

Scaling limits and the Schramm-Loewner evolution

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Abstract: These notes are from my mini-courses given at the PIMS summer school in 2010 at the University of Washington and at the Cornell probability summer school in 2011. The goal was to give an introduction to the Schramm-Loewner evolution to graduate students with background in probability. This is not intended to be a comprehensive survey of SLE.

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This is a combination of notes from two mini-courses that I gave. In neither course did I cover all the material here. The Seattle course covered Sections 1, 4–6, and the Cornell course covered Sections 2, 3, 5, 6. The first three sections discuss particular models. Section 1, which was the initial lecture in Seattle, surveys a number of models whose limit should be the Schramm-Loewner evolution. Sections 2–3, which are from the Cornell school, focus on a particular model, the loop-erased walk and the corresponding random walk loop measure. Section 3 includes a sketch of the proof of conformal invariance of the Brownian loop measure which is an important result for SLE. Section 4, which was covered in Seattle but not in Cornell, is a quick summary of important facts about complex analysis that people should know in order to work on SLE and other conformally invariant processes. The next section discusses the deterministic Loewner equation and the remaining sections are on SLE itself. I start with some very general comments about models in statistical mechanics intended for mathematicians with little experience with physics models.

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Introductory thoughts: Models in equilibrium statistical physics

The Schramm-Loewner evolution is a particularly powerful tool in the understanding of critical systems in statistical physics in two dimensions. Let me give a rather general view of equilibrium statistical physics and its relationship to

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probability theory. The typical form of a model is a collection of configurations γ and a base measure say m_1 . If the collection is finite, a standard base measure is counting measure, perhaps normalized to be a probability measure. If the collection is infinite, there can be subtleties in defining the base measure. There is also a function $E(\gamma)$ on configurations called the *energy* or *Hamiltonian*. There are also some parameters; let us assume there is one that we call $\beta \geq 0$. Many models choose β to be a constant times the reciprocal of temperature. For this reason large values of β are called “low temperature” and small values of β are called “high temperature”. The physical assumption is that the system in equilibrium tries to minimize energy. In the Gibbsian framework, this means that we consider the new measure m_2 which can be written as

$$dm_2 = e^{-\beta E} dm_1. \quad (1)$$

When mathematicians write down expressions like (1), it is implied that the measure m_2 is absolutely continuous with respect to m_1 . We might allow E to take on the value infinity, but m_2 would give measure zero to such configurations. However, physicists are not so picky. They will write expressions like this when the measures are singular and the energy E is infinite for all configurations. Let me give one standard example where m_1 is “Lebesgue measure on all functions g on $[0, 1]$ with $g(0) = 0$ ” and m_2 is one-dimensional Wiener measure. In that case we set $\beta = 1/2$ and

$$E(g) = \int_0^1 |g'(x)|^2 dx.$$

This is crazy in many ways. There is no “Lebesgue measure on all functions” and we cannot differentiate an arbitrary function — even a typical function in the measure m_2 . However, let us see how one can make some sense of this. For each integer N we consider the set of functions

$$g : \{1/N, 2/N, \dots, 1\} \rightarrow \mathbb{R},$$

and denote such a function as a vector (x_1, \dots, x_N) . The points have the density of Lebesgue measure in \mathbb{R}^N . The energy is given by a discrete approximation of $E(g)$:

$$E_N(g) = \frac{1}{N} \sum_{j=1}^N \left(\frac{x_j - x_{j-1}}{1/N} \right)^2,$$

where $x_0 = 0$, and hence

$$\exp \left\{ -\frac{1}{2} E_N(g) \right\} = \exp \left\{ -\frac{1}{2(1/N)} \sum_{j=1}^N (x_j - x_{j-1})^2 \right\}.$$

If we multiply both sides by a scaling factor of $(N/2\pi)^{N/2}$, then the right hand side becomes exactly the density of

$$(W_{1/N}, W_{2/N}, \dots, W_1)$$

with respect to Lebesgue measure on \mathbb{R}^N where W_t denotes a standard Brownian motion. Therefore as $N \rightarrow \infty$, modulo scaling by a factor that goes to infinity, we get (1).

