.$$

By (62), it suffices to show that

$$\int_0^{t^2} \frac{|X_s|\,ds}{X_s^2 + Y_s^2} \leq \phi(1/t).$$

Using (61) and (62), we have

$$\frac{|X_s|}{X_s^2 + Y_s^2} \leq \frac{\phi(1/s)}{\sqrt{s}},$$

and hence

$$\int_0^{t^2} \frac{|X_s|\,ds}{X_s^2 + Y_s^2} \leq \int_0^{t^2} \frac{\phi(1/s)\,ds}{\sqrt{s}} = \int_{t^{-2}}^{\infty} \phi(x)\,x^{-3/2}\,dx = \tilde{\phi}(1/t)\,t,$$

where

$$\tilde{\phi}(1/t) = t^{-1}\int_0^{t^2} \frac{\phi(1/s)\,ds}{\sqrt{s}}.$$

It is easy to check that $\tilde{\phi}$ is continuous and decays faster than x^ϵ for each ϵ. $\square$

Lemma 5.4. *If ψ is a subpower function, there is a $c > 0$ such that for every $0 < \delta \leq 1$, if T is ψ-good at δ and $Y_t = Y_t(\delta i)$, then*

$$Y_{\delta^2} \geq (1+c)\,\delta. \tag{66}$$

Proof. Using (61), (62) and the fact that Y_{t^2} increases with t, we see there is a $c_1 < \infty$ such that

$$|X_{t^2}| \leq c_1\,\delta \leq c_1\,Y_{t^2}, \quad 0 \leq t \leq \delta.$$

The Loewner equation (53) implies that

$$\partial_s Y_s = \frac{aY_s}{X_s^2 + Y_s^2} \geq \frac{a}{c_1^2+1}\,\frac{1}{Y_s}, \quad s \leq \delta^2,$$

from which (66) follows. ☐

Lemma 5.5. *For every subpower function ψ, there is a c such that if $0 < \delta \leq 1$ and T is ψ-good at δ, then*

$$\left|\tilde{h}_{4-t^2,t^2}(Z_{t^2}) - \tilde{h}_{4-t^2,t^2}(\delta i)\right| \geq c\,\hat{\psi}_\delta(t)^{-2}\,t^{1-\beta}, \quad \delta \leq t \leq 2.$$

Proof. Using (61) and (66), we see that there is a $c_1 > 0$ such that if $\mathcal{B}$ denotes the open disk of radius $c_1\,t\,\hat{\psi}_\delta(t)^{-1}$ about Z_{t^2}, then $\delta i \notin \mathcal{B}$. Using (64) and the Koebe (1/4)-theorem we see that $\tilde{h}_{4-t^2,t^2}(\mathcal{B})$ contains a disk of radius $(c_1/4)\,\hat{\psi}_\delta(t)^{-2}\,t^{1-\beta}$ about $\tilde{h}_{4-t^2,t^2}(Z_{t^2})$. Since $\tilde{h}_{4-t^2,t^2}(\delta i) \notin \mathcal{B}$, the result follows. ☐

Lemma 5.6. *For all subpower functions ψ, ϕ, there is a subpower function $\overline{\psi}$ such that if T is ψ-good at δ, then the following holds for all $0 \leq t \leq 2$. Suppose*

$$t\,\phi(1/t)^{-1} \leq y \leq t\,\phi(1/t), \tag{67}$$

$$|x| \leq [t+\delta]\,\phi(1/t). \tag{68}$$

Then

$$t^{-\beta}\,\overline{\psi}(1/t)^{-1} \leq \left|\tilde{h}'_{4-t^2,t^2}(x+iy)\right| \leq t^{-\beta}\,\overline{\psi}(1/t).$$

Proof. By Proposition 2.3 and (61) and (62),

$$\tilde{\psi}(1/t)^{-1}|\tilde{h}'_{4-t^2,t^2}(Z_{t^2})| \leq \left|\tilde{h}'_{4-t^2,t^2}(x+iy)\right| \leq \tilde{\psi}(1/t)\,|\tilde{h}'_{4-t^2,t^2}(Z_{t^2})|,$$

whre $Z_{t^2} = Z_{t^2}(i\delta)$. The result then follows from (64). ☐

Lemma 5.7. *For all subpower functions ψ, ϕ there is a subpower function $\overline{\psi}$ such that if T is ψ-good at δ, then the following holds for $1 \leq t \leq 2$. Suppose $w = x+iy$ with*

$$\delta \leq y \leq 1, \quad (x/y)^2 + 1 \leq \phi(1/y).$$

Then,

$$y^{-\beta}\,\overline{\psi}_\delta(1/y)^{-1} \leq |h'_{t^2}(w)| \leq y^{-\beta}\,\overline{\psi}_\delta(1/y).$$

