

SCALING LIMITS AND SLE

Lecture Notes, PIMS Summer School 2010

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1 Scaling limits of lattice models

The Schramm-Loewner evolution (SLE) is a measure on continuous curves that is a candidate for the scaling limit for discrete planar models in statistical physics. Although our lectures will focus on the continuum model, it is hard to understand *SLE* without knowing some of the discrete models that motivate it. In this lecture, I will introduce some of the discrete models. By assuming some kind of “conformal invariance” in the limit, we will arrive at some properties that we would like the continuum measure to satisfy.

1.1 Self-avoiding walk (SAW)

A self-avoiding walk (SAW) of length n in the integer lattice $\mathbb{Z}^2 = \mathbb{Z} + i\mathbb{Z}$ is a sequence of lattice points

$$\omega = [\omega_0, \dots, \omega_n]$$

with $|\omega_j - \omega_{j-1}| = 1, j = 1, \dots, n$, and $\omega_j \neq \omega_k$ for $j < k$. If J_n denotes the number of SAWs of length n with $\omega_0 = 0$, it is well known that

$$J_n \approx e^{\beta n}, \quad n \rightarrow \infty,$$

where e^β is the *connective constant* whose value is not known exactly. Here \approx means that $\log J_n \sim \beta n$ where $f(m) \sim g(m)$ means $f(m)/g(m) \rightarrow 1$. In fact, it is believed that there is an exponent, usually denoted γ , such that

$$J_n \asymp n^{\gamma-1} e^{\beta n}, \quad n \rightarrow \infty,$$

where \asymp means that each side is bounded by a constant times the other. The exponent ν is defined roughly by saying that the typical diameter (with respect to the uniform probability measure on SAWs of length n with $\omega_0 = 0$) is of order n^ν . The constant β is special to the square lattice, but the exponents ν and γ are examples of lattice-independent *critical exponents* that should be observable in a “continuum limit”. For example, we would expect the fractal dimension of the paths in the continuum limit to be $d = 1/\nu$.

To take a continuum limit we let $\delta > 0$ and

$$\omega^\delta(j\delta^d) = \delta \omega(j).$$

We can think of ω^δ as a SAW on the lattice $\delta\mathbb{Z}^2$ parametrized so that it goes a distance of order one in time of order one. We can use linear interpolation to make $\omega^\delta(t)$ a continuous curve. Consider the square in \mathbb{C}

$$D = \{x + iy : -1 < x < 1, -1 < y < 1\},$$

and let $z = -1, w = 1$. For each integer N we can consider a finite measure on continuous curves $\gamma : (0, t_\gamma) \rightarrow D$ with $\gamma(0+) = z, \gamma(t_\gamma) = w$ obtained as follows. To each SAW ω of length n in \mathbb{Z}^2 with $\omega_0 = -N, \omega_n = N$ and $\omega_1, \dots, \omega_{n-1} \in ND$ we give measure $e^{-\beta n}$. If we identify ω with $\omega^{1/N}$ as above, this gives a measure on curves in D from z to w . The total mass of this measure is

$$Z_N(D; z, w) := \sum_{\omega: Nz \rightarrow Nw, \omega \subset ND} e^{-\beta|\omega|}.$$

It is conjectured that there is a b such that as $N \rightarrow \infty$,

$$Z_N(D; z, w) \sim C(D; z, w) N^{-2b}. \quad (1)$$

Moreover, if we multiply by N^{2b} and take a limit, then there is a measure $\mu_D(z, w)$ of total mass $C(D; z, w)$ supported on simple (non self-intersecting) curves from z to w in D . The dimension of these curves will be $d = 1/\nu$.

Similarly, if D is another domain and $z, w \in \partial D$, we can consider SAWs from z to w in D . If ∂D is smooth at z, w , then (after taking care of the local lattice effects — we will not worry about this here), we define the measure as above, multiply by N^{2b} and take a limit. We conjecture that we get a measure $\mu_D(z, w)$ on simple curves from z to w in D . We write the measure $\mu_D(z, w)$ as

$$\mu_D(z, w) = C(D; z, w) \mu_D^\#(z, w),$$

where $\mu_D^\#(z, w)$ denotes a probability measure.

It is believed that the scaling limit satisfies some kind of “conformal invariance”. To be more precise we assume the following conformal *covariance* property: if $f : D \rightarrow f(D)$ is a conformal transformation and f is differentiable in neighborhoods of $z, w \in \partial D$, then

$$f \circ \mu_D(z, w) = |f'(z)|^b |f'(w)|^b \mu_{f(D)}(f(z), f(w)).$$

In other words the total mass satisfies the scaling rule

$$C(D; z, w) = |f'(z)|^b |f'(w)|^b C(f(D); f(z), f(w)),$$

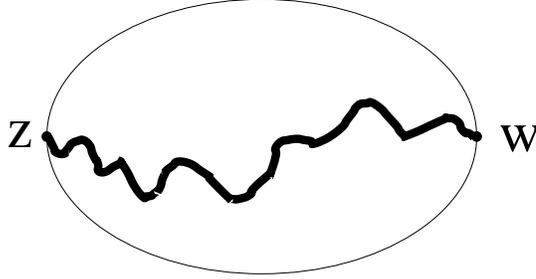


Figure 3: Scaling limit of SAW in a different domain

- **Ignore the parametrization.** We consider two curves equivalent if one is an (increasing) reparametrization of the other. In this case we do not need to specify how we parametrize $f \circ \gamma$.
- **Scaling by the dimension d .** If γ has the parametrization as given in the limit, then the amount of time need for $f \circ \gamma$ to traverse $f(\gamma[t_1, t_2])$ is

$$\int_{t_1}^{t_2} |f'(\gamma(s))|^d ds. \quad (2)$$

In either case, if we start with the probability measure $\mu_D^\#(z, w)$, the transformation $\gamma \mapsto f \circ \gamma$ induces a probability measure which we call $f \circ \mu_D^\#(z, w)$.

There are two more properties that we would expect the family of measures $\mu_D(z, w)$ to have. The first of these will be shared by all the examples in this section while the second will not. We just state the properties, and leave it to the reader to see why one would expect them in the limit.

- **Domain Markov property.** Consider the measure $\mu_D^\#(z, w)$ and suppose an initial segment of the curve $\gamma(0, t]$ is observed. Then the conditional distribution of the remainder of the curve given $\gamma(0, t]$ is the same as $\mu_{D \setminus \gamma(0, t]}^\#(\gamma(t), w)$.
- **Restriction property.** Suppose $D_1 \subset D$. Then $\mu_{D_1}(z, w)$ is $\mu_D(z, w)$ restricted to paths that lie in D_1 . In terms of Radon-Nikodym derivatives, this can be phrased as

$$\frac{d\mu_{D_1}(z, w)}{d\mu_D(z, w)}(\gamma) = 1_{\{\gamma(0, t_\gamma) \subset D_1\}}.$$

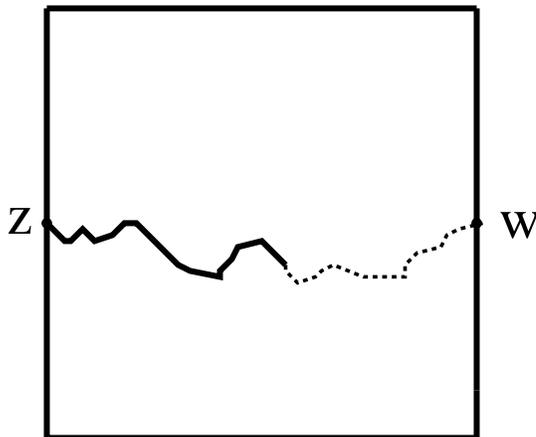


Figure 4: Domain Markov property

We have considered the case where $z, w \in \partial D$. We could consider $z \in \partial D, w \in D$. In this case the measure is defined similarly, but (1) becomes

$$Z_D(z, w) \sim C(D; z, w) N^{-b} N^{-\tilde{b}},$$

where \tilde{b} is a different exponent (see Lectures 5 and 6). The limiting measure $\mu_D(z, w)$ would satisfy the conformal covariance rule

$$f \circ \mu_D(z, w) = |f'(z)|^b |f'(w)|^{\tilde{b}} \mu_{f(D)}(f(z), f(w)).$$

Similarly we could consider $\mu_D(z, w)$ for $z, w \in D$.

1.2 Loop-erased random walk

We start with simple random walk. Let ω denote a nearest neighbor random walk from z to w in D . We no longer put in a self-avoidance constraint. We give each walk ω measure $4^{-|\omega|}$ which is the probability that the first n steps of an ordinary random walk in \mathbb{Z}^2 starting at z are ω . The total mass of this measure is the probability that a simple random walk starting at z immediately goes into the domain and then leaves the domain at w . Using the “gambler’s ruin” estimate for one-dimensional random walk, one can show that the total mass of this measure decays like $O(N^{-2})$; in fact

$$Z_N(D; z, w) \sim C(D; z, w) N^{-2}, \quad N \rightarrow \infty, \quad (3)$$

where $C(D; z, w)$ is the “excursion Poisson kernel”, $H_{\partial D}(z, w)$, defined to be the normal derivative of the Poisson kernel $H_D(\cdot, w)$ at z . In the notation of the previous

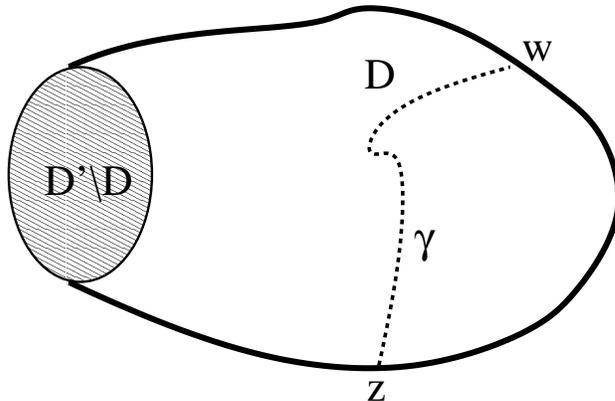


Figure 5: Illustrating the restriction property

section $b = 1$. For each realization of the walk, we produce a self-avoiding path by erasing the loops in chronological order.

