CONTINUITY OF RADIAL AND TWO-SIDED RADIAL SLE_κ AT THE TERMINAL POINT

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ABSTRACT. We prove that radial SLE_κ and two-sided radial SLE_κ are continuous at their terminal point.

1. INTRODUCTION

We answer a question posed by Dapeng Zhan about radial Schramm-Loewner evolution (SLE_κ) and discuss a similar question about two-sided SLE_κ that arose in work of the author with Brent Werness [4]. Radial SLE_κ was invented by Oded Schramm [8] and is a one-parameter family of random curves

$$\gamma : [0, \infty) \to \overline{\mathbb{D}}, \quad \gamma(0) \in \partial \mathbb{D},$$

where $\mathbb{D}$ denotes the unit disk. The definition implies that $\gamma(t) \neq 0$ for every t and

$$\liminf_{t \to \infty} |\gamma(t)| = 0.$$

Zhan asked for a proof that with probability one

$$\lim_{t \to \infty} \gamma(t) = 0. \tag{1}$$

For $\kappa > 4$, for which the SLE paths intersect themselves, this is not difficult to prove because the path makes closed loops about the origin. The harder case is $\kappa \leq 4$. Here we establish (1) for $\kappa \leq 4$ by proving a stronger result.

To state the result, let

$$\mathbb{D}_n = e^{-n}\mathbb{D} = \left\{ z \in \mathbb{C} : |z| < e^{-n} \right\},$$

$$\rho_n = \inf \left\{ t : |\gamma(t)| = e^{-n} \right\},$$

and let $\mathcal{G}_n$ denote the σ-algebra generated by $\{\gamma(s) : 0 \leq s \leq \rho_n\}$. We fix

$$\alpha = \frac{8}{\kappa} - 1,$$

which is positive for $\kappa < 8$.

Theorem 1. *For every $0 < \kappa < 8$, there exists $c > 0$ such that if γ is radial SLE_κ from 1 to 0 in $\mathbb{D}$ and j, k, n are positive integers, then*

$$\mathbb{P}\left\{\gamma[\rho_{n+k}, \infty) \subset \mathbb{D}_j \mid \mathcal{G}_{n+k}\right\} \geq \left[1 - c\, e^{-n\alpha/2}\right] 1\{\gamma[\rho_k, \rho_{n+k}] \subset \mathbb{D}_j\}. \tag{2}$$

Moreover, if $0 < \kappa \leq 4$, then

$$\mathbb{P}\left\{\gamma[\rho_{n+k}, \infty) \subset \mathbb{D}_k \mid \mathcal{G}_{n+k}\right\} \geq 1 - c\, e^{-n\alpha/2}. \tag{3}$$

There is another version of SLE, sometimes called two-sided radial SLE_κ which corresponds to chordal SLE_κ conditioned to go through an interior point. We consider the case of chordal SLE_κ in $\mathbb{D}$ from 1 to -1 conditioned to go through the origin stopped when it reaches the origin (see Section 3.3 for precise definitions).

Theorem 2. *For every $0 < \kappa < 8$, there exists $c > 0$ such that if γ is two-sided radial SLE_κ from 1 to -1 through 0 in $\mathbb{D}$ and j, k, n are positive integers, then*

$$(4) \qquad \mathbb{P}\{\gamma[\rho_{n+k}, \infty) \subset \mathbb{D}_j \mid \mathcal{G}_{n+k}\} \geq [1 - c\, e^{-n\alpha/2}]\, 1\{\gamma[\rho_k, \rho_{n+k}] \subset \mathbb{D}_j\}.$$

Using these theorem, we are able to obtain the following corollary. Unfortunately, we are not able to estimate the exponent u that appears.

Theorem 3. *For every $0 < \kappa < 8$, there exist $c < \infty, u > 0$ such that the following holds. Suppose γ is either radial SLE_κ from 1 to 0 in $\mathbb{D}$ or two-sided radial SLE_κ from 1 to -1 through 0 stopped when it reaches the origin. Then, for all nonnegative integers k, n,*

$$(5) \qquad \mathbb{P}\{\gamma[\rho_{n+k}, \infty) \cap \partial\mathbb{D}_k \neq \emptyset \mid \mathcal{G}_k\} \leq c\, e^{-un},$$

and hence

$$(6) \qquad \mathbb{P}\{\gamma[\rho_{n+k}, \infty) \cap \partial\mathbb{D}_k \neq \emptyset\} \leq c\, e^{-un}.$$

In particular, if γ has the radial parametrization, then with probability one,

$$\lim_{t \to \infty} \gamma(t) = 0.$$

Note that (5) is not as strong a result as (3). At the moment, we do not have uniform bounds for

$$\mathbb{P}\{\gamma[\rho_{n+k}, \infty) \cap \partial\mathbb{D}_k \neq \emptyset \mid \mathcal{G}_{n+k}\}$$

for radial SLE_κ with $4 < \kappa < 8$ or two-sided radial SLE_κ for $0 < \kappa < 8$.

♣ There is another, perhaps easier, way of obtaining (5) for radial SLE_κ, $4 < \kappa < 8$, by using the fact that the curve hits itself (and hence also forms closed loops about origin). This approach, however, does not work for $\kappa \leq 4$ or for two-sided radial for $\kappa < 8$ since in these cases the origin is not separated from $\partial\mathbb{D}$ in finite time.

1.1. **Outline of the paper.** When studying SLE, one uses many kinds of estimates: results for all conformal maps; results that hold for solutions of the (deterministic) Loewner differential equation; results about stochastic differential equations (SDE), often simple equations of one variable; and finally results that combine them all. We have separated the non-SLE results into a "preliminary" section with subsections emphasizing the different aspects.

We discuss three kinds of SLE_κ: radial, chordal, and two-sided radial. They are probability measures on curves (modulo reparametrization) in simply connected domains connecting, respectively: boundary point to interior point, two distinct boundary points, and two distinct boundary points conditioned to go through an interior point. In all three cases, the measures are conformally invariant and hence we can choose any convenient domain. For the radial equation, the unit disk $\mathbb{D}$ is most convenient and for this one gets the Loewner equation as originally studied by Loewner. For this equation a *radial parametrization* is used which depends on the interior point. For the chordal case, Schramm [8] showed that the half-plane with boundary points 0 and ∞ was most convenient, and the corresponding

Loewner equation is probably the easiest for studying fine properties. Here a *chordal parametrization* depending on the target boundary point (infinity) is most convenient. The two-sided radial, which was introduced in [7, 3] and can be considered as a type of $SLE(\kappa, \rho)$ process as defined in [6], has both an interior point and a boundary point. If one is studying this path up to the time it reaches the interior point, which is all that we do in this paper, then one can use either the radial or the chordal parametrization.

The three kinds of SLE_κ, considered as measures on curves modulo reparametrization, are locally absolutely continuous with respect to each other. To make this precise, it is easiest if one studies them simultaneously in a single domain with a single choice of parametrization. We do this here choosing the radial parametrization in the unit disk $\mathbb{D}$. We review the radial Loewner equation in Section 2.1. We write the equation slightly differently than in [8]. First, we add a parameter a that gives a linear time change. We also write a point on the unit circle as $e^{2i\theta}$ rather than $e^{i\theta}$; this makes the SDEs slightly easier and also shows the relationship between this quantity and the argument of a point in the chordal case. Indeed, if F is a conformal transformation of the unit disk to the upper half plane with $F(1) = 0$ and $F(e^{2i\theta}) = \infty$, then $\sin[\arg F(0)] = \sin\theta$.

The radial Loewner equation describes the evolution of a curve γ from 1 to 0 in $\mathbb{D}$. More precisely, if D_t denotes the connected component of $\mathbb{D} \setminus \gamma(0, t]$ containing the origin, and $g_t : D_t \to \mathbb{D}$ is the conformal transformation with $g_t(0) = 0, g_t'(0) > 0$, then the equation describes the evolution of g_t. At time t, the relevant information is $g_t(\gamma(t))$ which we write as e^{2iU_t}. To compare radial SLE_κ to chordal or two-sided radial SLE_κ with target boundary point $w = e^{2i\theta}$, we also need to keep track of $g_t(w)$ which we write as $e^{2i\theta_t}$.

Radial SLE_κ is obtained by solving the Loewner equation with $a = 2/\kappa$ and $U_t = -B_t$ a standard Brownian motion. If $X_t = \theta_t - U_t$, then X_t satisfies

$$dX_t = \beta \cot X_t \, dt + dB_t,$$

with $\beta = a$. Much of the study of SLE_κ in the radial parametrization can be done by considering the SDE above. In fact, the three versions of SLE_κ can be obtained by choosing different β. In Section 2.2 we discuss the properties of this SDE that we will need. We use the Girsanov theorem to estimate the Radon-Nikodym derivative of the measures on paths for different values of β.

Section 2.3 gives estimates for conformal maps that will be needed. The first two subsections discuss crosscuts and the argument of a point. If D is a simply connected subdomain of $\mathbb{D}$ containing the origin, then the intersection of D with the circle $\partial\mathbb{D}_k$ can contain many components. We discuss such crosscuts in Section 2.3.1 and state a simple topological fact, Lemma 2.3, that is used in the proofs of (2) and (4).

A classical conformally invariant measure of distance between boundary arcs is extremal distance or extremal length. We will only need to consider distance between arcs in a conformal rectangle for which it is useful to estimate harmonic measure, that is, hitting probabilities for Brownian motion. We discuss the general strategy for proving such estimates in Section 2.3.3. The following subsections give specific estimates that will be needed for radial and two-sided radial. The results in this section do not depend much at all on the Loewner equation — one fact that is used is that we are stopping a curve at the first time it reaches $\partial\mathbb{D}_n$ for some n.

The Beurling estimate (see [5, Section 3.8]) is the major tool for getting uniform estimates.