In particular,

$$y^{-\beta}\,\overline{\psi}_\delta(1/y)^{-1} \leq |h'_{t^2}(iy)| \leq y^{-\beta}\,\overline{\psi}_\delta(1/y). \tag{69}$$

Proof. We will do the case $t = 2$; the argument is similar for $1 \leq t \leq 2$. We let $\overline{\psi}$ denote a subpower function in this proof, but its value may change from line to line. Since $(x/y)^2 + 1 \leq \phi(1/y)$, we can see from Proposition 2.3 that

$$\overline{\psi}(1/y)^{-1}\, |h'_{t^2}(iy)| \leq |h'_{t^2}(w)| \leq \overline{\psi}(1/y)\, |h'_{t^2}(iy)|,$$

so we may assume $x = 0$. Let $s = y$. Using the Loewner equation (53), we can see that there is a $c < \infty$ such that

$$y \leq \mathrm{Im} h_{s^2}(iy) \leq cy,$$

$$|\mathrm{Re} h_{s^2}(iy)| \leq cy,$$

$$|h'_{s^2}(iy)| \leq c.$$

The last estimate and (59) imply that

$$|h'_4(y)| \asymp |h_{4-s^2,s^2}(Z_{s^2}(iy))|.$$

Using (65), we see that

$$|\mathrm{Re} Z_{s^2}(iy)| \leq y\,[c + \psi(1/y)].$$

By the previous result,

$$\overline{\psi}(1/y)^{-1}\, t^{-\beta} \leq |\tilde{h}'_{4-s^2,s^2}(Z_{s^2}(iy))| \leq \overline{\psi}(1/y)\, t^{-\beta}$$

□

Definition If n is a positive integer and $j = 1, \ldots, n^2$, we say that (j, n) is ψ-good if $T = 1 + (j-1)n^{-2}$ is ψ-good at n^{-1}. We let $E_{j,n}$ denote the event "(j, n) is ψ-good " and $E(j, n)$ denotes the indicator function of $E_{j,n}$.

It is important to note that on the event $E_{j,n}$, (69) implies that (46) holds, with perhaps a different choice of subpower function ψ. The main estimate for the lower bound is the following.

Proposition 5.8. *If $r < r_c$, here exists a subpower function ψ and $c > 0$ such that for all n and all $j = 1, 2, \ldots, n^2$,*

$$\mathbb{E}\left[|\hat{f}'_{j,n}(i/n)|^{\lambda}\, E_{j,n}\right] \geq c\, n^{-\zeta}. \tag{70}$$

Remark For fixed n, the expectation in (70) is the same for all j.

We will not include a proof of Proposition 5.8 because it has essentially appeared in [9, Theorem 10.8]; see also [5, Lemma 4.4]. We will point out that the assumption $r < r_c$ is crucial for the result. The proof in [9] and [5] uses a careful analysis of a relatively simple one-dimensional diffusion.

5.3. **Two point estimates.** In this subsection we fix a subpower function ψ such that Proposition 5.8 holds. If n is a positive integer, we write j,k for positive integers satisfying $1 \leq j < k \leq n^2$. We will consider $E_{j,n} \cap E_{k,n}$ with indicator function $E(j,n)\, E(k,n)$. If j,k,n are fixed we write

$$S = 1 + \frac{j-1}{n^2}, \qquad T = 1 + \frac{k-1}{n^2},$$

$$\tilde{h}_S = h_{t,S}, \quad h_t = h_{t,T}.$$

Proposition 5.9. *There is a subpower function ϕ such that for all $1 \leq j < k \leq n^2$*

$$\mathbb{E}\left[|\hat{f}'_T(i/n)|^\lambda \, |\hat{f}'_S(i/n)|^\lambda \, E(j,n)\, E(k,n)\right] \leq n^{-2\zeta} \left(\frac{n^2}{k-j}\right)^{\zeta/2} \phi\left(\frac{n^2}{k-j}\right).$$

Moreover, on the event $E_{j,n} \cap E_{k,n}$,

$$|\hat{f}_T(i/n) - \hat{f}_S(i/n)| \geq \left(\frac{k-j}{n^2}\right)^{\frac{1-\beta}{2}} \phi\left(\frac{n^2}{k-j}\right)$$

Proof. We write ϕ for a subpower function but we let its value vary from line to line; in the end we choose the maximum of all the subpower functions mentioned. Recall that $\hat{f}'_S(i/n) = \tilde{h}'_S(i/n), \hat{f}'_T(i/n) = h'_T(i/n)$ and

$$\hat{f}_T(i/n) - \hat{f}_S(i/n) = \tilde{h}_S(Z_{T-S}(i/n)) - \tilde{h}_S(i/n).$$

The second assertion of the proposition follows immediately from Lemma 5.5, so we need only show the first.