Again we are looking for a continuum limit $\mu_D(z, w)$ with paths of dimension d (not the same d as for SAW). The limit should satisfy

- Conformal covariance
- Domain Markov property

However, we would not expect the limit to satisfy the restriction property. The reason is that the measure given to each self-avoiding walk ω by this procedure is determined by the number of ordinary random walks which produce ω after loop erasure. If we make the domain smaller, then we lose some random walks that would produce ω and hence the measure would be smaller. In terms of Radon-Nikodym derivatives, we would expect

$$\frac{d\mu_{D_1}(z, w)}{d\mu_D(z, w)} < 1.$$

1.3 Percolation

Suppose that every point in the triangular lattice in the upper half plane is colored black or white independently with each color having probability $1/2$. A typical realization is illustrated in Figure 8 (if one ignores the bottom row).

We now put a boundary condition on the bottom row as illustrated — all black on one side of the origin and all white on the other side. For any realization of the coloring, there is a unique curve starting at the bottom row that has all white vertices

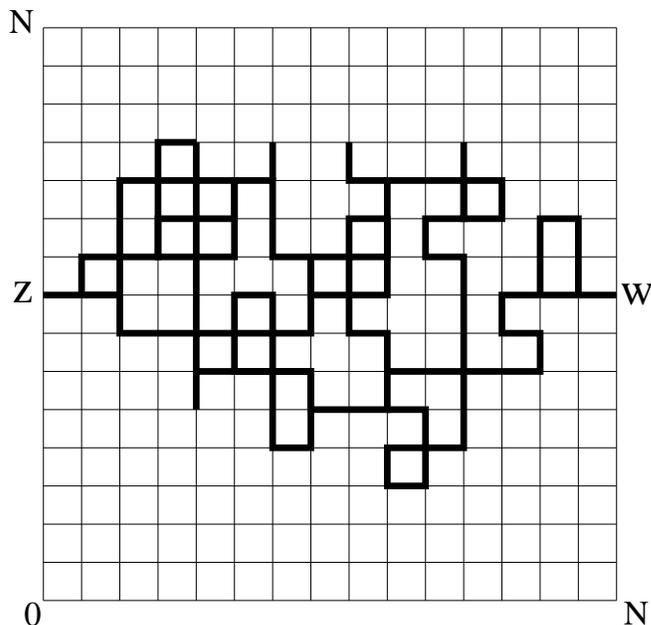


Figure 6: Simple random walk in D

on one side and all black vertices on the other side. This is called the *percolation exploration process*. Similarly we could start with a domain D and two boundary points z, w ; give a boundary condition of black on one of the arcs and white on the other arc; put a fine triangular lattice inside D ; color vertices black or white independently with probability $1/2$ for each; and then consider the path connecting z and w . In the limit, one might hope for a continuous interface. In comparison to the previous examples, the total mass of the lattice measures is one; another way of saying this is that $b = 0$. We suppose that the curve is conformally invariant, and one can check that it should satisfy the domain Markov property.

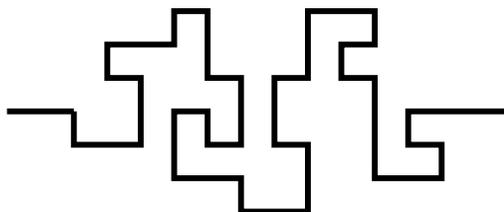


Figure 7: The walk obtained from erasing loops chronologically

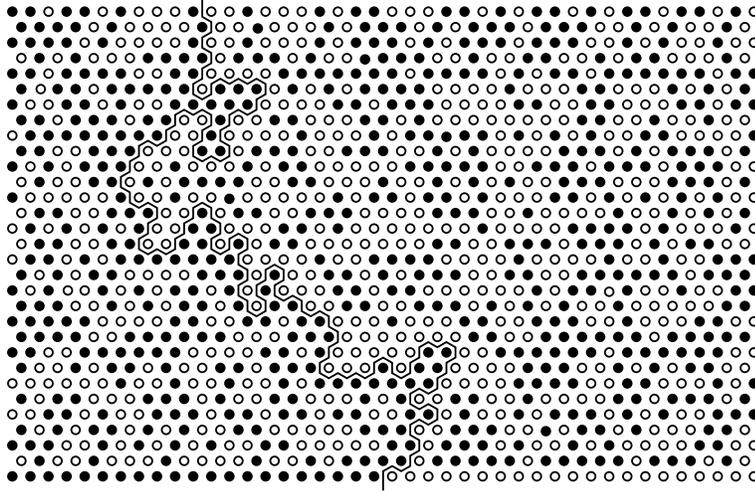


Figure 8: The percolation exploration process.

The scaling limit of percolation satisfies another property called the *locality property*. Suppose $D_1 \subset D$ and $z, w \in \partial D \cap \partial D_1$ as in Figure 5. Suppose that only an initial segment of γ is seen. To determine the measure of the initial segment, one only observes the value of the percolation cluster at vertices adjoining γ . Hence the measure of the path is the same whether it is considered as a curve in D_1 or a curve in D . The locality property is stronger than the restriction property which SAW satisfies. The restriction property is a similar statement that holds for the entire curve γ but not for all initial segments of γ .

1.4 Ising model

The Ising model is a simple model of a ferromagnet. We will consider the triangular lattice as in the previous section. Again we color the vertices black or white although we now think of the colors as spins. If x is a vertex, we let $\sigma(x) = 1$ if x is colored black and $\sigma(x) = -1$ if x is colored white. The measure on configurations is such that neighboring spins like to have the same sign.

It is easiest to define the measure for a finite collection of spins. Suppose D is a bounded domain in \mathbb{C} with two marked boundary points z, w which give us two boundary arcs. We consider a fine triangular lattice in D ; and fix boundary conditions $+1$ and -1 respectively on the two boundary arcs. Each configuration of spins is given energy

$$\mathcal{E} = - \sum_{x \sim y} \sigma(x) \sigma(y),$$

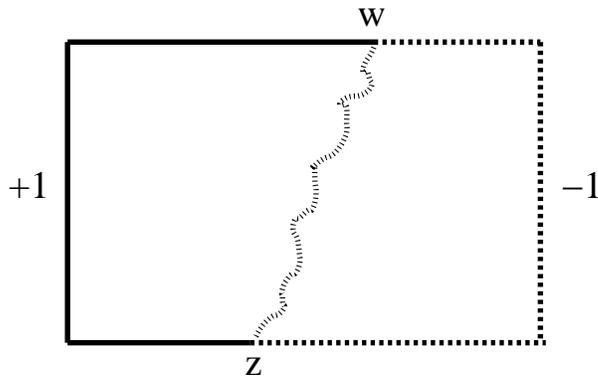


Figure 9: Ising interface.

where $x \sim y$ means that x, y are nearest neighbors. We then give measure $e^{-\beta\mathcal{E}}$ to a configuration of spins. If β is small, then the correlations are localized and spins separated by a large distance are almost independent. If β is large, there is long-range correlation. There is a critical β_c that separates these two phases.

For each configuration of spins there is a well-defined boundary between $+1$ spins and -1 spins defined in exactly the same way as the percolation exploration process. At the critical β_c it is believed that this gives an interesting fractal curve and that it should satisfy conformal covariance and the domain Markov property.

1.5 Assumptions on limits

Our goal is to understand the possible continuum limits for these discrete models. We will discuss the boundary to boundary case here but one can also have boundary to interior or interior to interior. (The terms “surface” and “bulk” are often used for boundary and interior.) Such a limit is a measure $\mu_D(z, w)$ on curves from z to w in D which can be written

$$\mu_D(z, w) = C(D; z, w) \mu_D^\#(z, w),$$

where $\mu_D^\#(z, w)$ is a probability measure. The existence of $\mu_D(z, w)$ assumes smoothness of ∂D near z, w , but the probability measure $\mu_D^\#(z, w)$ exists even without the smoothness assumption. The two basic assumptions are:

- Conformal covariance of $\mu_D(z, w)$ and conformal invariance of $\mu_D^\#(z, w)$.
- Domain Markov property.

The starting point for the Schramm-Loewner evolution is to show that *if we ignore the parametrization of the curves*, then there is only a one-parameter family of probability measures $\mu_D^\#(z, w)$ for *simply connected* domains D that satisfy conformal invariance and the domain Markov property. We will construct this family. The parameter is usually denoted $\kappa > 0$. By the Riemann mapping theorem, it suffices to construct the measure for one simply connected domain and the easiest is the upper half plane \mathbb{H} with boundary points 0 and ∞ . As we will see, there are a number of ways of parametrizing these curves.

2 Complex variables and conformal mappings

In order to study *SLE*, one needs to know some basic facts about conformal maps. We will discuss the main results here. For proofs and more details see, e.g., [2, 3, 4].

2.1 Review of complex analysis

Definition

- A *domain* in \mathbb{C} is an open, connected subset. Two main examples are the *unit disk*

$$\mathbb{D} = \{z : |z| < 1\}$$

and the *upper half plane*

$$\mathbb{H} = \{z = x + iy : y > 0\}.$$

- The *Riemann sphere* is the set $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ with the topology of the sphere which is to say that the open neighborhoods of ∞ are the complements of compact subsets of \mathbb{C} .
- A domain $D \subset \mathbb{C}$ is *simply connected* if its complement in \mathbb{C}^* is a connected subset of \mathbb{C}^* .
- A domain $D \subset \mathbb{C}$ is *finitely connected* if its complement in \mathbb{C}^* has a finite number of connected components.
- A function $f : D \rightarrow \mathbb{H}$ is *holomorphic* or *analytic* if the complex derivative $f'(z)$ exists at every point.

