Random Walk Problems Motivated by Statistical Physics

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This paper is dedicated to the memory of Ed Nelson.

Abstract. This paper is a an expanded version of a talk that I gave at the St. Petersburg summer school on probability and statistical physics in 2012. I discuss the state of research and open problems on three models that arise in equlibirum statistical physics: self-avoiding random walk, loop-erased random walk, and intersections of paths of simple random walks.

1. Introduction

A rich source of challenging problems in probability has been statistical physics. In this article, I will consider three related models of random walks with self-repulsion which have driven much of my research for many years:

- Self-avoiding walk
- Nonintersection probabilities for simple random walks
- Loop-erased walk

As is true for many lattice models in statistical physics, the definition of these models in finite regions is easy. The challenge is to understand the limit as the number of steps of the walks goes to infinity, or, essentially equivalently, to understand the scaling limit of the process as the lattice spacing goes to zero. There has been much progress in the last thirty-five years, but there are still many major problems that are open. In the course of this paper, I will define the models and discuss what is known rigorously and what is conjectured to hold for these walks. In most cases I will not give complete proofs, but I hope to give some ideas about how results are proved.

We start by stating some of the properties that relate these models to each other and to a number of models in "equilibrium statistical mechanics at criticality." These walks are nearest-neighbor paths on the integer lattice

$$\mathbb{Z}^d = \mathbb{Z}\underbrace{\times \cdots \times}_{d}\mathbb{Z}.$$

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- Using the square lattice is a convenience but should not be important in the long-time behavior. This lattice-independence is sometimes called *universality.*)
- The behavior depends strongly on the spatial dimension d.
- These models have an upper-critical dimension of 4. For all $d > 4$, the large scale behavior should be similar exhibiting "mean-field" behavior.
- For $d = 4$, there is still "mean-field" behavior but there are logarithmic corrections.
- For $d < 4$, the behavior is not mean-field, and there are nontrivial "critical exponents" that describe power-law correlations.
- The case $d = 1$ can be solved but should be considered a special case.
- For $d = 2$, the exponents that arise are rational numbers that can be computed exactly. The scaling limits of the walks exhibit some kind of conformal invariance.
- For $d = 3$, the critical exponents do not appear to be rational, and may never be given exactly. The least is known about this dimension.
- The critical exponents reflect the fractal properties of their scaling limits.

We will start with a more basic model, the simple random walk. Unlike the walks with interaction, the simple random walk is well understood, both on the lattice level and in its scaling limit, Brownian motion. There are two reasons to discuss this is some detail. First, the analysis of at least two of our models, intersection probabilities and the loop-erased walk, make strong use of our knowledge of simple random walk. Second, we use the returns to the origin as a warm-up toy model to motivate the more complicated models. This is a model with critical dimension $d = 2$.

Then we consider the self-avoiding walk which is a basic model for single polymers in a dilute solution. The definition is very simple, yet this has turned out to be a notoriously difficult problem to solve rigorously. Most of our treatment is in terms of conjectures[1] that can be understood probabilistically. However, probability theory has not been a particularly useful tool for rigorous analysis of this essentially combinatorial model.

The other two models in this paper arose as analogous models with some of the properties of the self-avoiding walk yet were tractable by probabilistic techniques. The intersection exponent for simple random walk is one of the easier (but still not easy!) models having the dimension dependent behavior as above. It is the simple random walk analogue of the self-avoiding walk intersection exponent, the latter of which is related to counting the number of self-avoiding walks. It also has the advantage that one can consider a continuous model, intersections of Brownian paths. Continuum models such as Brownian motion have the advantage that they are invariant under scaling, and in this case conformally invariant in two dimensions. It can be proved that the fractal dimension of certain exceptional sets on a Brownian motion can be given in terms of this exponent.

The loop-erased walk is a personal favorite as it goes back to my doctoral dissertation when my advisor Ed Nelson, motivated by the self-avoiding walk problem, suggested that I see what happens when one erases loops from a simple random

[1] Sometimes we prefer to use the word prediction rather than conjecture. The word prediction is used by a number of researchers to indicate a derivation that includes a serious heuristic mathematical justification but is not a rigorous proof.

walk. While it turned out that this model does not have the same behavior as the self-avoiding walk, it is an interesting model in itself. As others discovered later, it is related to a number of models such as the uniform spanning tree. This again is a difficult problem, but more has been proven for this model than for SAW.

We give a brief introduction to the Schramm-Loewner evolution which is a continuum limit of many critical models in two dimensions. We also give a heuristic about the self-avoiding walk that has not appeared before and helps relate that model to the other two models.

As already mentioned, critical exponents for lattice systems often reflect facts about the fractal dimensions of the scaling limit. We discuss two main ways of describing this, the Minkowski content (related to the box dimension) and the Hausdorff measure.

2. Simple random walk

2.1. Definition. Let $\mathbf{e}_j$ denote the unit vector in $\mathbb{Z}^d$ whose jth component equals one, and let $X_1, X_2, \ldots$ be independent random variables with

$$\mathbb{P}\{X_n = \mathbf{e}_j\} = \mathbb{P}\{X_n = -\mathbf{e}_j\} = \frac{1}{2d}, \quad j = 1, \ldots, d.$$

Then $S_n = X_1 + \cdots + X_n$ is called *simple (symmetric) random walk* in $\mathbb{Z}^d$ starting at the origin. This is a *kinetically growing* description of the process in the sense that it gives the conditional distribution of each step given the previous steps. We can also give a *configurational* description of the walk as a consistent collection of probability measures $\{\mathcal{P}_n\}$. Here $\mathcal{P}_n$ denotes the probability measure on nearest neighbor random walk paths of length n starting at the origin that gives measure $(2d)^{-n}$ to each such walk. We write such a path as

$$\omega = [\omega_0, \ldots, \omega_n]$$

with $\omega_0 = 0$ and $|\omega_j - \omega_{j-1}| = 1$. We write $|\omega| = n$ for the length (number of steps) of the path. There are $(2d)^n$ nearest neighbor paths starting at the origin of length n and each one gets measure $(2d)^{-n}$ under $\mathcal{P}_n$. The consistency condition (which is almost trivial to verify in this case) can be described as follows. Let us write $\omega \prec \omega'$ if $|\omega| \leq |\omega'|$ and the first $|\omega|$ steps of ω' agree with those of ω. Then, if $n \leq m$,

$$\mathcal{P}_n(\omega) = \sum_{\omega \prec \omega'} \mathcal{P}_m(\omega').$$

2.2. Local Central Limit Theorem. The techniques of classical probability are very well suited for studying simple random walk. Since S_n is the sum of independent, identically distributed random variables, the *central limit theorem (CLT)* implies that the distribution of $S_n/\sqrt{n}$ approaces that of a random variable $Z = (Z_1, \ldots, Z_d)$ with normal distribution with mean zero and covariance matrix $\Gamma = [\mathbb{E}(Z_j Z_k)] = d^{-1}\, I$. This distribution has density

$$g(x) = \left(\frac{d}{2\pi}\right)^{d/2} \exp\left\{-\frac{d|x|^2}{2}\right\}, \quad x \in \mathbb{R}^d.$$

More precise estimates are given by the *local central limit theorem (LCLT)*. If $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, let

$$\overline{p}(n, x) = n^{-d/2}\, g(x/\sqrt{n}).$$

The LCLT states roughly that for n large, with $n + x_1 + \cdots + x_d$ even,

$$\mathbb{P}\{S_n = x\} \sim 2\,\overline{p}(n,x). \tag{2.1}$$

Here the symbol $\sim$ indicates that the two sides are *asymptotic*, that is, the ratio converges to 1. The factor of 2 comes from the bipartite nature of simple random walk; after an even number of steps the sum of the components of S_n is always even. The relation (2.1) holds for "typical" x ($|x| \leq n^{1/2}$ say), but it cannot hold for all x, n. For example, if $|x| > n$, then the left-hand side is zero but the right-hand side is positive.

The standard proof of the LCLT uses the independence and identical distribution of the increments which allows one to compute the characteristic function (Fourier transform),

$$\phi_n(\theta) = \mathbb{E}\left[e^{i\theta\cdot S_n}\right] = \left(\mathbb{E}\left[e^{i\theta\cdot X_1}\right]\right)^n = \left[\frac{1}{d}\sum_{j=1}^{d}\cos\theta_j\right]^n.$$

The inversion formula

$$\mathbb{P}\{S_n = x\} = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} e^{-ix\cdot\theta}\,\phi_n(\theta)\,d\theta$$

gives an exact expression for the probability, and we are left only with estimating the integral. Estimating oscillatory integrals can be tricky, but in this case (see [**22**, Chapter 2] for details) it can be shown that there exists $\varepsilon > 0$ such that for $|x| \leq \varepsilon n$,

$$\mathbb{P}\{S_n = x\} = 2\,\overline{p}(n,x)\,\exp\left\{O\left(\frac{|x|^4}{n^3} + \frac{1}{n}\right)\right\}. \tag{2.2}$$

In particular, there exists $c < \infty$ such that if $|x| \leq n^{3/4}$,

$$\left|\frac{\mathbb{P}\{S_n = x\}}{2\,\overline{p}(n,x)} - 1\right| \leq c\,\frac{|x|^4 + n^2}{n^3}.$$

These expressions assume that x and n have the same parity. Note that this shows the validity of (2.1) for x much larger than typical. (The problems in statistical mechanics can often be phrased in terms of "large deviations" from a base measure such as simple random walk. Having sharp estimates for simple random walk away from the typical regime can be very useful.)

Convergence to the normal distribution is a *universal* phenomenon for sums of independent random variables. If $X_1, X_2, \ldots$ are i.i.d. random variables in $\mathbb{Z}^d$ with $\mathbb{E}[X_j] = 0, \mathbb{E}[|X_j|^2] < \infty$, and $S_n = X_1 + \cdots + X_n$, then $S_n/\sqrt{n}$ converges to a centered normal distribution with covariance matrix Γ the same as that of the X_j. Using the characteristic function, one can prove an LCLT of the form (2.1) (where the factor 2 is replaced with the appropriate periodicity factor — if the walk is aperiodic then the factor will be 1). This will be valid for $|x|$ of order $n^{1/2}$. Further moment and symmetry conditions are needed to give uniform estimates for the error. For example, the very precise estimates of the form (2.2) require that $\mathbb{E}[e^{b|X_1|}] < \infty$ for some $b > 0$ and that all the third moments of X_1 vanish.

2.3. Returns to the origin. Setting $x = 0$ in (2.2) gives the probability that the simple random walk is at the origin at time $2n$,

$$\mathbb{P}\{S_{2n} = 0\} = c_d\, n^{-d/2}\, [1 + O(n^{-1})], \tag{2.3}$$

where in this case $c_d = 2^{1-d}\,(d/\pi)^{\frac{d}{2}}$. Here $d/2$ is an example of a dimension dependent *critical exponent*. One does not need the full power of the LCLT to understand why we get the exponent $d/2$. In n steps the simple random walk tends to go distance $n^{1/2}$, and there are $O(n^{d/2})$ points that are distance $n^{1/2}$ from the origin. The probability of choosing a particular one, in this case the origin, should be of order $n^{-d/2}$. This exponent is a linear function of the spatial dimension d. (There are a number of ways to define simple random walks on sets of spatial dimension d where d is noninteger in a way that this exponent is $d/2$; we will not discuss them here.) We can think of $\nu = 1/2$ as a dimension independent exponent giving the rate of growth of $|S_n|$. This exponent is often defined as the *mean square displacement exponent* ν,

$$\mathbb{E}[|S_n|^2] = n^{2\nu}, \quad \nu = \frac{1}{2}.$$

A closely related quantity is the *expected number* of returns to the origin by time n,

$$E_n = \mathbb{E}\left[\sum_{j=0}^{n} 1\{S_j = 0\}\right] = \sum_{j=0}^{n} p_j(0).$$

Using the asymptotic expression (2.3) we see that

$$E_n = c\, n^{1-\frac{d}{2}} + O(n^{-\frac{d}{2}}), \quad d < 2,$$
$$E_n = c\,\log n + c' + O(n^{-1}), \quad d = 2,$$
$$E_n = c + c' n^{1-\frac{d}{2}} + O(n^{-\frac{d}{2}}), \quad d > 3.$$

Here, and throughout this paper, we use c, c' for for dimension dependent constants whose values will differ in different places. Even though there is only one positive integer less than 2, we have written the exponent as $1 - \frac{d}{2}$ to show how the formula generalizes to noninteger d. Here we have a critical exponent that is piecewise linear, changing at a critical dimension 2. At the critical exponent, there is a logarithmic correction.

The estimation of E_n is straightforward given our estimates of $p_n(0)$. However, estimating probabilities of events can be more difficult. Let

$$p(n) = \mathbb{P}\{S_j \neq 0 : j = 1, \ldots, n\}.$$
$$q(n) = \mathbb{P}\{S_j = 0 \text{ for some } j = n+1, \ldots, 2n\}.$$

We claim that

$$p(n) \asymp E_n^{-1} \asymp \begin{cases} n^{\frac{d}{2}-1}, & d < 2, \\ (\log n)^{-1}, & d = 2, \\ 1, & d > 2. \end{cases},$$

and

$$q(n) \asymp \begin{cases} 1, & d < 2, \\ (\log n)^{-1}, & d = 2, \\ n^{1-\frac{d}{2}}, & d > 2. \end{cases}, \tag{2.4}$$

Here, and throughout this paper, we use $\asymp$ to indicate that two sides are *comparable*, that is, each side is bounded above by a positive constant times the other side.

We sketch the proof. We start with $q(n)$ for which we can do all dimensions at the same time. Let σ_n be the smallest $j > n$ such that $S_j = 0$, so that

$$q(n) = \sum_{j=n+1}^{2n} \mathbb{P}\{\sigma_n = j\}.$$

Let $V(n,m)$ be the number of visits to the origin between times $n+1$ and m,

$$V(n,m) = \sum_{j=n+1}^{m} 1\{S_j = 0\}.$$

The strong Markov property implies that if $n < j \leq m$, then

$$\mathbb{E}\left[V(n,m) \mid \sigma = j\right] = E_{m-j},$$

and hence

$$\mathbb{E}\left[V(n,m)\right] = \sum_{j=n+1}^{m} \mathbb{P}\{\sigma_n = j\}\, E_{m-j}.$$

In particular,

$$\mathbb{E}\left[V(n,2n)\right] \leq q(n)\, E_n \leq \mathbb{E}\left[V(n,3n)\right].$$

The asymptotics for E_n then imply (2.4). If we are at or above the critical dimension we can do better with a little more work. Let us consider $d = 2$. If $n+1 \leq j \leq 2n$ and is not too close to $2n$, then E_{2n-j} is about $c\,\log n + O(1)$. Also,

$$\mathbb{E}\left[V(n,2n)\right] = c\,\log 2 + O(n^{-1}).$$

In this case we get

$$q(n) = \frac{\log 2}{\log n}\left[1 + O\left(\frac{1}{\log n}\right)\right].$$

For the lower bound on $p(n)$ we use what is known as a *last-exit decomposition* to show that

$$q(n) \asymp p(n)\, \mathbb{E}[V(n,2n)]. \tag{2.5}$$

On the event $\{V(n,2n) > 0\}$, let η denote the largest $k \leq 2n$ with $S_k = 0$. Then,

$$\mathbb{P}\{\eta = k\} = \mathbb{P}\{S_k = 0\}\mathbb{P}\{\eta = k \mid S_k = 0\} = p(n-k)\, \mathbb{P}\{S_k = 0\}.$$

$$q(n) = \mathbb{P}\{V(n,2n) > 0\} = \sum_{j=n}^{2n}\{\eta = j\} \geq p(n) \sum_{j=n}^{2n}\{S_j = 0\},$$

which gives one direction of (2.5). Similarly, we can see that

$$p(n) \geq \frac{\mathbb{P}\{V(n,2n) > 0, V(2n,3n) = 0\}}{\mathbb{E}[V(n,2n)]}.$$

To finish the proof we need to show that

$$\mathbb{P}\{V(n,2n) > 0, V(2n,3n) = 0\} \asymp \mathbb{P}\{V(n,2n) > 0\}.$$

For $d = 1$, the probability on the left-hand side is bounded above by the probability that

$$S_n \geq 0, \quad S_{2n} \leq -\sqrt{n}, \quad S_j - S_{2n} < \sqrt{n},\ j = 2n, \ldots, 3n.$$

It is a simple exercise to show that this probability is bounded below by a constant independent of n. For $d \geq 2$, we can show a stronger estimate,

$$q(n) \sim p(n)\,\mathbb{E}[V(n,2n)].$$

A useful technique to relate expectations and probabilities of rare events is generating functions. Generating functions often can be viewed in terms of expectations for random walks that are killed at a geometric rate. Let $\lambda \in (0,1)$ and let T be a random variable independent of the random walk S_n with a geometric distribution,

$$\mathbb{P}\{T \geq n\} = \lambda^n, \quad n = 0,1,2,\ldots. \tag{2.6}$$

We think of T as the time that the random walk is "killed"; from time $T+1$ onward the walker lives in what is sometimes called a cemetery site. Let E_λ, p_λ denote the analogues of $E_n, p(n)$ replacing n with the random time T,

$$E_\lambda = \mathbb{E}\left[\sum_{n=0}^{\infty} 1\{S_j = n, T \geq n\}\right] = \sum_{n=0}^{\infty} \lambda^n\,\mathbb{P}\{S_n = 0\},$$

$$p_\lambda = \mathbb{P}\{S_j \neq 0,\ j = 1,\ldots,T\} = \sum_{n=0}^{\infty}(1-\lambda)\,\lambda^n\,p(n).$$

The strong Markov property gives the simple relation,

$$E_\lambda = 1 + (1-p_\lambda)\,E_\lambda,$$

or

$$p_\lambda = \frac{1}{E_\lambda}. \tag{2.7}$$

It is straightforward from (2.3) to get the asymptotics of E_λ as $\lambda \uparrow 1$, and hence also of p_λ. One cannot always deduce the asymptotics of the coefficients of a generating function from the asymptotics of the function, but in this case the monotonicity of $p(n)$ allows it to be done.