Since T is ψ-good at $1/n$, we know from (64) that

$$|h'_T(i/n)| \leq |h'_{T-S}(i/n)| \left(\frac{n^2}{k-j}\right)^{\beta} \phi\left(\frac{n^2}{k-j}\right).$$

Therefore,

$$\mathbb{E}\left[|\hat{f}'_T(i/n)|^\lambda \, |\hat{f}'_S(i/n)|^\lambda \, E(j,n)\, E(k,n)\right] \leq$$

$$\left(\frac{n^2}{k-j}\right)^{\lambda\beta} \phi\left(\frac{n^2}{k-j}\right) \mathbb{E}\left[|h'_{T-S}(i/n)|^\lambda \, |\tilde{h}'_S(i/n)|^\lambda \, E(j,n)\, E(k,n)\right].$$

Let $\overline{E}(k,n)$ be the indicator function of the event

$$Y_{T-S} \geq \left(\frac{k-j}{n^2}\right)^{1/2} \psi\left(\frac{\sqrt{k-j}}{n}\right).$$

Then (61) implies that $\overline{E}(k,n) \geq E(k,n)$. Also $|h'_{T-S}(i/n)|^\lambda \, \overline{E}(k,n)$ and $|\tilde{h}'_S(i/n)|^\lambda \, E(j,n)$ are independent random variables. Therefore,

$$\mathbb{E}\left[|h'_{T-S}(i/n)|^\lambda \, |\tilde{h}'_S(i/n)|^\lambda \, E(j,n)\, E(k,n)\right] \leq$$

$$\mathbb{E}\left[|\tilde{h}'_S(i/n)|^\lambda \, E(j,n)\right] \, \mathbb{E}\left[|h'_{T-S}(i/n)|^\lambda \, \overline{E}(k,n)\right].$$

We now apply Theorem 4.1 to say that the right hand side above is bounded above by

$$n^{-\zeta}\,(k-j)^{-\zeta/2}\,\phi\left(\frac{n^2}{k-j}\right) = n^{-2\zeta}\,\left(\frac{n^2}{k-j}\right)^{\zeta/2}\,\phi\left(\frac{n^2}{k-j}\right).$$

□

6. Proof of Theorem 3.3

In this section we will use the forward Loewner flow to prove Theorem 3.3. Throughout we will fix $\kappa = 2/a < 8$. We will write u rather than α (to avoid having both α and a in formulas). and we fix $\alpha_+ < u < \alpha_c$. To prove the theorem it suffices to show that for every bounded domain $D \subset \mathbb{H}$ bounded away from the real line, there is a set V_D with $\dim_h[V_D] = F(u)$ and such that (32) holds for $t \in V_D$ with $\gamma(t) \in D$. We fix such a D and allow constants to depend on D. The basic strategy is typical for establishing upper bounds for multifractal spectrum. We estimate a particular moment of $|g_\tau'(z)|$ for an appropriate stopping time, use Chebyshev's inequality to get an estimate on probabilities, and use this estimate to bound the dimension of a well chosen covering V. We warn the reader that some of the notation in this section is not consistent with that in other sections.

We parametrize SLE_κ so that the conformal maps g_t satisfy

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{71}$$

where $U_t = -B_t$ is a standard Brownian motion. This is valid for $z \in \mathbb{C}\setminus\{0\}$ up to time $T_z \in (0,\infty]$. We let H_t be the unbounded component of $\mathbb{H}\setminus\gamma(0,t]$.