The main results of this paper can be found in Section 3. The first three subsections define the three types of SLE_κ, radial, chordal, two-sided radial, in terms of radial. (To be more precise, it defines these processes up to the time the path separates the origin from the boundary point w). Section 4.1 contains the hardest new result in this paper. It is an analogue for to radial SLE_κ of a known estimate for chordal SLE_κ on the probability of hitting a set near the boundary. This is the main technical estimate for Theorem 1. A different estimate is proved in Section 4.2 for two-sided radial. The final section finishes the proof Theorem 3 by using a known technique to show exponential rates of convergence.

I would like to thank Dapeng Zhan for bringing up the fact that this result is not in the literature and Joan Lind and Steffen Rohde for useful conversations.

1.2. **Notation.** We let

$$\mathbb{D} = \{|z| < 1\}, \quad \mathbb{D}_n = e^{-n}\,\mathbb{D} = \{|z| < e^{-n}\}.$$

If γ is a curve, then

$$\rho_n = \inf\{t : \gamma(t) \in \partial\mathbb{D}_n\}.$$

If γ is random, then $\mathcal{F}_t$ denotes the σ-algebra generated by $\{\gamma(s) : s \le t\}$ and $\mathcal{G}_n = \mathcal{F}_{\rho_n}$ is the σ-algebra generated by $\{\gamma(t) : t \le \rho_n\}$. Let D_t be the connected component of $\mathbb{D} \setminus \gamma(0,t]$ containing the origin and

$$H_n = D_{\rho_n}.$$

If D is a domain, $z \in D$, $V \subset \partial D$, we let $h_D(z, V)$ denote the harmonic measure starting at z, that is, the probability that a Brownian motion starting at z exits D at V.

When discussing SLE_κ we will fix κ and assume that $0 < \kappa < 8$. We let

$$a = \frac{2}{\kappa}, \quad \alpha = \frac{8}{\kappa} - 1 = 4a - 1 > 0.$$

2. Preliminaries

2.1. **Radial Loewner equation.** Here we review the radial Loewner differential equation; see [5] for more details. The radial Loewner equation describes the evolution of a curve from 1 to 0 in the unit disk $\mathbb{D}$. Let $a > 0$, and let $U_t : [0, \infty) \to \mathbb{R}$ be a continuous function with $U_0 = 0$. Let g_t be the solution to the initial value problem

$$(7) \qquad \partial_t g_t(z) = 2a\, g_t(z) \frac{e^{2iU_t} + g_t(z)}{e^{2iU_t} - g_t(z)}, \quad g_0(z) = z.$$

For each $z \in \overline{\mathbb{D}} \setminus \{1\}$, the solution of this equation exists up to a time $T_z \in (0, \infty]$. Note that $T_0 = \infty$ and $g_t(0) = 0$ for all t. For each $t \ge 0$, D_t, as defined above, equals $\{z \in \mathbb{D} : T_z > t\}$, and g_t is the unique conformal transformation of D_t onto $\mathbb{D}$ with $g_t(0) = 0, g_t'(0) > 0$. By differentiating (7) with respect to z, we see that $\partial_t g_t'(0) = 2a g_t'(0)$ which implies that $g_t'(0) = e^{2at}$.

If we define $h_t(z)$ to be the continuous function of t such that

$$g_t(e^{2iz}) = \exp\{2ih_t(z)\}, \quad h_0(z) = z,$$

then the Loewner equation becomes

$$(8) \qquad \partial_t h_t(z) = a \, \cot(h_t(z) - U_t), \quad h_0(z) = z.$$

We will consider this primarily for real $z = x \in (0, \pi)$. Note that if $x \in (0, \pi)$ and D_t agrees with $\mathbb{D}$ in a neighborhood of e^{2ix}, then

$$(9) \qquad |g_t'(e^{2ix})| = h_t'(x).$$

The radial equation can also be used to study curves whose "target" point is a boundary point $w = e^{2i\theta_0}, 0 < \theta_0 < \pi$. If we let $\theta_t = h_t(\theta_0)$, then (8) becomes

$$\partial_t \theta_t = a \, \cot(\theta_t - U_t),$$

which is valid for $t < T_w$. Using (9), we get

$$|g_t'(w)| = h_t'(\theta_0) = \exp\left\{ -a \int_0^t \frac{ds}{\sin^2(\theta_s - U_s)} \right\}.$$

♣ The radial Loewner equation as in [8] or [5] is usually written with $a = 1/2$. Also, the $2U_t$ in the exponent in (7) is usually written as U_t. We choose to write $2U_t$ so that the equation (8) is simpler, and because $\theta_t - U_t$ corresponds to an angle when we map the disk to the upper half plane, see Section 2.3.2.

We say that g_t is generated by γ if $\gamma : [0, \infty) \to \overline{\mathbb{D}}$ is a curve such that for each t, D_t is the connected component of $\mathbb{D} \setminus \gamma(0, t]$ containing the origin. Not every continuous U_t yields conformal maps g_t generated by a curve, but with probability one SLE_κ is generated by a curve (see [9] for a proof for $\kappa \neq 8$ which is all that we need in this paper). For ease, we will restrict our discussion to g_t that are generated by curves.

Definition

- A curve arising from the Loewner equation will be called a *Loewner curve*. Two such curves are equivalent if one is obtained from the other by increasing reparametrization.
- A Loewner curve has the a-radial parametrization if $g_t'(0) = e^{2at}$.

Recall that $\rho_n = \inf\{t : |\gamma(t)| = e^{-n}\}$. A simple conseqence of the Koebe 1/4-theorem is the existence of $c < \infty$ such that for all n

$$(10) \qquad \rho_{n+1} \leq \rho_n + c.$$

2.2. **Radial Bessel equation.** Analysis of radial SLE leads to studying a simple one-dimensional SDE (12) that we call the *radial Bessel equation*. This equation can be obtained using the Girsanov theorem by "weighting" or "tilting" a standard Brownian motion as we now describe. Suppose X_t is a standard one-dimensional Brownian motion defined on a probability space $(\Omega, \mathbb{P})$ with $0 < X_0 < \pi$ and let $\tau = \inf\{t : \sin X_t = 0\}$. Roughly speaking, the radial Bessel equation with parameter β (up to time τ) is obtained by weighting the Brownian motion locally by $(\sin X_t)^\beta$. Since $(\sin X_t)^\beta$ is not a local martingale, we need to compensate it by a C^1 (in time) process e^{Φ_t} such that $e^{-\Phi_t}(\sin X_t)^\beta$ is a local martingale. The appropriate compensator is found easily using Itô's formula; indeed,

$$M_t = M_{t,\beta} = (\sin X_t)^\beta \, e^{\beta^2 t/2} \exp\left\{ \frac{(1-\beta)\beta}{2} \int_0^t \frac{ds}{\sin^2 X_s} \right\}, \quad 0 \leq t < \tau,$$

is a local martingale satisfying

$$(11) \qquad\qquad dM_t = \beta\, M_t\, \cot X_t\, dX_t.$$

In particular, for every $\epsilon > 0$ and $t_0 < \infty$, there exists $C = C(\beta, \epsilon, t_0) < \infty$ such that if $\tau_\epsilon = \inf\{t : \sin X_t \leq \epsilon\}$, then

$$C^{-1} \leq M_t \leq C, \quad 0 \leq t \leq t_0 \wedge \tau_\epsilon.$$

Let $\mathbb{P}_\beta$ denote the probability measure on paths $X_t, 0 \leq t < \tau$ such that for each $\epsilon > 0, t_0 < \infty$, the measure $\mathbb{P}_\beta$ on paths $X_t, 0 \leq t \leq t_0 \wedge \tau_\epsilon$ is given by

$$d\mathbb{P}_\beta = \frac{M_{t_0 \wedge \tau_\epsilon}}{M_0}\, d\mathbb{P}.$$

The Girsanov theorem states that

$$B_t = B_{t,\beta} := X_t - \beta \int_0^t \cot X_s\, ds, \quad 0 \leq t < \tau$$

is a standard Brownian motion with respect to the measure $\mathbb{P}_\beta$. In other words,

$$dX_t = \beta\, \cot X_t\, dt + dB_t, \quad 0 \leq t < \tau.$$

We call this the radial Bessel equation (with parameter β). By comparison with the usual Bessel equation, we can see that

$$\mathbb{P}_\beta\{\tau = \infty\} = 1 \text{ if and only if } \beta \geq \frac{1}{2}.$$

♣ Although the measure $\mathbb{P}_\beta$ is mutually absolutely continuous with respect to $\mathbb{P}$ when one restricts to curves $X_t, 0 \leq t \leq \tau_\epsilon \wedge t_0$, it is possible that the measure $\mathbb{P}_\beta$ on curves $X_t, 0 \leq t < \tau \wedge t_0$ has a singular part with respect to $\mathbb{P}$.