2.4. Scaling limit. When studying the long-time behavior of a discrete process there are two complementary approaches. One is to study directly the quantities of interest such as $p(n), q(n)$ in the previous section. The other is to first find a *scaling limit* of the process and then to analyze the continuous process. Both are valid approaches and very often one can derive properties of one from the other. However, one must take care in doing so — proofs often involve interchanges of limits that must be justified.

For the examples we will be doing there is a standard way to define scaling limits. Suppose $\mathbb{P}_n$ is a probability measure on paths $\omega = [\omega_0,\ldots,\omega_n]$ in $\mathbb{Z}^d$. If the typical diameter of such a path is of order $f(n)$, then we can consider $\mathbb{P}_n$ as a measure on scaled paths

$$\omega^{(n)}(t) = \frac{\omega_{tn}}{f(n)}, \quad 0 \leq t \leq 1. \tag{2.8}$$

(This is valid for t such that tn is an integer and is extended to other t by linear interpolation.) If we can take a limit of these measures, we have a probability measure on paths $\omega : [0,1] \to \mathbb{R}^d$. Similarly, if we have a probability measure $\mathbb{P}$ on infinite paths $[\omega_0, \omega_1, \ldots]$ we can hope to take the scaling limit of the measure on scaled paths as above and get a measure on continuous paths $\omega : [0,\infty) \to \mathbb{R}^d$.

The scaling limit of the simple random walk is well understood. By taking $f(n) = \sqrt{n/d}$, we get convergence to standard d-dimensional Brownian motion. (The factor $1/\sqrt{d}$ is included for convenience.) The "probability of being at the origin at time t" is given by the density of the standard normal with covariance tI,

$$p(t,x) = c_d\, t^{-d/2}\, e^{-|x|^2/2t}.$$

The following can be shown directly for a d-dimensional Brownian motion B_t.

- If $d < 2$ (that is, $d = 1$), the path is *(point) recurrent*, that is, there are arbitrarily large times t with $B_t = 0$.
- If $d = 2$, the path is *neighborhood recurrent* but not *point recurrent*. With probability one, $B_t \neq 0$ for all $t > 0$, but
$$\limsup_{t\to\infty} |B_t| = 0.$$
- If $d > 2$, the path is *transient*, that is, with probability one,
$$\lim_{t\to\infty} |B_t| = \infty.$$

If $d \geq 2$, the distribution of Brownian motion conditioned so that it does not return to the origin is trivially the same as the unconditioned distribution since the conditioning is on an event of probability one. This includes the critical dimension $d = 2$ and gives plausibility (but is not a proof of) the fact that two-dimensional random walk conditioned to avoid the origin has the same long-range behavior as the unconditioned probability. Indeed, one can show that the scaling limit of 2-dimensional random walk conditioned not to return to the origin is the same as that of the unconditioned walk.

If $d = 1$, one cannot talk about Brownian motion starting at the origin conditioned never to return because with probability one there are positive times t arbitrarily close to 0 with $B_t = 0$. However, there are several ways to define a probability measure Q_1 on paths that corresponds to Brownian motion "conditioned to stay positive for $0 < t \leq 1$". For example, we can take $\varepsilon > 0$ and consider the measure on paths B_t with $B_0 = \varepsilon$ conditioned that $B_t > 0$ for $0 < t \leq 1$. This is a well-defined probability measure and then one hopes (and in fact can succeed!) to take the limit of these probability measures as $\varepsilon \downarrow 0$. There is a similar, but different, probability measure Q_2 that can be considered as Brownian motion "conditioned *never* to return to zero". As a measure on paths $\omega : [0,1] \to \mathbb{R}$ it can be constructed as follows.

- For each $\varepsilon > 0$ consider the measure on paths taken by letting $B_0 = \varepsilon$ and conditioning that $B_t > 0$ for $0 \leq t \leq 1/\varepsilon$.
- View this as a measure on paths $\omega : [0,1] \to \mathbb{R}$ by restricting to these times.

This is not the same measure as obtained by conditioning on no intersection up to time 1.

A little bit of stochastic calculus is useful for understanding these measures. The gambler's ruin estimate for Brownian motion states that if $B_t = x \in (0,R)$, then the probability that the Brownian motion reaches level R before hitting the origin is $x/R = B_t/R$. The measure Q_2 is given by Brownian motion *weighted* or *tilted* by the martingale $M_t = B_t$ which satisfies

$$dM_t = \frac{1}{B_t}\, M_t\, dB_t.$$

The Girsanov theorem states that

$$dB_t = \frac{dt}{B_t} + dW_t,$$

where W_t is a standard Brownian motion with respect to the tilted measure. The measure Q_1 is absolutely continuous with respect to Q_2 with

$$\frac{dQ_1}{dQ_2} = \frac{c}{B_1}.$$

3. Self-avoiding walk

3.1. Definition. A *self-avoiding walk (SAW)* of length n in $\mathbb{Z}^d$ is a nearest neighbor walk $\omega = [\omega_0, \ldots, \omega_n]$ such that $\omega_j \neq \omega_k$ for $0 \leq j < k \leq n$. Self-avoiding walks were introduced by Flory [**13**] as a model of polymers. Although the definition is simple, mathematical analysis of the problem has proved to be very difficult and many problems remain open.

Let $\tilde{\Omega}_n$ denote the set of SAWs of length n starting at the origin, let $C_n = \#(\tilde{\Omega}_n)$ denote the number of such SAWs, and let $C_n(x)$ denote the number of such SAWs with $\omega_n = x$. Let $\tilde{\mathcal{P}}_n$ be the uniform measure on $\tilde{\Omega}_n$ which gives measure C_n^{-1} to each $\omega \in \tilde{\Omega}_n$. Unlike the simple random walk measures $\{\mathcal{P}_n\}$, the measures $\{\tilde{\mathcal{P}}_n\}$ are not consistent. Let

$$\tilde{p}_n(x) = \tilde{\mathcal{P}}_n\{\omega_n = x\} = \frac{C_n(x)}{C_n}$$

denote the probability density function for the endpoint of an n-step SAW.

Although this model has a simple description, there is little that is known rigorously except for $d > 4$ for which one can give a perturbative argument based on the *lace expansion* [**37**]. None of the standard techniques from probability theory seem to apply this this essentially combinatorial problem. However, we can use probabilisitic ideas to describe the *conjectured* behavior of the paths, and we will do this. In this section, except when we say explicitly otherwise, descriptions are only conjectures.

If $n \leq m$, let $\tilde{\mathcal{P}}_{n,m}$ be the measure on $\tilde{\Omega}_n$ obtained from projecting the uniform measure on $\tilde{\Omega}_m$ onto the first n steps of the path,

$$\tilde{\mathcal{P}}_{n,m}(\omega) = \sum_{\omega' \in \tilde{\Omega}_m, \omega \prec \omega'} \tilde{\mathcal{P}}_m(\omega') = \frac{\#\{\omega' \in \tilde{\Omega}_m : \omega \prec \omega'\}}{C_m}.$$

The *infinite self-avoiding walk* is the consistent set of measures $\{\tilde{\mathcal{P}}_{n,\infty}\}$ given by

$$\tilde{\mathcal{P}}_{n,\infty}(\omega) = \lim_{m \to \infty} \tilde{\mathcal{P}}_{n,m}(\omega).$$

To show that this is well defined, we need to show that the limit exists for all $\omega \in \tilde{\Omega}_n$. At the moment this limit is known to exist only for $d > 4$, but we will it assume it exists for all d.

The number C_n is not known. Since any SAW of $n + m$ steps is obtained by concatenating SAWs of length n and m, we get the submultiplicativity relation

$$C_{n+m} \leq C_n \, C_m.$$

In other words, $\log C_n$ is a subadditive function, and it follows from general facts about subadditive functions that

$$\lim_{n\to\infty} \frac{\log C_n}{n} = \inf_{n\to\infty} \frac{\log C_n}{n} := \beta.$$

In other words, $C_n \approx e^{n\beta}$. The number e^β, which is called the *connective constant*, is not known. For $d = 2$, easy arguments give $2 \leq e^\beta \leq 3$ and it is believed that it is about 2.64. It is unknown whether or not e^β is an algebraic number. However, for SAWs on the honeycomb lattice, Nienhuis [**35**] first predicted and later Duminil-Copin and Smirnov [**9**] proved that $e^\beta = \sqrt{2+\sqrt{2}}$.

3.2. Universal behavior. The connective constant depends on the lattice and does not appear in the scaling limit. We will be more interested in the following *universal* quantities whose qualitative behavior depend on the spatial dimension but not on the particular lattice. These quantities should describe aspects of the scaling limit.

- Let $\tilde{\mathcal{S}}_n^2$ denote the mean square displacement of a SAW,
$$\tilde{\mathcal{S}}_n^2 = \sum_{x\in\mathbb{Z}^d} |x|^2\, \tilde{p}_n(x).$$
Roughly speaking, the typical displacement of a walk in $\tilde{\Omega}_n$ is comparable to $\tilde{\mathcal{S}}_n$. Since there are of order $\tilde{\mathcal{S}}_n^d$ points within distance $\tilde{\mathcal{S}}_n$ of the origin, we predict that
$$\tilde{p}_n(x) \asymp \tilde{\mathcal{S}}_n^{-d} \quad \text{if} \quad |x| \asymp \tilde{\mathcal{S}}_n. \tag{3.1}$$
- Define $\phi(n)$ by $C_n = e^{\beta n}\,\phi(n)$ and $\tilde{q}(n,m)$ by
$$\tilde{q}(n,m) = \frac{C_{n+m}}{C_n\, C_m} = \frac{\phi(n+m)}{\phi(n)\,\phi(m)}.$$
In other words, $\tilde{q}(n,n) = 1/\phi(n)$ is the probability that two independent SAWs chosen from $\tilde{\Omega}_n$ have no intersection other than their starting point and hence can be concatenated and translated to give a walk in $\tilde{\Omega}_{2n}$.
- Consider the probability that a SAW of length $2n$ is at the origin at time $2n$. This does not quite make sense, so instead we say that a path $\omega = [\omega_0,\ldots,\omega_{2n-1}] \in \tilde{\Omega}_{2n-1}$ is a *self-avoiding loop (rooted at the origin)* of length $2n$ if $|\omega_{2n-1}| = 1$. If we attach the edge from ω_{2n-1} to the origin, we get a path that is self-avoiding except for the final return to the starting point. Then, "the probability that a SAW is at the origin at time $2n$" is
$$\frac{\#\{\omega\in\tilde{\Omega}_{2n-1} : |\omega_{2n-1}| = 1\}}{C_{2n-1}} = \sum_{|\mathbf{e}|=1} \tilde{p}_{2n-1}(\mathbf{e}).$$
(Perhaps we should divide by C_{2n} instead but this should only change this quantity by a multiplicative constant.)
- Let T_R be the first time that the infinite SAW leaves the ball of radius R about the origin, and let $J_R = \mathbb{E}[T_R]$. Roughly speaking, the infinite SAW contains on order J_R points in the ball of radius R. If $J_R \asymp n$, then $R \asymp \tilde{\mathcal{S}}_n$. In other words we expect J_R to be comparable to the inverse of the function $\tilde{\mathcal{S}}_n$,
$$J_{\tilde{\mathcal{S}}_n} \asymp n.$$

- Let $g(x)$ denote the *infinite SAW Green's function* which is defined to be the probability that the infinite SAW goes through the point x. Since the walk visits on order J_R points in the ball of radius R, and the number of points in the ball is comparable to R^d, we expect that

$$g(x) \asymp |x|^{-d} \, J_{|x|}. \tag{3.2}$$

- The *SAW Green's function (evaluated at the critical value)*, which is not the same as $g(x)$, is defined by

$$G(x) = \sum_{n=0}^{\infty} C_n(x) e^{-n\beta} = \sum_{n=0}^{\infty} \tilde{p}_n(x) \, \phi(n)^{-1}. \tag{3.3}$$

  A major difference between this and the infinite SAW Green's function is that x is chosen to be an endpoint of the SAW rather than a point in the middle. It is not obvious that this sum is finite but it is believed to be. More generally, the SAW Green's generating function is defined by

$$G(x;s) = \sum_{n=0}^{\infty} C_n(x) \, s^n.$$

  This is easily seen to be finite for $s < e^{-\beta}$ and $G(x) = G(x; e^{-\beta})$.

3.3. Displacement and fractal dimension. The mean square displacement exponent ν is defined by $\tilde{\mathcal{S}}_n \approx n^\nu$. Flory gave a heuristic argument that suggested that

$$\tilde{\mathcal{S}}_n \asymp n^\nu, \quad \nu = \nu_d = \max\left\{\frac{1}{2}, \frac{3}{2+d}\right\}.$$

This prediction can also be written in terms of the "fractal dimension" $D = 1/\nu$ of the scaling limit,

$$J_R \asymp R^D, \quad D = \min\left\{2, \frac{2+d}{3}\right\}.$$

Although there were flaws in the argument, the predictions are close to what is actually believed now. The prediction for the dimension D as a function of d is piecewise linear. It gives the correct answer for $d = 1$ (which is a trivial case) and suggests that once one reaches the critical dimension 4, the behavior is the same as for simple random walk. There is a heuristic reason why we might guess that there is a "dimension phase transition" at $d = 4$. Putting a self-avoidance contraint requires that at each vertex on the SAW two paths avoid each other, the "past" and the "future". Since random walk paths are two-dimensional objects, this constraint might not be too strong in the spatial dimensions for which two two-dimensional paths do not intersect each other.

This heuristic has been proved to be correct for $d > 4$, but the proof is not easy. The main technique is the *lace expansion* which was introduced by Brydges and Spencer and then further developed by Hara and Slade to prove that $\tilde{\mathcal{S}}_n \sim cn$ for $d \geq 5$. In fact, the normalized paths approach Brownain motion. We will not discuss this method in this paper —see [**37**] for an introduction.