6.1. Preliminaries.

Let

$$Z_t = Z_t(z) = X_t + iY_t = g_t(z) - U_t.$$

If $z \in \mathbb{H}$, let

$$\Delta_t = |g_t'(z)|, \quad \Upsilon_t = \frac{Y_t}{|g_t'(z)|}, \quad \Theta_t = \arg Z_t, \quad S_t = \sin\Theta_t.$$

Υ_t equals $1/2$ times the conformal radius of H_t about z (or we can think of it as the conformal radius normalized so that the conformal radius of $\mathbb{H}$ about i equals 1). The Koebe-1/4 theorem implies that

$$\frac{\Upsilon_t}{2} \le \operatorname{dist}(z, \partial H_t) \le 2\,\Upsilon_t. \tag{72}$$

Straightforward computations using (71) show that for $z \in \mathbb{H}$,

$$\partial_t \Delta_t = \Delta_t\,\frac{a(Y_t^2 - X_t^2)}{|Z_t|^4}, \quad \partial_t\Upsilon_t = -\Upsilon_t\,\frac{2aY_t^2}{|Z_t|^4}.$$

There exists $0 < c_1 < c_2 < \infty$ such that for $z \in D$,

$$c_1 \le \Upsilon_0 \le c_2, \quad c_1 \le S_0 \le 1. \tag{73}$$

Let
$$r = r(u) = \frac{1}{2} - 2a - \frac{1}{4u}, \tag{74}$$
$$\lambda = \lambda_r = \frac{r^2}{2a} + r\left(1 - \frac{1}{2a}\right), \quad \xi = \xi_r = \frac{r^2}{4a} = \frac{\lambda}{2} - \frac{r}{2}\left(1 - \frac{1}{2a}\right). \tag{75}$$
Note that r increases with u, and since $a > 1/4$,
$$r < r(u_c) = 1 - 4a < \min\left\{\frac{1}{2} - 2a, 2 - 3a\right\},$$
$$r < 0, \quad r + \lambda > 0.$$
The following is a straightforward Itô's formula calculation that we omit.

Proposition 6.1. *Suppose $r \in \mathbb{R}$ and λ, ξ are as in* (75). *If $z \in \mathbb{H}$, and*
$$M_t = M_t(z) = |Z_t|^r \, Y_t^{\xi} \, \Delta_t^{\lambda} = S_t^{-r} \, \Upsilon_t^{\xi + r} \, \Delta_t^{\lambda + r},$$
then M_t is a local martingale satisfying
$$dM_t = M_t \, \frac{r \, X_t}{|Z_t|^2} \, dB_t.$$

Let $\mathcal{D}_n$ denote the of dyadic rationals in $\mathbb{C}$
$$z = \frac{j}{2^n} + i \frac{k}{2^n}, \quad j, k \in \mathbb{Z}.$$
Note that if $w \in \mathbb{C}$, then there exists $z \in \mathcal{D}_n$ with $|z - w| \leq 2^{-n}$ and hence $\mathcal{B}(w, 2^{-n}) \subset \mathcal{B}(z, 2^{-n+1})$.

6.2. **Basic strategy.** Let
$$\tau_{n,z} = \inf\left\{s : \Upsilon_s(z) \leq 2^{-n+3}\right\}.$$
We will only consider n sufficiently large so that $2^{-n+4} \leq c_1$ where c_1 is the constant in (73). Note that $\mathbb{P}\{\tau_{n,z} = \infty\} > 0$. If $\tau_{n,z} < \infty$, (72) implies
$$2^{-n+2} \leq \mathrm{dist}(z, \partial H_t) = \mathrm{dist}(z, \gamma(0, t]) \leq 2^{-n+4}.$$
In particular, if $|w - z| \leq 2^{-n}$, $\mathrm{dist}(w, \partial H_t) \geq 2^{-n+1}$. Let $\mathrm{hm}_{n,z} = \mathrm{hm}_{\tau_{n,z}}$. If $|z - \gamma(t)| \leq 2^{-n}$, then $\tau_{n,z} \leq t$ and hence by monotonicity of harmonic measure,
$$\mathrm{hm}_{n,z}\left[\mathcal{B}(z, 2^{-n+1})\right] \geq \mathrm{hm}_t\left[\mathcal{B}(z, 2^{-n+1})\right] \geq \tilde{\mu}(t, 2^{-n}).$$

Let $\mathcal{D}_n(D)$ denote the set of $z \in \mathcal{D}$ such that $\mathrm{dist}(z, D) \leq 2^{-n}$ and
$$A_m^u = A_m^u(D) = \bigcup_{n=m}^{\infty} \bigcup_{z} \mathcal{B}(z, 2^{-n+1}),$$
where the inner union is over all $z \in \mathcal{D}_n(D)$ satisfying
$$\mathrm{hm}_{n,z}\left[\mathcal{B}(z, 2^{-n+1})\right] \geq 2^{-nu}. \tag{76}$$
Then if $\gamma(t) \in D \setminus A_m^u$, for all n sufficiently large
$$\mathrm{hm}_t\left[\mathcal{B}(\gamma(t), 2^{-n})\right] \leq 2^{-nu}.$$