This example shows that even though the expression (1) is meaningless, there is “truth” in it. The way to make it precise is in terms of a limit of objects that are well defined. This is the general game in taking scaling limits of systems. The basic outline is as follows.

- Consider a sequence of simple configurations. In many cases, and we will assume it here, for each N , there is a set of configurations of cardinality $c_N < \infty$ ($c_N \rightarrow \infty$) and a well defined energy $E_N(\gamma)$ defined for each configuration γ . For ease, let us take counting measure as our base measure $m_{1,N}$.
- Define $m_{2,N}$ using (1).
- Define the *partition function* Z_N to be the total mass of the measure $m_{2,N}$. If m_1 is counting measure

$$Z_N = \sum_{\gamma} e^{-\beta E_N(\gamma)}.$$

- Find scaling factors r_N and hope to show that $r_N m_{2,N}$ has a limit as a measure. (One natural choice is $r_N = 1/Z_n$ in which case we hope to have a probability measure in the limit. However, this not the only important possibility.)

All of this has been done with β fixed. Critical phenomena studies systems where the behavior of the scaling limit changes dramatically at a critical value β_c . (The example we did above with Brownian motion does not have a critical value; changing β only changes the variance of the final Brownian motion.) We will be studying systems at this critical value. A nonrigorous (and, frankly, not precise) prediction of Belavin, Polyakov, and Zamolodchikov [1, 2] was that the scaling limits of many two-dimensional systems at criticality are conformally invariant. This idea was extended by a number of physicists using nonrigorous ideas of conformal field theory. This was very powerful, but it was not clear how to make it precise. A big breakthrough was made by Oded Schramm [22] when he introduced what he called the stochastic Loewner evolution (SLE). It can be considered the missing link (but not the only link!) in making rigorous many predictions from physics.

Probability naturally rises in studying models from statistical physics. Indeed, any nontrivial finite measure can be made into a probability measure by normalizing. Probabilistic techniques can then be very powerful; for example, the study of Wiener measure is much, much richer when one uses ideas such as the strong Markov property. However, some of the interesting measures in statistical physics are not finite, and even for those that are one can lose information if one always normalizes. SLE, as originally defined, was a purely probabilistic technique but it has become richer by considering nonprobability measures given by (normalized) partition functions.

We will be going back and forth between two kinds of models:

- *Configurational* where one gives weights to configurations. This is the standard in equilibrium statistical mechanics as well as combinatorics.
- *Kinetic (or kinetically growing)* where one builds a configuration in time. For deterministic models, this gives differential equations and for models with randomness we are in the comfort zone for probabilists.

It is very useful to be able to go back and forth between these two approaches. The models in Section 1 and Section 2 are given as configurational models. However, when one has a finite measure one can create a probability measure by normalization and then one can create a kinetic model by conditioning. For the case of the loop-erased walk discussed in Section 2, the kinetically growing model is the Laplacian walk. Brownian motion, at least as probabilists generally view it, is a kinetic model. However, one can get very important configurational models and Section 3 takes this viewpoint. The Schramm-Loewner evolution, as introduced by Oded Schramm, is a continuous kinetic model derived from continuous configurational models that are conjectured to be the scaling limit for discrete models. In some sense, SLE is an artificial construction — it describes a random configuration by giving random dynamics even though the structure we are studying did not grow in this fashion. Although it is artificial, it is a very powerful technique. However, it is useful to remember that it is only a partial description of a configurational model. Indeed, our definition of SLE here is really configurational.