For $d = 2$ there is strong theoretical and numerical evidence that the Flory prediction is correct. However, the most that has been proved is that $\tilde{\mathcal{S}}_n/n \to 0$ [**8**]. For $d = 4$, the prediction is expected to be correct with a logarithmic correction:

$$\tilde{\mathcal{S}}_n \sim c \, n^{1/2} \, (\log n)^u, \quad d = 4.$$

In fact, the prediction is $u = 1/8$ but we will write it as above and give a heuristic derivation of the exact value later. Numerical evidence suggests that Flory's guess is not quite right in three dimensions,

$$\tilde{\mathcal{S}}_n \sim c\, n^{\nu}, \quad \nu = .588\cdots .$$

In order to simplify some of our formulas later, we will make the followng convention for the remainder of the paper.

- **Notation**. We write
$$L_n = \begin{cases} \log n, & d = 4 \\ 1, & d \neq 4. \end{cases}$$

Then, we can write

$$\tilde{S}_n \sim c\, L_n^u\, n^{\nu}, \quad J_R \sim c\, L_R^{-uD}\, R^D.$$

Using (3.2), we get

$$g(x) \asymp L_{|x|}^{-uD}\, |x|^{D-d}.$$

The critical exponent ν (or D) has the following properties that are characteristic of most of the models we discuss in this paper.

- There is a critical dimension above which the value of the exponent is a linear function of the dimension (in this case, the value is constant).
- This value holds at the critical dimension with logarithmic corrections.
- The value below the critical dimension is not linear in d.
- For $d = 2$ the exponent takes on a rational value.
- For $d = 3$, the value is not known exactly and perhaps will never be given explicitly.

3.4. Number of walks and nonintersection probabilities. The exponents $\gamma = \gamma_d, u' = u'_4$ are defined by saying that the correction term ϕ in C_n satisfies

$$\phi(n) \asymp L_n^{u'}\, n^{\gamma-1},$$

or, equivalently,

$$\tilde{q}(n,n) \asymp \phi(n)^{-1} \asymp L_n^{-u'}\, n^{1-\gamma}.$$

When comparing to continuum models, it is often easier to use the displacement R of the SAW rather than the number of steps as the variable. We can say that the probability that two SAWs of displacement R starting at the origin do not intersect is comparable to

$$L_R^{-u'}\, R^{-\xi}\, L_R^{\xi}, \quad \xi = \xi_d = D(\gamma - 1).$$

For $d \geq 4$, in analogy with a similar quantity for simple random walk, ξ is predicted to equal zero and hence we can write this as just $L_R^{-u'}$. For $d \geq 5$, this has been proved using the lace expansion. Nienhuis [**35**] first predicted that $\gamma_2 = 43/32$ and hence $\xi_2 = 11/24$. Numerical simulations are consistent with this, and this exponent can also be computed for the conjectured scaling limit. In three dimensions, the value is not known. We will later give a heuristic argument as to why $u' = 1/4$. This prediction is reflected in the behavior of the Green's function $G(x)$ in (3.3). The sum should be dominated by n of order $J_{|x|}$ and there are $O(J_{|x|})$ such values. Hence

$$G(x) \asymp J_{|x|}^{1-d}\, \hat{\phi}(|x|)^{-1} \asymp |x|^{D-d+\xi}\, L_{|x|}^{u'-uD} \asymp g(x)\, \hat{\phi}(|x|)^{-1}.$$

The predictions $u' = 1/4, u = 1/8, D = 2$ for $d = 4$ show that the logarithmic term drops out in this case giving

$$G(x) \asymp |x|^{D-d+\xi}.$$

While this last fact has not been proved for the (strict) SAW in $d = 4$, it has been established for some related four dimensional models, see [**4**].

3.5. Loop probabilities and self-avoiding polygons. A difference between simple and self-avoiding walks is that the relation (3.1) is not expected to hold for x near the origin. Indeed, one would expect $\tilde{p}_n(x)$ to be much smaller than $\tilde{\mathcal{S}}_n^{-d}$ for small x. If ω is a SAW of length $2n-1$ with $|\omega_{2n-1} - \omega_0| = 1$, then by adding the edge from ω_{2n-1} to ω_0 we obtain a *(rooted, oriented) self-avoiding loop (SAL)* of length $2n$. Let $\tilde{\Omega}^L_{2n,x} = \tilde{\Omega}^L_{2n,x,d}$ denote the set of SALs in $\mathbb{Z}^d$ of length $2n$ with $\omega_0 = x$ and

$$\tilde{\Omega}^L_{2n} = \bigcup_{x\in\mathbb{Z}^d} \tilde{\Omega}^L_{2n,x}, \quad \tilde{\Omega}^L = \bigcup_{n=1}^{\infty} \tilde{\Omega}^L_{2n}.$$

By lattice symmetry, $\#(\tilde{\Omega}^L_{2n,x}) = \#(\tilde{\Omega}^L_{2n,0}) = 2d\, C_{2n-1}(\mathbf{e})$. Let μ^L denote the measure on SALs that gives measure $e^{-2n\beta}$ to each SAL of length $2n$ where e^β is the connective constant.

A SAL can be viewed as a pair of SAWs of length n — the first n steps (traversed in usual order) and the last n steps (traversed in reverse order). We need these walks to avoid each other, and this puts a significant constraint near the origin and near ω_n. The probability of no intersection near the origin is of order $\tilde{q}(n,n)$ and similarly for intersection near ω_n. Hence we guess that

$$e^{2n\beta}\, \mu^L(\tilde{\Omega}^L_{2n,0}) \asymp \sum_{x\in\mathbb{Z}^d} C_n(x)^2\, \tilde{q}(n,n)^2 \asymp \sum_{|x|\leq n^\nu\, L_n^u} C_n(x)^2\, \tilde{q}(n,n)^2.$$

The last equality uses the fact that the typical displacement is of order $n^\nu\, L_n^u$. A typical point x in the last sum has $|x| \asymp n^\nu\, L_n^u$ with $C_n(x) \asymp n^{-d\nu}\, L_n^{-du}\, C_n$. Since the number of terms in the last sum is comparable to $n^{d\nu}\, L_n^{du}$, we see that we should expect $\#(\tilde{\Omega}^L_{2n,0})$ to be comparable to

$$n^{-d\nu}\, L_n^{-du}\, C_n^2\, \tilde{q}(n,n)^2 \asymp n^{-d\nu} L_n^{-du} C_{2n}\, \tilde{q}(n,n) \asymp n^{-d\nu} L_n^{-du} e^{2n\beta},$$

and hence for each x,

$$\mu^L(\tilde{\Omega}^L_{2n,x}) = \mu^L(\tilde{\Omega}^L_{2n,0}) \asymp n^{-d\nu} L_n^{-du}. \tag{3.4}$$

An important thing to note is that the exponents γ, u' do not appear in this expression. These exponents describe the behavior of a SAW near an endpoint. Loops do not have have endpoints; all points on a loop look like "middle points" of the loop. For this reason loops can be easier to study than walks with endpoints.

When considering scaling limits, it is often more natural to consider SAWs or SALs of a particular diameter. Let $\tilde{\Omega}^L(R)$ denote the set of SALs rooted at 0 with with diameter between R and $2R$. The number of steps in a typical SAL of diameter R is $J_R \asymp L_R^{-uD}\, R^D$. If we sum (3.4) for n between $L_R^{-uD}\, R^D$ and $2\, L_R^{-uD}\, R^D$, we see that we predict

$$\begin{aligned} \mu^L[\tilde\Omega^L(R)] &\asymp L_R^{-uD}\, R^D\, [L_R^{-uD}\, R^D]^{-d\nu}\, L_n^{-du} \\ &\asymp [L_R^{-uD}\, R^D]\, R^{-d} \\ &\asymp [\#\text{ of steps}]\, R^{-d} \end{aligned} \tag{3.5}$$

An *(oriented) self-avoiding polygon (SAP)* is an equivalence class of rooted self-avoiding loops under the equivalence relation

$$[\omega_0,\ldots,\omega_{2n-1}] \sim [\omega_j-\omega_j \ldots,\omega_{2n-1}-\omega_j,\omega_0-\omega_j,\ldots,\omega_{j-1}-\omega_j].$$

(Self-avoiding polygons as defined in [**31**] are unoriented; there is an obvious two-to-one relationship between oriented and unoriented polygons.) Let $\tilde\Omega'_{2n,x}$ denote the set of SAPs that include x among its vertices, and let

$$\tilde\Omega'_{2n} = \bigcup_{x\in\mathbb{Z}^d} \tilde\Omega'_{2n,x}, \quad \tilde\Omega' = \bigcup_{n=1}^{\infty} \tilde\Omega'_{2n}.$$

Each SAP in $\tilde\Omega'_{2n}$ has $2n$ representatives in $\tilde\Omega^L_{2n}$. There is an obvious bijection between $\tilde\Omega'_{2n,0}$ and $\tilde\Omega^L_{2n,0}$ obtained by choosing the rooted representative of the polygon that is rooted at the origin. Define the self-avoiding polygon measure μ' on $\tilde\Omega^L$ by

$$\mu'(\omega) = (2n)^{-1}\,\mu^L(\omega) = (2n)^{-1}\, e^{-2n\beta} \quad \text{if} \quad \omega\in\tilde\Omega^L_{2n}.$$

This also gives a measure on $\tilde\Omega'$ by

$$\mu'(\omega) = \sum_{\omega'} \mu'(\omega') = e^{-2n\beta} \quad \text{if} \quad \omega\in\tilde\Omega'_{2n},$$

where the sum is over all $\omega'\in\tilde\Omega^L_{2n}$ that are representatives of the SAP ω. Note that (3.5) predicts that $\mu'[\tilde\Omega^L(R)]\asymp R^{-d}$. By summing over $|x|\le 2R$, we get the following simple *hyperscaling* relation.

- **Hyperscaling**. Let V_R denote the set of SAPs of diameter at least R contained in the disk of radius $2R$ about the origin. Then $\mu'(V_R)\asymp 1$.

In fact, we predict that $\mu'(V_R)$ has a limiting value. Note that this relation has no critical exponents and hence no normalization is needed when we scale.

3.6. Infinite two-sided SAW. Let $\tilde{\mathcal{P}}'_n$ denote the uniform measure on SALs of length $2n$ rooted at the origin,

$$\omega = [\omega_0 = 0,\ldots,\omega_{2n-1},\omega_{2n}=0].$$

We can also view such a SAL as a pair $(\omega^1,\omega^2)\in\tilde\Omega_n\times\tilde\Omega_n$

$$\omega^1 = [\omega_0,\omega_1,\ldots,\omega_n], \quad \omega^2 = [\omega_{2n},\omega_{2n-1},\omega_{2n-2},\ldots,\omega_n].$$

Hence $\tilde{\mathcal{P}}'_n$ is $\tilde{\mathcal{P}}_n\times\tilde{\mathcal{P}}_n$ conditioned on the event that the only intersections of the paths occur at times 0 and n. If $n<m$, we can view $\tilde{\mathcal{P}}'_m$ as a measure $\tilde{\mathcal{P}}'_{n,m}$ on $\tilde\Omega_n\times\tilde\Omega_n$ by restricting to the first n steps of the two paths. This is supported on pairs of SAWs that have no intersection except at the initial point. The *two-sided self-avoiding walk* or *infinite self-avoiding loop* is the consistent set of measures $\tilde{\mathcal{P}}'_{n,\infty}$ on $\tilde\Omega_n\times\tilde\Omega_n$ obtained by taking the limit as $m\to\infty$ with n fixed. We expect that this is the same limit that one would get by fixing n, taking the measure $\tilde{\mathcal{P}}_{2m}$ on paths

$$[\omega_0,\omega_1,\ldots,\omega_{2m}]$$

and considering this as a measure on pairs of paths (ω^1, ω^2),

$$\omega^1 = [\omega_m - \omega_m, \omega_{m-1} - \omega_m, \ldots, \omega_{m-n} - \omega_m]$$

$$\omega^2 = [\omega_m - \omega_m, \omega_{m+1} - \omega_m, \ldots, \omega_{m+n} - \omega_m].$$

In other words, it is the limit of the distribution of the "middle" of a long SAW. We can also obtained this measure as the limit of the measures $\tilde{\mathcal{P}}_m \times \tilde{\mathcal{P}}_m$ conditioned on the event that the paths do not intersect.

3.7. Scaling limits. The scaling limits for SAW are obtained by considering scaled paths as in (2.8) where

$$f(n) = L_n^u \, n^\nu.$$

If $d \geq 4$, the prediction is that the scaled paths have the same limit as simple random walks, that is, they converge to Brownian motion with covariance matrix σI for some $\sigma = \sigma_d$ The lace expansion techniques establish this for $d > 4$, but is still open for $d = 4$.

For $d < 4$, it is guessed that the scaling limit should not be Brownian motion but rather should be a probability measure that is supported on paths with Hausdorff dimension $D = 1/\nu$. The limits should be slightly different depending on which of the measures $\tilde{\mathcal{P}}_n, \tilde{\mathcal{P}}_{n,\infty}$, or $\tilde{\mathcal{P}}'_{n,\infty}$ we are studying. These measures should concentrate on paths that look the same in the middle (often called the bulk) of the path but have different behavior at the endpoints. For $d = 2$, the conjectured limit process has been constructed, a particular case of the Schramm-Loewner evolution (SLE). No three dimensional process has not been constructed as a potential candidate for the limit.

Perhaps the nicest scaling limit should come from the measure μ' on self-avoiding loops or self-avoiding polygons. If we scale the loops as in (2.8), then the limit should be a σ-finite measure on continuous loops. Let us be more precise. We will phrase it in terms of rooted loops in $\tilde{\Omega}^L$, but we can also view it as a limit of the measure on the unrooted polygons in $\tilde{\Omega}'$. For each n, let $\mu'_{(n)}$ denote the measure μ' viewed as a measure on loops scaled as in (2.8). For each $R > 0$, let $\mu'_{(n),R}$ denote $\mu'_{(n)}$ restricted to (scaled) loops of diameter at least R^{-1} that are contained in the ball of radius R about the origin. Then it is conjectured that as $n \to \infty$, the limit $\mu'_{(n),R}$ exists and gives a measure μ'_R on loops of total mass $m(R) < \infty$. We now let $R \to \infty$ and hope to obtain a σ-finite measure on loops μ'_∞. In this interchange of limits only the "macrosopic" loops are retained. The small loops around points disappear (this can be considered an example of "subtracting away infinity" that is often needed to find nontrivial scaling limits).

4. Nonintersection probabilities for simple walks

The self-avoiding walk gives many problems that are too difficult for us to solve at the moment. We will consider some analogous problems for simple random walk. It turns out that these problems are difficult, but progress has been made on them. We will consider the analogue of $\tilde{q}(n, n)$ for simple random walk. There were two interpretations of this quantity — one in terms of the number of SAWs (for which the simple random walk analogue is trivial) and the other in terms of the probability that two walks do not intersect (which is nontrivial for simple random walk).

Let S^1, S^2 be independent simple random walks starting at the origin in $\mathbb{Z}^d$. We write

$$S^j[m,n] = \{S^j_k : m \leq k \leq n\}.$$

Define the nonintersection probability,

$$q(n) = \mathbb{P}\{S^1[0,n] \cap S^2[1,n] = \emptyset\}.$$

This turns out to be a difficult quantity to estimate. As an easier quantity, let us consider the *expected number* of intersections, $\mathbb{E}[I_n]$, where

$$I_n = \sum_{j=0}^{n} \sum_{k=1}^{n} 1\{S^1_j = S^2_k\}. \tag{4.1}$$

Using the local central limit theorem, we see that $\mathbb{P}\{S^1_j = S^2_k\} = \mathbb{P}\{S_{j+k} = 0\} \sim c\,(j+k)^{-d/2}$ (provided that $j+k$ is even), and hence

$$\mathbb{E}[I_n] = \sum_{j=0}^{n} \sum_{k=1}^{n} \mathbb{P}\{S_{j+k} = 0\} \sim c\, n^{\frac{4-d}{2}}\, L_n.$$

Here as in the previous section, we write $L_n = 1$ if $d \neq 4$ and $L_n = \log n$ if $d = 4$.