Hence for each m, A^u_m is a cover of $D \cap K_u$ where K_u is the set of $\gamma(t)$ that do not satisfy (32). Let $N_n = N_{n,u}(D)$ be the cardinality of the set of $z \in \mathcal{D}_n(D)$ satisfying (76). Then for all s,

$$\mathcal{H}^s\left[D \cap K_u\right] \leq \lim_{m\to\infty} \mathcal{H}^s\left(A^u_m\right) \leq c \lim_{m\to\infty} \sum_{n=m}^{\infty} N_n\, 2^{-ns}.$$

The following proposition follows immediately.

Proposition 6.2. *Suppose* $u, s > 0$ *and*

$$N_{n,u}(D) \preccurlyeq 2^{ns}, \quad n \to \infty.$$

Then

$$\dim_h\left[D \cap K_u\right] \leq s.$$

Proof. The argument above shows that for all $s' > s$, $\mathcal{H}^{s'}\left[D \cap K_u\right] = 0$. □

In order to show that with probability one for SLE that

$$\dim_h\left[D \cap K_u\right] \leq s.$$

it suffices to show that

$$\mathbb{E}\left[N_{n,u}(D)\right] \preccurlyeq 2^{ns}, \quad n \to \infty.$$

Indeed, this relation and a simple Borel-Cantelli argument imply that with probability one for all $s' > s$, $N_{n,u}(D) \leq 2^{ns'}$. Note that

$$\mathbb{E}\left[N_{n,u}(D)\right] \leq c_D\, 2^{2n} \sup_{\operatorname{dist}(z,D)\leq 2^{-n}} \mathbb{P}\left\{\tau_{n,z} < \infty;\ \mathrm{hm}_{n,z}[\mathcal{B}(z, 2^{-n+1})] \geq 2^{-nu}\right\}.$$

For SLE_κ, conformal invariance of harmonic measure and distortion estimates imply that on the event $\tau_{n,z} < \infty$,

$$\mathrm{hm}_{n,z}\left[\mathcal{B}(z, 2^{-n+1})\right] \asymp 2^{-n}\, |g'_{\tau_{n,z}}(z)|.$$

Indeed, $g_{\tau_{n,z}}(\mathcal{B}(z, 2^{-n+1}))$ is a connected set whose diameter is comparable to $2^{-n}\,|g'_{\tau_{n,z}}(z)|$ and whose distance from the real axis is comparable to $2^{-n}\,|g'_{\tau_{n,z}}(z)|$. Hence, there exists $c < \infty$ such that

$$\mathbb{E}\left[N_{n,u}(D)\right] \leq c_D\, 2^{2n} \sup_{\operatorname{dist}(z,D)\leq 2^{-n}} \mathbb{P}\left\{\tau_{n,z} < \infty; |g'_{\tau_{n,z}}(z)| \geq c\, 2^{-n(u-1)}\right\}.$$

In the remainder of this section we will show that there exists $c = c_D < \infty$ such that for all n sufficiently large and all z with $\operatorname{dist}(z, D) \leq 2^{-n}$,

$$\mathbb{P}\left\{\tau_{n,z} < \infty; |g'_{\tau_{n,z}}(z)| \geq c\, 2^{-n(u-\frac{1}{2})}\right\} \leq 2^{-n\rho(u)}, \tag{77}$$

where

$$\rho(u) = \left[\frac{1}{8a} + 2a - 1\right] u + \left[\frac{1}{2} - \frac{1}{8a}\right] + \frac{1}{32au}.$$

Then from what he have said we know that with probability one,

$$\dim_h\left[D \cap \gamma(\overline{\Theta}^{\mathrm{hm}}_u)\right] \leq 2 - \rho(u - \frac{1}{2}) = F(u).$$

The second equality is a simple calculation. The remainder of this section is devoted to establishing (77).