2.2.1. *An estimate.* Here we establish an estimate (14) for the radial Bessel equation which we will use in the proof of continuity of two-sided radial SLE. Suppose that X_t satisfies

$$(12) \qquad\qquad dX_t = \beta\, \cot X_t\, dt + dB_t, \quad 0 \leq t < \tau,$$

where $\beta \in \mathbb{R}$, B_t is a standard Brownian motion, and $\tau = \inf\{t : \sin X_t = 0\}$. Let

$$F(x) = F_\beta(x) = \int_x^{\pi/2} (\sin t)^{-2\beta}\, dt, \quad 0 < x < \pi,$$

which satisfies

$$(13) \qquad\qquad F''(x) + 2\beta\, (\cot x)\, F'(x) = 0.$$

Lemma 2.1. *For every $\beta > 1/2$, there exists $c_\beta < \infty$ such that if $0 < \epsilon < x \leq \pi/2$, X_t satisfies (12) with $X_0 = x$, and*

$$\tau_\epsilon = \inf\{t \geq 0 : X_t = \epsilon \text{ or } \pi/2\},$$

then

$$\mathbb{P}\{X_{\tau_\epsilon} = \epsilon\} \leq c_\beta\, (\epsilon/x)^{2\beta - 1}.$$

Proof. Itô's formula and (13) show that $F(X_{t \wedge \tau_\epsilon})$ is a bounded martingale, and hence the optional sampling theorem implies that

$$F(x) = \mathbb{P}\{X_{\tau_\epsilon} = \epsilon\} F(\epsilon) + \mathbb{P}\{X_{\tau_\epsilon} = \pi/2\} F(\pi/2) = \mathbb{P}\{X_{\tau_\epsilon} = \epsilon\} F(\epsilon).$$

Therefore,

$$\mathbb{P}\{X_{\tau_\epsilon} = \epsilon\} = \frac{F(x)}{F(\epsilon)}.$$

If $\beta > 1/2$, then

$$F(\epsilon) \sim \frac{1}{2\beta - 1} \epsilon^{1-2\beta}, \quad \epsilon \to 0+,$$

from which the lemma follows. $\square$

Lemma 2.2. *For every $\beta > 1/2, t_0 < \infty$, there exists $c = c_{\beta, t_0} < \infty$ such that if X_t satisfies (12) with $X_0 \in (0, \pi)$, then*

$$(14) \qquad \mathbb{P}\left\{\min_{0 \le t \le t_0} \sin X_t \le \epsilon \sin X_0\right\} \le c\, \epsilon^{2\beta - 1}.$$

Proof. We allow constants to depend on β, t_0. Let $r = \sin X_0$. It suffices to prove the result when $r \le 1/2$. Let $\sigma = \inf\{t : \sin X_t = 1 \text{ or } \epsilon r\}$ and let $\rho = \inf\{t > \sigma : \sin X_t = r\}$. Using the previous lemma we see that

$$\mathbb{P}\{\sin X_\sigma = \epsilon r\} \le c\, \epsilon^{2\beta - 1}.$$

Since $r \le 1/2$ and there is positive probability that the process started at $\pi/2$ stays in $[\pi/4, 3\pi/4]$ up to time t_0, we can see that

$$\mathbb{P}\{\rho > t_0 \mid \sin X_\sigma = 1\} \ge c_1,$$

Hence, if q denotes the probability on the left-hand side of (14), we get

$$q \le c\, \epsilon^{2\beta - 1} + (1 - c_1)\, q.$$

$\square$

2.3. Deterministic lemmas.

2.3.1. *Crosscuts in $\partial \mathbb{D}_k$.*

Definition A *crosscut* of a domain D is the image of a simple curve $\eta : (0, 1) \to D$ with $\eta(0+), \eta(1-) \in \partial D$.

Recall that H_n is the connected component of $\mathbb{D} \setminus \gamma(0, \rho_n]$ containing the origin. Let

$$\partial_n^0 = \partial H_n \setminus \gamma[0, \rho_n],$$

which is either empty or is an open subarc of $\partial \mathbb{D}$.

For each $0 < k < n$, let $V_{n,k}$ denote the connected component of $H_n \cap \mathbb{D}_k$ that contains the origin, and let $\partial_{n,k} = \partial V_{n,k} \cap H_n$. The connected components of $\partial_{n,k}$ comprise a collection $\mathcal{A}_{n,k}$ of open subarcs of $\partial \mathbb{D}_k$. Each arc $l \in \mathcal{A}_{n,k}$ is a crosscut of H_n such that $H_n \setminus l$ has two connected components. Let $V_{n,k,l}$ denote the component of $H_n \setminus l$ that does not contain the origin; note that these components are disjoint for distinct $l \in \mathcal{A}_{n,k}$. If $\partial_n^0 \ne \emptyset$, there is a unique arc $l^* = l_{n,k}^* \in \mathcal{A}_{n,k}$ such that $\partial_n^0 \subset \partial V_{n,k,l^*}$.

♣ Note that each $l \in \mathcal{A}_{n,k}$ is a connected component of $\partial \mathbb{D}_k \cap H_n$; however, there may be components of $\partial \mathbb{D}_k \cap H_n$ that are not in $\mathcal{A}_{n,k}$. In particular, it is possible that $V_{n,k,l} \cap \mathbb{D}_k \ne \emptyset$. The arc l^* is the unique arc in $\mathcal{A}_{n,k}$ such that each path from 0 to ∂_n^0 in H_n must pass through

l^*. One can construct examples where there are other components l of $\partial \mathbb{D}_k \cap H_n$ with the property that every path from 0 to ∂_n^0 in H_n must pass through l. However, these components are not in $\mathcal{A}_{n,k}$.

If $k < n$ and $\gamma[\rho_n, \infty) \cap \partial \mathbb{D}_k \neq \emptyset$, then the first visit to $\partial \mathbb{D}_k$ after time ρ_n must be to the closure of the one of the crosscuts in $\mathcal{A}_{n,k}$. In this paper we will estimate the probability of hitting a given crosscut. Since there can be many crosscuts, it is not immediate how to use this estimate to bound the probability of hitting any crosscut. This is the technical issue that prevents us from extending (3) to all $\kappa < 8$. The next lemma, however, shows that if the curve has not returned to $\partial \mathbb{D}_j$ after time ρ_k, then there is only one crosscut in $\mathcal{A}_{n,k}$ from which one can access $\partial \mathbb{D}_j$.

Lemma 2.3. *Suppose $j < k < n$ and γ is a Loewner curve in $\mathbb{D}$ starting at 1 with $\rho_n < \infty$, $H_n \not\subset \mathbb{D}_j$, and $\gamma[\rho_k, \rho_n] \subset \mathbb{D}_j$. Then there exists a unique crosscut $l \in \mathcal{A}_{n,k}$ such that if $\eta : [0, 1) \to H_n \cap \mathbb{D}_j$ is a simple curve with $\eta(0) = 0, \eta(1-) \in \partial \mathbb{D}_j$ and*

$$s_0 = \inf\{s : \eta(s) \in \partial \mathbb{D}_k\},$$

then $\eta(s_0) \in l$. If $\partial_n^0 \neq \emptyset$, then $l = l_{n,k}^*$.

Proof. Call $l \in \mathcal{A}_{n,k}$ *good* if there exists a curve η as above with $\eta(s_0) \in l$. Since $H_n \not\subset \mathbb{D}_j$, there exists at least one good l. Also, if $-1 \in \partial H_n$, then $l_{n,k}^*$ is good. Hence, we only need to show there is at most one good $l \in \mathcal{A}_{n,k}$. Suppose η^1, η^2 are two such curves with times s_0^1, s_0^2 and let $z_j = \eta(s_0^j)$. We need to show that z_1 and z_2 are in the same crosscut in $\mathcal{A}_{n,k}$. If $z_1 = z_2$ this is trivial, so assume $z_1 \neq z_2$. Let l^1, l^2 denote the two subarcs of $\partial \mathbb{D}_k$ obtained by removing z_1, z_2 (these are not crosscuts in $\mathcal{A}_{n,k}$). Let l^1 denote the arc that contains $\gamma(\rho_k)$ and let U denote the connected component of $(H_k \cap \mathbb{D}_j) \setminus (\eta^1 \cup \eta^2)$ that contains $\gamma(\rho_k)$. Our assumptions imply that $\gamma[\rho_k, \rho_n] \subset U$. In particular, $l^2 \cap \gamma[\rho_k, \rho_n] = \emptyset$. Therefore l^2, z_1, z_2 lie in the same component of H_n and hence in the same crosscut of $\mathcal{A}_{n,k}$. $\qquad\square$

2.3.2. *Argument.*

Definition If γ is a Loewner curve in $\mathbb{D}$ starting at 1, $w \in \partial \mathbb{D} \setminus \{1\}$, and $t < T_w$, then

$$S_t = S_{t,0,w} = \sin \arg F_t(0),$$

where $F_t : D_t \to \mathbb{H}$ is a conformal transformation with $F_t(\gamma(t)) = 0, F_t(w) = \infty$.

If $z \in \mathbb{H}$, let $h_+(z) = h_{\mathbb{H}}(z, (0, \infty))$ denote the probability that a Brownian motion starting at z leaves $\mathbb{H}$ at $(0, \infty)$ and let $h_-(z) = 1 - h_+(z)$ be the probability of leaving at $(-\infty, 0)$. Using the explicit form of the Poisson kernel in $\mathbb{H}$, one can see that $h_-(z) = \arg(z)/\pi$. Using this, we can see that

$$S_0 = \sin \theta_0$$

and

$$(15) \qquad\qquad \sin \arg(z) \asymp \min\{h_+(z), h_-(z)\},$$

where $\asymp$ means each side is bounded by an absolute constant times the other side.

If $t < T_w$, we can write $\partial D_t = \{\gamma(t)\} \cup \{w\} \cup \partial_t^+ \cup \partial_t^-$ where ∂_t^+ (∂_t^-) is the part of ∂D_t that is sent to the positive (resp., negative) real axis by F_t. Using conformal invariance and (15), we see that

$$(16) \qquad\qquad S_t \asymp \min\{h_{D_t}(0, \partial_t^+), h_{D_t}(0, \partial_t^-)\}.$$

2.3.3. *Extremal length.* The proofs of our deterministic lemmas will use estimates of extremal length. These can be obtained by considering appropriate estimates for Brownian motion which are contained in the next lemma. Let $\mathcal{R}_L$ denote the open region bounded by a rectangle,

$$\mathcal{R}_L = \{x + iy \in \mathbb{C} : 0 < x < L, 0 < y < \pi\}.$$

We write $\partial \mathcal{R}_L = \partial_0 \cup \partial_l \cup \partial_L^+ \cup \partial_L^-$ where

$$\partial_0 = [0, i\pi], \quad \partial_L = [L, L + i\pi], \quad \partial_L^+ = (i\pi, L + i\pi), \quad \partial_L^- = (0, L).$$

If D is a simply connected domain and A_1, A_2 are disjoint arcs on ∂D, then the π-*extremal distance* (π times the usual extremal distance or length) is the unique L such that there is a conformal transformation of D onto $\mathcal{R}_L$ mapping A_1, A_2 onto ∂_0 and ∂_L, respectively. Estimates for the Poisson kernel in $\mathcal{R}_L$ are standard, see, for example, [5, Sections 2.3 and 5.2]. The next two lemmas which we state without proof give the estimates that we need.