Similarly, we could assume that the walks S^1, S^2 have independent killing times T^1, T^2 distributed as in (2.6) and let

$$I_\lambda = \sum_{j=0}^{T^1} \sum_{k=1}^{T^2} 1\{S^1_j = S^2_k\}.$$

Then a similar estimate shows that as $\lambda \uparrow 1$,

$$\mathbb{E}\left[I_\lambda\right] \sim c\,\lambda^{\frac{d-4}{2}}\, L_{1/(1-\lambda)}.$$

If

$$q_\lambda = \mathbb{P}\{S^1[0,T^1] \cap S^2[1,T^2] = \emptyset\} = \mathbb{P}\{I_\lambda = 0\},$$

we might hope in analogy with (2.7) to show that $q_\lambda = 1/\mathbb{E}[I_\lambda]$, or at least $q_\lambda \asymp 1/\mathbb{E}[I_\lambda]$. Here is how a heuristic argument might go. On the event $\{S^1[0,T^1] \cap S^2[1,T^2] \neq \emptyset\}$ consider the first intersection of the paths. After the paths intersect, the walks look like two walks starting at the same point. Using the Markov property we might hope to get

$$\mathbb{E}\left[I_\lambda \mid I_\lambda \geq 1\right] = 1 + \mathbb{E}\left[I_\lambda\right].$$

Since $\mathbb{E}[I_\lambda] = \mathbb{P}\{I_\lambda \geq 1\}\,\mathbb{E}\left[I_\lambda \mid I_\lambda \geq 1\right]$, we can solve this and get

$$\mathbb{P}\{I_\lambda = 0\} = \frac{1}{\mathbb{E}[I_\lambda]}.$$

Unfortunately, this argument is not valid because there is not a well-defined "first" time that the paths hit. This is because there are two times scales involved, that of each of the two paths. For example, if $S^1_1 = S^2_1$ and $0 = S^1_0 = S^2_2$, then the "first" intersection could be considered at $j = 1, k = 1$ or at $j = 0, k = 2$. It turns out this is not just a technical difficulty. As we will show, it is *not* the case that $q_\lambda \asymp 1/\mathbb{E}[I_\lambda]$. Determining the asymptotics of q_λ or $q(n,n)$ is a difficult problem.

To show that the conclusions of this heuristic argument are wrong, consider the $d = 1$ case. Suppose S^1, S^2 are simple random walks in $\mathbb{Z}$ starting at the origin. if $S^1[0,n] \cap S^2[1,n] = \emptyset$, then either $S^1[0,n]$ lies in the nonnegative integers and $S^2[1,n]$ in the negative integers, or $S^1[0,n]$ lies in the nonpositive integers and $S^2[1,n]$ in the positive integers. A version of the "gambler's ruin" estimate shows that the probability that $S^1[0,n]$ stays in the nonnegative integers is comparable to $n^{-1/2}$ and hence $q(n,n) \asymp n^{-1}$. Since $\mathbb{E}[I_n] \asymp n^{3/2}$, we see that $q(n,n) \not\asymp \mathbb{E}[I_n]$.

For $d \geq 5$, we see that $\mathbb{E}[I_\infty] < \infty$. It is not very difficult, and we leave as an exercise, to show that in fact $\mathbb{P}\{I_\infty = 0\} > 0$, that is, $q(n,n) \asymp 1$. In the remainder of this section we focus on the difficult dimensions, $d = 2, 3, 4$. We will state the main result now.

- There exists an exponent $\zeta = \zeta_d$ called the *intersection exponent* such that
$$q(n,n) \asymp n^{-\zeta} \, L_n^{-1/2}.$$
If $d = 2$, $\zeta_2 = 5/8$. If $d = 4$, $\zeta = 0$. If $d = 3$, then the value is not known precisely. Rigorously, it is known that $1/4 < \zeta_3 < 1/2$ and numerical simulations suggest that $\zeta_3 \approx .29$.

4.1. Long-range intersection. We start our analysis with an easier esimate, the probability of a "long-range intersection", say
$$Q(n) = \mathbb{P}\{S^1[0,n] \cap S^2[n+1, 2n] \neq \emptyset\}.$$
We will show that
$$Q(n) \asymp n^{-\alpha} \, L_n^{-1}, \qquad \alpha = \max\left\{0, \frac{d-4}{2}\right\}. \tag{4.2}$$
The proof uses a second-moment argument. The second-moment method is a standard way to show that a random variable has a good chance to take on a value close to its expectation. Roughly speaking, if X is a nonnegative random variable with $\mathbb{E}[X^2] \asymp \mathbb{E}[X]^2$, then there is a good chance that $X \geq \mathbb{E}[X]/2$. The precise statement is in following estimate which can be derived easily from the Cauchy-Schwarz inequality:
$$\mathbb{P}\left\{X \geq \frac{1}{2}\,\mathbb{E}(X)\right\} \geq \frac{[\mathbb{E}(X)]^2}{4\,\mathbb{E}(X^2)}. \tag{4.3}$$
Let J_n denote the number of intersections,
$$J_n = \sum_{j=0}^{n} \sum_{k=n+1}^{2n} 1\left\{S_j^1 = S_k^2\right\}.$$
By expanding the sum, bringing the expectation inside, and using the local central limit theorem, one can show without too much difficulty that
$$\mathbb{E}\left[J_n\right] \asymp n^{\frac{4-d}{2}}, \qquad \mathbb{E}\left[J_n^2\right] \leq c\, n^{\beta}\, L_n, \qquad \beta = \max\left\{4-d, \frac{4-d}{2}\right\}.$$
Combining these estimates with (4.3) gives the lower bound $Q(n) \geq c\, n^{-\alpha}\, L_n^{-1}$. The upper bound in (4.2) is trivial for $d \leq 3$, and for $d \geq 5$ follows from $Q(n) = \mathbb{P}\{J_n > 0\} \leq \mathbb{E}[J_n]$.

This leaves the tricky case of proving the upper bound for $d = 4$. It is obtained by showing that
$$\mathbb{E}\left[J_n \mid J_n \geq 1\right] \geq c\, L_n, \tag{4.4}$$
and hence,
$$\mathbb{P}\{J_n \geq 1\} \leq \frac{\mathbb{E}[J_n]}{\mathbb{E}\left[J_n \mid J_n \geq 1\right]} \leq c\, L_n^{-1}.$$

To establish (4.4) one considers the first intersection where "first" is defined with respect to the first path,

$$\tau_n = \min\left\{j : S^1_j \in S^2[n+1, 2n]\right\}.$$

$$\sigma_n = \min\{k \geq n : S^2_k = S^1_{\tau_n}\}.$$

The heuristic argument is the following. The strong Markov property implies that $S^1_j, j \geq \tau_n$ is a simple random walk starting at $S^2_{\sigma_n}$. One would hope that near $S^2_{\sigma_n}$ the path $S^2[n, 2n]$ would look like two random walk starting at $S^2_{\sigma_n}$. However, this is not immediately obvious since the definition of the time τ_n required looking at the entire path $S^2[n+1, 2n]$. The example in the previous section shows that one needs to be careful about such arguments. In this case, however, one can justify the claim with some work; we will not give the argument.

The long-range estimate (4.2) strongly suggests something about the scaling limit (Brownian motion) that was first proved by Dvoretsky, Erdős, and Kakutani [**11**]. If B^1_t, B^2_t are independent Brownian motions in $\mathbb{R}^d$ then for $0 < s < t < \infty$,

$$\mathbb{P}\{B^1[0,t] \cap B^2[s,t] \neq \emptyset\} \begin{cases} > 0, & d < 4 \\ = 0, & d \geq 4. \end{cases} \tag{4.5}$$

In four dimensions, the Brownian motions just barely avoid hitting each other. This is analogous to the fact that two-dimensional Brownian motions just barely avoid returning to the origin. A precise formulation of this is the following. With probability one, the following holds for all $s > 0$.

- If $d < 4$, there exists $t < \infty$ such that $B^1[s,t] \cap B^2[s,t] \neq \emptyset$.
- If $d = 4$, for every $t < \infty$, $B^1[s,t] \cap B^2[s,t] = \emptyset$, but
$$\text{dist}\left[B^1[s,\infty), B^2[s,\infty)\right] = 0.$$
- If $d > 4$,
$$\text{dist}\left[B^1[s,\infty), B^2[s,\infty)\right] > 0.$$

4.2. An easier probability to estimate. The long-range estimate can be used to give the asymptotics of a different probability, the probability that a walk avoids *two* other walks, or equivalently, the probability that a (one-sided) walk avoids a two-sided walk. Let S^3 be a third simple random walk independent of S^1, S^2 starting at the origin. If we set

$$S_n = \begin{cases} S^2_n, & n \geq 0 \\ S^3_{-n}, & n \leq 0, \end{cases}$$

then S_n can be viewed as a two-sided random walk. Let

$$p(n) = \mathbb{P}\left\{S^1[0,n] \cap \left(S^2[1,n] \cup S^3[1,n]\right) = \emptyset\right\}.$$

Then for $d \leq 4$, we claim that

$$p(n) \asymp n^{\frac{d-4}{2}}\, L_n^{-1}. \tag{4.6}$$

For $d = 1$, we can see this by noting that there is no intersection if and only if S^1 stays on one side of the origin and S^2 and S^3 stay on the other side. Each one of these events has probability of order $n^{-1/2}$ so the probability that they all happen is comparable to $n^{-3/2}$.

Let us sketch the argument that works for $d > 1$. Consider the long-range intersection event $\{S^1[0,n] \cap S^2[n+1, 2n] \neq \emptyset\}$ and let τ_n, σ_n be as in the previous section. Suppose $0 < j < n$ and $n < k < 2n$ and consider the event $E_{j,k} = E_{j,k,n} = \{\tau_n = j, \sigma_n = k\}$. Then we can write

$$\mathbb{P}\{S^1[0,n] \cap S^2[n+1,2n] \neq \emptyset\} = \sum_{j=0}^{n} \sum_{k=n+1}^{2n} \mathbb{P}[E_{j,k}].$$

In order for the event $E_{j,k}$ to occur we need the following:

- $S^1_j = S^2_k$;
- $S^1[0,j-1] \cap S^2[n,k] \neq \emptyset, \;\; S^j[0,j-1] \cap S^2[k,2n] \neq \emptyset.$

The probability that $S^1_j = S^2_k$ is comparable to $(j+k)^{-d/2} \asymp n^{-d/2}$. Given $S^1_j = S^2_k$, and supposing that k is not too close to n or $2n$, we would guess (and can make precise) that the probability that the second event occurs is comparable to $p(n)$. Therefore,

$$\mathbb{P}\{\tau_n = j, \; \sigma_n = k\} \asymp n^{-d/2}\, p(n).$$

There are n^2 pairs (j,k), and hence we expect (for $d = 2,3,4$),

$$\begin{aligned} L_n^{-1} &\asymp \mathbb{P}\{S^1[0,n] \cap S^2[n+1,2n] \neq \emptyset\} \\ &= \sum_{j=0}^{n} \sum_{k=n+1}^{2n} \mathbb{P}\{\tau_n = j, \; \sigma_n = k\} \\ &\asymp n^2\, n^{-d/2}\, p(n). \end{aligned}$$

This gives (4.6).

Let us define the random variable,

$$Y_n = \mathbb{P}\left\{S^1[0,n] \cap S^2[1,n] = \emptyset \mid S^1[0,n]\right\},$$

which is measurable with respect to the random walk S^1. Since S^2 and S^3 are independent, we can see that

$$q(n) = \mathbb{E}\left[Y_n\right], \quad p(n) = \mathbb{E}\left[Y_n^2\right].$$

Using the Cauchy-Schwartz inequality and the fact that $0 \leq Y_n \leq 1$, we see that

$$p(n)^2 \leq q(n)^2 \leq p(n).$$

If it were the case that $\mathbb{E}[Y_n^2] \asymp (\mathbb{E}[Y_n])^2$, we would have $p(n) \asymp q(n)^2$. We will say that the model is *mean-field* if the relationship $\mathbb{E}[Y_n^\lambda] \asymp (\mathbb{E}[Y_n])^\lambda$ holds. The general principle can then be stated as follows.

- Mean-field behavior is valid at the critical dimension but is not valid below the critical dimension.

We will consider these two cases separately, but let us now illustrate the major difference. Let E_n denote the event that $S^1[0,2^n] \cap S^2[1,2^n] \neq \emptyset$, and $F(n) = \mathbb{P}(E_n) = q(2^n)$. Then

$$F(n+1) = F(n)\, \mathbb{P}(E_{n+1} \mid E_n).$$

- In the critical dimension $d = 4$, $\mathbb{P}(E_{n+1} \mid E_n)$ is close to one and is almost the same as the unconditioned probability which is

$$\begin{aligned} \mathbb{P}\{S^1[2^n+1, 2^{n+1}] \cap S^2[1, 2^{n+1}] = \emptyset, \\ S^1[0, 2^{2n}] \cap S^2[2^n+1, 2^{n+1}] = \emptyset\}. \end{aligned} \tag{4.7}$$

  Both can be written as

$$1 - \frac{\alpha}{n} + O(\varepsilon_n),$$

where $\sum \varepsilon_n < \infty$ and α is a constant to be determined. This implies that

$$F(n) = F(1) \prod_{j=1}^{n-1} \frac{F(j+1)}{F(j)} \sim c\, n^{-\alpha} = \frac{c\,(\log 2)^{\alpha}}{(\log 2^n)^{\alpha}}.$$

In fact, the probability in (4.7) can be estimated to show that $\alpha = 1/2$. This is somewhat analogous to the problem of returns to the origin in the critical dimension two. Although two-dimensional random walk is recurrent, if we condition random walk not to return to the origin in the first n steps, the conditional distribution of the path has the same scaling limit as the unconditional distribution.

- if $d = 2, 3$, then one expects that $\mathbb{P}(E_{n+1} \mid E_n)$ converges to a constant $\theta \in (0,1)$ that is *not* the same as the limiting value for the unconditioned probability as in (4.7). In this case

$$F(n) \sim c\,\theta^n = c\,(2^n)^{-\zeta}, \quad \zeta = -\log_2 \theta.$$

The probability θ can be given in terms of Brownian motion. If B^1, B^2 are independent Brownian motions starting at the origin in $\mathbb{R}^d$ $(d = 2, 3)$, then

$$\theta = \mathbb{P}\{B^1[0,2] \cap B^2[0,2] = \emptyset \mid B^1[0,1] \cap B^2[0,1] = \emptyset\}.$$

This last expression does not make sense as written because the event $\{B^1[0,1] \cap B^2[0,1] = \emptyset\}$ has zero probability. However, one can make sense in terms of it as a limit. The distribution on $B^1[0,1], B^2[0,1]$ given $B^1[0,1] \cap B^2[0,1] = \emptyset$ is not the same as the unconditioned distribution. This is analogous to the problem of returns to the origin in $d = 1$; the distribution of a one-dimensional random walk or Brownian motion given that it has avoided the origin is not the same as the unconditioned distribution.

4.3. Four dimensions. In this subsection we fix $d = 4$. For ease, let us consider a slight variation of I_n from (4.1),

$$\tilde{I}_n = \sum_{j=0}^{\infty} \sum_{k=1}^{n} 1\{S_j^1 = S_k^2\}.$$

Clearly $\tilde{I}_n \geq I_n$, and

$$\mathbb{E}\left[\tilde{I}_n - I_n\right] = \sum_{j=n+1}^{\infty} \sum_{k=1}^{n} \mathbb{P}\{S_j^1 = S_k^2\} \leq c \sum_{j=n+1}^{\infty} \sum_{k=1}^{n} (j+k)^{-2} = O(1).$$

Let Y_n denote the conditional expectation of $\tilde{I}_{2^n}$ given the second path S^2. Then

$$Y_n = \mathbb{E}[\tilde{I}_{2^n} \mid S^2] = \sum_{m=1}^{n} Z_k, \quad \text{where} \quad Z_k = \sum_{k=2^{m-1}-1}^{2^m} G(S_j^2).$$

We know that $G(x) \sim C_4\,|x|^{-2}$. We have written it this way to focus on the random variable Z_k with

$$\mathbb{E}[Z_k] = \sum_{k=2^{m-1}-1}^{2^m} G(S_j^2) \sim C_4 \sum_{k=2^{m-1}-1}^{2^m} k^{-2} \sim C_4\,\log 2.$$

In fact, as $k \to \infty$, the distribution of Z_k converges to that of a random variable Z with mean $C_4 \log 2$ and such that $\mathbb{E}[e^{sZ}] < \infty$ for s near the origin. The random variables $Z_1, Z_2, \ldots$ are not independent but they have "short-range correlations", indeed, $\mathrm{Cov}(Z_j, Z_k) \leq c\, e^{-a|j-k|}$. The upshot is that one can prove a law of large numbers for Y_n showing that with high probability $Y_n = c_4 \log 2^n \,[1 + o(1)]$.