1. Scaling limits of lattice models

The Schramm-Loewner evolution (SLE) is a measure on continuous curves that is a candidate for the scaling limit for discrete planar models in statistical physics. Although my lectures will focus on the continuum model, it is hard to understand *SLE* without knowing some of the discrete models that motivate it. In this lecture, I will introduce some of the discrete models. By assuming some kind of “conformal invariance” in the limit, we will arrive at some properties that we would like the continuum measure to satisfy.

1.1. Self-avoiding walk (SAW)

A self-avoiding walk (SAW) of length n in the integer lattice $\mathbb{Z}^2 = \mathbb{Z} + i\mathbb{Z}$ is a sequence of lattice points

$$\omega = [\omega_0, \dots, \omega_n]$$

with $|\omega_j - \omega_{j-1}| = 1, j = 1, \dots, n$, and $\omega_j \neq \omega_k$ for $j < k$. If J_n denotes the number of SAWs of length n with $\omega_0 = 0$, it is well known that

$$J_n \approx e^{\beta n}, \quad n \rightarrow \infty,$$

where e^β is the *connective constant* whose value is not known exactly. Here \approx means that $\log J_n \sim \beta n$ where $f(m) \sim g(m)$ means $f(m)/g(m) \rightarrow 1$. In fact, it

is believed that there is an exponent, usually denoted γ , such that

$$J_n \asymp n^{\gamma-1} e^{\beta n}, \quad n \rightarrow \infty,$$

where \asymp means that each side is bounded by a constant times the other. Another exponent ν is defined roughly by saying that the typical diameter (with respect to the uniform probability measure on SAWs of length n with $\omega_0 = 0$) is of order n^ν . The constant β is special to the square lattice, but the exponents ν and γ are examples of lattice-independent *critical exponents* that should be observable in a “continuum limit”. For example, we would expect the fractal dimension of the paths in the continuum limit to be $d = 1/\nu$.

To take a continuum limit we let $\delta > 0$ and

$$\omega^\delta(j\delta^d) = \delta \omega(j).$$

We can think of ω^δ as a SAW on the lattice $\delta\mathbb{Z}^2$ parametrized so that it goes a distance of order one in time of order one. We can use linear interpolation to make $\omega^\delta(t)$ a continuous curve. Consider the square in \mathbb{C}

$$D = \{x + iy : -1 < x < 1, -1 < y < 1\},$$

and let $z = -1, w = 1$. For each integer N we can consider a finite measure on continuous curves $\gamma : (0, t_\gamma) \rightarrow D$ with $\gamma(0+) = z, \gamma(t_\gamma) = w$ obtained as follows. To each SAW ω of length n in \mathbb{Z}^2 with $\omega_0 = -N, \omega_n = N$ and $\omega_1, \dots, \omega_{n-1} \in ND$ we give measure $e^{-\beta n}$. If we identify ω with $\omega^{1/N}$ as above, this gives a measure on curves in D from z to w . The total mass of this measure is

$$Z_N(D; z, w) := \sum_{\omega: Nz \rightarrow Nw, \omega \subset ND} e^{-\beta|\omega|}.$$

It is conjectured that there is a b such that as $N \rightarrow \infty$,

$$Z_N(D; z, w) \sim C(D; z, w) N^{-2b}. \quad (2)$$

Moreover, if we multiply by N^{2b} and take a limit, then there is a measure $\mu_D(z, w)$ of total mass $C(D; z, w)$ supported on simple (non self-intersecting) curves from z to w in D . The dimension of these curves will be $d = 1/\nu$.

Similarly, if D is another domain and $z, w \in \partial D$, we can consider SAWs from z to w in D . If ∂D is smooth at z, w , then (after taking care of the local lattice effects — we will not worry about this here), we define the measure as above, multiply by N^{2b} and take a limit. We conjecture that we get a measure $\mu_D(z, w)$ on simple curves from z to w in D . We write the measure $\mu_D(z, w)$ as

$$\mu_D(z, w) = C(D; z, w) \mu_D^\#(z, w),$$

where $\mu_D^\#(z, w)$ denotes a probability measure.