Lemma 2.4. *There exist $0 < c_1 < c_2 < \infty$ such that the following holds. Suppose $L \geq 2$, and V is the closed disk of radius $1/4$ about $1 + (\pi/2)i$.*

- *If $z \in V$,*

$$(17) \qquad c_1 \leq h_{\mathcal{R}_L}(z, \partial_0), h_{\mathcal{R}_L}(z, \partial_L^+), h_{\mathcal{R}_L}(z, \partial_L^-) \leq c_2,$$

- *If $z \in V$ and $A \subset \partial_L$, then*

$$(18) \qquad h_{\mathcal{R}_L}(z, A) \leq c_2 \, e^{-L} \, |A|,$$

  where $|\cdot|$ denotes length.
- *If B_t is a standard Brownian motion, $\tau_L = \inf\{t : B_t \notin \mathcal{R}_L\}$, $\sigma = \inf\{t : \mathrm{Re}(B_t) = 1\}$, then if $0 < x < 1/2$ and $0 < y < \pi$,*

$$(19) \qquad \mathbb{P}^{x+iy}\{B_\sigma \in V \mid \sigma < \tau_L\} \geq c_1.$$

Lemma 2.5. *For every $\delta > 0$, there exists $c > 0$ such that if $L \geq \delta$ and $z \in \mathcal{R}_L$ with $\mathrm{Re}(z) \leq \delta/2$, then*

$$h_{\mathcal{R}_L}(z, \partial_L) \leq c \, e^{-L} \, \min\left\{ h_{\mathcal{R}_L}(z, \partial_L^+), h_{\mathcal{R}_L}(z, \partial_L^-) \right\}.$$

We explain the basic idea on how we will use these estimates. Suppose D is a domain and l is a crosscut of D that divides D into two components D_1, D_2. Suppose D_2 is simply connected and A is a closed subarc of ∂D_2 with $\partial D_2 \cap \bar{l} = \emptyset$. Let ∂^+, ∂^- denote the connected components of $\partial D_2 \setminus \{\bar{l}, A\}$. We consider $A, \partial^+, \partial^-$ as arcs of ∂D in the sense of prime ends. Let $F : D_2 \to \mathcal{R}_L$ be a conformal transformation sending l to ∂_0 and A to ∂_L and suppose that $L \geq 2$. Let $l_1 = F^{-1}(1 + i(0, \pi))$. Let $\tau = \inf\{t : B_t \notin D\}, \sigma = \inf\{t : B_t \in l_1\}$. Then if $z \in D_1$ and $A_1 \subset A$,

$$h_D(z, \partial^+), h_D(z, \partial^-) \geq c \, \mathbb{P}^z\{\sigma < \tau\}.$$

$$h_D(z, A_1) \leq c \, \mathbb{P}^z\{\sigma < \tau\} \, e^{-L} \, |F(A_1)|.$$

In particular, there exists $c < \infty$ such that for $z \in D_1, A_1 \subset A$,

$$h_D(z, A_1) \leq c \, e^{-L} \, |F(A_1)| \, \min\left\{ h_D(z, \partial^+), h_D(z, \partial^-) \right\}.$$

2.3.4. *Radial case.* We will need some lemmas that hold for all curves γ stopped at the first time they reach the sphere of a given radius or the first time they reach a given vertical line. If D is a domain and $\eta : (0,1) \to D$ is a crosscut, we write η for the image $\eta(0,1)$ and $\overline{\eta} = \eta[0,1]$.

♣ The next lemma is a lemma about Loewner curves, that is, curves modulo reparametrization. To make the statement nicer, we choose a parametrization such that $\rho_{n+k} = 1$. Although the parametrization is not important, it is important that we are stopping the curve at the first time it reaches $\partial\mathbb{D}_{n+k}$.

Lemma 2.6. *There exists $c < \infty$ such that the following is true. Suppose $k > 0, n \geq 4$ and $\gamma : [0,1] \to \mathbb{D}$ is a Loewner curve with $\gamma(0) = 1; |\gamma(1)| = e^{-n-k}$; and $e^{-n-k} < |\gamma(t)| < 1$ for $0 < t < 1$. Let D be the connected component of $\mathbb{D} \setminus \gamma(0,1]$ containing the origin, and let*

$$\eta = \left\{ e^{-k+i\theta} : \theta_1 < \theta < \theta_2 \right\} \in \mathcal{A}_{n+k,k}$$

be a crosscut of D contained in $\partial\mathbb{D}_k$.

Let $F : D \to \mathbb{D}$ be the unique conformal transformation with $F(0) = 0, F(\gamma(1)) = 1$. Suppose that we write $\partial\mathbb{D}$ as a disjoint union

$$\partial\mathbb{D} = \{1\} \cup V_1 \cup V_2 \cup V_3,$$

where V_3 is the closed interval of $\partial\mathbb{D}$ not containing 1 whose endpoints are the images under F of $\eta(0+), \eta(1-)$ and V_1, V_2 are connected, open intervals. Then

$$\operatorname{diam}[F(\eta)] \leq c\, e^{-n/2}\, (\theta_2 - \theta_1)\, \min\left\{|V_1|, |V_2|\right\},$$

where $|\cdot|$ denotes length.

♣ It is important for our purposes to show not only that $F(\eta)$ is small, but also that it is smaller than both V_1 and V_2. When we apply the proposition, one of the intervals V_1, V_2 may be very small.

Proof. Let U denote the connected component of $D \setminus \eta$ that contains the origin and note that U is simply connected. Let

$$U^* = U \cap \left\{ |z| > e^{-n-k} \right\}.$$

Since $\gamma(0,1) \subset \{|z| > e^{-n-k}\}$ and $|\gamma(1)| = e^{-n-k}$, we can see that U^* is simply connected with $\eta \cup \partial\mathbb{D}_{n+k} \subset \partial U^*$. Let

$$g : \mathcal{R}_L \longrightarrow U^*$$

be a conformal transformation mapping ∂_0 onto $\partial\mathbb{D}_{n+k}$ and ∂_L onto $\overline{\eta}$. Such a transformation exists for only one value of L, the π-extremal distance between $\partial\mathbb{D}_{n+k}$ and $\overline{\eta}$ in U^*. Since $\eta \cap \mathbb{D}_k = \emptyset$, and the complement of U^* contains a curve connecting $\partial\mathbb{D}_k$ and $\partial\mathbb{D}_{n+k}$, see that $L \geq n/2 \geq 2$ (this can be done by comparison with an annulus, see. e.g., [5, Example 3.72]). We write

$$\partial U^* = \partial\mathbb{D}_{n+k} \cup \overline{\eta} \cup \partial_- \cup \partial_+$$

where ∂_- (∂_+) is the image of ∂_L^- (resp., ∂_L^+) under g. Here we are considering boundaries in terms of prime ends, e.g., if γ is simple then each point on $\gamma(0,1)$

corresponds to two points in ∂D. Note that $\{F(\partial_-), F(\partial_+)\}$ is $\{V_1, V_2\}$, so we can rewrite the conclusion of the lemma as

$$(20) \qquad h_U(0, \eta) \leq c\, e^{-n/2}\, (\theta_2 - \theta_1) \min \{h_U(0, \partial_-), h_U(0, \partial_+)\}.$$

Let $\ell = g(1 + i(0, \pi))$ which separates $\partial \mathbb{D}_{n+k}$ from η, and hence also separates the origin from η in U. Let B_t be a Brownian motion starting at the origin and let

$$\sigma = \inf\{t : B_t \in \ell\}, \quad \tau = \inf\{t : B_t \notin U\}.$$

Using conformal invariance and (17), we can see that if $z \in g(V)$, the probability that a Brownian motion starting at z exits U^* at $\partial \mathbb{D}_{n+k}$ is at least c_1. However, the Beurling estimate implies that this probability is bounded above by $c\, [e^{-(n+k)}/|z|]^{1/2}$. From this we conclude that there exists j such that $g(V)$ is contained in $\mathbb{D}_{n+k-j}$. We claim that there exists c such that the probability that a Brownian motion starting at $z \in D_{n+k-j}$ exits U^* at η is bounded above by $ce^{-n/2}\, (\theta_2 - \theta_1)$. Indeed, the Beurling estimate implies that the probability to reach $\partial \mathbb{D}_{k+1}$ without leaving U^* is $O(e^{-n/2})$, and using the Poisson kernel in the disk we know that the probability that a Brownian motion starting on $\partial \mathbb{D}_{k+1}$ exits $\mathbb{D}_k$ at η is bounded above by $c\, (\theta_2 - \theta_1)$. Using (18), we conclude that

$$\mathbb{P}\{B_\tau \in \eta \mid \sigma < \tau\} \leq c \max_{z \in g(V)} h_{U^*}(z, \eta) \leq c\, (\theta_t - \theta_1)\, e^{-n/2}.$$

Since the event $\{B_\tau \in \eta\}$ is contained in the event $\{\sigma < \tau\}$, we see that

$$(21) \qquad \mathbb{P}\{B_\tau \in \eta\} \leq c\, (\theta_t - \theta_1)\, e^{-n/2}\, \mathbb{P}\{\sigma < \tau\}.$$

Using (19) and conformal invariance, we can see that

$$\mathbb{P}\{B_\sigma \in g(V) \mid \sigma < \tau\} \geq c_1,$$

and combining this with (17) we see that

$$\min \{\mathbb{P}\{B_\tau \in \partial_- \mid \sigma < \tau\},\, \mathbb{P}\{B_\tau \in \partial_+ \mid \sigma < \tau\}\} \geq c_1^2.$$

In particular, there exists c such that

$$\min \{\mathbb{P}\{B_\tau \in \partial_-\}, \mathbb{P}\{B_\tau \in \partial_+\}\} \geq c\, \mathbb{P}\{\sigma < \tau\}.$$

Combining this with (21), we get (20). □

2.3.5. *An estimate for two-sided radial.* Recall that $\psi_{n,k}$ is the first time after ρ_n that the curve γ intersects $l^*_{n,k}$, the crosscut defining $V^*_{n,k}$.