If $0 \in D$, and f is holomorphic on D then we can write

$$f(z) = \sum_{j=0}^{\infty} a_j z^j,$$

where the radius of convergence is at least $\text{dist}(0, \partial D)$. In particular, if $f(0) = 0$, then either f is identically zero, or there exists a nonnegative integer n such that

$$f(x) = z^n g(z),$$

where g is holomorphic in a neighborhood of 0 with $g(0) \neq 0$. In particular, if $f'(0) \neq 0$, then f is locally one-to-one, but if $f'(0) = 0$, it is not locally one-to-one.

If we write a holomorphic function $f = u + iv$, then the functions u, v are harmonic functions and satisfy the Cauchy-Riemann equations

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v.$$

Conversely, if u is a harmonic function on D and $z \in D$, then we can find a unique holomorphic function $f = u + iv$ with $v(z) = 0$ defined in a neighborhood of z by solving the Cauchy-Riemann equations. It is not always true that f can be extended to all of D , but there is no problem if D is simply connected.

Proposition 1. *Suppose $D \subset \mathbb{C}$ is simply connected.*

- *If u is a harmonic function on D and $z \in D$, there is a unique holomorphic function $f = u + iv$ on D with $v(z) = 0$.*
- *If f is a holomorphic function on D with $f(z) \neq 0$ for all z , then there exists a holomorphic function g on D with $e^g = f$. In particular, if $w \in \mathbb{C} \setminus \{0\}$ and $h = e^{g/w}$, then $h^w = f$.*

The *Cauchy integral formula* states that if f is holomorphic in a domain containing the closed unit disk, then

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(z) dz}{z^{n+1}}.$$

In particular, if f is holomorphic on \mathbb{D} ,

$$|f^{(n)}(0)| \leq n! \|f\|_{\infty}.$$

Here $\|f\|_{\infty}$ denotes the maximum of f on $\overline{\mathbb{D}}$ which (by the $n = 0$ case which is called the *maximum principle*) is the same as the maximum on $\partial \mathbb{D}$.

Proposition 2. Suppose f is a holomorphic function on a domain D . Suppose $z \in D$, and let

$$d_z = \text{dist}(z, \partial D), \quad M_z = \sup\{|f(w)| : |w - z| < d_z\}.$$

Then,

$$|f^{(n)}(z)| \leq n! d_z^{-n} M_z.$$

Proof. Consider $g(w) = f(z + d_z w)$. □

A similar estimate exists for harmonic functions in \mathbb{R}^d . It can be proved by representing a harmonic function in the unit ball in terms of the Poisson kernel.

Proposition 3. For all positive integers d, n , there exists $C(d, n) < \infty$ such that if u is a harmonic function on the unit ball $U = \{x \in \mathbb{R}^d : |x| < 1\}$ and D denotes an n th order mixed partial,

$$|Du(0)| \leq C(d, n) \|u\|_\infty.$$

Derivative estimates allow one to show “equicontinuity” results. We write $f_n \implies f$ if for every compact $K \subset D$, f_n converges to f uniformly on K . We state the following for holomorphic functions, but a similar result holds for harmonic functions.

Proposition 4. Suppose D is a domain.

- If f_n is a sequence of holomorphic functions on D , and $f_n \implies f$, then f is holomorphic.
- If f_n is a sequence of holomorphic functions on D that is locally bounded, then there exists a subsequence f_{n_j} and a (necessarily, holomorphic) function f such that $f_{n_j} \implies f$.

Proposition 5 (Schwarz lemma). If $f : \mathbb{D} \rightarrow \mathbb{D}$ is holomorphic with $f(0) = 0$, then $|f(z)| \leq |z|$ for all z . If f is not a rotation, $|f'(0)| < 1$ and $|f(z)| < |z|$ for all $z \neq 0$.

Proof. Let $g(z) = f(z)/z$ with $g(0) = f'(0)$ and use the maximum principle. □

2.2 Conformal transformations

Definition

- A holomorphic function $f : D \rightarrow D_1$ is called a *conformal transformation* if it is one-to-one and onto.
- Two domains D, D_1 are *conformally equivalent* if there exists a conformal transformation $f : D \rightarrow D_1$.

It is easy to verify that “conformally equivalent” defines an equivalence relation. A necessary, but not sufficient, condition for a holomorphic function f to be a conformal transformation onto $f(D)$ is $f'(z) \neq 0$ for all z . Functions that satisfy this latter condition can be called *locally conformal*. The function $f(z) = e^z$ is a locally conformal transformation on \mathbb{C} that is not a conformal transformation. Proving global injectiveness can be tricky, but the following lemma gives a very useful criterion.

Proposition 6 (Hurwitz). *Suppose f_n is a sequence of one-to-one holomorphic functions on a domain D and $f_n \implies f$. Then either f is constant or f is one-to-one.*

This is the big theorem.

Theorem 1 (Riemann mapping theorem). *If $D \subset \mathbb{C}$ is a proper, simply connected domain, and $z \in D$, there exists a unique conformal transformation*

$$f : \mathbb{D} \rightarrow D$$

with $f(0) = z, f'(0) > 0$.

Proof. The hard part is existence. We will not discuss the details, but just list the major steps. The necessary ingredients to fill in the details are Propositions 1, 2, 4, 5, and 6. Consider the set \mathcal{A} of conformal transformations $g : D \rightarrow g(D)$ with $g(z) = 0, g(D) \subset \mathbb{D}$. Then one proceeds to show:

- \mathcal{A} is nonempty.
- $M := \sup\{g'(z) : g \in \mathcal{A}\} < \infty$.
- There exists $\hat{g} \in \mathcal{A}$ with $\hat{g}'(z) = M$.
- $\hat{g}(D) = \mathbb{D}$

□

If D is a simply connected domain, then to specify the conformal transformation $f : \mathbb{D} \rightarrow D$ one needs to specify two quantities: $z = f(0)$ and the argument of $f'(0)$. We can think of this as “three real degrees of freedom”. Similarly, to specify the map it suffices to specify where three boundary points are sent.

The Riemann mapping theorem does not say anything about the limiting behavior of $f(z)$ as $|z| \rightarrow 1$. One needs to assume more assumptions in order to obtain further results.

Proposition 7. *Suppose D is a simply connected domain and $f : \mathbb{D} \rightarrow D$ is a conformal transformation.*

- *If $\mathbb{C} \setminus D$ is locally connected, then f extends to a continuous function on \mathbb{D} .*
- *If ∂D is a Jordan curve (that is, homeomorphic to the unit circle), then f extends to a homomorphism of $\overline{\mathbb{D}}$ onto \overline{D} .*

2.3 Univalent function

Definition

- A function f is *univalent* if f is holomorphic and one-to-one.
- A univalent function f on \mathbb{D} with $f(0) = 0$, $f'(0) = 1$ is called a *schlicht* function. Let \mathcal{S} denote the set of schlicht functions.

The Riemann mapping theorem implies that there is a one-to-one correspondence between proper simply connected domains D containing the origin and $(0, \infty) \times \mathcal{S}$. Any $f \in \mathcal{S}$ has a power series expansion at the origin

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m.$$

Much of the work of classical function theory of the twentieth century was focused on estimating the coefficients a_n of the schlicht functions. Three early results are:

- [Bieberbach] $|a_2| \leq 2$
- [Loewner] $|a_3| \leq 3$
- [Littlewood] For all n , $|a_n| < en$.

Bieberbach's conjecture was that the coefficients were maximized when the simply connected domain D was as large as possible under the constraint $f'(0) = 1$. A good guess would be that a maximizing domain would be of the form

$$D = \mathbb{C} \setminus (-\infty, -x]$$

for some $x > 0$, where x is determined by the condition $f'(0) = 1$. It is not hard to show that the *Koebe function*

$$f_{\text{Koebe}}(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n,$$

maps \mathbb{D} conformally onto $\mathbb{C} \setminus (-\infty, -1/4]$. This led to Bieberbach's conjecture which was proved by de Branges.

Theorem 2 (de Branges). *If $f \in \mathcal{S}$, then $|a_n| \leq n$ for all n .*

To study *SLE*, it is not necessary to use as powerful a tool as de Branges' theorem. Indeed, the estimates of Bieberbach and Littlewood above suffice for most problems. A couple of other simpler results are very important for analysis of conformal maps. To motivate the first, suppose that $f : \mathbb{D} \rightarrow f(D)$ is a conformal transformation with $f(0) = 0$ and $z \in \mathbb{C} \setminus f(D)$ with $|z| = \text{dist}(0, \partial f(D))$. If one wanted to maximize $f'(0)$ under these constraints, then it seems that the best choice for $f(D)$ would be the complex plane with a radial line to infinity starting at z removed. This indeed is the case which shows that the Koebe function is a maximizer.

Theorem 3 (Koebe-1/4). *If $f \in \mathcal{S}$, then $f(D)$ contains the open disk of radius 1/4 about the origin.*

Uniform bounds on the coefficients a_n give uniform bounds on the rate of change of $|f'(z)|$. The optimal bounds are given in the next theorem.

Theorem 4 (Distortion). *If $f \in \mathcal{S}$ and $|z| = r < 1$,*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}.$$

The distortion theorem can be used to analyze conformal transformations of non-simply connected domains since such domains are "locally simply connected". Suppose D is a domain. Let us write $z \sim w$ if

$$|z - w| < \frac{1}{2} \max \{ \text{dist}(z, \partial D), \text{dist}(w, \partial D) \}.$$

The distortion theorem implies that if $z \sim w$, then

$$\frac{|f'(z)|}{12} \leq |f'(w)| \leq 12 |f'(z)|.$$

The adjacency relation induces a metric ρ on D by $\rho(z, w) = n$ where n is the minimal length of a sequence

$$z = z_0, z_1, \dots, z_n = w,$$

with $z_j \sim z_{j-1}, j = 1, \dots, n$. We then get the inequality

$$12^{-\rho(z,w)} |f'(z)| \leq |f'(w)| \leq 12^{\rho(z,w)} |f'(z)|.$$

For simply connected domains, one can get better estimates than this using the distortion theorem directly. However, this kind of estimate applies to nonsimply connected domains (and also to estimates for positive harmonic functions in \mathbb{R}^d).