It is believed that the scaling limit satisfies some kind of “conformal invariance”. To be more precise we assume the following conformal *covariance*

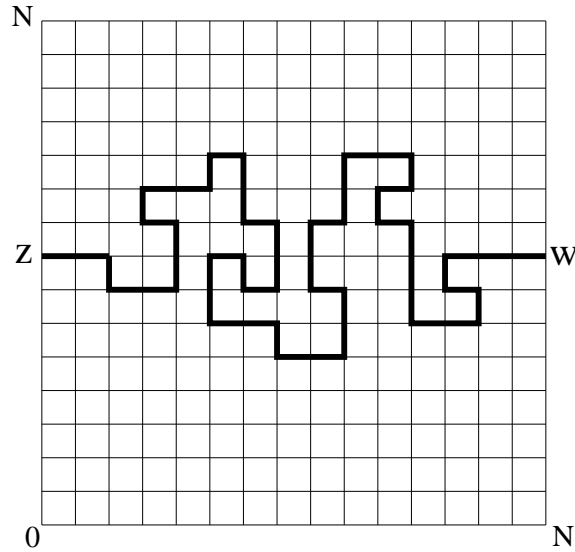


FIG 1. Self-avoiding walk in a domain.

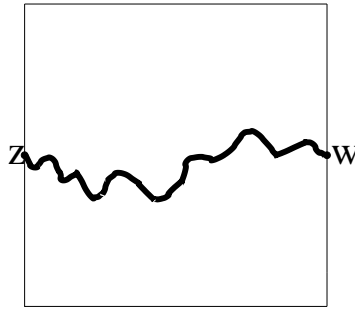


FIG 2. Scaling limit of SAW.

property: if $f : D \rightarrow f(D)$ is a conformal transformation and f is differentiable in neighborhoods of $z, w \in \partial D$, then

$$f \circ \mu_D(z, w) = |f'(z)|^b |f'(w)|^b \mu_{f(D)}(f(z), f(w)).$$

In other words the total mass satisfies the scaling rule

$$C(D; z, w) = |f'(z)|^b |f'(w)|^b C(f(D); f(z), f(w)), \tag{3}$$

and the corresponding probability measures are conformally *invariant*:

$$f \circ \mu_D^\#(z, w) = \mu_{f(D)}^\#(f(z), f(w)).$$

Note that (3) is consistent with (2).

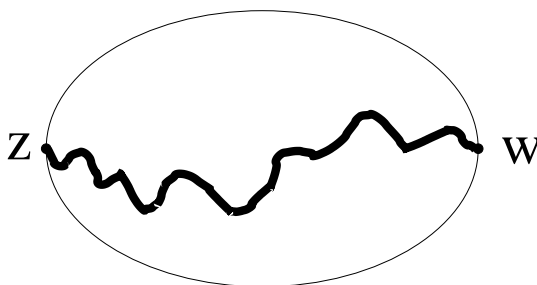


FIG 3. Scaling limit of SAW in a different domain.

Let us be a little more precise about the definition of $f \circ \mu_D^\#(z, w)$. Suppose $\gamma : (0, t_\gamma) \rightarrow D$ is a curve with $\gamma(0+) = z, \gamma(t_\gamma-) = w$. For ease, let us assume that γ is simple. Then the curve $f \circ \gamma$ is the corresponding curve from $f(z)$ to $f(w)$. At the moment, we have not specified the parametrization of $f \circ \gamma$. We will consider two possibilities:

- **Ignore the parametrization.** We consider two curves equivalent if one is an (increasing) reparametrization of the other. In this case we do not need to specify how we parametrize $f \circ \gamma$.
- **Scaling by the dimension d .** If γ has the parametrization as given in the limit, then the amount of time need for $f \circ \gamma$ to traverse $f(\gamma[t_1, t_2])$ is