Lemma 2.7. *There exists $c < \infty$ such that if $0 < k < n$ and $\psi = \psi^*_{n,k} < \infty$, then*

$$S_{D_\psi}(0) \leq c\, e^{(k-n)/2}\, S_{H_n}(0)$$

Proof. Let $\eta = l^*_{n,k}$ and let U^* be as in the proof of Lemma 2.6. Since η disconnects -1 from 0, we can see that when we write

$$\partial U^* = \partial \mathbb{D}_{n+k} \cup \overline{\eta} \cup \partial_- \cup \partial_+,$$

then $\partial_- \subset \partial_- U, \partial_+ \subset \partial_+ U$ (or the other way around). We also have a universal lower bound on $h_{H_n \backslash \eta}(0, \eta)$. Hence from Lemma 2.5 and (16) we see that

$$h_{H_n \backslash \eta}(0, \eta) \leq c\, e^{(k-n)/2}\, S_{H_n}(0).$$

There is a crosscut l of D_ψ that is contained in l^*, has one of its endpoints equal to $\gamma(\psi)$, and such that 0 is disconnected from -1 in D_ψ by l. If V denotes the

connected component of $D_{\psi_n} \setminus l$ containing the origin, then $\partial V \cap \partial D_\psi$ (considered as prime ends) is contained in either $\partial_+ D_\psi$ or $\partial_- D_\psi$. Therefore,

$$S_{D_\psi}(0) \leq c\, h_{D_\psi \setminus l}(0, l) \leq c\, h_{H_n \setminus \eta}(0, \eta) \leq c\, e^{(k-n)/2}\, S_{H_n}(0).$$

$\square$

♣ This proof uses strongly the fact that l^* separates -1 from 0 in H_n. The reader may wish to draw some pictures to see that for other crosscuts $l \in \mathcal{A}_{n,k}$, $S_{D_\psi}(0)$ need not be small.

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3. Schramm-Loewner evolution (SLE)

Suppose D is a simply connected domain with two distinct boundary points w_1, w_2 and one interior point z. There are three closely related versions of SLE_κ in D: chordal SLE_κ from w_1 to w_2; radial from w_1 to z; and two-sided radial from w_1 to w_2 going through z. The last of these can be thought of as chordal SLE_κ from w_1 to w_2 conditioned to go through z. All of these processes are conformally invariant and are defined only up to increasing reparametrizations. Usually chordal SLE_κ is parametrized using a "half-plane" or "chordal" capacity with respect to w_2 and radial and two-sided radial SLE_κ are defined with a radial parametrization with respect to z, but this is only a convenience. If the same parametrization is used for all three processes, then they are mutually absolutely continuous with each other if one stops the process at a time before which that paths separate z and w_2 in the domain.

We now give precise definitions. For ease we will choose $D = \mathbb{D}$, $z = 0$, $w_1 = 1$ and $w_2 = w = e^{2i\theta_0}$ with $0 < \theta_0 < \pi$. We will use a radial parametrization. We first define radial SLE_κ (for which the point w plays no role in the definition) and then define chordal SLE_κ (for which the point 0 is irrelevant when one considers processes up to reparametrization but is important here since our parametrization depends on this point) and two-sided SLE_κ in terms of radial. The definition using the Girsanov transformation is really just one example of a general process of producing "$SLE(\kappa, \rho)$ processes".

Let $h_t(x)$ be the solution of (8) with $h_0(x) = \theta_0$ and let

$$(22) \qquad\qquad X_t = h_t(w) - U_t, \quad S_t = \sin X_t.$$

Note that S_t is the same as defined in Section 2.3.2 and

$$(23) \qquad\qquad h_t'(w) = \exp\left\{ -a \int_0^t \frac{ds}{S_s^2} \right\}.$$

Let

$$\tau_\epsilon = \tau_\epsilon(w) = \inf\{t \geq 0 : S_t = \epsilon\},$$

$$\tau = \tau_0(w) = \inf\{t \geq 0 : S_t = 0\} = \inf\left\{t : \operatorname{dist}(w, \mathbb{D} \setminus D_t) = 0\right\}.$$

3.1. **Radial SLE_κ.** If $\kappa > 0$, then *radial SLE_κ (parametrized so that $g'_t(0) = e^{4t/\kappa}$)* is the solution of the Loewner equation (7) or (8) with $a = 2/\kappa$ and $U_t = -B_t$ where B_t is a standard Brownian motion. This definition does not reference the point w. However, if we define X_t by (22), we have

$$dX_t = a \cot X_t \, dt + dB_t.$$

Suppose that $(\Omega, \mathcal{F}, \mathbb{P}_0)$ is a probability space under which X_t is a Brownian motion. Then, see Section 2.2, for each $\beta \in \mathbb{R}$ there is a probability $\mathbb{P}_\beta$ such that

$$B_{t,\beta} = X_t - \beta \int_0^t \cot X_s \, ds, \quad 0 \le s < \tau,$$

is a standard Brownian motion. In other words,

$$dX_t = \beta \cot X_t \, dt + dB_{t,\beta}.$$

In particular, $B_t = B_{t,a}$. We call this radial SLE_κ *weighted locally by $S_t^{\beta - a}$*, where $S_t = \sin X_t$. Radial SLE_κ is obtained by choosing $\beta = a$. Using (23) we can write the local martingale in (11) as

$$M_{t,\beta} = S_t^\beta \, e^{t\beta^2/2} \, h'_t(w)^{\frac{\beta(\beta-1)}{2a}}.$$

We summarize the discussion in Section 2.2 as follows. If σ is a stopping time, let $\mathcal{F}_\sigma$ denote the σ-algebra generated by $\{X_{s \wedge \sigma} : 0 \le s < \infty\}$.

Lemma 3.1. *Suppose σ is a stopping time with $\sigma \le \tau_\epsilon$ for some $\epsilon > 0$. Then the measures $\mathbb{P}_\alpha$ and $\mathbb{P}_\beta$ are mutually absolutely continuous on $(\Omega, \mathcal{F}_\sigma)$. More precisely, if $t_0 < \infty$, there exists $c = c(\epsilon, t_0, \alpha, \beta) < \infty$ such that if $\sigma \le \tau_\epsilon \wedge t_0$,*

$$(24) \qquad \frac{1}{c} \le \frac{d\mathbb{P}_\alpha}{d\mathbb{P}_\beta} \le c.$$

♣ Clearly we can give more precise estimates for the Radon-Nikodym derivative, but this is all we will need in this paper.

Different values of β given different processes; chordal and two-sided radial SLE_κ correspond to particular values.

3.2. **Chordal SLE_κ: $\beta = 1 - 2a$.** Chordal SLE_κ (from 1 to w in $\mathbb{D}$ in the radial parametrization stopped at time T_w) is obtained from radial SLE_κ by weighting locally by S_t^{1-3a}. In other words,

$$(25) \qquad dX_t = (1 - 2a) \cot X_t \, dt + dB_{t,1-2a},$$

where $B_{t,1-2a}$ is a Brownian motion with respect to $\mathbb{P}_{1-2a}$.

This is not the usual way chordal SLE_κ is defined so let us relate this to the usual definition. SLE_κ from 0 to ∞ in $\mathbb{H}$ is defined by considering the Loewner equation

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t},$$

where $U_t = -B_t$ is a standard Brownian motion. There is a random curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ such that the domain of g_t is the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$. SLE_κ connected boundary points of other simply connected domains is defined (modulo time change) by conformal transformation. One can use Itô's formula to check that our definition agrees (up to time change) with the usual definition.

If D is a simply connected domain and w_1, w_2 are boundary points at which ∂D is locally smooth, the chordal SLE_κ partition function is defined (up to an unimportant multiplicative constant) by

$$H_\mathbb{H}(x_1, x_2) = |x_2 - x_1|^{-2b},$$

and the scaling rule

$$H_D(w_1, w_2) = |f'(w_1)|^b |f'(w_2)|^b H_{f(D)}(f(w_1), f(w_2)),$$

where $b = (3a - 1)/2$ is the boundary scaling exponent. To obtain SLE_κ from 0 to x in $\mathbb{H}$ one can take SLE_κ from 0 to ∞ and then weight locally by the value of the partition function between $g_t(x)$ and U_t, i.e., by $|g_t(x) - U_t|^{-2b}$. A simple computation shows that

$$H_\mathbb{D}(e^{2i\theta_1}, e^{2i\theta_2}) = |\sin(\theta_1 - \theta_2)|^{-2b} = |\sin(\theta_1 - \theta_2)|^{1-3a}.$$

Hence we see that chordal SLE_κ in $\mathbb{D}$ is obtained from radial SLE_κ by weighting locally by the chordal partition function.

3.3. **Two-sided radial SLE_κ: $\beta = 2a$.** If $\kappa < 8$, *Two-sided radial SLE_κ (from 1 to w in $\mathbb{D}$ going through 0 stopped when it reaches 0)* is obtained by weighting chordal SLE_κ locally by $(\sin X_t)^{(4a-1)}$. Equivalently, we can think of this as weighting radial SLE_κ locally by $(\sin X_t)^a$. It should be considered as chordal SLE_κ from 1 to w conditioned to go through 0.

♣ If $\kappa \geq 8$, SLE_κ paths are plane-filling and hence conditioning the path to go through a point is a trivial conditioning. For this reason, the discussion of two-sided radial is restricted to $\kappa < 8$.

