STAT 25100

HW8 (due May 31)

FINAL EXAMINATION

- **Graduating seniors**. Wednesday, May 30, 7 – 9 pm, 117 Eckhart. If you cannot make this time you must talk to me ahead of time to schedule a different time.
- **Others.** Thursday, June 7, 6:00 – 8:00 pm, Kent 120.

The final examination will strongly emphasize material covered in the last four homeworks but there may be multi-part problems that require knowledge from the earlier part of the class. The final will be closed-book. You may bring two 8 1/2 x 11 pages of notes and formulas to aid you; the one you prepared for the midterm and another for material after the midterm. You should also bring a calculator.

1. (3pt) Suppose we flip a fair coin.

- Suppose we stop when we have flipped three consecutive heads. What is the expected number of flips until we stop?
- Suppose we stop whenver we have flipped THH, i.e., a tails followed by two consecutve heads. What is the expected number of flips until we stop?

2. (2pt) (based on Ross, p. 412, #42). A group of 24 people — 14 men and 10 women — are randomly arranged into 12 pairs of 2 each. Find the expectation and variance of the number of pairs that consist of a man and a woman.

3. (2pt) (based on Ross, p. 417 #69,70 and class) Suppose a coin is selected with probability p of turning up heads. Suppose that p is a random variable with $U(0,1)$ distribution. Suppose that after the coin in selected, the coin is flipped n times and let X be the number of heads obtained. Show that

$$\mathbf{P}\{X = j\} = \frac{1}{n+1}, \quad j = 0, 1, \ldots, n.$$

4. (3pt) (based on Ross, p. 414, #53) A prisoner is trapped in a cell containing 4 doors. The first door leads to a tunnel that returns him to the his cell after 2 days arrive. the second door and third doors do similarly except that it takes 4 days to arrive back to the cell. The fourth door leads to freedom in one day.

- Assuming that the prisoner never remembers which door he has tried (except for the one he has just come out of) and randomly chooses with probability 1/3 for the other three doors door, what is the expected amount of time until the prisoner obtains freedom?
- Assuming that the prisoner does remember the doors he has tried and at each time chooses randomly among the untried doors, what is the expected amount of time until the prisoner obtains freedom?

5. (3pt) (2006 Putnam exam, A-4) Let $S = \{1, 2, \ldots, n\}$ for some integer $n > 1$. Say a permutation π of S has a local maximum at $k \in S$ if

(i) $\pi(k) > \pi(k+1)$ if $k = 1$;

(ii) $\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1)$ if $1 < k < n$;

(iii) $\pi(k-1) < \pi(k)$if $k = n$.

(For example, if $n = 5$ and π takes values at $1, 2, 3, 4, 5$ of $2, 1, 4, 5, 3$ then π has a local maximum of 2 at $k = 1$, and a local maximum of 5 at $k = 4$.) If the permutation is chosen uniformly from the set of all $n!$ permutations, what is the expected value of the number of local maxima?

6. (2pt) Suppose X is a $N(0, 1)$ random variable. Find $E[X^n]$ for all integers n by using the moment generating function. (Hint: you may want to write the moment generating function in a Taylor series before you start differentiating.)

7. (2pt) Consider the following variation on the secretary problem. The assumptions are the same in class except that you will be happy if you choose either of the top two candidates. Suppose you do the strategy as discussed in class: let the fraction p of candidates go without any hires and then choose the first person better than all the people you have already seen. Assuming the number of candidates N is very large find:

- For fixed p, the probability that you get one of the top two candidates.
- The p that maximizes this probability.

8. (2pt) Consider the following population model: at time one there is one individual. Suppose there are X_n individuals at time n. Each individual (independently) either dies or creates a new individual, each with probability 1/2. If at any time, all the individuals die, then we say that the population has become extinct. Find the probability that the population eventually becomes extinct. (Hint: let $a(k)$ be the probability that the population becomes extinct given that there are k individuals in the population. Express $a(k)$ in terms of $a(1)$ and then find an equation for $a(1)$.)

9.* (5pt). Consider a random walker moving as in class, i.e., at each time step the random walker moves either one step up or one step down each with probability 1/2. Let N be a positive integer and let T be the first time the walker is at N or at $-N$. We are interested in $E(T)$.

1. Let $e(x)$ be the expected number of steps the walker needs to take assuming the walker starts at x. By definition $e(N) = e(-N) = 0$. Our problem is to find $e(0)$. Find an equation for $e(x)$ in terms of $e(x-1)$ and $e(x+1)$.
2. If you did the last part correctly, you should have $2N-1$ linear equations in $2N-1$ unknowns, $e(-N+1), e(-N+2), \ldots, e(N-1)$. (In fact, there are really only N different equations in N unknowns because symmetry implies $e(x) = e(-x)$.) Prove that there exists a unique solution to these equations.
3. Find the solution.
4. If the general problem is too hard, you can do the entire problem with $N = 3$ for partial credit.