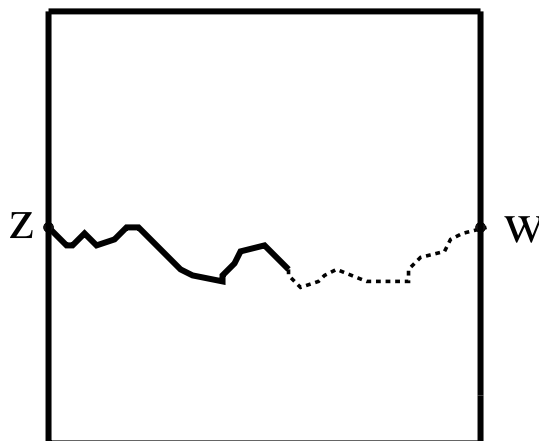
$$\int_{t_1}^{t_2} |f'(\gamma(s))|^d ds. \quad (4)$$

In either case, if we start with the probability measure $\mu_D^\#(z, w)$, the transformation $\gamma \mapsto f \circ \gamma$ induces a probability measure which we call $f \circ \mu_D^\#(z, w)$.

There are two more properties that we would expect the family of measures $\mu_D(z, w)$ to have. The first of these will be shared by all the examples in this section while the second will not. We just state the properties, and leave it to the reader to see why one would expect them in the limit.

- **Domain Markov property.** Consider the measure $\mu_D^\#(z, w)$ and suppose an initial segment of the curve $\gamma(0, t]$ is observed. Then the conditional distribution of the remainder of the curve given $\gamma(0, t]$ is the same as $\mu_{D \setminus \gamma(0, t]}^\#(\gamma(t), w)$.
- **Restriction property.** Suppose $D' \subset D$. Then $\mu_{D'}(z, w)$ is $\mu_D(z, w)$ restricted to paths that lie in D' . In terms of Radon-Nikodym derivatives, this can be phrased as

$$\frac{d\mu_{D'}(z, w)}{d\mu_D(z, w)}(\gamma) = 1_{\{\gamma(0, t_\gamma) \subset D'\}}.$$

FIG 4. *Domain Markov property.*

We have considered the case where $z, w \in \partial D$. We could consider $z \in \partial D, w \in D$. In this case the measure is defined similarly, but (2) becomes

$$Z_D(z, w) \sim C(D; z, w) N^{-b} N^{-\tilde{b}},$$

where \tilde{b} is a different exponent (see Lectures 5 and 6). The limiting measure $\mu_D(z, w)$ would satisfy the conformal covariance rule

$$f \circ \mu_D(z, w) = |f'(z)|^b |f'(w)|^{\tilde{b}} \mu_{f(D)}(f(z), f(w)).$$

Similarly we could consider $\mu_D(z, w)$ for $z, w \in D$.

1.2. Loop-erased random walk

We start with simple random walk. Let ω denote a nearest neighbor random walk from z to w in D . We no longer put in a self-avoidance constraint. We give each walk ω measure $4^{-|\omega|}$ which is the probability that the first n steps of an ordinary random walk in \mathbb{Z}^2 starting at z are ω . The total mass of this measure is the probability that a simple random walk starting at z immediately goes into the domain and then leaves the domain at w . Using the “gambler’s ruin” estimate for one-dimensional random walk, one can show that the total mass of this measure decays like $O(N^{-2})$; in fact

$$Z_N(D; z, w) \sim C(D; z, w) N^{-2}, \quad N \rightarrow \infty, \quad (5)$$

where $C(D; z, w)$ is the “excursion Poisson kernel”, $H_{\partial D}(z, w)$, defined to be the normal derivative of the Poisson kernel $H_D(\cdot, w)$ at z . In the notation of