The definition comes from the Green's function for chordal SLE_κ. If γ is a chordal SLE_κ curve from 0 to ∞ and $z \in \mathbb{H}$, let R_t denote the conformal radius of the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$ with respect to z, and let $R = \lim_{t\to\infty} R_t$. The Green's function $G(z) = G_\mathbb{H}(z; 0, \infty)$ can be defined (up to multiplicative constant) by the relation

$$\mathbb{P}\{R_t \leq \epsilon\} \sim c\,\epsilon^{2-d}\, G(z), \quad z \to \infty.$$

where $d = \max\{1 + \frac{\kappa}{8}, 2\}$ is the Hausdorff dimension of the paths. Roughly speaking, the probability that a chordal SLE_κ in $\mathbb{H}$ from 0 to ∞ gets within distance ϵ of z looks like $c\, G(z)\, \epsilon^{2-d}$. For other simply connected domains, the Green's function is obtained by conformal covariance

$$G_D(z; w_1, w_2) = |f'(z)|^{2-d}\, G_{f(D)}(f(z); f(w_1), f(w_2)),$$

assuming smoothness at the boundary. In particular, one can show that (up to an unimportant multiplicative constant)

$$G_\mathbb{D}(0; 1, e^{i\theta}) = (\sin \theta)^{4a-1}, \quad \kappa < 8.$$

4. Proofs of main results

4.1. **Continuity of radial SLE.** The key step to proving continuity of radial SLE_κ is an extension of an estimate for chordal SLE_κ to radial SLE_κ. The next lemma gives the analogous estimate for chordal SLE_κ; a proof can be found in [1]. Recall that $\alpha = (8/\kappa) - 1$.

Lemma 4.1. *For every $0 < \kappa < 8$, there exists $c < \infty$ such that if η is a crosscut in $\mathbb{H}$ and γ is a chordal SLE_κ curve from 0 to ∞ in $\mathbb{H}$, then*

$$(26) \qquad \mathbb{P}\{\gamma(0, \infty) \cap \eta \neq \emptyset\} \leq c \left[\frac{\operatorname{diam}(\eta)}{\operatorname{dist}(0, \eta)}\right]^\alpha.$$

We will prove the corresponding result for radial SLE_κ. We start by establishing the estimate up to a fixed time (this is the hardest estimate), and then extending the result to infinite time.

Lemma 4.2. *For every $t < \infty$, there exists $C_t < \infty$ such that the following holds. Suppose η is a crosscut of $\mathbb{D}$ and γ is a radial SLE_κ curve from 1 to 0 in $\mathbb{D}$. Then*

$$\mathbb{P}\{\gamma(0, t] \cap \eta \neq \emptyset\} \leq C_t \left[\frac{\operatorname{diam}(\eta)}{\operatorname{dist}(1, \eta)}\right]^\alpha.$$

Proof. Fix a positive integer n sufficiently large so that $\gamma(0, t] \cap \mathbb{D}_n = \emptyset$. All constants in this proof may depend on n (and hence on t).

Since $\operatorname{dist}(1, \eta) \leq 2$, it suffices to prove the lemma for crosscuts satisfying $\operatorname{diam}(\eta) < 1/100$ and $\operatorname{dist}(1, \eta) > 100 \operatorname{diam}(\eta)$. Such crosscuts do not disconnect 1 from 0 in $\mathbb{D}$.

Let $V = V_\eta$ denote the connected component of $\mathbb{D} \setminus \eta$ containing the origin, and let $F = F_\eta$ be a conformal transformation of V onto $\mathbb{D}$ with $F(0) = 0$. We write ∂V as a disjoint union:

$$\partial V = \{1\} \cup \eta[0, 1] \cup \partial_1 \cup \partial_2,$$

where ∂_1, ∂_2 are open connected subarcs of $\partial \mathbb{D}$. Let

$$L(\eta) = \frac{1}{2\pi} |F(\eta)| = h_{V_\eta}(0, \eta),$$

$$L^*(\eta) = \frac{1}{2\pi} \min\{|F(\partial_1)|, |F(\partial_2)|\} = \min\{h_{V_\eta}(0, \partial_1), h_{V_\eta}(0, \partial_2)\},$$

where $|\cdot|$ denotes length. Note that

$$\operatorname{diam}(\eta) \asymp L(\eta) \qquad \operatorname{dist}(1, \eta) \asymp L^*(\eta),$$

and hence we can write the conclusion of the lemma as

$$(27) \qquad \mathbb{P}\{\gamma(0, t] \cap \eta \neq \emptyset\} \leq c \left[\frac{L(\eta)}{L^*(\eta)}\right]^\alpha,$$

which is what we will prove.

Let γ be a radial SLE_κ curve. If $\gamma(0, t] \cap \eta = \emptyset$ and $\eta(0, 1) \subset D_t$, let V_t be the connected component of $D_t \setminus \eta$ containing the origin with corresponding maps F_t. We write

$$\partial V_t = \{\gamma(t)\} \cup \eta[0, 1] \cup \partial_{1, t} \cup \partial_{2, t}$$

where the boundaries are considered in terms of prime ends. Let

$$L_t(\eta) = \frac{1}{2\pi} |F_t(\eta)| = h_{V_t}(0, \eta),$$

$$L_t^*(\eta) = \frac{1}{2\pi} \min\left\{|F_t(\partial_{1,t})|, |F_t(\partial_{2,t})|\right\} = \min\left\{h_{V_t}(0, \partial_{1,t}), h_{V_t}(0, \partial_{2,t})\right\}.$$

Note that $L_t(\eta)$ decreases with t but $L_t^*(\eta)$ is not monotone in t.

As before, let $\rho = \rho_n$ be the first time s that $|\gamma(s)| \leq e^{-n}$; our assumption on n implies that $\rho \geq t$. Let $\sigma = \sigma_n$ be the first time s that $\text{Re}[\gamma(s)] \leq e^{-2n}$. Our proof will include a series of claims each of which will be proved after their statement.

- **Claim 1**. There exists $u > 0$ (depending on n), such that

(28) $$\mathbb{P}\{\sigma \wedge \rho \geq t\} \geq u.$$

Deterministic estimates using the Loewner equation show that if U_t stays sufficiently close to 0, then $\rho < \sigma$. Therefore, since $\rho \geq t$,

$$\mathbb{P}\{\sigma \wedge \rho > t\} \geq \mathbb{P}\{\rho < \sigma\} > 0.$$

- **Claim 2**. There exists $c < \infty$ such that

(29) $$\mathbb{P}\{\gamma(0, \sigma] \cap \eta \neq \emptyset\} \leq c \left[\frac{\text{diam}(\eta)}{\text{dist}(0, \eta)}\right]^\alpha$$

To show this we compare radial SLE_κ from 1 to 0 with chordal SLE_κ from 1 to -1. Note by (10) that σ is uniformly bounded. Straightforward geometric arguments show that there exists c (recall that constants may depend on n) such that $c^{-1} \leq h_\sigma'(-1) \leq c$ and $\sin X_\sigma \geq c^{-1}$. By (24) the Radon-Nikodym derivative of radial SLE_κ with respect to chordal SLE_κ is uniformly bounded away from 0 and ∞ and therefore if $\tilde{\gamma}$ denotes a chordal SLE_κ path from 1 to -1,

$$\mathbb{P}\{\gamma(0, \sigma] \cap \eta = \emptyset\} \asymp \mathbb{P}\{\tilde{\gamma}(0, \sigma] \cap \eta = \emptyset\}.$$

Hence (29) follows from (26).

- **Claim 3**. There exists $\delta > 0$ such that if $L(\eta), L^*(\eta) \leq \delta$, then on the event

(30) $$\{\gamma(0, \sigma] \cap \eta = \emptyset\},$$

    we have

(31) $$\frac{L_\sigma(\eta)}{L_\sigma^*(\eta)} \leq \frac{L(\eta)}{L^*(\eta)}.$$

It suffices to consider η with $L(\eta), L^*(\eta) \leq 1/10$, and without loss of generality we assume that η is "above" 1 in the sense that its endpoints are $e^{i\theta_1}, e^{i\theta_2}$ with $0 < \theta_1 \leq \theta_2 < 1/4$. Let $\gamma(\sigma) = e^{-2n} + iy$, and let $V_\sigma = V_{\eta,\sigma}$ be the connected component of $V \setminus \gamma(0, \sigma]$ containing the origin. Suppose $\gamma(0, \sigma] \cap \eta = \emptyset$ and

(32) $$\eta \subset V_\sigma,$$

(If $\eta \not\subset V_\sigma$, then $L_\sigma(\eta) = 0$.) As before we write

$$\partial V_\sigma = \{\gamma(\sigma)\} \cup \eta[0, 1] \cup \partial_{1,\sigma} \cup \partial_{2,\sigma},$$

where we write $\partial_{1,\sigma}$ for the component of the boundary that includes -1. Note that (30) and (32) imply that $\partial_{1,\sigma}$ in fact contains $\{e^{i\theta} : \theta_2 < \theta < 3\pi/2\}$. Let ℓ denote the crosscut of V_σ given by the vertical line segment whose lowest point is $\gamma(\sigma)$ and whose highest point is on $\{e^{i\theta} : 0 < \theta < \pi/2\}$. Note that $V_\sigma \setminus \ell$ has two connected components, one containing the origin and the other, which we denote by $V^* = V_{\sigma,\eta}^*$, with $\eta \subset \partial V^*$. Let ϵ denote the length of ℓ and for the moment assume that $\epsilon < 1/4$. Topological considerations using (30) and (32) imply that all

the points in $\partial V^* \cap \{z \in \mathbb{D} : e^{-2n} < \text{Re}(z) < e^{-1}\}$ (considered as prime ends) are in $\partial_{2,\sigma}$.

We consider another crosscut ℓ' defined as follows. Let $x = e^{-2n} + \epsilon$. Start at $x + i\sqrt{1-x^2} \in \partial\mathbb{D}$ and take a vertical segment downward of length 2ϵ to the point $z' = x + i(\sqrt{1-x^2} - 2\epsilon)$. From z' take a horizontal segment to the left ending at $\{\text{Re}(z) = e^{-2n}\}$. This curve, which is a concatentation of two line segments, must intersect the path $\gamma(0,\sigma]$ at some point; let ℓ' be the crosscut obtained by stopping this curve at the first such intersection. Let V' be the connected component of $V^* \setminus \ell'$ that contains ℓ on its boundary. The key observations are: $\text{dist}(\ell, \ell') \geq c\epsilon$, $\text{diam}(\ell) = O(\epsilon), \text{diam}(\ell') = O(\epsilon)$ and $\text{area}(V') = O(\epsilon^2)$. In particular (see, e.g., [5, Lemma 3.74]) the π-extremal distance between ℓ and ℓ' is bounded below by a positive constant c_1 independent of ϵ.