With this observation we return to the argument at the end of Section 4.1. Recall that $\mathbb{E}[J_n \mid J_n \geq 1]$ a priori depends strongly on what the path S^2 looks like near $S^2_{\sigma_n}$. The fact that Y_n above satisfies a strong law, however, can be used to show that $\mathbb{E}[J_n \mid J_n \geq 1]$ does *not* depend strongly on the choice of σ_n (we are omitting the important estimates that make this precise). We can show that

$$\mathbb{E}[J_n \mid J_n \geq 1] = \frac{2\,\mathbb{E}[J_n]}{\log n}\,[1 + o(1)].$$

Here we use a couple of facts. First, the logarithm is slowly varying so that $\log(2n) \sim \log n \sim \log(n/2)$ so the exact number of steps in the paths is not so important. Also, assuming that σ_n is not too close to n or $2n$, there are *two* random walk paths coming out. Hence the expected number of intersections should look like twice the number for two random walks starting at the origin. We then use this to see that

$$\mathbb{P}\{J_n \geq 1\} = \frac{\log 2}{2 \log n}\,[1 + o(1)].$$

There is one more key idea in the proof. Let V_n denote the event

$$V_n = \left\{ S^1[0, 2^{2n}] \cap S^2[1, 2^n] \neq \emptyset \right\}.$$

If we look at paths conditioned on the event V_n, then the requirement that the paths do not intersect has a large effect on the early part of the paths but has very little effect on the bulk of the paths. This is analogous to the fact that conditioning a two-dimensional random walk to avoid the origin does not change the long-range behavior of the path. To be more precise, one shows that

$$\mathbb{P}[V_{n+1} \mid V_n] = 1 - \frac{1}{2n} + \varepsilon_n, \tag{4.8}$$

where $\sum \varepsilon_n < \infty$. (In fact, $\varepsilon_n = O(n^{-2} \log n)$ can be proved) It is an easy exercise to show that (4.8) implies that there exists $0 < c < \infty$ such that $\mathbb{P}(V_n) \sim c\, n^{-1/2} = c\,(\log 2)^{1/2}\,[\log 2^n]^{-1/2}|$, and hence $q(n) \sim c\,\sqrt{\log 2}\,(\log n)^{-1/2}$.

4.4. Two or three dimensions. Below the critical dimension, the paths of Brownian motions intersect. It turns out to be easier to discuss nonintersection probabilities for Brownian motion and then to return to random walk. One of the advantages of Brownian motion over random walk is that Brownian motion satisfies an exact scaling relationship. For this section we will assume $d = 2$ or $d = 3$.

Let B^1, B^2 be independent d-dimensional Brownian motions which we can assume are defined on different probability spaces $(\Omega_1, \mathbb{P}_1), (\Omega_2, \mathbb{P}_2)$. We write $(\Omega, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathbb{P}_1 \times \mathbb{P}_2)$. We will let the paths run until the first time they reach a prescribed distance from the origin. It will be useful to use geometric scales, so we let

$$T^j_n = \inf\{t : |B^j_t| = e^n\}.$$

Let A_n denote the event

$$A_n = \{B^1[T^1_0, T^1_n] \cap B^2[T^2_0, T^2_n] = \emptyset\},$$

and let $q_{n,x,y} = \mathbb{P}[A_n \mid B^1_0 = x, B^2_0 = y]$, $q_n = q_{n,0,0}$. Using (4.5) it is easy to see that $q_n \downarrow 0$ as $n \to \infty$. Our goal is to determine the rate of decay. The next proposition will use the following fact which can be derived from the Harnack inequality: there exists a constant c such that if $|x|, |y| \leq 1/2$, then

$$c^{-1}\, q_n \leq q_{n,x,y} \leq c\, q_n. \tag{4.9}$$

Proposition 4.1. *There exists $\xi = \xi_d \leq 2$ such that as $n \to \infty$,*

$$q_n \approx e^{-n\xi}$$

where $\approx$ means that the logarithms of the two sides are asymptotic. Moreover, there exists $\delta > 0$ such that

$$q_n \geq \delta\, e^{-n\xi}.$$

Proof. The observation is that there exists a c such that for all n, m,

$$q_{n+m} \leq c\, q_n\, q_m. \tag{4.10}$$

We will show the slightly weaker statement $q_{n+m+1} \leq c\, q_n\, q_m$. Consider the event

$$V_{n,m} = \{B^1[T^1_{n+1}, T^1_{n+1+m}] \cap B^2[T^2_{n+1}, T^2_{n+1+m}] = \emptyset\}.$$

The event A_{n+m+1} is contained in $A_n \cap V_{n,m}$. The scaling property of Brownian motion implies that $\mathbb{P}[V_{n,m}] = q_m$. If we condition on the event A_n, that only affects the distribution of the paths up to the first visit of the sphere of radius e^n. Using scaling and (4.9), we can see that $\mathbb{P}[V_{n,m} \mid A_n] \leq c\, q_m$, and hence

$$\mathbb{P}(A_{n+m+1}) \leq \mathbb{P}(A_n \cap V_{n,m}) = \mathbb{P}(A_n)\, \mathbb{P}(V_{n,m} \mid A_n) \leq c\, q_n q_m.$$

This gives (4.10). If we set $b_n = \log(cq_n)$, we see that $b_{n+m} \leq b_n + b_m$ and hence by a standard result about subadditive functions, the limit

$$\lim_{n\to\infty} \frac{b_n}{n} = \inf_n \frac{b_n}{n} = -\log \xi,$$

exists, and $q_n \geq c^{-1}\, e^{-n\xi}$. To get the bound on ξ, we can consider the event U_n that the first component of B^1_t is positive for $T^1_0 \leq t \leq T^1_n$ and that the first component of B^2_t is negative for $T^2_0 \leq t \leq T^2_n$. Clearly $U_n \subset A_n$, and the gambler's ruin estimate for one-dimensional Brownian motion shows that $\mathbb{P}(U_n) \geq ce^{-2n}$. □

The proof does not give the value of the *intersection exponent* ξ. Subadditivity or submultiplicity is a good technique to give simple proofs of existence of exponents. However, they do not give the exponent and also do not give a good rate of convergence. In this case, we can also establish a *super*multiplicity estimate that allows for up-to-constants estimates.

Proposition 4.2. *There exist $0 < c_1 < c_2 < \infty$ such that as $n \to \infty$,*

$$c_1\, e^{-n\xi} \leq q_n \leq c_2\, e^{-n\xi}.$$

Sketch of proof. The key step is to prove a supermultiplicativity estimate: there exists $c > 0$ such that for all n, m,

$$q_{n+m} \geq c\, q_n\, q_m. \tag{4.11}$$

The key two steps in proving this are the following.

- **Separation lemma**. Let K_n be the event that the first component of B_t^1 is greater than e^{n-2} for $T_{n-1}^1 \leq t \leq T_n^1$ and that the first component of B_t^2 is less than $-e^{n-2}$ for $T_{n-1}^2 \leq t \leq T_n^2$. (This is just one of many possible "separation events" that we could define.) Then there exists $c > 0$ such that
$$\mathbb{P}(K_n \cap A_n) \geq c\,\mathbb{P}(A_n).$$
In other words, conditioned that the two Brownian paths do not intersect each other, there is a good chance that the ends of the paths are "separated".
- **Separation at the beginning.** Let K_n' be the event that the intersection of $B^1[T_0^1, T_n^1]$ with the unit ball is contained in $U := \{z : (z_1, z_2, \ldots, z_d) : z_1 \geq e^{-2}\}$ and the intersection of $B^2[T_0^2, T_n^2]$ with the unit ball is contained in $-U$. (This is just one of many possible "separation at the beginning events" that we could define.) Then there exists $c > 0$ such that
$$\mathbb{P}(K_n' \cap A_n) \geq c\,\mathbb{P}(A_n).$$

These estimates require some work, and we will not do it here. However once one has them, roughly speaking, one attaches an event of the first type with a (scaled version) of an event of the second type to get (4.11).

□

Determining the value of the intersection exponent is a difficult problem. It is useful in the analysis to generalize the definition. The exponent $\xi(j,k) = \xi_d(j,k)$ describes the probabilities that j Brownian paths avoid k other Brownian paths. To be precise, let $B_t^1, \ldots, B_t^{j+k}$ be independent Brownian motions starting at the origin with corresponding stopping times $T_n^1, \ldots, T_n^{j+k}$. Let
$$\Gamma_n^1 = B^1[T_0^1, T_n^1] \cup \cdots \cup B^j[T_0^j, T_n^j],$$
$$\Gamma_n^2 = B^{j+1}[T_0^{j+1}, T_n^{j+1}] \cup \cdots \cup B^{j+k}[T_0^{j+k}, T_n^{j+k}].$$
We assume that $B^1, \ldots, B^j$ are defined on $(\Omega, \mathbb{P})$, $B^{j+1}, \ldots, B^{j+k}$ are defined on $(\Omega', \mathbb{P}')$, and let $(\Omega^*, \mathbb{P}^*) = (\Omega \times \Omega', \mathbb{P} \times \mathbb{P}')$. Then the exponent $\xi(j,k)$ is defined by the relation
$$\mathbb{P}^*\{\Gamma_n^1 \cap \Gamma_n^2 = \emptyset\} \asymp e^{-n\xi(j,k)}.$$
The existence of an exponent satisfying these conditions can be proven as in the previous two propositions. We can generalize more, by noting that if we define the random variable on Ω,
$$\mathcal{H}_n = \mathcal{H}_{n,j} = -\log \mathbb{P}'\{B^{j+1}[T_0^{j+1}, T_n^{j+1}] \cap \Gamma_n^1 = \emptyset\},$$
then
$$\mathbb{P}^*\{\Gamma_n^1 \cap \Gamma_n^2 = \emptyset\} = \mathbb{E}\left[e^{-k\mathcal{H}_n}\right].$$
Considered this way, there is no need to restrict k to an integer, so we define $\xi(\beta) = \xi_d(j,\beta)$ by
$$\mathbb{E}\left[e^{-\beta\mathcal{H}_n}\right] \asymp e^{-n\xi(\beta)}.$$
We have written it this way in order show the analogy with expressions arising in statistical physics. The quantity $\mathcal{H}_n$ is called the *Hamiltonian* or *energy*, the

exponent β is the analogue of the *inverse temperature*, the expectation $\mathbb{E}\left[e^{-\beta \mathcal{H}_n}\right]$ is the *partition function*, and the exponent

$$\xi(\beta) = - \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[e^{-\beta \mathcal{H}_n}\right]$$

is the *free energy*.

Jensen's inequality shows that the function $\beta \mapsto \xi(\beta)$ is concave. No more than this can be concluded just from existence of the limit, but for the Brownian exponent it is known [**18**] that ξ is C^2 for $\beta > 0$ with $\xi''(\beta) < 0$. (For $d = 2$, it has been shown to be real analytic [**26**], and it is conjectured that this is true for $d = 3$.) The starting point for the proof is the up-to-constants estimates from Proposition 4.2. The analysis of the behavior of the exponent goes under various names in the literature, most commonly *large deviations* and *multifractal behavior*. This is a nontrivial case in which a "multifractal formalism" which we now describe can be established. Let us define the function $\rho(s)$ by the relation

$$\mathbb{P}\{\mathcal{H}_n \geq e^{-sn}\} \approx e^{-\rho(s)n}, \quad n \to \infty.$$

Since

$$\mathbb{E}\left[e^{-\beta \mathcal{H}_n}\right] \geq e^{-\beta s n}\, \mathbb{P}\{\mathcal{H}_n \geq e^{-sn}\},$$

we see that for every s,

$$\xi(\beta) \leq \beta s + \rho(s).$$

In fact, it can be shown that

$$\rho(s) = \sup_{\beta} [\xi(\beta) - \beta s],$$

$$\xi(\beta) = \inf_{s} [\beta s + \rho(s)].$$

These relations are sometimes called *Legendre transforms*. Since $\xi''(\beta) < 0$, we can define β_s by $\xi'(\beta_s) = s$, and the relation $s \leftrightarrow \beta_s$ or $\beta \leftrightarrow s_\beta$ is bijective for $\beta > 0$ and

$$\rho(s) = \xi(\beta_s) - s\,\beta_s, \quad \xi(\beta) = \beta\, s_\beta + \rho(s_\beta).$$

Roughly speaking, the expectation $\mathbb{E}\left[e^{-\beta \mathcal{H}_n}\right]$ is carried on an event with $\mathcal{H}_n \approx e^{-s_\beta n}$ and different values of β give different values of s_β.

None of the discussion above actually computes the value of the exponent $\xi(j, \beta)$. One value that can be computed easily as in Section 4.2 is

$$\xi_d(2, 1) = \xi_d(1, 2) = \frac{4-d}{2},$$

which combined with the argument above implies that

$$\frac{4-d}{4} < \xi_d(1, 1) < \frac{4-d}{2}. \tag{4.12}$$

Here we have used the nontrivial fact that the exponent is a strictly concave function. This is essentially the best rigorous estimate for $\xi_3(1, 1)$, but as we discuss below, the exponent $\xi_2(1, 1)$ has been determined.

4.5. Exceptional sets on the Brownian path. A time $t \in (0,1)$ is called a *cut time* and B_t is called a *cut point* for $B[0,1]$ if $B[0,t) \cap B(t,1] = \emptyset$. If $d < 4$, one can show that for each t, with probability one, t is not a cut time. However, this does not answer the question whether or not there exists exceptional times that are cut times. In fact, such times do exist as was first proved by K. Burdzy [**5**]. An easier argument to prove the existence of cut points does more by showing that the Hausdorff dimension of the set of cut times is strictly positive. The heuristic is the following. By scaling, the probability that $B[0, \frac{1}{2} - \varepsilon] \cap B[\frac{1}{2} + \varepsilon, 1] = \emptyset$ is comparable to $\varepsilon^{\xi/2}$ (the factor 1/2 in the exponent comes from the fact that $|B(\frac{1}{2}) - B(\frac{1}{2} - \varepsilon)|$ is about $\varepsilon^{1/2}$. If we call time $k2^{-n}$ an n-cut time if

$$B[0,(k-1)2^{-n}] \cap B[k2^{-n}, 1] = \emptyset,$$

then the expected number of n-cut times is $(2^n)^{1-\frac{\xi}{2}}$. This indicates that the dimension of the set of cut times should be $1 - \frac{\xi}{2}$. This can be proved using ideas from Proposition 4.2 and general techniques for computing dimension, see Section 7. This theorem can be proved without knowing the value of ξ. Note that (4.12) shows that $\xi_d < 2$ for $d = 2, 3$, and hence the theorem implies the existence of cut points.

THEOREM 4.3. *If $d = 2, 3$, then with probability one, the Hausdorff dimension of the set of cut times of $B[0,1]$ is $(2 - \xi_d)/2$ and the Hausdorff dimension of the set of cut points of $B[0,1]$ is $2 - \xi_d$.*

This theorem is also true for $d = 1$ where $\xi_1 = 2$. In fact, a stronger fact is true for one-dimensional Brownian motion: with probability one, for all $0 < t < 1$, $B[0,t) \cap B(t,1] \neq \emptyset$ (see [**6**] for a short proof of this). Another exceptional set was first investigated numerically by Benoit Mandelbrot [**32**]. Suppose B_t is a two-dimensional Brownian bridge, that is, a Brownian motion conditioned so that $B_0 = B_1$. (Although this is conditioning on an event of probability zero, there is no problem in making it precise.) The *outer boundary* or *frontier* is the boundary of the unbounded component of the complement of the path. Mandelbrot observed that the frontier was a self-avoiding loop, and simulations suggested that the dimension of this loop was 4/3, the same as conjectured by Flory for self-avoiding walk. One can describe this dimension in terms of an intersection exponent. A typical point on the frontier locally looks like two Brownian motions (the past and the future) conditioned so that they do not disconnect the point from infinity. The exponent that governs this probability is called the 2-disconnection exponent, which is the same as $\xi(2,0) := \xi(2,0+)$. The following is proved similarly to the previous theorem.