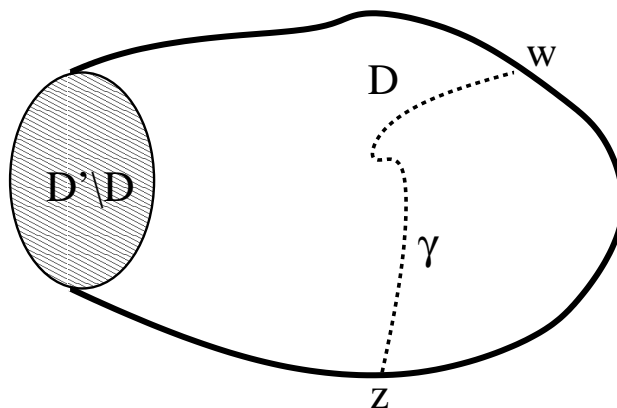


FIG 5. Illustrating the restriction property.

the previous section $b = 1$. For each realization of the walk, we produce a self-avoiding path by erasing the loops in chronological order.

Again we are looking for a continuum limit $\mu_D(z, w)$ with paths of dimension d (not the same d as for SAW). The limit should satisfy

- Conformal covariance
- Domain Markov property

However, we would not expect the limit to satisfy the restriction property. The reason is that the measure given to each self-avoiding walk ω by this procedure is determined by the number of ordinary random walks which produce ω after loop erasure. If we make the domain smaller, then we lose some random walks that would produce ω and hence the measure would be smaller. In terms of Radon-Nikodym derivatives, we would expect

$$\frac{d\mu_{D_1}(z, w)}{d\mu_D(z, w)} < 1.$$

We discuss this process more in Section 2.

1.3. Percolation

Suppose that every point in the triangular lattice in the upper half plane is colored black or white independently with each color having probability $1/2$. A typical realization is illustrated in Figure 8 (if one ignores the bottom row).

We now put a boundary condition on the bottom row as illustrated — all black on one side of the origin and all white on the other side. For any realization of the coloring, there is a unique curve starting at the bottom row that has all white vertices on one side and all black vertices on the other side. This is called the *percolation exploration process*. Similarly we could start with a domain D

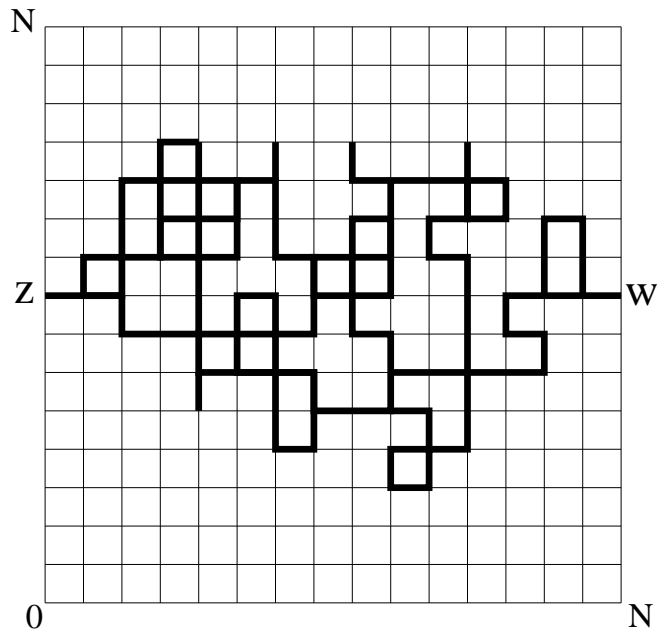
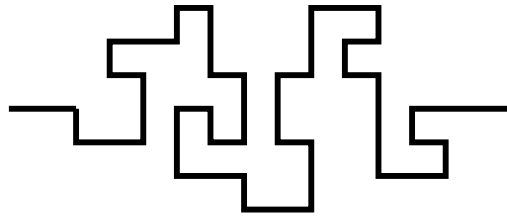
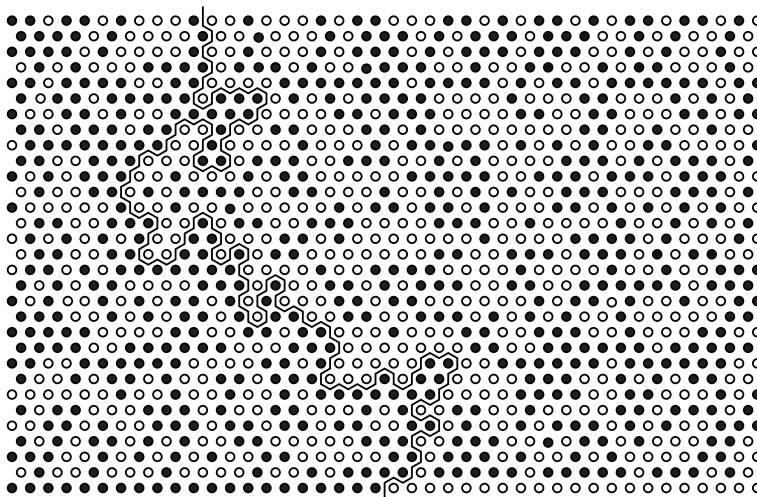
FIG 6. Simple random walk in D .