2.4 Harmonic measure and the Beurling estimate

Harmonic measure is the hitting measure by Brownian motion. If D is a domain, and B_t is a (standard) complex Brownian motion, let

$$\tau_D = \inf\{t \geq 0 : B_t \notin D\}.$$

Definition

- The harmonic measure (in D from $z \in D$) is the probability measure supported on ∂D given by

$$\text{hm}_D(z, V) = \mathbf{P}^z \{B_{\tau_D} \in V\}.$$

- The Poisson kernel, if it exists, is the function $H_D : D \times \partial D \rightarrow [0, \infty)$ such that

$$\text{hm}_D(z, V) = \int_V H_D(z, w) |dw|.$$

Conformal invariance of Brownian motion implies conformal invariance of harmonic measure and conformal covariance of the Poisson kernel. To be more specific, if $f : D \rightarrow f(D)$ is a conformal transformation,

$$\text{hm}_D(z, V) = \text{hm}_{f(D)}(f(z), f(V)),$$

$$H_D(z, w) = |f'(w)| H_{f(D)}(f(z), f(w)).$$

The latter equality requires some smoothness assumptions on the boundary; we will only need to use it when ∂D is analytic in a neighborhood of w , and hence (by Schwarz reflection) the map f can be analytically continued in a neighborhood of w . Two important examples are

$$H_{\mathbb{D}}(z, w) = \frac{1}{2\pi} \frac{1 - |z|^2}{|z - w|^2}, \quad H_{\mathbb{H}}(x + iy, \tilde{x}) = \frac{1}{\pi} \frac{y}{(x - \tilde{x})^2 + y^2}.$$

The Poisson kernel for any simply connected domain can be determined by conformal transformation and the scaling rule above. Finding explicit formulas for nonsimply connected domains can be difficult. Using the strong Markov property, we can see that if $D_1 \subset D$ and $w \in \partial D \cap \partial D_1$,

$$H_D(z, w) = H_{D_1}(z, w) + \int_{\partial D_1 \setminus \partial D} H_{D_1}(z, z_1) H_D(z_1, w) |dz_1|.$$

Example Let $D = \{z \in \mathbb{H} : |z| > 1\}$. Then

$$f(z) = z + \frac{1}{z}, \quad f'(z) = 1 - \frac{1}{z^2} = \frac{1}{z} \left[z - \frac{1}{z} \right],$$

is a conformal transformation of D onto \mathbb{H} . Therefore

$$H_D(z, e^{i\theta}) = |f'(e^{i\theta})| H_{\mathbb{H}} \left(z + \frac{1}{z}, f(e^{i\theta}) \right) = 2 [\sin \theta] H_{\mathbb{H}} \left(z + \frac{1}{z}, f(e^{i\theta}) \right)$$

If we write $z = re^{i\psi}$, then we can see that for $r \geq 2$,

$$H_D(re^{i\psi}, e^{i\theta}) = \frac{2 \sin \psi \sin \theta}{\pi r} [1 + O(r^{-1})].$$

Theorem 5 (Beurling projection theorem). *Suppose $D \subset \mathbb{D}$ is a domain containing the origin and let*

$$V = \{r < 1 : re^{i\theta} \notin D \text{ for some } 0 \leq \theta < 2\pi\}.$$

Then,

$$\text{hm}_D(0, \partial\mathbb{D}) \leq \text{hm}_{\mathbb{D} \setminus V}(0, \partial\mathbb{D}).$$

This theorem is particular important when $\mathbb{D} \setminus D$ is a connected set connected to $\partial\mathbb{D}$. By conformal invariance, one can show that

$$\text{hm}_{\mathbb{D} \setminus [r, 1)}(0, \partial D) \asymp r^{-1/2}.$$

This leads to the following corollary.

Proposition 8 (Beurling estimate). *There exists $c < \infty$ such that if D is a simply connected domain containing the origin, B_t is a standard Brownian motion starting at the origin, and $r = \text{dist}(0, \partial D)$, then*

$$\mathbf{P} \{B[0, \tau_D) \not\subset \mathbb{D}\} \leq cr^{1/2}.$$

In particular,

$$\text{hm}_D(0, \partial D \setminus \mathbb{D}) \leq cr^{1/2}.$$

2.5 Multiply connected domains

Since conformal transformations are also topological homeomorphisms, topological properties must be preserved. In particular, nonsimply connected domains are not conformally equivalent to simply connected domains. In fact, for nonsimply connected domains, topological equivalence is not sufficient for conformal equivalence. When considering domains, isolated points in the complement are not interesting because they can be added to the domain. Let \mathcal{R} denote the set of domains that are proper subsets of the Riemann sphere and whose boundary contains no isolated points. (In particular, \mathbb{C} is not in \mathcal{R} because its complement is an isolated point in the sphere; if we add this point to the domain, then the domain is no longer a proper subset.)

Definition A domain $D \in \mathcal{R}$ is *k-connected* if its complement consists of $k + 1$ connected components. Let \mathcal{R}_k denote the set of k -connected domains in \mathcal{R} .

The Riemann mapping theorem states that all domains in \mathcal{R}_0 are conformally equivalent. Domains in \mathcal{R}_1 are called *conformal annuli*. One example of such a domain is

$$A_r = \{z \in \mathbb{C} : r < |z| < 1\}, \quad 0 < r < 1.$$

The next theorem states that there is a one-parameter family of equivalence classes of 1-connected domains.

Theorem 6. *If $0 < r_1 < r_2 < \infty$, then A_{r_1} and A_{r_2} are not conformally equivalent. If $D \in \mathcal{R}_1$, then there exists a (necessarily unique) r such that D is conformally equivalent to A_r .*

The next theorem shows that the equivalence classes for \mathcal{D}_k are parameterized by $3k - 2$ variables. Let \mathcal{R}_k^* denote the set of domains D in \mathcal{R}_k of the form

$$D = \mathbb{H} \setminus \bigcup_{j=1}^n I_j, \quad I_j = [x_j^- + iy_j, x_j^+ + iy_j].$$

Here $x_j^-, x_j^+ \in \mathbb{R}, y_j > 0$. The set \mathcal{R}_k^* is parameterized by $3k$ variables. However, if $D \in \mathcal{R}_k^*, x \in \mathbb{R}$ and $r > 0$, then $x + D$ and rD are clearly conformally equivalent to D .

Theorem 7. *If $D_1, D_2 \in \mathcal{R}_k^*$ are conformally equivalent, then $D_1 = rD_2 + x$ for some $r > 0, x \in \mathbb{R}$. Every k -connected domain is conformally equivalent to a domain in \mathcal{R}_k^* .*

3 The Loewner differential equation

3.1 Half-plane capacity

Definition

- Let \mathcal{D} denote the set of simply connected subdomains D of \mathbb{H} such that $K = \mathbb{H} \setminus D$ is bounded.
- We call $K = \mathbb{H} \setminus D$ a (*compact*) \mathbb{H} -*hull*
- Let \mathcal{D}_0 denote the set of $D \in \mathcal{D}$ with $\text{dist}(0, K) > 0$.

If $D \in \mathcal{D}$, let g_D denote a conformal transformation of D onto \mathbb{H} . Such a transformation is not unique; indeed, if f is an conformal transformation of \mathbb{H} onto \mathbb{H} , then $f \circ g_D$ is also a transformation. In order to specify, g_D uniquely we will specify the following conditions:

$$\lim_{z \rightarrow \infty} g_D(z) - z = 0.$$

If we do this, then we can expand g_D at infinity as

$$g_D(z) = z + \frac{a(D)}{z} + O(|z|^{-2}). \quad (4)$$

Definition If K is a \mathbb{H} -hull, the half-plane capacity of K , denoted $\text{hcap}(K)$ is defined by

$$\text{hcap}(K) = a(D),$$

where $a(D)$ is the coefficient in (4).

Examples

- $K = \{z \in \mathbb{H} : |z| \leq 1\}$, $g_D(z) = z + \frac{1}{z}$, $\text{hcap}(K) = 1$.
- $K = (0, i]$, $g_D(z) = \sqrt{z^2 + 1} = z + \frac{1}{2z} + \dots$, $\text{hcap}(K) = \frac{1}{2}$.

Half-plane capacity satisfies the scaling rule

$$\text{hcap}(rK) = r^2 \text{hcap}(K).$$

The next proposition gives a characterization of hcap in terms of Brownian motion killed when it leaves D .

Proposition 9. *If K is a \mathbb{H} -hull then*

$$\begin{aligned} \text{Im}g_D(z) &= \text{Im}(z) - \mathbf{E}^z[\text{Im}(B_{\tau_D})], \\ \text{hcap}(K) &= \lim_{y \rightarrow \infty} y \mathbf{E}^{iy}[\text{Im}(B_{\tau_D})]. \end{aligned}$$

Proof. Write $g_D = u + iv$. Then v is a positive harmonic function on D that vanishes on ∂D and satisfies

$$v(x + iy) = y - a(D) \frac{y}{|z|^2} + O(|z|^{-2}), \quad z \rightarrow \infty. \quad (5)$$

In particular, $\text{Im}(z) - v(z)$ is a bounded harmonic function on D , and the optional sampling theorem implies that

$$\text{Im}(z) - v(z) = \mathbf{E}^z [\text{Im}(B_{\tau_D}) - v(B_{\tau_D})] = \mathbf{E}^z [\text{Im}(B_{\tau_D})].$$

This gives the first equality, and the second follows from (5). □

Definition The *radius (with respect to zero)* of a set is

$$\text{rad}(K) = \sup\{|w| : w \in K\}.$$

More generally, $\text{rad}(K, z) = \sup\{|w - z| : z \in K\}$.