THEOREM 4.4. *With probability one, the Hausdorff dimension of the planar Brownian frontier is $2 - \xi(2,0)$.*

More generally, the intersections exponents $\xi(2,\beta)$ are related to the multifractal spectrum of harmonic measure (the hitting distribution by another Brownian motion starting at infinity) of a Brownian loop.

4.6. Values of the exponent. The value of the intersection exponent ζ_3 is not known and the best rigorous bounds are as mentioned before,

$$\frac{1}{4} < \zeta_3 = \frac{\xi_3(1,1)}{2} < \frac{1}{2}.$$

Numerics suggest ζ_3 is around .29. An interesting open question is whether or not we will ever know the value: can it be given explicitly or at least implicitly, say, as the value of an integral?

In two dimensions, the values of the exponents can be given exactly using conformal invariance. This was first predicted by Duplantier and Kwon [**10**] who first gave the value $\zeta_2 = 5/8$, and this conjecture was extended to all values of $\xi_2(j,\lambda)$ in [**30**]. If $\xi(j,\lambda) = \xi_2(j,\lambda)$, then the predictions were

$$\xi(1,\lambda) = \frac{1}{8} + \frac{\lambda}{2} + \frac{1}{8}\sqrt{24\lambda+1},$$

$$\xi(2,\lambda) = \frac{11}{24} + \frac{\lambda}{2} + \frac{5}{24}\sqrt{24\lambda+1}.$$

Particularly important values are

$$\xi(1,1) = 2\zeta_2 = \frac{5}{4}, \quad \xi(2,0) = \frac{2}{3}, \quad \xi(1,2) = \xi(2,1) = 2.$$

These values have been established rigorously using the Schramm-Loewner evoltution (SLE) [**24, 25**]. In particular, the set of cut points of the Brownian motion has Hausdorff dimension $2 - \frac{5}{4} = \frac{3}{4}$ and the frontier has dimension $2 - \frac{2}{3} = \frac{4}{3}$, establishing Mandelbrot's conjecture.

4.7. Random walk. Althought it is not obvious, it is true that the random walk exponent turns out to be the same as the Brownian intersection exponent. In fact, the only way that the only proofs of the existence of the random walk exponent show that it is the same as the Brownian exponent.

THEOREM 4.5. *Suppose S^1, S^2 are independent simple random walks starting at the origin in $\mathbb{Z}^d, d = 2,3$. There exist $0 < c_1 < c_2 < \infty$ such that for all $n \geq 1$,*

$$c_1\, n^{-\zeta} \leq \mathbb{P}\{S^1[0,n] \cap S^2[1,n] = \emptyset\} \leq c_2\, n^{-\zeta},$$

where $\zeta = \zeta_d = \xi_d(1,1)/2$.

We will not discuss the proof of this, but we will give discuss one of the ingredients, strong approximation or coupling. This is a process of defining simple random walk and Brownian motion on the same probability space so that their paths are close. For one-dimensional Brownian motions, the following *Skorokhod embedding* can be used. If B_t is a one-dimensional Brownian motion and stopping times are defined by $\tau_0 = 0$ and

$$\tau_n = \inf\left\{t > \tau_{n-1} : |B_t - B_{\tau_{n-1}}| = 1\right\}.$$

Then $S_n := B_{\tau_n}$ is a one-dimensional simple random that is "embedded" in the Brownian motion. We can give a heuristic argument (that can be made rigorous) to see how close the paths of the random walk are to the Brownian motion. Roughly speaking, $\tau_n = n + O(n^{1/2})$ and $|B_{n+O(n^{1/2})} - B_n| \approx n^{1/4}$ and hence $|B_n - S_n|$ is of order $n^{1/4}$.

For $d > 1$, we cannot embed a simple random walk into the Brownian motion since with probability one the Brownian motion does not visit points in $\mathbb{Z}^d$ after time zero. However, we can combine the embedding with some extra randomness. Let $\theta_n = (\theta_n^1, \ldots, \theta_n^d)$ denote a "directed one-dimensional random walk", that is, a process which at each time chooses uniformly one of the d directions and then moves one step in the positive direction. If $B_n^1, \ldots, B_n^d$ are independent one-dimensional Brownian motion with corresponding random walks $S_n^1, \ldots, S_n^d$ defined by the Skorokhod embedding, then

$$S_n = \left(S^1_{\theta^1_n}, \ldots, S^d_{\theta^d_n} \right)$$

is a d-dimensional random walk for which we can show that $|B_n - S_n|$ is of order $n^{1/4}$.

Another issue that needs to be dealt with is to show that when Brownian motions or random walks get close then they probably intersect. (Strong approximations only imply that when Brownian motions intersect the random walk paths get very close, and vice versa.) If $d = 2$, there is a uniform bound for any continuous curve, called the *Beurling estimate*. For $d = 3$, there is no uniform estimate for every continuous curve (since straight lines in $d = 3$ are not hit), so one needs to do more work.

5. Loop-erased walk

The loop-erased random walk (LERW) is a different measure on self-avoiding paths in $\mathbb{Z}^d$. The definition is easiest for $d \geq 3$ so we will first assume this. The process is obtained from simple random walk by erasing the loops chronologically. To be precise, let S_n denote a simple random walk starting at the origin and define times σ_n by

$$\sigma_0 = \max\{j : S_j = 0\},$$

and for $n > 0$,

$$\sigma_n = \max\left\{ j > \sigma_{n-1} : S_j = S_{\sigma_{n-1}+1} \right\}.$$

Transience of simple random walk for $d \geq 3$ implies that these times are well defined and we can define $\hat{S}_n$ by

$$\hat{S}_n = S_{\sigma_n}.$$

It is immediate from the definition that $\hat{S}_n$ is a path with no self intersections.

We can also describe $\hat{S}_n$ in terms of its (nonMarkovian) transition probabilities. Suppose that $[\hat{S}_0, \ldots, \hat{S}_n] = \omega = [\omega_0, \ldots, \omega_n]$. Then we know the following.

- The loop erasure of $[S_0, \ldots, S_{\sigma_n}]$ is ω.
- $[S_{\sigma_n+1}, \infty) \cap \omega = \emptyset$.

Using the second condition we can see that if $|x - \omega_n| = 1$, then

$$\mathbb{P}\{\hat{S}_{n+1} = x \mid [\hat{S}_0, \ldots, \hat{S}_n] = \omega\} = \frac{f_\omega(x)}{\sum_{|y-\omega_n|=1} f_\omega(y)}, \tag{5.1}$$

where $f_\omega(y)$ is the probability that a simple random walk starting at y never visits ω. The function $y \mapsto f_\omega(y)$ is the unique function that is (discrete) harmonic on $\mathbb{Z}^d \setminus \omega$ with boundary value 0 on ω and 1 at infinity. For this reason the loop-erased walk is also called the *Laplacian random walk*.

We need to adjust this definition for $d = 2$ since the random walk is recurrent. It turns out that all reasonable ways of doing sao give the same answer. Three equivalent definitions are the following.

- Take a simple random walk until it goes a distance R from the origin. Erase loops on this finite path and look at the first n steps. This gives a measure on SAWs of length n. For fixed n, the limit $R \to \infty$ of these measures exists and gives a consistent set of measures on SAWs.

- Use the transition probabilities (5.1) where $f_\omega(y)$ is the unique harmonic function on $\mathbb{Z}^2 \setminus \omega$ with boundary value 0 on ω and satisfying $f_\omega(y) \sim \log|y|$ as $|y| \to \infty$. In fact, there exists a (known) constant c such that

$$f_\omega(y) = c \lim_{m\to\infty} (\log m)\, \mathbb{P}^y\{S[0,m] \cap \omega = \emptyset\}.$$

- Take a simple random walk conditioned so that it never returns to the origin. By this we mean the space-inhomogeneous Markov process that takes steps as in (5.1) where $f_\omega(x)$ is replaced by $f_{\{0\}}(x)$. (This latter function is sometimes called the *potential kernel* of the random walk.) This process is transient and one can erase loops from the infinite path. One can use this definition as well for $d \geq 3$.

5.1. Displacement. To find out how far the loop-erased walk has gone in k steps, we need to know what fraction of points remain on the simple path after loop erasure. Let S be a simple random walk and let $\hat{S}$ be the corresponding self-avoiding path obtained by erasing the loops. (If $d = 2$, we let S be the random walk conditioned to avoid the origin after time 0.) Then we can write

$$\hat{S}_k = S_{\sigma(k)},$$

where $\sigma(k) = \max\{n : S_n = \hat{S}_k\}$. From this we get the (provable) relation

$$\mathbb{E}\left[|\hat{S}_k|^2\right] = \mathbb{E}\left[|S_{\sigma(k)}|^2\right] \asymp \mathbb{E}\left[\sigma(k)\right].$$

Let us define $\rho(n)$ to be the inverse of σ in the sense that $\rho(n) = k$ if $\sigma(k) \leq n < \sigma(k+1)$. In other words, $\rho(n)$ of the first n steps of S remain after loop-erasure. Let V_n denote the the event that the point S_n is not erased, that is, $n = \sigma(k)$ for some k. Then

$$\mathbb{E}[\rho(n)] = \sum_{j=0}^{n} \mathbb{P}\{S_j \text{ is not erased}\} \asymp n\, \mathbb{P}[V_n].$$

Th point S_n is not erased if and only if after time n the random walk does not hit the loop erasure of what it has already seen. That is,

$$V_n = \{LE\,(S[0,n]) \cap S[n+1,\infty) = \emptyset\}\,, \tag{5.2}$$

where $LE(\omega)$ denotes the chronological loop-erasure of a finite path which is always well defined.

We need to answer the following question. Suppose $x \in \mathbb{Z}^d$. What is the probability that the loop-erased walk goes through the point x? Equivalently, what is the probability that there exists a time n on the simple random walk path such that $S_n = x$ and (5.2) holds? We can write this probability as

$$\sum_{\omega:0\to x} \hat{P}(\omega)\, \mathbb{P}^x\{S[1,\infty) \cap \omega = \emptyset\}, \tag{5.3}$$

where the sum is over all SAWs $\omega = [\omega_0, \ldots, \omega_k]$ with $\omega_0 = 0, \omega_k = x$, and

$$\hat{P}(\omega) = \sum_{n=0}^{\infty} \mathbb{P}\{LE(S[0,n]) = \omega\}.$$

A careful look at the loop-erasing procedure allows one us to calculate $\hat{P}(\omega)$ for $\omega = [\omega_0, \ldots, \omega_k]$,

$$\hat{P}(\omega) = \left(\frac{1}{2d}\right)^k F(\omega), \quad F(\omega) = \prod_{j=0}^{k} G_{A_{j-1}}(\omega_j),$$

where $G_A(x)$ denotes the expected number of return to x before visiting A of a simple random walk starting at x, and $A_{j-1} = \{\omega_0, \ldots, \omega_{j-1}\}$. (If $d = 2$, we omit the $j = 0$ term which is infinite — omitting this is equivalent to conditioning that the random walk never returns to the origin.) A cute exercise in Markov chains shows that if $x, y \notin A$, then

$$G_A(x)\, G_{A\cup\{x\}}(y) = G_{A\cup\{y\}}(x)\, G_A(y),$$

from which we can conclude that $F(\omega)$ is a function only of the vertices $\{\omega_0, \ldots, \omega_k\}$ and not on the order in which they appear. In particular, $\hat{P}(\omega) = \hat{P}(\omega^R)$ where ω^R denotes the reversed walk

$$\omega^R = [\omega_k - \omega_k, \omega_{k-1} - \omega_k, \ldots, \omega_0 - \omega_k].$$

This allows us to write (5.3) as

$$\sum_{\omega:0\to -x} \hat{P}(\omega)\, \mathbb{P}\{S[1,\infty) \cap \omega = \emptyset\},$$

and by lattice symmetry we can replace $-x$ with x.

Another expression for F that takes a little more work is

$$F(\omega) = \exp\left\{\sum_{x\in\mathbb{Z}^d} \frac{1}{2j} \sum_{j=1}^{\infty} \mathbb{P}^x\{S_{2j} = x, S[0,2j] \cap \omega \neq \emptyset\}\right\}.$$

(If $d = 2$, we replace the probability with $\mathbb{P}^x\{S_{2j} = x, S[0,2j]\cap\omega \neq \emptyset, 0 \notin S[0,2j]\}$.) This can also be written as

$$F(\omega) = \exp\left\{\sum_{\eta\cap\omega\neq\emptyset} \overline{\mu}(\eta)\right\},$$

were $\overline{\mu}$ denotes the *random walk loop measure* on unrooted loops η. It is defined by $\overline{\mu}(\eta) = [b(\eta)/2j]\, 4^{-2j}$ if η is an unrooted loop of length $2j$ and $b(\eta)$ denotes the number of rooted loops that generate η. If η is a self-avoiding polygon, then $b(\eta) = 2j$; however; if η has double points, it is possible for $b(\eta)$ to be a strictly smaller divisor of $2j$. If $d = 2$ we restrict to loops that do not intersect the origin. This expression shows that F depends only on the points visited. One can also use this expression to show the deep relationship between loop-erased walks and other models such as the uniform spanning tree.

Anyway, this leads to studying the function

$$\hat{q}(n) = \mathbb{P}\{LE(S^1[0,n]) \cap S^2[1,n] = \emptyset\},$$

which is similar to the function $q(n)$ in Section 4. Clearly, $\hat{q}(n) \geq q(n)$. In analogy to what we have seen before, we might expect that

$$\hat{q}(n) \asymp n^{-\hat{\zeta}}\, L_n^{-\hat{u}},$$

where $\hat{\zeta} = \hat{\zeta}_d$ is an appropriate critical exponent that vanishes for $d \geq 4$. Roughly speaking, m steps of a simple walk correspond to $m\,\hat{q}(m)$ steps of a loop-erased walk, or n steps of the loop-erased walk correspond to $n^{2\hat{\nu}}\, L_n^{\hat{u}}$ steps of a simple random walk where $2\hat{\nu} = 1/(1-\hat{\zeta})$. This leads to the conjecture

$$\mathbb{E}\left[|\hat{S}_n|^2\right] \asymp n^{2\hat{\nu}}\, L_n^{\hat{u}}, \quad 2\hat{\nu} = \frac{1}{1-\hat{\zeta}}.$$

with $\tilde{\nu}_d = 1/2$ for $d \geq 4$. Also $\hat{q}(n) \geq q(n)$ implies that

$$\hat{\zeta}_d \leq \zeta_d,\ d = 2, 3, \qquad \hat{u} \leq \frac{1}{2}.$$

We will approach $\hat{q}(n)$ similarly to $q(n)$. We first consider long-range intersections, and then use that to find an easier probability to estimate.

5.2. Long-range intersection. Suppose S^1, S^2 are independent simple random walks starting distance $\sqrt{n}$ apart. Consider the probability that the loop-erasure of the first intersects the second,

$$\mathbb{P}\{LE(S^1[0,n]) \cap S^2[0,n] \neq \emptyset\}.$$

Clearly this is smaller than the probability obtained by replacing $LE(S^1[0,n])$ with $S^1[0,n]$ which was considered in Section 4.1. It turns out, however, the two probabilities are comparable, that is, the probability above is comparable to 1 for $d < 4$, $(\log n)^{-1}$ for $d = 4$, and $n^{(4-d)/2}$ if $d > 4$. For $d = 2$, this is not difficult since $LE(S^1[0,n])$ is a continuous path of diameter of order $\sqrt{n}$ and topological considerations imply that a simple random walk has a good chance to hit it.