Let B_t be a standard complex Brownian motion starting at the origin and let

$$\tau = \inf\{t : B_t \notin V_\sigma\}, \quad \xi = \inf\{t : B_t \in \ell'\}.$$

Then using Lemma 2.4 we see that

(33) $$\min\{\mathbb{P}\{B_\tau \in \partial_{1,\sigma}\}, \mathbb{P}\{B_\tau \in \partial_{2,\sigma}\}\} \geq c_2\, \mathbb{P}\{\xi < \tau\}.$$

Also, we claim that

$$\mathbb{P}\{B_\tau \in \eta \mid \xi < \tau\} \leq c\,\epsilon\, L(\eta).$$

To justify this last estimate, note that $\text{dist}(B_\xi, \partial\mathbb{D}) = O(\epsilon)$. It suffices to consider the probability that a Brownian motion starting at B_ξ hits η before leaving $\mathbb{D}$. The gambler's ruin estimate implies that the probability that a Brownian motion starting at B_ξ reaches $\{\text{Re}(z) = 1/2\}$ before leaving $\mathbb{D}$ is $O(\epsilon)$. Given that we reach $\{\text{Re}(z) = 1/2\}$, the probability to hit η before leaving $\mathbb{D}$ is $O(L(\eta))$. Therefore,

(34) $$\mathbb{P}\{B_\tau \in \eta\} \leq c\,\epsilon\, L(\xi)\, \mathbb{P}\{\xi < \tau\}.$$

By combining (33) and (34), we see that we can choose $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$,

$$\mathbb{P}\{B_\tau \in \eta\} \leq L(\eta)\min\{\mathbb{P}\{B_\tau \in \partial_{1,\sigma}\}, \mathbb{P}\{B_\tau \in \partial_{2,\sigma}\}\},$$

and hence

(35) $$L_\sigma(\eta) \leq L(\eta)\, L_\sigma^*(\eta) \quad \text{if} \quad \epsilon \leq \epsilon_0.$$

One we have fixed ϵ_0, we note there exists $c = c(\epsilon_0) > 0$ such that if $\epsilon \geq \epsilon_0$,

$$L_\sigma^*(\eta) = \min\{\mathbb{P}\{B_\tau \in \partial_{1,\sigma}\}, \mathbb{P}\{B_\tau \in \partial_{2,\sigma}\}\} \geq c_2.$$

Indeed, to bound $\mathbb{P}\{B_\tau \in \partial_{1,\sigma}\}$ from below we consider Brownian paths starting at the origin that leave $\partial\mathbb{D}$ before reaching $\{z : \text{Re}(z) \geq e^{-2n}\}$. To bound $\mathbb{P}\{B_\tau \in \partial_{2,\sigma}\}$ consider Brownian paths in the disk that start at the origin, go through the crosscut l (defined using $\epsilon = \epsilon_0$), and then make a clockwise loop about $\gamma(\sigma)$ before leaving $\mathbb{D}$ and before reaching $\{\text{Re}(z) \geq 1/2\}$. Topological considerations show that these paths exit V_σ at $\partial_{2,\sigma}$. Combining this with (35) and the estimate $L_\sigma(\eta) \leq L(\eta)$, we see that there exists $c_1 > 0$ such that for all η,

$$L_\sigma(\eta) \leq c_1^{-1}\, L(\eta)\, L_\sigma^*(\eta).$$

In particular,

$$\frac{L_\sigma(\eta)}{L_\sigma^*(\eta)} \leq \frac{L(\eta)}{L^*(\eta)} \quad \text{if} \quad L^*(\eta) \leq c_1.$$

From this we conclude (31).

Fix δ such that (31) holds, and let $\phi(r)$ be the supremum of

$$\mathbb{P}\{\gamma(0,t] \cap \eta \neq \emptyset\}$$

where the supremum is over all η with

$$L(\eta) \leq r \, \min\{\delta, L^*(\eta)\}.$$

- **Claim 4**. If $r < \delta$, then $\phi(r)$ equals $\tilde{\phi}(r)$ which is defined to be the supremum of

$$\mathbb{P}\{\gamma(0,t] \cap \eta \neq \emptyset\}$$

where the supremum is over all η with

$$L(\eta) \leq \min\{\delta, r \, L^*(\eta)\}.$$

To see this, suppose η is a curve with $L(\eta) \leq r\delta, L^*(\eta) > \delta$. Let S be the first time s such that $L^*_s(\eta) = \delta$. Note that $S < \inf\{s : \gamma(s) \in \eta\}$. Since $L_S(\eta) \leq L(\eta) \leq r\delta$, we see that

$$\mathbb{P}\{\gamma(0,t] \cap \eta \neq \emptyset\} \leq \mathbb{P}\{\gamma(0,t] \cap \eta \neq \emptyset \mid S < \infty\} \leq \tilde{\phi}(r).$$

This establishes the claim.

To finish the proof of the lemma, suppose $r < \delta$. Since $\tilde{\phi}(r) = \phi(r)$, we can find a crosscut η with $L(\eta) \leq r \, L^*(\eta)$ and $L^*(\eta) \leq \delta$ such that

$$\phi(r) = \mathbb{P}\{\gamma(0,t] \cap \eta \neq \emptyset\}.$$

(For notational ease we are assuming the supremum is obtained. We do not need to assume this, but could rather take a sequence of crosscuts η_j with $\mathbb{P}\{\gamma(0,t] \cap \eta_j \neq \emptyset\} \to \phi(r)$.) Using Claim 3, we see that if $\gamma(0,\sigma] \cap \eta = \emptyset$, then

$$L_\sigma(\eta) \leq L(\eta) \leq r, \quad L^*_\sigma(\eta) \leq L_\sigma(\eta).$$

Therefore,

$$\mathbb{P}\{\gamma(0,t] \cap \eta \neq \emptyset \mid \gamma(0,\sigma \wedge \rho] \cap \eta = \emptyset\} \leq \phi(r).$$

Hence, using (28),

$$\phi(r) = \mathbb{P}\{\gamma(0,t] \cap \eta \neq \emptyset\} \leq \mathbb{P}\{\gamma(0,\sigma \wedge \rho] \cap \eta \neq \emptyset\} + (1-u)\,\phi(r),$$

which implies

$$\phi(r) \leq \frac{1}{1-u}\,\mathbb{P}\{\gamma(0,\sigma \wedge \rho] \cap \eta \neq \emptyset\}.$$

Combining this with (29), we get

$$(36) \qquad\qquad \phi(r) \leq \frac{c}{1-u}\left[\frac{L(\eta)}{L^*(\eta)}\right]^\alpha \leq c'\,r^\alpha.$$

$\square$

Proposition 4.3. *If $0 < \kappa < 8$, there exists $c < \infty$ such that the following holds. Suppose η is a crosscut of $\mathbb{D}$ and γ is a radial SLE_κ curve from 1 to 0 in $\mathbb{D}$. Then*

$$\mathbb{P}\{\gamma(0,\infty) \cap \eta \neq \emptyset\} \leq c \left[\frac{\mathrm{diam}(\eta)}{\mathrm{dist}(1,\eta)}\right]^\alpha.$$

Proof. We may assume that $\eta \cap \mathbb{D}_1 = \emptyset$. By Lemma 2.6, for $n \geq 5$, conditioned on $\gamma[0, \rho_n] = \emptyset$, we know that

$$L_{\rho_n} \leq c\, L[\eta]\, e^{-n/2} L^*_{\rho_n}.$$

Since $\rho_{n+1} - \rho_n$ is uniformly bounded in n, we can use Lemma 4.2 to conclude that

$$\mathbb{P}\{\gamma[0, \rho_5] \cap \eta \neq \emptyset\} \leq c \left[\frac{L[\eta]}{L^*[\eta]} \right]^\alpha,$$

and for $n \geq 5$,

$$\mathbb{P}\{\gamma[0, \rho_{n+1}) \cap \eta \neq \emptyset \mid \gamma[0, \rho_n] \cap \eta = \emptyset\} \leq c\, L[\eta]^\alpha\, e^{-n\alpha/2} \leq c\, e^{-n\alpha/2} \left[\frac{L[\eta]}{L^*[\eta]} \right]^\alpha.$$

By summing over n we get the proposition.

$\square$

Proof of Theorem 1. We start by proving the stronger result for $\kappa \leq 4$. Note that $\partial \mathbb{D}_k \cap H_n$ is a disjoint union of crosscuts $\eta = \{ e^{-k+i\theta} : \theta_{1,\eta} < \theta < \theta_{2,\eta} \}$. For each η, we use Lemma 2.6 and Proposition 4.3 to see that

$$\mathbb{P}\left\{ \gamma[\rho_{n+k}, \infty) \cap \eta \neq \emptyset \mid \mathcal{F}_{\rho_n} \right\} \leq c\, e^{-n\alpha/2} (\theta_2 - \theta_1)^\alpha.$$

However, since $\alpha \geq 1$ (here we use the fact that $\kappa \leq 4$),

$$(37) \qquad \sum_\eta (\theta_{2,\eta} - \theta_{1,\eta})^\alpha \leq \left[\sum_\eta (\theta_{2,\eta} - \theta_{1,\eta}) \right]^\alpha \leq (2\pi)^\alpha.$$

We will now prove (2) assuming only $\kappa < 8$. Let $E = E_{j,k,n}$ denote the event $\gamma[\rho_k, \rho_{n+k}] \subset \mathbb{D}_j$. Lemma 2.3 implies that on the event E, there is a unique crosscut $l \in \mathcal{A}_{n+k,k}$ such that every curve from the origin to $\partial \mathbb{D}_j$ in H_{n+k} intersects l. Hence, on E

$$\mathbb{P}\left\{ \gamma[\rho_{n+k}, \infty) \not\subset \mathbb{D}_j \mid \mathcal{G}_{n+k} \right\}$$

is bounded above by the supremum of

$$\mathbb{P}\left\{ \gamma[\rho_{n+k}, \infty) \cap l \neq \emptyset \mid \mathcal{G}_{n+k} \right\},$$

where the supremum is over all $l \in \mathcal{A}_{n+k,k}$. For each such crosscut l, we use Lemma 2.6 and Proposition 4.2 to see that

$$\mathbb{P}\left\{ \gamma[\rho_{n+k}, \infty) \cap l \neq \emptyset \mid \mathcal{G}_n \right\} \leq c\, e^{-n\alpha/2}.$$

$\square$

4.2. **Two-sided radial SLE_κ.** In order to prove that two-sided radial SLE_κ is continuous at the origin, we will prove the following estimate. It is the analogue of Proposition 4.3 restricted to the crosscut that separates the origin from -1.