Proposition 10. *Suppose K is an \mathbb{H} -hull and $|z| \geq 2\text{rad}(K)$. Then*

$$\mathbf{E}^z [\text{Im}(B_{\tau_D})] = \text{hcap}(K) [\pi H_{\mathbb{H}}(z, 0)] \left[1 + O\left(\frac{\text{rad}(K)}{|z|}\right) \right].$$

Sketch. By scaling, we may assume that $\text{rad}(K) = 1$. Let $\xi = \tau_{\mathbb{D}_+} = \inf\{t : B_t \in \mathbb{R} \text{ or } |B_t| = 1\}$. Note that $\xi \leq \tau_D$; indeed, any path from z that exits D at K must first visit a point in $\partial\mathbb{D}_+$. By the strong Markov property,

$$\mathbf{E}^z [\text{Im}(B_{\tau_D})] = \int_0^\pi H_{\mathbb{D}_+}(z, e^{i\theta}) \mathbf{E}^{e^{i\theta}} [\text{Im}(B_{\tau_D})] d\theta.$$

The Poisson kernel $H_{\mathbb{D}_+}(z, e^{i\theta})$ can be computed exactly. For our purposes we need the estimate

$$H_{\mathbb{D}_+}(z, e^{i\theta}) = 2 H_{\mathbb{H}}(z, 0) \sin \theta [1 + O(|z|^{-1})].$$

Therefore,

$$\frac{H_{\mathbb{D}_+}(z, e^{i\theta})}{\pi H_{\mathbb{H}}(z, 0)} = [1 + O(|z|^{-1})] \int_0^\pi \mathbf{E}^{e^{i\theta}} [\text{Im}(B_{\tau_D})] \frac{2}{\pi} \sin \theta d\theta.$$

Using the Poisson kernel $H_{\mathbb{D}_+}(iy, e^{i\theta})$, we see that

$$\text{hcap}(K) = \lim_{y \rightarrow \infty} y \mathbf{E}^{iy} [\text{Im}(B_{\tau_D})] = \int_0^\pi \mathbf{E}^{e^{i\theta}} [\text{Im}(B_{\tau_D})] \frac{2}{\pi} \sin \theta d\theta. \quad \square$$

3.2 Loewner differential equation in \mathbb{H}

Definition A *curve* is a continuous function of time. It is *simple* if it is one-to-one.

Suppose $\gamma : (0, \infty) \rightarrow \mathbb{H}$ is a simple curve with $\gamma(0+) = 0$. Let

$$K_t = \gamma(0, t], \quad D_t = \mathbb{H} \setminus K_t, \quad g_t = g_{D_t}, \quad a(t) = \text{hcap}(K_t).$$

It is not hard to show that $t \mapsto a(t)$ is a continuous, strictly increasing, function. We will also assume that $a(\infty) = \infty$.

Theorem 8 (Loewner differential equation). *Suppose γ is a simple curve as above such that a is continuous differentiable. Then for $z \in \mathbb{H}$, $g_t(z)$ satisfies the differential equation*

$$\partial_t g_t(z) = \frac{\dot{a}(t)}{g_t(z) - U_t}, \quad g_0(z) = z,$$

where $U_t = g_t(\gamma(t)) = \lim_{w \rightarrow \gamma(t)} g_t(w)$. Moreover, the function $t \mapsto U_t$ is continuous. If $z \notin \gamma(0, \infty)$, then the equation is valid for all t . If $z = \gamma(s)$, then the equation is valid for $t < s$.

Although we do not give the details, let us show where the equation arises by computing the right-derivative at time $t = 0$. Let $r_t = \text{rad}(K_t)$. If we write $g_t = u_t + iv_t$, then Proposition (10) implies

$$v_t(z) - \text{Im}(z) = -\mathbf{E}^z[\text{Im}(B_{\tau_{D_t}})] = -a(t) [\pi H_{\mathbb{H}}(z, 0)] [1 + O(r_t/|z|)].$$

Note that

$$\text{Im}[1/z] = -\pi H_{\mathbb{H}}(z, 0).$$

Hence (with a little care on the real part) we can show that

$$g_t(z) - z = a(t) [1/z] [1 + O(r_t/|z|)],$$

which implies

$$\lim_{t \rightarrow 0+} \frac{g_t(z) - z}{t} = \frac{\dot{a}(t)}{z}.$$

We see that it is convenient to parameterize the curve γ so that $a(t)$ is differentiable, and, in fact, and if we are going to go through the effort, we might as well make $a(t)$ linear.

Definition The curve γ is *parameterized by (half-plane) capacity with rate $a > 0$* if $a(t) = at$.

The usual choice is $a = 2$. In this case, if the curve is parameterized by capacity, then the Loewner equation becomes

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z. \quad (6)$$

Example Suppose $U_t = 0$ for all t . Then the Loewner equation becomes

$$\partial_t g_t(z) = \frac{2}{g_t(z)}, \quad g_0(z) = z,$$

which has the solution

$$g_t(z) = \sqrt{z^2 + 4t}, \quad K_t = (0, i2\sqrt{t}].$$

If we start with a simple curve, then the function U_t which is called the *driving function* is continuous. Let us go in the other direction. Suppose $t \mapsto U_t$ is a continuous, real-valued function. For each $z \in \mathbb{H}$ we define $g_t(z)$ as the solution to (6), then standard methods in differential equations show that the solution exists up to some time $T_z \in (0, \infty]$. We define

$$D_t = \{z : T_z > t\}.$$

Then it can be shown that g_t is a conformal transformation of D_t onto \mathbb{H} with $g_t(z) - z = o(1)$ as $z \rightarrow \infty$. We would like to define a curve γ by

$$\gamma(t) = g_t^{-1}(U_t) = \lim_{y \rightarrow 0^+} g_t^{-1}(U_t + iy). \quad (7)$$

The quantity $g_t^{-1}(U_t + iy)$ always makes sense, but it is not true that the limit can be taken for every continuous U_t . The “problem” functions U_t have the property that they move faster along the real line than the hull is growing. From the simple example above, we see that if the driving function remains constant, then in time $O(t)$ the hull grows at rate $O(\sqrt{t})$. If $U_t = o(\sqrt{t})$ for small t , then we are fine. In fact, the following holds.

Theorem 9. [12]

- There exists $c_0 > 0$ such that if U_t satisfies

$$|U_{t+s} - U_t| \leq c_0 \sqrt{s},$$

for all s sufficiently small, then the curve γ exists and is a simple curve.

- There exists $c_1 < \infty$ and a function U_t satisfying

$$|U_{t+s} - U_t| \leq c_1 \sqrt{s},$$

for all t, s for which the limit (7) does not exist for some t .

Definition Suppose $t \rightarrow U_t$ is a driving function. We say that U_t generates the curve $\gamma : [0, \infty) \rightarrow \overline{\mathbb{H}}$ if for each t , D_t is the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$.

3.3 Radial parametrization

The half-plane capacity parametrization is convenient for curves going from one boundary point to another (∞ is a boundary point of \mathbb{H}). When considering paths going from a boundary point to an interior point, it is convenient to consider the *radial parametrization* which is another kind of capacity parametrization. Suppose D is a simply connected domain and $z \in D$.

Definition If D is a simply connected domain and $z \in D$, then the *conformal radius* of z in D is defined to be $|f'(0)|$ where $f : \mathbb{D} \rightarrow D$ is a conformal transformation with $f(0) = z$. We let $\Upsilon_D(z)$ denote one-half times the conformal radius.

By definition $\Upsilon_{\mathbb{D}}(0) = 1/2$ and a straightforward calculation shows that $\Upsilon_{\mathbb{H}}(z) = \text{Im}(z)$. (We put the factor $1/2$ in the definition of $\Upsilon_D(z)$ so that the latter equation holds.)

If $\gamma : (0, \infty) \rightarrow D \setminus \{z\}$ is a simple curve with $\gamma(0+) = w, \gamma(\infty) = z$. Let $D_t = D \setminus \gamma(0, t]$.

Definition The curve γ has a *radial parametrization* (with respect to z) if

$$\log \Upsilon_{D_t}(z) = -at + r$$

for some a, r .

3.4 Radial Loewner differential equation in \mathbb{D}

Suppose γ is a simple curve as in the last subsection with $D = \mathbb{D}, |w| = 1$, and $z = 0$. For each t , let g_t be the unique conformal transformation of D_t onto \mathbb{D} with $g_t(0) = 0, g_t'(0) > 0$. We assume that the curve has the radial parametrization with $\log[2\Upsilon_{D_t}(0)] = -at/2$. In other words, $g_t'(0) = e^{at/2}$. The following is proved similarly to Theorem 8.

Theorem 10 (Loewner differential equation). *Suppose γ is a simple curve as above. Then for $z \in \mathbb{D}$, $g_t(z)$ satisfies the differential equation*

$$\partial_t g_t(z) = \frac{a}{2} g_t(z) \frac{g_t(z) + e^{i2U_t}}{g_t(z) - e^{i2U_t}}, \quad g_0(z) = z,$$

where $e^{i2U_t} = g_t(\gamma(t)) = \lim_{w' \rightarrow \gamma(t)} g_t(w')$. Moreover, the function $t \mapsto U_t$ is continuous. If $z \notin \gamma(0, \infty)$, then the equation is valid for all t . If $z = \gamma(s)$, then the equation is valid for $t < s$.

For ease, let us assume $a = 2, t = 0, U_t = 0$, for which the equation becomes

$$\partial_t g_t(z)|_{t=0} = z \frac{g_t(z) + 1}{g_t(z) - 1},$$

or

$$\partial_t [\log g_t(z)] = \frac{g_t(z) + 1}{g_t(z) - 1}.$$

The function on the right hand side is (a multiple of) the complex form of the Poisson kernel $H_{\mathbb{D}}(z, 1)$. In \mathbb{H} we considered separately the real and imaginary parts; in \mathbb{D} we consider separately the radial and angular parts, or equivalently, the real and imaginary parts of the logarithm.

When analyzing the radial equation, it is useful to consider the function $h_t(z)$ defined by

$$g_t(e^{2iz}) = e^{2ih_t(z)}.$$

Then the Loewner equation becomes

$$\partial_t h_t(z) = a \cot(h_t(z) - U_t).$$

If $h_t(z) - U_t$ is near zero, then

$$\cot(h_t(z) - U_t) \sim \frac{1}{h_t(z) - U_t},$$

and hence this can be approximated by the chordal Loewner equation.