For $d \geq 3$, the upper bound follows from the estimate with $S^1[0,n]$ replacing $LE(S^1[0,n])$. For the lower bound we estimate by the probability that S^2 hits a cut point for the random walk $S^1[0,n]$. Note that cut points for a walk are not erased. The formal argument uses a second moment estimate. If $d = 3$, the fundamental fact is that that set of cut *points* is a set of dimension $2(1-\zeta) > 1$ and hence a two-dimensional set ($S^2[0,n]$) has a good chance to hit it. For $d \geq 4$, being a cut time is a "local" property and hence if two points are separated the events that S^1 visits these points at a cut time are nearly independent.

5.3. Two easier probabilities to estimate. We consider the event

$$\{LE(S^1[0,n]) \cap S^2[0,n] \neq \emptyset\},$$

from the previous subsection with $|S_0^1 - S_0^2|$ of order $\sqrt{n}$. On this event, we will take the first intersection of the paths. Since there are two paths, there are two ways to define "first", and these will give us different estimates.

We first consider what happens when we choose the first time on the loop-erased path. Let

$$\sigma = \min\left\{j : S_j^1 \text{ is not erased}, S_j^1 \in S^2[0,n]\right\},$$
$$\tau = \min\{k : S_k^2 = S_\sigma^1\}.$$

Let

$$\omega^- = S^1[0,\sigma], \quad \omega^+ = S^1[\sigma+1,n],$$
$$\eta^- = S^2[0,\tau], \quad \eta^+ = S^2[\tau+1,n].$$

Then we have the following relations

$$S_\sigma^1 = S_\tau^2, \quad LE(\omega^-) \cap \omega^+ = \emptyset, \quad LE(\omega^-) \cap \eta^- = \emptyset, \quad LE(\omega^-) \cap \eta^+ = \emptyset.$$

This leads us to define

$$\hat{p}(n) = \mathbb{P}\{LE(S[0,n]) \cap (S^2[1,n] \cup S^3[1,n] \cup S^4[1,n]) = \emptyset\},$$

where S, S^2, S^3, S^4 are simple random walks starting at the origin, and suggests that if y is a typical point with $|y| \asymp n^{1/2}$,

$$\mathbb{P}\{LE(S^1[0,n]) \cap S^2[0,n] \neq \emptyset,\ S^1_\sigma = S^2_\tau = y\} \asymp n^{2-d}\,\hat{p}(n).$$

The term n^{2-d} is comparable to the probability that both S^1 and S^2 visit y. Since there are of order $n^{d/2}$ such y, we sum over y to get

$$\mathbb{P}\{LE(S^1[0,n]) \cap S^2[1,n] = \emptyset\} \asymp n^{(4-d)/2}\,\hat{p}(n).$$

This argument can be carried out to give

$$\hat{p}(n) \asymp \begin{cases} n^{(d-4)/2}, & d < 4 \\ L_n^{-1}, & d = 4 \end{cases} . \tag{5.4}$$

Now we will see what happens if we choose the first time along the second path. Let

$$\tau = \min\{k : S^2_k \in LE(S^1[0,n])\},$$
$$\sigma = \max\{j \leq n : S^1_j = S^2_k\}.$$

If we define $\omega^-, \omega^+, \eta^-, \eta^+$ as above, then the necessary relations are

$$S^1_\sigma = S^2_\tau, \quad \eta^- \cap LE(\omega^-) = \emptyset, \quad \omega^+ \cap LE(\omega^-) = \emptyset, \quad \eta^- \cap LE(\omega^+) = \emptyset.$$

This leads us to define another event for simple random walks S^1, S^2, S^3 starting at the origin. Let E_n denote the event that

$$LE(S^1[0,n]) \cap S^2[1,n] = \emptyset,$$

and

$$[LE(S^1[0,n]) \cup LE(S^2[0,n])] \cap S^3[1,n] = \emptyset.$$

Then arguing as in the previous paragraph, we have

$$\mathbb{P}(E_n) \asymp \begin{cases} n^{(d-4)/2}, & d < 4 \\ L_n^{-1}, & d = 4 \end{cases}$$

This event corresponds to "3-arm points" in uniform spanning trees.

5.4. Intersection exponent. Let us define the random variable Z_n which depends on the random walk S^1 by

$$\hat{\mathcal{H}}_n = -\log \mathbb{P}\left\{S^2[1,e^{2n}] \cap LE(S^1[1,e^{2n}]) = \emptyset \mid S^1[1,e^{2n}]\right\}.$$

We have chosen this notation to emphasize the analogy with $\mathcal{H}_n$ in the previous section. With this notation

$$\hat{q}(e^{2n}) = \mathbb{E}\left[e^{-\mathcal{H}_n}\right].$$

The estimate (5.4) shows that for $d = 2, 3, 4$,

$$\mathbb{E}\left[e^{-3\mathcal{H}_n}\right] \asymp e^{-\lambda\,(4-d)\,n}\,L_{e^{2n}}^{-1}.$$

Using the Hölder inequality, we get

$$\mathbb{E}\left[e^{-3\mathcal{H}_n}\right] \leq \mathbb{E}\left[e^{-\mathcal{H}_n}\right] \leq \mathbb{E}\left[e^{-3\mathcal{H}_n}\right]^{1/3}.$$

As in the case for intersections of random walks, it can be shown that "mean-field behavior" holds in the critical dimension, that is, if $d = 4$.

$$\mathbb{E}\left[e^{-\mathcal{H}_n}\right] \asymp \left(\mathbb{E}\left[e^{-3\mathcal{H}_n}\right]\right)^{1/3} \asymp n^{-1/3}.$$

The mean-square displacement grows like

$$\mathbb{E}\left[|\hat{S}_n|^2\right] \asymp n\,(\log n)^{1/3}.$$

For $d = 2, 3$, we define the loop-erased walk intersection exponent $\hat{\xi}(\lambda) = \hat{\xi}_d(\lambda)$ by

$$\mathbb{E}\left[e^{-\lambda \mathcal{H}_n}\right] \approx n^{-\hat{\xi}(\lambda) n}.$$

We let $\hat{\xi} = \hat{\xi}_d(1)$ which is the same as $2\hat{\zeta}$. Using the relation above we see that the mean-square displacement exponent is

$$\hat{\nu} = \frac{1}{2 - \hat{\xi}},$$

and the fractal dimension of the paths is $2 - \hat{\xi}$. The estimate above implies that $\xi_d(3) = 4 - d$, and Hölder's inequality gives

$$\frac{4-d}{3} \leq \hat{\xi}_d(1) \leq 4 - d,$$

and the belief that mean-field behavior does not hold strongly suggests that the inequalities are strict.

Kenyon [**15**] gave the first proof that $\hat{\xi}_2 = 3/4$. This value had been predicted in the physics literature using nonrigorous methods. This gives the displacement exponent $\hat{\nu}_2 = 4/5$ and the fractal dimension of the paths is $5/4$. It is believed, and has been proved for the scaling limit SLE_2, that

$$\hat{\xi}_2(\lambda) = \frac{\lambda}{2} + \frac{1}{8}\sqrt{8\lambda + 1}.$$

For $d = 3$, there is no proof that the exponent $\hat{\xi}_3$ exists although from the estimate above and the estimate for the intersection exponent ξ_3 we have the bounds

$$\frac{1}{3} \leq \hat{\xi}_3 \leq \hat{\xi}_3 < 1.$$

The lower bound should be closer to the real value, but the proof suggests that it is not exactly equal. This translates to the following estimates for the displacement exponent

$$\frac{3}{5} \leq \hat{\nu}_3 < 1,$$

with a strong suggestion that the lower bound is a strict inequality. It is possible that this exponent will never be known exactly. Numerical simulations [**39**] suggest that $\hat{\nu} = .6157\cdots$.

One may note that for $d = 2, 3, 4$, the loop-erased walk tends to go farther than what is conjectured for the SAW. (If $d = 3$, the lower bound for the loop-erased walk is the Flory exponent which is now expected to be strictly between the values for the SAW and the loop-erased walk.) There is a heuristic reason why one should expect this to hold. Recall the transition probabilities (5.1) for LERW viewed as a Laplacian random walk. The probabilities for the next step are proportion to the probabilities that a *simple random walk* avoids the current path. Assuming it exists, the infinite self-avoiding walk has a similar transition probability except that one considers the probabilities that *self-avoiding walks* avoid the current path. If we believe that SAW are thinner than simple random walks, then one should expect that LERW weights more heavily those paths that are getting away from the current path, and hence LERW would have thinner paths than SAW.

5.5. Scaling limit. If $d \geq 4$ it is has been shown that the scaling limit of loop-erased random walk is Brownian motion. For $d = 2$, the scaling limit is SLE_κ with $\kappa = 2$. This is the unique value of κ that gives a path of Hausdorff dimension 5/4. The theorem as we stated is has not been proved. However, a different version where the path is parametrized by "capacity" has been proved [**28**].

There are still many questions to be answered if $d = 3$. Gady Kozma [**16**] showed that the LERW has a scaling limit. Unfortunately, this proof does not give much information about the scaling limit.

6. Other topics

6.1. Schramm-Loewner evolution (SLE). It was first predicted (nonrigorously and, in fact, not precisely) by Belavin, Polyakov, and Zamolodchikov [**2, 3**] that the scaling limit of critical statistical mechanical models in two dimensions is conformally invariant. One example of this was already well known. Suppose B_t is a standard two-dimensional Brownian motion which can be considered as taking values in the complex numbers $\mathbb{C}$. By using Itô's formula (or, essentially equivalently, by using the fact that harmonic functions are preserved by conformal maps), the following fact can be proved. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a nonconstant holomorphic function and

$$Y_t = f(B_t).$$

Then Y_t is a *time change* of a Brownian motion. To be more precise, if σ_t is defined by

$$\int_0^{\sigma_t} |f'(B_s)|^2 \, ds = t,$$

then Y_{σ_t} is a standard Brownian motion. The exponent 2, which comes from the scaling relation for Brownian motion, reflects the two-dimensionality of Brownian paths.

In searching for possible scaling limits for processes such as the loop-erased walk, Oded Schramm chose to make conformal invariance one of the defining properties. To simplify things he considered paths modulo reparametrization. Suppose we have a probability measure on simple curves $\gamma : (-\infty, \infty) \to \mathbb{C}$ with $\gamma(-\infty) = 0, \gamma(\infty) = \infty$. For each t, let $H_t = \mathbb{C}^* \setminus \gamma(-\infty, t]$ where $\mathbb{C}^*$ denotes the Riemann sphere. This is a simply connected subdomain of the sphere, and hence by the Riemann mapping theorem, there is a unique conformal transformation

$$g_t : H_t \to \mathcal{O},$$

where $\mathcal{O} = \{z \in \mathbb{C} : |z| > 1\}$ such that $g_t(\gamma(t)) = 1, g_t(\infty) = \infty$. Schramm made the following assumption which is called the *conformal domain Markov property*:

- μ is a probability measure on simple curves $\gamma : (0, \infty) \to \mathcal{O}$ with $\gamma(0+) = 1, \gamma(\infty) = \infty$, such that for every t, the conditional distribution of $\gamma^{(t)}(s) := g_t[\gamma(t + s)]$ given $\gamma(-\infty, t]$ is μ.

It turns out that there is only a one-parameter family of such measures μ which are now called the *(radial) Schramm-Loewner evolution with parameter* $\kappa > 0$ or SLE_κ. If we insist on simple curves, the parameter must satisfy $\kappa \leq 4$ and this gives curves of Hausdorff dimension $D = 1 + \frac{\kappa}{8}$. Radial SLE_κ is defined for other simply connected domains by conformal invariance.

If the self-avoiding walk has a conformally invariant limit, then it should also satisfy a property called the *restriction property*. Roughly speaking, a process satisfies this property if the conditional distribution given that the process stays in a simply connected subdomain is the same as the distribution in that subdomain (which is defined by conformal invariance). It has been shown [**27, 29**] that $\kappa = 8/3$ is the only value of $\kappa \leq 4$ such that SLE_κ satisfies the restriction property. Therefore, *if SAW has a conformally invariant limit*, the limit process is be $SLE_{8/3}$. These curves have fractal dimension 4/3 which gives another derivation of the displacement exponent $\nu = 3/4$. One can also compute an intersection exponent for $SLE_{8/3}$ and rederive the exponent $\gamma = 43/32$. While this is very nice, it is still open to establish that the SAW has a limit, let alone, that the limit is conformally invariant. Why should we expect a conformally invariant limit? The physical arguments suggest plausibility but are not very convincing. However, now we have another reason to believe it. If one assumes conformal invariance, then the limit must be $SLE_{8/3}$ and we can test this conjecture numerically. Tom Kennedy (see, e.g., [**14**]) has done extensive simulations that strongly support the conjecture.

The Schramm-Loewner evolution has also been used to show that what Mandelbrot saw as the outer boundary of Brownian motion is probably correct. Indeed, it is now a theorem that the frontier of Brownian motion looks locally like an $SLE_{8/3}$ path. If the latter is the limit of self-avoiding loops then so is the first.

It has been proved [**28**] that in a sense the loop-erased random walk converges to SLE_2 which has dimension 5/4. More precisely, the LERW with the discrete analogue of the capacity parametrization converges to SLE_2 with the capacity parametrization. It is still open to show the stronger fact that we get convergence in the "natural parametrization" as in Section 3.7. Masson [**33**] was able to give another derivation of the exponent $\hat{\xi} = 3/4$ using SLE calculations.

6.2. An easier probability for SAW. In analyzing the nonintersections of random walks and the loop-erased walk, we were able to find an easier probabilty to estimate by considering long-range intersections. We will discuss something similar here for self-avoidng walk in dimensions $d = 2, 3, 4$. Unfortunately, like most of what has been discussed for SAW, we cannot prove these results. We will use the notation from Section 3.

Let $\Delta_n = n^\nu \, L_n^u$ be the typical displacement of walks of length n. Consider two independent SAWs

$$\omega^j = [\omega_0^j, \ldots, \omega_n^j], \quad j = 1, 2,$$

of n steps starting distance Δ_n apart. To be precise, let U_n denote the set of ordered pairs of SAWs (ω^1, ω^2) with $\omega_0^1 = 0, \Delta_n/2 \leq |\omega_0^2| \leq 2\,\Delta_n$. For each choice of starting point there are $e^{\beta n}\,\phi(n)$ choices for each walk. Also the number of choices for the starting point ω_0^2 is comparable to Δ_n^d. Therefore, $\#(U_n) \asymp e^{2\beta n}\,\phi(n)^2\,\Delta_n^d$. Consider the set $V_n = \{(\omega^1, \omega^2) \in U_n : \omega^1 \cap \omega^2 \neq \emptyset\}$. In analogy with the long-range probabilities we saw in the random walk intersection and loop-erased walk problems, we expect that the probability of V_n is comparable to L_n^{-1}, that is

$$\#(V_n) \asymp e^{2\beta n}\,\phi(n)^2\,\Delta_n^d\,L_n^{-1}.$$

On the event V_n, let σ denote the time of the first intersection where first is defined on the first path,

$$\sigma = \min\left\{j : \omega_j^1 \in \{\omega_0^2, \ldots, \omega_n^2\}\right\},$$

and let τ be the (unique) k such that $\omega_k^2 = \omega_\sigma^1$. Let $V_n(j,k) = V_n \cap \{\sigma = j, \tau = k\}$, and consider $n/4 \leq j, k \leq 3n/4$. Since there are $n/2$ choices for each of j, k, we expect that

$$e^{-2\beta n}\,\phi(n)^{-2}\,\#\,[V_n(j,k)] \asymp n^{-2}\,\Delta_n^d\,L_n^{-1} \asymp \begin{cases} n^{d\nu-2}, & d=2,3 \\ (\log n)^{4u-1}, & d=4. \end{cases}$$

At the moment the paths are set so that $\omega_0^1 = 0$. By translating, we can set $\omega_\sigma^1 = \omega_\tau^2 = 0$. Define the paths $\eta_-^1, \eta_+^1, \eta_-^2, \eta_+^2$ of lengths $j, n-j, k, n-k$, respectively, by

$$\eta_-^1 = [\omega_j^1 - \omega_j^1, \omega_{j-1}^1 - \omega_j^1, \ldots, \omega_0^1 - \omega_j^1],$$
$$\eta_+^1 = [\omega_j^1 - \omega_j^1, \omega_{j+1}^1 - \omega_j^1, \ldots, \omega_n^1 - \omega_j^1],$$

and η_-^2, η_+^2, similarly. We have four SAWs starting at the origin and the definition of the time (σ,τ) implies that

$$\eta_-^1 \cap \eta_+^1 = \emptyset, \quad \eta_-^2 \cap \eta_+^2 = \emptyset, \tag{6.1}$$

$$\eta_-^1 \cap \eta_-^2 = \emptyset, \quad \eta_-^1 \cap \eta_+^2 = \emptyset. \tag{6.2}$$

The number of SAWs $(\eta_-^1, \eta_+^1, \eta_-^2, \eta_+^2)$ of lengths $j, n-j, k, n-k$, respectively is comparable to $e^{2\beta n}\,\phi(n)^4$. If we impose the conditions in (6.1), this reduces to $e^{2\beta n}\,\phi(n)^2$. With this restriction, the distributions of (η_-^1, η_+^1) and (η_-^2, η_+^2) near the origin look like independent two-sided SAWs. This leads to the following conjecture which we state in two different ways.