FIG 7. The walk obtained from erasing loops chronologically.

and two boundary points z, w ; give a boundary condition of black on one of the arcs and white on the other arc; put a fine triangular lattice inside D ; color vertices black or white independently with probability $1/2$ for each; and then consider the path connecting z and w . In the limit, one might hope for a continuous interface. In comparison to the previous examples, the total mass of the lattice measures is one. This implies that the scaling exponent should take on the value $b = 0$. We suppose that the curve is conformally invariant, and one can check that it should satisfy the domain Markov property.

The scaling limit of percolation satisfies another property called the *locality property*. Suppose $D_1 \subset D$ and $z, w \in \partial D \cap \partial D_1$ as in Figure 5. Suppose that only an initial segment of γ is seen. To determine the measure of the initial segment,

FIG 8. *The percolation exploration process.*

one only observes the value of the percolation cluster at vertices adjoining γ . Hence the measure of the path is the same whether it is considered as a curve in D_1 or a curve in D . The locality property is stronger than the restriction property which SAW satisfies. The restriction property is a similar statement that holds for the entire curve γ but not for all initial segments of γ .

1.4. *Ising model*

The Ising model is a simple model of a ferromagnet. We will consider the triangular lattice as in the previous section. Again we color the vertices black or white although we now think of the colors as spins. If x is a vertex, we let $\sigma(x) = 1$ if x is colored black and $\sigma(x) = -1$ if x is colored white. The measure on configurations is such that neighboring spins like to have the same sign.

It is easiest to define the measure for a finite collection of spins. Suppose D is a bounded domain in \mathbb{C} with two marked boundary points z, w which give us two boundary arcs. We consider a fine triangular lattice in D ; and fix boundary conditions $+1$ and -1 respectively on the two boundary arcs. Each configuration of spins is given energy

$$\mathcal{E} = - \sum_{x \sim y} \sigma(x) \sigma(y),$$

where $x \sim y$ means that x, y are nearest neighbors. We then give measure $e^{-\beta \mathcal{E}}$ to a configuration of spins. If β is small, then the correlations are localized and spins separated by a large distance are almost independent. If β is large, there is long-range correlation. There is a critical β_c that separates these two phases.

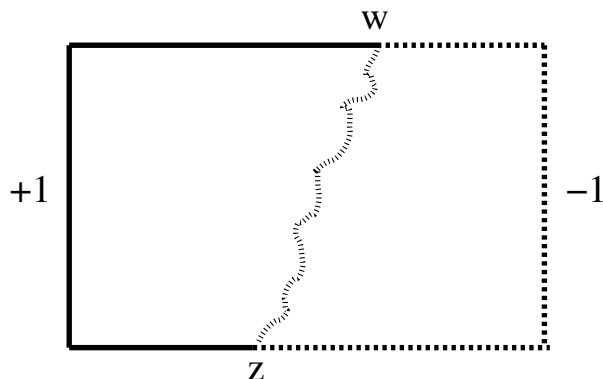


FIG 9. Ising interface.