4 Schramm-Loewner evolution (SLE_{κ})

4.1 Chordal SLE_{κ}

Suppose we have a family of probability measures $\mu_D^{\#}(z, w)$ on simple curves (modulo reparametrization) connecting distinct boundary points in simply connected domains. By considering the measure on curves in \mathbb{H} from 0 to ∞ , we see that this induces a probability measure on driving functions U_t . This measure satisfies:

- For every $s < t$, the random variable $U_t - U_s$ is independent of $\{U_r : 0 \leq r \leq s\}$ and has the same distribution as U_{t-s} .

Since U_t is also a continuous process, a well known result in probability tells us that U_t must be a one-dimensional Brownian motion. Using the fact that the measure is invariant under dilations, we can see that the drift of the Brownian motion must equal zero. This leaves one parameter κ , the variance parameter of the Brownian motion.

This gives Schramm’s definition. (Note: Schramm used the term stochastic Loewner evolution; it was renamed the Schramm-Loewner evolution by others in honor of Schramm.)

Definition The *chordal Schramm-Loewner evolution with parameter κ* (SLE_κ) (from 0 to ∞ in \mathbb{H}) is the solution to the Loewner evolution

$$\partial_t \tilde{g}_t(z) = \frac{2}{\tilde{g}_t(z) - \tilde{U}_t}, \quad \tilde{g}_0(z) = z,$$

where \tilde{U}_t is a one-dimensional Brownian motion with variance parameter κ .

As before, we let \tilde{D}_t be the domain of \tilde{g}_t and $\tilde{K}_t = \mathbb{H} \setminus \tilde{D}_t$. Under this definition, $\text{hcap}(\tilde{K}_t) = 2t$. For computation purposes, it is useful to consider $g_t = \tilde{g}_{t/\kappa}$ which satisfies

$$\partial_t g_t(z) = \frac{(2/\kappa)}{g_t(z) - \tilde{U}_{t/\kappa}}.$$

Since $\tilde{U}_{t/\kappa}$ is a standard Brownian motion, we get an alternative definition. This is the definition we will use; it is a linear time change of Schramm’s original definition. Throughout these notes we will write $a = 2/\kappa$.

Definition The *chordal Schramm-Loewner evolution with parameter κ* (SLE_κ) (from 0 to ∞ in \mathbb{H}) is the solution to the Loewner evolution

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{8}$$

where $U_t = -B_t$ is a standard one-dimensional Brownian motion and $a = 2/\kappa$. Under this parametrization, $\text{hcap}(K_t) = at$.

A Brownian motion path is Hölder continuous of all orders less than 1/2, but is not Hölder continuous of order 1/2. Hence, we cannot use the criterion of Theorem 9 to assert that this gives a random measure on curves. However, this is the case and we state this as a theorem. In many of the statements below we leave out the phrase “with probability one”.

Theorem 11. *Chordal SLE_κ is generated by a (random) curve.*

This was proved in [13] for $\kappa \neq 8$. The $\kappa = 8$ is more delicate, and the only proof involves showing that the measure is obtained as a limit of measures on discrete curves, see [6]. For $\kappa \neq 8$, the curve γ is Hölder continuous of some order (depending on κ), but for $\kappa = 8$ it is not Hölder continuous of any order $\alpha > 0$. When we speak of Hölder continuity here, we mean *with respect to the capacity parametrization*. It turns out that this is not the parametrization that gives the optimal modulus of continuity. The next theorem describes the phases of SLE .

Theorem 12.

- If $\kappa \leq 4$, then γ is a simple curve with $\gamma(0, \infty) \subset \mathbb{H}$.
- If $4 < \kappa < 8$, then γ has double points and $\gamma(0, \infty) \cap \mathbb{R} \neq \emptyset$. The curve is not plane-filling, that is to say, $\mathbb{H} \setminus \gamma(0, \infty) \neq \emptyset$.
- If $\kappa \geq 8$, then the curve is space-filling, that is, $\gamma[0, \infty) = \overline{\mathbb{H}}$.

Sketch of proof. If $\gamma(s) = \gamma(t)$ for some $s < t$, and $s < r < t$, then the image of the curve under g_r has the property that it hits the real line. Hence, whether or not there are double points is equivalent to whether or not the real line is hit. Let T be the first time that $\gamma(t) \in [x, \infty)$ where $x > 0$. Let

$$X_t = X_t(x) = g_t(x) - U_t.$$

If $T < \infty$, then $X_T = 0$. By (8), we get

$$dX_t = \frac{a}{X_t} dt + dB_t, \quad X_0 = x.$$

This is a Bessel equation and is the same equation satisfied by the absolute value of a d -dimensional Brownian motion where $a = (d - 1)/2$. It is well known that solutions of this equation avoid the origin if and only if $a \geq 1/2$ which corresponds to $\kappa \leq 4$.

To see if the curve is plane-filling, let us fix $z \in \mathbb{H}$ and ask if $\gamma(t) = z$ for some t . Let

$$Z_t = Z_t(z) = g_t(z) - U_t,$$

which satisfies

$$dZ_t = \frac{a}{Z_t} dt + dB_t.$$

(A little care must be taken in reading this equation — Z_t is complex but B_t is a real-valued Brownian motion.) We reparametrize the curve using the radial parametrization with respect to z ,

$$\hat{Z}_t = Z_{\sigma(t)}.$$

In this new parametrization, the lifetime will be finite if the curve stays away from z (for then the conformal radius does not go to zero), and the lifetime will be infinite if the curve goes to z . For ease assume $\text{Im}[z] = 1$, and let $\Theta_t = \arg[\hat{Z}_t]$. Then with the aid of some standard stochastic calculus, we can see that Θ_t satisfies the equation

$$d\Theta_t = (1 - 2a) \cot \Theta_t dt + dW_t, \quad \Theta_0 = \arg z, \tag{9}$$

where W_t is a standard Brownian motion. Whether or not the lifetime is finite in the radial parametrization boils down to whether or not a process satisfying (9) ever

reaches $\{0, \pi\}$. Recalling that $\cot \theta \sim 1/\theta$ for small θ , by comparison with the Bessel process we find that the process avoids the origin if and only if $1 - 2a \geq 1/2$ which corresponds to $\kappa \geq 8$.

□

Two other interesting facts can be derived from (9).

- The process $\arg Z_t$ is a martingale if and only if $\kappa = 4$ ($1 - 2a = 0$). Note that it does not matter which parametrization we use when we want to see if a process is a martingale. $\kappa = 4$ is related to the harmonic explorer and the Gaussian free field.
- If $\kappa < 8$, and

$$\phi(\theta) = \mathbf{P}\{\Theta_T = \pi \mid \Theta_0 = \theta\},$$

then $\phi(\Theta_{t \wedge T})$ is a martingale, and hence by Itô's formula, F satisfies

$$\frac{1}{2} \phi''(\theta) + (1 - 2a) \phi'(\theta) = 0.$$

Solving this equation, with appropriate boundary conditions gives

$$\phi(\theta) = c \int_0^\theta \sin^{4a-2} x \, dx, \quad c = \left[\int_0^\pi \sin^{4a-2} x \, dx \right]^{-1}.$$

Note that $\kappa < 8$ implies $4a - 2 > -1$, and hence the integral is finite. One can check that $\phi(\arg z)$ represents the probability that the curve γ goes to the right of z .

4.2 Dimension of the path

If $\kappa \geq 8$, the *SLE* curve is plane-filling and hence has dimension two. For this section, we assume $\kappa < 8$. Roughly speaking, the fractal dimension of the curve $\gamma[s, t]$ is given by d , where the number of balls of radius ϵ needed to cover $\gamma[s, t]$ grows like ϵ^{-d} as $\epsilon \rightarrow 0$. For fixed z , let $p(z, \epsilon)$ denote the probability that the curve gets within distance ϵ of z . Then a “back of the envelope” calculation shows that the *expected* number of balls of radius ϵ needed to cover, say,

$$\gamma(0, \infty) \cap \{z \in \mathbb{H} : |z - i| \leq 1/2\}$$

should decay like $\epsilon^{-2} p(i, \epsilon)$, or we would expect that $p(i, \epsilon) \approx \epsilon^{2-d}$.

Let us be more precise. Let $\Upsilon_t(z) = \Upsilon_{D_t}(z)$ where, as before, $\Upsilon_D(z)$ equals one-half times the conformal radius. Let $\Upsilon(z) = \lim_{t \rightarrow \infty} \Upsilon_t(z)$. Then we might expect that there is a function $\tilde{G}(z)$ and a dimension d such that

$$\mathbf{P}\{\Upsilon(z) \leq \epsilon\} \sim \tilde{G}(z) \epsilon^{2-d}, \quad \epsilon \rightarrow 0.$$

Again, let $Z_t = Z_t(z) - g_t(z)$. Then, if such a function existed, one can show that

$$|g'_t(z)|^{2-d} \hat{G}(Z_t),$$

would have to be a local martingale. Using Itô's formula, we can show that this implies that $\hat{G} = cG$ where G is the *chordal SLE $_\kappa$ Green's function*

$$G(z) = \text{Im}(z)^{2-d} [\arg z]^{4a-1}, \quad d = 1 + \frac{1}{4a} = 1 + \frac{\kappa}{8}.$$

The proof of the following is discussed in [5].

Theorem 13. *If $\kappa < 8$ and $z \in \mathbb{H}$,*

$$\lim_{\epsilon \rightarrow 0^+} \epsilon^{d-2} \mathbf{P}\{\Upsilon(z) \leq \epsilon\} = c_* G(z), \quad c_* = \left[\int_0^\pi \sin^{4a} x \, dx \right]^{-1}.$$

This “one-point” estimate is not good enough to compute the Hausdorff dimension of the path. A more difficult “two-point” estimate of the form

$$\mathbf{P}\{\Upsilon(z) \leq \epsilon \Upsilon(w) \leq \epsilon\} \asymp \epsilon^{2(2-d)} |z - w|^{d-2},$$

was proved by Beffara [1] from which he concluded the following.