- Suppose $\omega^1, \ldots, \omega^4$ are independent SAWs of length n starting at the origin. Then

$$\mathbb{P}\{\omega_1 \cap (\omega_3 \cup \omega_4) = \{0\} \mid \omega_1 \cap \omega_2 = \omega_3 \cap \omega_4 = \{0\}\}$$
$$\asymp \begin{cases} n^{d\nu-2}, & d=2,3 \\ (\log n)^{4u-1}, & d=4. \end{cases}$$

.

- Suppose (ω^1, ω^2) and (ω^3, ω^4) are independent two-sided SAWs starting at the origin. Then the probability that the only intersection of ω^1 with $\omega^3 \cup \omega^4$ in the ball of radius R occurs at the origin is comparable to $R^{d-(2/\nu)}$ for $d = 2,3$ and to $(\log R)^{4u-1}$ for $d = 4$.

Mean-field behavior at the critical dimension $d = 4$ should imply that if $\eta_-^1, \eta_+^1, \eta_-^2, \eta_+^2$ are independent SAWs at the origin, then the four events in (6.1) and (6.2) are independent up to a multiplicative constant, that is, the probability that they all occur is comparable to the product of the individual probabilities. Thus we expect

$$\begin{aligned} \mathbb{P}\{\omega_1 \cap (\omega_3 \cup \omega_4) = \{0\} \mid \omega_1 \cap \omega_2 = \omega_3 \cap \omega_4 = \{0\}\} \\ \asymp\ & \mathbb{P}\{\omega_1 \cap (\omega_3 \cup \omega_4) = \{0\}\} \\ \asymp\ & [\mathbb{P}\{\omega_1 \cap \omega_3 = \{0\}\}]^2 \\ \asymp\ & (\log n)^{-2u'}. \end{aligned}$$

This implies that $2u' + 4u = 1$. This is not enough to determine the values of u', u. However, if we can show that $G(x) \asymp |x|^{-2}$ (with no logarithmic corrections), then, as in Section 3.4, we get another equation $u' = 2u$ from which we conclude that $u' = 2u = 1/4$. While it has not been proved that $G(x) \asymp |x|^{-2}$, this has been established recently for some weakly SAWs that are expected to have the same qualitative behavior. See [**4**] for an introduction to this work.

7. Fractal Dimension

The terms "fractal" and "fractal dimension" were coined by Benoit Mandelbrot. Although the terms originally had some precise meaning, it has become more useful not to be specific. However, in order to have theorems, one must have exact definitions. Here we discuss a few of the definitions and some of the techniques for proving them. The examples we have in mind are (continuous) curves $\gamma : [0,T] \to \mathbb{R}^d$ that arise as scaling limits of discrete curves as in Section 3.7 where we expect that fractal dimension to be $\alpha > 1$. If γ is such a curve we will write γ_T for the set of points $\{\gamma(s) : 0 \leq s \leq T\}$.

The form of the scaling limit gives a natural parametrization to the continuous process which is called Minkowksi content. Suppose A is a bounded subset of $\mathbb{R}^d$. For $\varepsilon > 0$, let $V(A,\varepsilon)$ denote the volume of the ε-"sausage" $A_\varepsilon = \{z \in \mathbb{R}^d : \operatorname{dist}(z,A) \leq \varepsilon\}$. The *upper α-Minkowski content* is defined by

$$\mathrm{Cont}^+_\alpha(A) = \limsup_{\varepsilon \downarrow 0} \varepsilon^{\alpha-d}\, V(A,\varepsilon).$$

The *lower α-Minkowski content* $\mathrm{Cont}^-_\alpha(A)$ is defined similarly by replacing lim sup with lim inf, and if the two values agree, then we call the value the *α-Minkowski content* $\mathrm{Cont}_\alpha(V)$. There is a unique $\alpha \in [0,d]$ called the *upper box (or box-counting or Minkowski) dimension* such that

$$\mathrm{Cont}^+_\beta(A) = \begin{cases} \infty, & \beta < \alpha \\ 0, & \beta > \alpha \end{cases}.$$

The *lower box dimension* is defined similarly, and if the two dimensions agree, we call this the *box dimension* $\dim_b(V)$. Note that

$$\mathrm{Cont}_\beta(A) = \begin{cases} \infty, & \beta < \dim_b(V) \\ 0, & \beta > \dim_b(V) \end{cases},$$

but nothing is said about the value, or even the existence, of $\mathrm{Cont}_\alpha(V)$ for $\alpha = \dim_b(V)$.

When possible, it is good to prove the existence and nontriviality of the Minkowski content in the appropriate dimension. It is known to exist for Brownian motion if $d \geq 3$ (for $d \leq 2$, the dimension of the set of double points is the same as the dimension of the set of points and the content is zero in this case). It has recently been shown to exist for the Schramm-Loewner evolution [**23**].

Often, it is too hard to show the content exists. Another measure of dimension is *Hausdorff dimension* which arises from *Hausdorff measure*. This definition is more complicated, but it has the advantage that it is defined for all sets. Suppose $0 \leq \alpha \leq d$. If $\varepsilon > 0$, let

$$\mathcal{H}^\varepsilon_\alpha(V) = \inf \sum [\mathrm{diam}(U_j)]^\alpha,$$

where the infimum is over all covers of V by sets $U_1, U_2, \ldots$ each with diameter at most ε. As $\varepsilon \downarrow 0$, we get fewer possible coverings. Hence $\mathcal{H}^\varepsilon_\alpha(V)$ is monotone in ε and the α-Hausdorff measure

$$\mathcal{H}_\alpha(V) = \lim_{\varepsilon \downarrow 0} \mathcal{H}^\varepsilon_\alpha(V)$$

is well defined. The term "measure" here is appropriate; in fact, $\mathcal{H}_\alpha$ is a Borel measure in $\mathbb{R}^d$. It is not hard to see that there is a unique α such that

$$\mathcal{H}_\beta(A) = \begin{cases} \infty, & \beta < \alpha \\ 0, & \beta > \alpha \end{cases},$$

and we call α the Hausdorff dimension $\dim_h(V)$. As in the case for box dimension, we say nothing about $\mathcal{H}_{\dim_h(A)}(A)$. It is not hard to show that the Hausdorff dimension is no larger than the lower box dimension, but it can be smaller. Typically, for random sets, at the critical dimension the Hausdorff measure is zero so this does not give a parametrization; however, one can sometimes define Hausdorff measures with different gauge functions.

If A is a random set, then the standard way to give an upper bound on its Hausdorff dimension is by bounding the lower box dimension. For ease, let us assume that A is contained in a compact set K (or equivalently, that we are considering $A \cap K$). Since

$$V(A, \varepsilon) = \int_{K_\varepsilon} 1\{\text{dist}(z, A) \le \varepsilon\} \, dz,$$

where we write dz for $d^d z$, we see that

$$\mathbb{E}\,[V(A,\varepsilon)] = \int_{K_\varepsilon} \mathbb{P}\{\text{dist}(z, A) \le \varepsilon\} \, dz.$$

If we can find α, c such that for all ε sufficiently small,

$$\mathbb{E}\,[V(A,\varepsilon)] \le c\,\varepsilon^{d-\alpha},$$

then a straightforward argument shows that with probability one for all $\beta > \alpha$, $\text{Cont}_\beta(V) = 0$, and hence $\dim_h(A) \le \alpha$.

Lower bounds on Hausdorff dimension can be trickier. One of the most useful techniques goes back to Frostman. The basic idea is that if one can put an α-dimensional measure on a set, then the set is at least α-dimensional. To be more precise, suppose that $V \subset \mathbb{R}^d$ is compact, and μ is a positive measure supported on V with $0 < \mu(V) < \infty$ and

$$\mathcal{E}_\alpha(\mu) := \int_{V \times V} \frac{\mu(dx)\,\mu(dy)}{|x-y|^\alpha} < \infty.$$

Then $\dim_h(V) \ge \alpha$. To get a feeling for this condition, suppose that μ is d-dimensional Lebesgue measure restricted to V. Then this integral is finite if and only if $\alpha < d$.

THEOREM 7.1. *Suppose $V \subset \mathbb{R}^d$ is compact and A is a random, closed subset of V. Let $I(z,\varepsilon)$ denote the indicator function of the event $\{\text{dist}(z,A) \le \varepsilon\}$. Suppose that there exist $0 < c_1 < c_2 < \infty$ such that for all $x, y \in V$,*

$$c_1\,\varepsilon^{d-\alpha} \le \mathbb{E}\,[I(x,\varepsilon)] \le c_2\,\varepsilon^{d-\alpha},$$

$$\mathbb{E}\,[I(x,\varepsilon)\,I(y,\varepsilon)] \le c_2\,\varepsilon^{2(d-\alpha)}\,|x-y|^{\alpha-d}.$$

Then $\mathbb{P}\{\dim_h(A) \le \alpha\} = 1$ and $\mathbb{P}\{\dim_h(A) = \alpha\} > 0$.

Let us sketch the proof. The upper bound follows as above, so we will focus on the lower bound. Let μ_ε denote the (random) measure whose density with respect to d-dimensional Lebesgue measure is $\varepsilon^{\alpha-d}\, I(z,\varepsilon)$. Then using the estimates as above, we can see that there exist $0 < c_3 < c_4 < \infty$ such that

$$c_3^2 \leq (\mathbb{E}\,[\mu_\varepsilon(V)])^2 \leq \mathbb{E}\left[\mu_\varepsilon(V)^2\right] \leq c_4.$$

Moreover, for every $\beta < \alpha$, there exists $C_\beta < \infty$ such that

$$\mathbb{E}\,[\mathcal{E}_\beta(\mu_\varepsilon)] \leq C_\beta.$$

The second moment estimate gives

$$\mathbb{P}\left\{\mu_\varepsilon(V) \geq \frac{c_3}{2}\right\} \geq \frac{c_3^2}{c_4},$$

and the Markov inequality gives

$$\mathbb{P}\left\{\mathcal{E}_\beta(\mu_\varepsilon) \geq R_\beta\right\} \leq \frac{C_\beta}{R_\beta} = \frac{c_3^2}{2c_4}, \quad \text{where } R_\beta = \frac{2c_4\, C_\beta}{c_3^2}.$$

Hence,

$$\mathbb{P}\left\{\mu_\varepsilon(V) \geq \frac{c_3}{2}, \mathcal{E}_\beta(\mu_\varepsilon) \leq R_\beta\right\} \geq \frac{c_3^2}{2c_4}.$$

In particular,

$$\mathbb{P}\left\{\mu_{2^{-n}}(V) \geq \frac{c_3}{2}, \mathcal{E}_\beta(\mu_{2^{-n}}) \leq R_\beta \text{ i.o.}\right\} \geq \frac{c_3^2}{2c_4}.$$

If the event on the left-hand side occurs we can find a subsequential limit of the measures $\mu_{2^{-n}}$ to obtain a measure μ supported on A with $\mu(V) \geq c_3/2$ and $\mathcal{E}_\beta(\mu) \leq R_\beta$. This implies for every $\beta < \alpha$,

$$\mathbb{P}\left\{\dim_h(A) \geq \beta\right\} \geq \frac{c_3^2}{c_4},$$

and hence,

$$\mathbb{P}\left\{\dim_h(A) \geq \alpha\right\} \geq \frac{c_3^2}{c_4}.$$

As an example, let us show that the path of a Brownian motion in $\mathbb{R}^d$, $d \geq 3$ equals two. Let $A = B[0,\infty)$. We will show that if V is a closed ball not containing the origin, then $\mathbb{P}\{\dim_h(A \cap V) \leq 2\} = 1$ and $\mathbb{P}\{\dim_h(A \cap V) = 2\} > 0$. A simple scaling argument, which we leave to the reader, can use this to show that $\mathbb{P}\{\dim_h(A) = 2\} = 1$. We will use one standard fact about Brownian motion: if $\varepsilon < |x|$, and $q(x)$ denotes the probability that a Brownian motion starting at x every hits the ball of radius ε about the origin, then $q(x) = (\varepsilon/|x|)^{d-2}$. This is obtained by noting that q is a radially symmetric harmonic function on $\{|x| > \varepsilon\}$ with boundary values $q(\varepsilon) = 1, q(\infty) = 0$. Using the notation in the theorem, we see that $\mathbb{E}[I(x,\varepsilon)] \asymp \varepsilon^{d-2}$ (with the implicit constants depending on the set V). To get the two-point estimate, we see that the probability that the path gets within distance ε of y after the first time it gets with distance ε of x is bounded above by $q(|x| - \varepsilon)$. If the path gets close to both x and y, then either it gets close to x and then close to y or the other way around (it is possible for both to happen). This gives the estimate

$$\mathbb{E}\,[I(x,\varepsilon)\, I(y,\varepsilon)] \leq c\,\varepsilon^{d-2}\,[\varepsilon/|x-y|]^{d-2}.$$

We can now use the theorem.

As another example, let us consider the intersection of two Brownian paths in $\mathbb{R}^3$. Let B^1, B^2 be independent Brownian motions with corresponding indicator functions $I^1(x,\varepsilon), I^2(x,\varepsilon)$. Let $I(x,\varepsilon) = I^1(x,\varepsilon)\, I^2(x,\varepsilon)$. Then arguing as in the previous paragraph we see that $\mathbb{E}[I(x,\varepsilon)] \asymp \varepsilon^2$ and

$$\mathbb{E}\left[I(x,\varepsilon)\, I(y,\varepsilon)\right] \leq c\, \varepsilon^2 \left[\varepsilon/|x-y|\right]^2.$$

If we continue as in the proof of the rheorem, we see that with positive probability we get a measure μ with strictly positive measure satisfying $\mathcal{E}_\beta(\mu) < \infty$ for $\beta < 1$. Moreover, from the contruction we can see that it is supported on $B^1[0,\infty) \cap B^2[0,\infty) \cap V$. Hence, on this event of positive measure,

$$\dim_h \left[B^1[0,\infty) \cap B^2[0,\infty) \cap V\right] = 1.$$

n particular, it is not empty. We have shown that a set is nonempty by showing it has positive dimension!

Further reading

If you have gotten this far, you realize that we have left out may details in this paper! Here is a little guide for more reading. The reference list below is nowhere near a complete listing of the work on the problems in this area; one should look at the references of the books and survey papers below to get further references. For a detailed description of the simple random walk, see [**22**]. This also includes discussion on the loop-erased walk. For an older treatment (before the breakthroughs in two dimensions) focusing on the intersection properties see [**19**]. For the state of the art on self-avoiding walks by the early nineties (which is not too different than the state of the art today!) try [**31**]. This can be supplemented with [**37**] for more on $d > 4$ and [**4**] for $d = 4$. I suggest [**38**] to see the strong relationship between uniform spanning trees and loop-erased walk. See [**34**] for a good treatment of Brownian motion including some discussion of the exceptional sets. For the general properties of fractal dimensions, I recommend the classic [**32**] as an enjoyable read, and [**12**] for an accessible rigorous introduction. There are many sources for the recent work on the Schramm-Loewner evolution. A good start is the original paper [**36**]. For recent surveys you can try [**1, 20, 21**].

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