6.3. **Weighting by the martingale.** The local martingale M_t is not a martingale because it "blows up" on the event of measure zero that z is on the path $\gamma(0,\infty)$. However, if we choose stopping times τ such as $\tau_{n,z}$ which prevent the path from getting too close to z, then the stopped process $M_{t\wedge\tau}$ is a martingale. Let $\mathbb{P}^*, \mathbb{E}^*$ denote probabilities and expectations with respect to the weighted measure. The Girsanov theorem implies that

$$dB_t = \frac{r\,X_t}{|Z_t|^2}\,dt + dW_t, \quad 0 \le t < \tau,$$

where W_t is a standard Brownian motion with respect to the measure $\mathbb{P}^*$. In particular,

$$d\Theta_t = \frac{(1-2a-r)\,X_t\,Y_t}{|Z_t|^4}\,dt - \frac{Y_t}{|Z_t|^2}\,dW_t.$$

It is useful to use a "radial" parametrization $\sigma(t)$. We write $\hat{Z}_t = Z_{\sigma(t)}$, $X_t = X_{\sigma(t)}$, etc. The radial parametrization is defined by

$$\hat{\Upsilon}_t := \Upsilon_{\sigma(t)} = e^{-2at}.$$

Note that

$$-2a\,\hat{\Upsilon}_t = \partial_t \hat{\Upsilon}_t = -2a\,\frac{\hat{Y}_t^2}{|\hat{Z}_t|^4}\,\partial_t\sigma(t),$$

which implies

$$\partial_t\sigma(t) = \frac{|\hat{Z}_t|^4}{Y_t^2}.$$

Note that

$$d\hat{\Theta}_t = (1-2a)\,\cot\hat{\Theta}_t\,dt + d\hat{B}_t,$$

and the local martingale M_t satisfies

$$d\hat{M}_t = -r\,\hat{M}_t\,\cot\hat{\Theta}_t\,d\hat{B}_t.$$

$$d\hat{\Theta}_t = (1-2a-r)\,\cot\hat{\Theta}_t\,dt + d\hat{W}_t, \tag{78}$$

where $\hat{B}_t, \hat{W}_t$ are standard Brownian motions with respect to $\mathbb{P}$ and $\mathbb{P}^*$ respectively. Since $1-2a-r > 1/2$, in the measure $\mathbb{P}^*$, $\hat{\Theta}_t$ never reaches $\{0,\pi\}$ and we can show that $\hat{M}_t$ is actually a martingale. Also, the invariant probability density for the SDE (78) equals

$$f(\theta) = c\,\sin^{2(1-2a-r)}\theta.$$

Note that $\sin^r$ is integrable with respect to $f(\theta)\,d\theta$. The important fact for us, is that there is a c such that if $\hat{\Theta}_t$ satisfies (78) with $\sin\hat{\Theta}_0 \ge c_1$, then for all $t > 0$,

$$\mathbb{E}^*[\hat{S}_t^r] \le c. \tag{79}$$

(Here we use $r < 2 - 3a$.)

Let

$$\tau_s = \inf\left\{t : \Upsilon_t = e^{-2as}\right\}.$$

For $r < \frac{1}{2} - 2a$, with have for all s,

$$\mathbb{P}^*\{\tau_s < \infty\} = 1.$$

Then, using (79),

$$\begin{aligned}
\mathbb{E}\left[|g'_{\tau_s}(z)|^{\lambda+r}; \tau_s < \infty\right] &= \mathbb{E}\left[M_{\tau_s}\, S^r_{\tau_s}\, \Upsilon^{-\xi-r}_{\tau_s}; \tau_s < \infty\right] \\
&= e^{2as(\xi+r)}\, \mathbb{E}\left[M_{\tau_s}\, S^r_{\tau_s}; \tau_s < \infty\right] \\
&= e^{2as(\xi+r)}\, M_0(s)\, \mathbb{E}^*\left[\hat{S}^r_s\right] \\
&\leq c\, e^{2as(\xi+r)}.
\end{aligned}$$

Since $\lambda + r > 0$,

$$\mathbb{P}\left\{\tau_s < \infty; |g'_{\tau_s}(z)| \geq \epsilon^{u-\frac{1}{2}}\right\} \leq \epsilon^{-(u-\frac{1}{2})(\lambda+r)}\, \mathbb{E}[|g'_{\tau_s}(z)|^{\lambda+r}; \tau_s < \infty] \leq c\, \epsilon^{\rho(u)}$$

where

$$\rho(u) = -(u - \frac{1}{2})(\lambda + r) - (r + \xi).$$

Doing the algebra, we get

$$\rho(u) = \left[\frac{1}{8a} + 2a - 1\right] u + \left[\frac{1}{2} - \frac{1}{8a}\right] + \frac{1}{32au}.$$

This proves (77).

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