Theorem 14. *If $\kappa < 8$, the SLE $_\kappa$ paths have Hausdorff dimension*

$$d = 1 + \frac{\kappa}{8}.$$

In particular, for every $1 < d < 2$, there exists a unique κ that produces paths of dimension d .

4.3 SLE in simply connected domains

The starting assumption for chordal SLE $_\kappa$ was that it was a conformally invariant family of probability measures connecting distinct boundary points. Using this, Schramm derived that there was only a one-parameter family of possible measures which he defined as SLE $_\kappa$. The definition was given in the upper half-plane, but one can then *define* SLE $_\kappa$ connecting boundary points w_1, w_2 in a simply connected domain as the image of SLE $_\kappa$ in \mathbb{H} under a conformal transformation $F : \mathbb{H} \rightarrow D$ with $F(0) = w_1, F(\infty) = w_2$.

Remarks

- This definition is really a measure on curves *modulo reparametrization*.
- The map F is not unique but if \tilde{F} is another such map one can show that $\tilde{F}(z) = F(rz)$ for some $r > 0$. Using a scale invariance (modulo reparametrization) of SLE_κ , which follows from a scaling rule for Brownian motion, we see that the definition is independent of the choice of F .

In the calculation below, two important parameters will appear.

Definition ;

- The *central charge* $\mathbf{c} = \mathbf{c}(\kappa)$ is defined by

$$\mathbf{c} = \frac{(6 - \kappa)(3\kappa - 8)}{2\kappa} = \frac{(3a - 1)(3 - 4a)}{a}$$

- The *boundary scaling exponent* $b = b(\kappa)$ is defined by

$$b = \frac{6 - \kappa}{2\kappa} = \frac{3a - 1}{2}.$$

- For any κ , its *dual value* κ' is defined by $\kappa \kappa' = 16$.

We note that $\mathbf{c} \in (-\infty, 1]$ and $\mathbf{c}(\kappa) = \mathbf{c}(\kappa')$. The relationship $\kappa \leftrightarrow \mathbf{c}(\kappa)$ two-to-one with a double at $\kappa = 4, \mathbf{c} = 1$. As κ increases from 0 to ∞ , $b(\kappa)$ decreases from ∞ to $-1/2$.

4.4 Subdomains of \mathbb{H}

Suppose that $D = \mathbb{H} \setminus K \in \mathcal{D}$ with $\text{dist}(0, K) > 0$. Let $F = F_D$ be the conformal transformation of \mathbb{H} onto D with $F(0) = 0, F(\infty) = \infty, F'(\infty) = 1$, and let $\Phi = F^{-1}$. If γ is an SLE_κ curve in \mathbb{H} , then $\gamma^*(t) = F \circ \gamma(t)$ is (a time change of) SLE_κ in D from $F(0)$ to infinity. Let us write $\gamma_t = \gamma(0, t], \gamma_t^* = \gamma_t^*(0, t]$. Since γ_t^* is a curve in \mathbb{H} , we can write its Loewner equation,

$$\partial_t g_t^*(z) = \frac{\dot{a}^*(t)}{g_t^*(z) - U_t^*},$$

where $a^*(t) = \text{hcap}[\gamma_t^*]$. Let $F_t = g_t^* \circ F \circ g_t^{-1}$, and note that $F_t(U_t) = U_t^*$ and that F_t is the corresponding conformal transformation of \mathbb{H} onto $g_t^*(D)$ with $F_t(U_t) =$

$U_t^*, F_t(\infty) = \infty, F_t'(\infty) = 1$. By using the scaling rule for the half-plane capacity, we can show that

$$\partial_t a^*(t) = a F'(U_t)^2,$$

and hence

$$\partial_t g_t^*(z) = \frac{a F'(U_t)^2}{g_t^*(z) - U_t^*}.$$

With the aid of some careful chain rule and stochastic calculus computations, one can find the stochastic differential equation satisfied by the driving function U_t^* . It turns out to be nicer if one reparametrizes so that the half-plane capacity of γ^* grows linearly at rate a . Let $\hat{U}_t = U_{\sigma(t)}^*$ denote the driving function in the time change and let $\Phi_t = F_{\sigma(t)}^{-1}$.

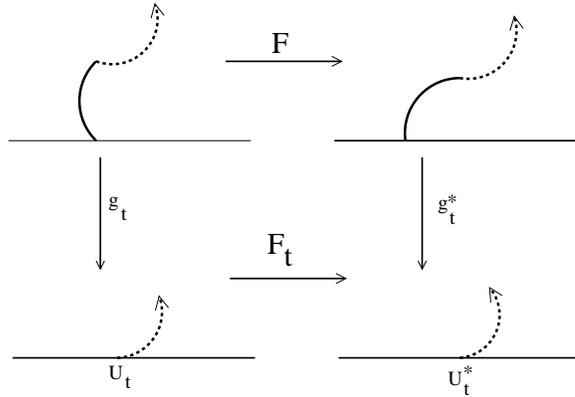


Figure 10: The maps

Proposition 11. *Under the time change, the driving function \hat{U}_t satisfies*

$$d\hat{U}_t = b \frac{\Phi_t''(\hat{U}_t)}{\Phi_t'(\hat{U}_t)} dt + dW_t,$$

where W_t is a standard Brownian motion. This is valid up to the first time that the curve hits $K = \mathbb{H} \setminus D$.

This proposition implies that there is another way to define SLE_κ in D . Consider a solution to the Loewner equation (8) where U_t satisfies the SDE

$$dU_t = b \frac{\Phi_t''(U_t)}{\Phi_t'(U_t)} dt + dW_t.$$

Here Φ_t is a conformal transformation of $g_t(D)$ onto \mathbb{H} with $\Phi_t(\infty) = \infty$, $\Phi_t'(\infty) = 1$. Let γ denote the corresponding curve and let

$$\tau = \inf\{t : \gamma(t) \in K\}.$$

Then this gives the distribution of SLE_κ in D up to time τ .

The drift term is nontrivial unless $b = 0$ which corresponds to $\kappa = 6$. This particular property of SLE_6 is called *locality*. Note that the percolation exploration process satisfies a discrete analogue of the locality property, and this is one way to see that $\kappa = 6$ should correspond to percolation.

Proposition 12 (Locality). *If γ is an SLE_6 curve in D , then the distribution of γ is the same as that of SLE_6 in \mathbb{H} up to the first time that the curve hits $\mathbb{H} \setminus D$.*

4.5 Fundamental local martingale

We will start this section by stating an important computation which first appeared in [7]. We will not motivate it now, but we will discuss implication below. Suppose γ is an SLE_κ curve in \mathbb{H} with driving function $U_t = -B_t$, and D is a domain as above. Let Φ_t, τ be defined as in the previous subsection, and recall the values of b, c . Let S denote the *Schwarzian derivative*,

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3f''(z)^2}{2f'(z)^2}.$$

For $t < \tau$, we define

$$M_t = \Phi_t'(U_t)^b \exp \left\{ -\frac{ac}{12} \int_0^t S\Phi_s(U_s) ds \right\}.$$

Note that we can write M_t as

$$M_t = C_t \Phi_t'(U_t)^b$$

where C_t has paths that are continuously differentiable in t .

Proposition 13. *M_t is a local martingale satisfying*

$$dM_t = b \frac{\Phi_t''(U_t)}{\Phi_t'(U_t)} M_t dU_t, \quad t < \tau$$

It is not hard to show that $\Phi'_t(U_t) \leq 1$, and hence for $\kappa \leq 8/3$, M_t is a bounded martingale.

An important tool in understanding the curve is *Giranosov's Theorem*. The form that we will need it is the following. Suppose M_t is a bounded martingale satisfying

$$dM_t = M_t J_t dU_t.$$

Then we can define a new probability measure by stating that if E is an event depending only on $\{U_s : 0 \leq s \leq t\}$, then

$$Q(E) = M_0^{-1} \mathbf{E}[M_t 1_E].$$

The theorem states that

$$W_t = U_t - \int_0^t J_s ds,$$

is a standard Brownian motion with respect to Q , or equivalently,

$$dU_t = J_t dt + dW_t.$$

This is a theorem about martingales, but we can apply this theorem to positive *local* martingales by choosing an increasing collection of stopping times ξ_n such that $M_{t \wedge \xi_n}$ are bounded martingales. Comparing this with Proposition 11 gives us the following.

Proposition 14. *SLE_κ weighted (in the sense of the Girsanov theorem) by M_t has the same distribution as SLE_κ in D (up to the first time that the curve hits $\mathbb{H} \setminus D$).*

Recall that for $\kappa \leq 4$, the SLE_κ curve in \mathbb{H} never hits the real line. This implies that the SLE_κ curve in D never hits $\mathbb{H} \setminus D$ and the last proposition is valid for all time. With a little more argument, one can let $t \rightarrow \infty$ and prove the following. As $t \rightarrow \infty$, $g_t(D)$ looks more and more like \mathbb{H} and from this one gets $\Phi'_t(U_t) \rightarrow 1$.

Theorem 15. *If $\kappa \leq 4$,*

$$\Phi'(0)^b = \mathbf{E}[M_0] = \mathbf{E}[M_\infty] = \mathbf{E} \left[1_{\{\gamma(0, \infty) \subset D\}} \exp \left\{ -\frac{ac}{12} \int_0^\infty S\Phi_t(U_t) dt \right\} \right].$$

4.6 Brownian loop measure

The “compensator” term

$$\exp \left\{ -\frac{ac}{12} \int_0^t S\Phi_s(U_s) ds \right\}$$

from last subsection comes from a direct calculation. However, there is a nice interpretation of this quantity in terms of loops of Brownian motions [8].