Proposition 4.4. *If $\kappa < 8$ there exist c' such if γ is two-sided radial from 1 to -1 through 0 in $\mathbb{D}$, then for all $k, n > 0$, if $l = l^*_{n+k.k}$,*

$$(38) \qquad \mathbb{P}\{\gamma[\rho_{n+k}, \infty) \cap \bar{l} \neq \emptyset \mid \mathcal{G}_{n+k}\} \leq c'\, e^{-n\alpha/2}.$$

Proof. Let $\rho = \rho_{n+k}$ and as in Lemma 2.7, let $\psi = \psi^*_{n+k.k}$ be the first time $t \geq \rho$ that $\gamma(t) \in \bar{l}$. It suffices to show that

$$\mathbb{P}\{\psi < \rho_{n+k+1} \mid \mathcal{G}_{n+k}\} \leq c\, e^{-n\alpha/2},$$

for then we can iterate and sum over n. By Lemma 2.7, we know that

$$(39) \qquad\qquad S_\psi(0) \le c\,e^{-n/2}\,S_\rho(0).$$

Also, (10) gives $\rho_{n+k+1} - \rho \le c_1$ for some uniform $c_1 < \infty$. Recalling that two-sided SLE_κ corresponds to the radial Bessel equation (11) with $\beta = 2a$, we see from Lemma 2.2, that

$$\mathbb{P}\left\{ \min_{\rho \le t \le \rho + c_1} S_t(0) \le \epsilon\, S_\rho(0) \mid \mathcal{G}_{n+k} \right\} \le c\epsilon^{4a-1} = c\,\epsilon^\alpha.$$

Combining this with (39) gives the first inequality. $\qquad\qquad\square$

Proof of Theorem 2. To prove (4), we recall Lemma 2.3 which tells us that if $\gamma[\rho_k, \rho_{n+k}] \subset \mathbb{D}_j$, then in order for $\gamma[\rho_{n+k}, \infty)$ to intersect $\mathbb{D}_j$ is is necessary for it to intersect $\bar{l}$. $\qquad\qquad\square$

4.3. **Proof of Theorem 3.** Here we finish the proof of Theorem 3. We have already proved the main estimates (2) and (4). The proof is essentially the same for radial and two-sided radial; we will do the two-sided radial case. We will use the following lemma which has been used by a number of authors to prove exponential rates of convergence, see, e.g., [2]. Since it is not very long, we give the proof. An important thing to note about the proof is that it does not give a good estimate for the exponent u.

Lemma 4.5. *Let ϵ_j be a decreasing sequence of numbers in $[0, 1)$ such that*

$$(40) \qquad\qquad \limsup_{n \to \infty} \epsilon_n^{1/n} < 1.$$

Then there exist c, u such that the following holds. Let X_n be a discrete time Markov chain on state space $\{0, 1, 2 \ldots\}$ with transition probabilities

$$p(j, 0) = 1 - p(j, j+1) \le \epsilon_j.$$

Then,

$$\mathbb{P}\{X_n < n/2 \mid X_0 = 0\} \le c\,e^{-nu}.$$

♣ The assumption that ϵ_j decrease is not needed since one can always consider $\delta_j = \min\{\epsilon_1, \ldots, \epsilon_j\}$ but it makes the coupling argument described below easier.

Proof. We will assume that $p(j, 0) = \epsilon_j$. The more general result can be obtained by a simple coupling argument defining (Y_n, X_n) on the same space where

$$\mathbb{P}\{Y_{j+1} = 0 \mid Y_j = n\} = 1 - \mathbb{P}\{Y_{j+1} = n+1 \mid Y_j = n\} = \epsilon_n,$$

in a way such that $Y_n \le X_n$ for all n.

Let $p_n = \mathbb{P}\{X_n = 0 \mid X_0 = 0\}$, with corresponding generating function

$$G(\xi) = \sum_{n=0}^\infty p_n\,\xi^n.$$

Let

$$\delta = \mathbb{P}\{X_n \ne 0 \text{ for all } n \ge 1 \mid X_0 = 0\} = \prod_{n=0}^\infty [1 - p(n, n+1)] = \prod_{n=0}^\infty [1 - \epsilon_n] > 0.$$

For $n \geq 1$, let
$$\mathbb{P}\{X_n = 0; X_j \neq 0, 1 \leq j \leq n-1 \mid X_0 = 0\}$$
with generating function
$$F(\xi) = \sum_{n=1}^{\infty} q_n \, \xi^n.$$
Note that
$$q_n = p(0,1) \, p(1,2) \cdots p(n-2, n-1) \, p(n-1, 0) \leq \epsilon_{n-1}.$$
Therefore, (40) implies that the radius of convergence of F is strictly greater than 1. Since $F(1) = 1 - \delta < 1$, we can find $t > 1$ with $F(t) < 1$, and hence
$$G(t) = [1 - F(t)]^{-1} < \infty,$$
In particular, if $e^{2u} < t$, then there exists $c < \infty$ such that for all n,
$$p_n \leq c \, e^{-2un}.$$
Let A_n be the event that $X_m = 0$ for some $m \geq n/2$. Then,
$$\mathbb{P}(A_n) \leq \sum_{j \geq n/2} p_j \leq c' \, e^{-un}.$$
But on the complement of A_n, we can see that $X_n \geq n/2$.

$\square$

Proof of Theorem 3. The proof is the same for radial or two-sided SLE_κ. Let us assume the latter. The important observation is that for every $0 < k < m < \infty$, we can find $\epsilon > 0$ such that for all n,
$$\mathbb{P}\{\gamma[\rho_{n+5+k}, \rho_{n+m+5+k}] \subset \mathbb{D}_{n+5} \mid \mathcal{G}_n\} \geq \epsilon.$$
(This can be shown by considering the event that the driving function stays almost constant for a long interval of time after ρ_n.. We omit the details.) By combining this with Proposition 4.4, we can see that there exists m, ϵ such that
(41) $$\mathbb{P}\{\gamma[\rho_{n+m+5}, \infty) \subset \mathbb{D}_{n+5} \mid \mathcal{G}_n\} \geq \epsilon.$$

To finish the argument, let us fix k. Let $c', u = \alpha/2$ be the constants from (4) and let m be sufficiently large so that $c'e^{-nu} \leq 1/2$ for $n \geq m$. For positive integer n define L_n to be the largest integer j such that
$$\gamma[\rho_{n+k-j}, \rho_{n+k}] \subset \mathbb{D}_k.$$
The integer j exists but could equal zero. From (4), we know that
$$\mathbb{P}\{L_{n+k+1} = L_{n+k} + 1 \mid \mathcal{G}_{n+k}\} \geq 1 - c' \, e^{-nL_{n+k}u},$$
and if $L_{n+k} \geq m$, the right-hand side is greater than $1/2$.

We see that the distribution of L_{n+k} is stochastically bounded below by that of a Markov chain X_n of the type in Lemma 4.5. Using this we see that there exists C', δ such that
$$\mathbb{P}\{L_{n+k} \leq n/2 \mid \mathcal{G}_k\} \leq C' \, e^{-\delta n}.$$
On the event $\mathbb{P}\{L_{n+k} \geq n/2\}$, we can use (38) to conclude that the conditional probability of returning to $\partial \mathbb{D}_k$ after time ρ_{n+k} given $L_{n+k} \geq n/2$ is $O(e^{-n\alpha/4})$. This completes the proof with $u = \min\{\delta, \alpha/4\}$.

$\square$

References

[1] Tom Alberts and Michael J. Kozdron. Intersection probabilities for a chordal SLE path and a semicircle. *Electron. Commun. Probab.*, 13:448–460, 2008.

[2] Xavier Bressaud, Roberto Fernández, and Antonio Galves. Decay of correlations for non-Hölderian dynamics. A coupling approach. *Electron. J. Probab.*, 4:no. 3, 19 pp. (electronic), 1999.

[3] G. Lawler. Schramm-Loewner evolution (SLE). In *Statistical mechanics*, volume 16 of *IAS/Park City Math. Ser.*, pages 231–295. Amer. Math. Soc., Providence, RI, 2009.

[4] G. Lawler and B. Werness. Multi-point Green's function and an estimate of Beffara. preprint.

[5] G.F. Lawler. *Conformally invariant processes in the plane.* Amer Mathematical Society, 2008.

[6] Gregory Lawler, Oded Schramm, and Wendelin Werner. Conformal restriction: the chordal case. *J. Amer. Math. Soc.*, 16(4):917–955 (electronic), 2003.

[7] Gregory F. Lawler and Joan R. Lind. Two-sided $SLE_{8/3}$ and the infinite self-avoiding polygon. In *Universality and renormalization*, volume 50 of *Fields Inst. Commun.*, pages 249–280. Amer. Math. Soc., Providence, RI, 2007.

[8] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel Journal of Mathematics*, 118(1):221–288, 2000.

[9] O. Schramm and S. Rohde. Basic properties of SLE. *Annals of mathematics*, 161(2):883, 2005.

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