A (*rooted*) loop ω is a continuous curve $\omega : [0, t_\omega] \rightarrow \mathbb{C}$ with $\omega(0) = \omega(t_\omega)$. Such a loop can also be considered as a periodic function of period t_ω . An *unrooted* loop is a rooted loop with the root forgotten. To specify a rooted loop, one can give a triple $(z, t, \tilde{\omega})$ where $z \in \mathbb{C}$ is the root, t is the time duration, and $\tilde{\omega}$ is a loop of time duration one. To obtain ω from the triple, one uses Brownian scaling to convert $\tilde{\omega}$ to a loop of time duration t and then translates the loop so it has root z .

Definition

- The *rooted Brownian loop measure* $\tilde{\nu} = \tilde{\nu}_{\mathbb{C}}$ is the measure on rooted loops $(z, t, \tilde{\omega})$ given by

$$\text{area} \times \frac{dt}{2\pi t^2} \times \text{Brownian bridge},$$

where Brownian bridge denotes the probability measure associated to Brownian motions $B_t, 0 \leq t \leq 1$ conditioned so that $B_0 = B_1$.

- The (*unrooted*) *Brownian loop measure in \mathbb{C}* , $\nu = \nu_{\mathbb{C}}$, is the measure on unrooted loops obtained from $\tilde{\nu}_{\mathbb{C}}$ by forgetting the roots.
- If $D \subset \mathbb{C}$, the measures $\tilde{\nu}_D$ and ν_D are obtained by restricting $\tilde{\nu}$ and ν , respectively, to loops staying in D .

By definition the Brownian loop measure satisfies the “restriction property”. It is an infinite measure since small loops have large measure. However, if D is a bounded domain its importance comes from the fact that it also satisfies conformal invariance. The following theorem only holds for the measure on *unrooted* loops.

Theorem 16. *If $f : D \rightarrow f(D)$ is a conformal transformation, then*

$$f \circ \nu_D = \nu_{f(D)}.$$

The relationship between the Brownian loop measure and the compensator is as follows. If D is a domain and $V_1, V_2 \subset D$, let $\Lambda(D; V_1, V_2)$ denote the Brownian loop measure of the set of loops in D that intersect both V_1 and V_2 . Then, by analyzing the Brownian loop measure infinitesimally, we have the following.

Proposition 15. *Suppose γ is a curve in \mathbb{H} parameterized so that $\text{hcap}(\gamma_t) = at$ and let $D = \mathbb{H} \setminus K \in \mathcal{D}$. Then if*

$$t < \tau := \inf\{s : \gamma(s) \in K\},$$

we have

$$-\frac{a}{6} \int_0^t S\Phi_s(U_s) ds = \Lambda(D; \gamma_t, K).$$

Therefore, the fundamental local martingale can be written as

$$M_t = \Phi'_t(U_t)^b \exp \left\{ \frac{\mathbf{c}}{2} \Lambda(D; \gamma_t, K) \right\},$$

and if $\kappa \leq 4$, we have a limiting value

$$M_\infty = 1\{\gamma \subset D\} \exp \left\{ \frac{\mathbf{c}}{2} \Lambda(D; \gamma, K) \right\},$$

where $\gamma = \gamma_\infty$. Combining this we get the main theorem comparing SLE_κ in D to $SLE_\kappa \in \mathbb{H}$.

Theorem 17. *Suppose $D = \mathbb{H} \setminus K \in \mathcal{D}$ with $\text{dist}(0, K) > 0$ and $\kappa \leq 4$. Let γ be an SLE_κ curve from 0 to ∞ in \mathbb{H} defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and let*

$$M_\infty = 1\{\gamma \subset D\} \exp \left\{ \frac{\mathbf{c}}{2} \Lambda(D; \gamma, K) \right\}.$$

Then,

$$\mathbf{E}[M_\infty] = \Phi'_D(0)^b.$$

Moreover, if Q is defined by

$$dQ = \frac{M_\infty}{\Phi'_D(0)^b} d\mathbf{P},$$

then has the distribution of γ with respect to Q is that of SLE_κ in D from 0 to ∞ .

4.7 SLE_κ as a nonprobability measure

For this subsection, we assume that $\kappa \leq 4$ so that SLE_κ is supported on simple paths. If D is a simply connected domain, we say that $z, w \in \partial D$ are *smooth (boundary points of D)* if ∂D is locally analytic near z, w . For any such D , there is a unique conformal transformation

$$F = F_D : \mathbb{H} \rightarrow D$$

with $F(0) = z, F(\infty) = w, |F'(\infty)| = 1$. Here we abuse notation somewhat, to write $F(\infty) = w, |F'(\infty)| = 1$ to denote that as $w' \rightarrow \infty$,

$$F(w') = w - \frac{1}{w'} \mathbf{n},$$

where \mathbf{n} denotes the inward unit normal in D at w . We also call $0, \infty$ smooth boundary points for domains $D \in \mathcal{D}$. In this case, the map F is the same as the F defined in Section 4.4.

Definition Suppose D is a simply connected domain and z, w are smooth. If $\kappa \leq 4$, the (unparametrized) SLE_κ measure $\mu_D(z, w)$ is defined by

$$\mu_D(z, w) = C_D(z, w) \mu_D^\#(z, w),$$

where:

- $\mu_D^\#(z, w)$ is the probability measure on paths (modulo reparametrization) obtained as the image of SLE_κ in \mathbb{H} under F .
- $C_D(z, w) = |F'(z)|^b$.

In particular, we have normalized the measure so that $C_{\mathbb{H}}(0, \infty) = 1$. In fact, using the scaling rule for the Poisson kernel, we can see that for all simply connected domains

$$C_D(z, w) = [\pi H_{\partial D}(z, w)]^b,$$

where $H_D(z, w)$ is the *excursion or boundary Poisson kernel* defined as the normal derivative of $H_D(\cdot, w)$ at z . In particular,

$$C_{\mathbb{H}}(0, x) = |x|^{-2b}.$$

We can summarize a number of the results in this section as follows.

Theorem 18. *Assume $\kappa \leq 4$, D, D' are simply connected domains, and z, w are smooth boundary points.*

- **Conformal covariance.** *If $f : D \rightarrow f(D)$ is a conformal transformation and $f(z), f(w)$ are smooth boundary points of $f(D)$, then*

$$f \circ \mu_D(z, w) = |f'(z)|^b |f'(w)|^b \mu_{f(D)}(f(z), f(w)).$$

- **Boundary perturbation.** *If $D_1 \subset D$ and $\partial D, \partial D_1$, then $\mu_{D_1}(z, w)$ is absolutely continuous with respect to $\mu_D(z, w)$ with*

$$\frac{d\mu_{D_1}(z, w)}{d\mu_D(z, w)}(\gamma) = Y(\gamma) \tag{10}$$

where

$$Y(\gamma) = Y(D_1, D; z, w)(\gamma) = 1\{\gamma \subset D_1\} \exp \left\{ \frac{c}{2} \Lambda(D; \gamma, D \setminus D_1) \right\}.$$

We can also state (10) in terms of probability measures,

$$\frac{d\mu_{D_1}^\#(z, w)}{d\mu_D^\#(z, w)}(\gamma) = \frac{Y(\gamma)}{E(Y)}$$

where E denotes expectation with respect to $\mu_D^\#(z, w)$. This formulation has the advantage that it does not require z, w to be smooth boundary points. Note that if $f : D \rightarrow f(D)$ is a conformal transformation, then

$$Y(f(D_1), f(D); f(z), f(w))(f \circ \gamma) = Y(D_1, D; z, w)(\gamma).$$

The definition of $\mu_D(z, w)$ very much used the fact that the curve was going from z to w . However, many of the discrete examples indicate that the measure should be reversible, that is, the measure $\mu_D(w, z)$ can be obtained from $\mu_D(z, w)$ by reversing the paths. This is not easy to show from the definition of SLE_κ , but fortunately, Zhan [15] has given a proof.

Theorem 19. *For $\kappa \leq 4$, $\mu_D(w, z)$ can be obtained from $\mu_D(z, w)$ by reversing the paths.*

4.8 Natural parametrization (length)

Everything we have done so far has considered SLE_κ in the capacity parametrizaion or “up to time change”. The *natural parametrization or length* should be a d -dimensional measure where d is the Hausdorff dimension of the path. Here we show how to define it. Let γ denote an SLE_κ curve in \mathbb{H} from 0 to ∞ and let Θ_t denote the amount of time to traverse $\gamma[0, t]$ in the natural parameterization. If $\kappa \geq 8$, the path is space filling and we define

$$\Theta_t = \text{area}(\gamma_t).$$

For the remainder of this section, assume that $\kappa < 8$.

The starting point for the definition is the belief that the expected amount of time spend in a bounded domain V should be (up to an arbitrary constant multiple) equal to

$$\int_V G(z) dA(z),$$

where G denotes the Green’s function and dA denotes intergration with respect to area. More generally, the expected amount of time spent in V after time t , given the path γ_t should be given by

$$\Psi_t(V) := \int_V G_{D_t}(z; \gamma(t), \infty) dA(z).$$

For each z , the process $M_t(z) = G_{D_t}(z; \gamma(t), \infty)$ is a positive local martingale and hence is a supermartingale. Using this, we see that $\Psi_t(V)$ is a supermartingale. This leads to the following definition which comes from the Doob-Meyer decomposition of a supermartingale.

Definition The natural parametrization $\Theta_t(V)$ is the unique increasing adapted process $\Theta_t(V)$ such that

$$\Theta_t(V) + \Psi_t(V)$$

is a martingale. The natural parameterization is given by

$$\Theta_t = \lim_{n \rightarrow \infty} \Theta_t(V_n),$$

where V_n is an increasing sequence of sets whose union is \mathbb{H} .

Work needs to be done to show that this is well defined and nontrivial [10, 11]. Indeed, if $\Psi_t(V)$ were a local martingale, then we would not have a nontrivial process.

This defines the natural length in \mathbb{H} ; there are two ways to define in subdomains. Suppose $D \in \mathcal{D}$. Then for each curve γ lying in D , we can consider as a curve in \mathbb{H} and compute its length, or we could use the conformal covariance rule (2). Fortunately, they give the same answer [9].

It may be surprising at first, but the capacity parameterization and the natural parameterization are singular with respect to each other.

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