For each configuration of spins there is a well-defined boundary between $+1$ spins and -1 spins defined in exactly the same way as the percolation exploration process. At the critical β_c it is believed that this gives an interesting fractal curve and that it should satisfy conformal covariance and the domain Markov property.

1.5. Assumptions on limits

Our goal is to understand the possible continuum limits for these discrete models. We will discuss the boundary to boundary case here but one can also have boundary to interior or interior to interior. (The terms “surface” and “bulk” are often used for boundary and interior.) Such a limit is a measure $\mu_D(z, w)$ on curves from z to w in D which can be written

$$\mu_D(z, w) = C(D; z, w) \mu_D^\#(z, w),$$

where $\mu_D^\#(z, w)$ is a probability measure. The existence of $\mu_D(z, w)$ assumes smoothness of ∂D near z, w , but the probability measure $\mu_D^\#(z, w)$ exists even without the smoothness assumption. The two basic assumptions are:

- Conformal covariance of $\mu_D(z, w)$ and conformal invariance of $\mu_D^\#(z, w)$.
- Domain Markov property.

The starting point for the Schramm-Loewner evolution is to show that *if we ignore the parametrization of the curves*, then there is only a one-parameter family of probability measures $\mu_D^\#(z, w)$ for *simply connected* domains D that satisfy conformal invariance and the domain Markov property. We will construct this family. The parameter is usually denoted $\kappa > 0$. By the Riemann mapping theorem, it suffices to construct the measure for one simply connected domain and the easiest is the upper half plane \mathbb{H} with boundary points 0 and ∞ . As we will see, there are a number of ways of parametrizing these curves.

2. Loop measures and loop-erased random walk

This section will focus on two closely related models: the loop-erased random walk and a measure on simple random walk loops. They are also closely related to the enumeration of spanning trees on a graph. Although much of what we say here can be generalized to Markov chains, we will only consider the case of random walk in \mathbb{Z}^2 . We will not give complete proofs; a reference for more details is [11, Chapter 9].

We write

$$\omega = [\omega_0, \omega_1, \dots, \omega_n], \quad \omega_j \in \mathbb{Z}^2, \quad |\omega_j - \omega_{j-1}| = 1, j = 1, \dots, n,$$

for a (*nearest neighbor*) path in \mathbb{Z}^2 . We write $|\omega| = n$ for the number of steps of the path. We call the path a (*rooted*) loop if $\omega_0 = \omega_n$. For each path ω , we define

$$q(\omega) = (1/4)^{|\omega|}.$$

Note that $q(\omega)$ is exactly the probability that a simple random walk in \mathbb{Z}^2 starting at ω_0 traverses ω in its first n steps. We view q as a measure on the set of paths of finite length.

◆ More generally, we can define $q(\omega) = \lambda^{|\omega|}$ for some $\lambda > 0$, but $1/4$ is the “critical” value of the parameter and most interesting to us.

A path ω is *self-avoiding* if $\omega_j \neq \omega_k$ for $j \neq k$. For every path ω there is a unique self-avoiding subpath

$$L(\omega) = [\eta_0, \eta_1, \dots, \eta_k]$$

called its (*chronological*) *loop-erasure* satisfying the following.

- $\eta_0 = \omega_0, \eta_k = \omega_n$.
- For each j , $\eta_j \in \omega$. Moreover, if

$$\sigma(j) = \max\{m : \omega_m = \eta_j\},$$

then $\sigma(0) < \sigma(1) < \dots < \sigma(k)$ and

$$\{\eta_0, \dots, \eta_j\} \cap \{\omega_{\sigma(j)+1}, \dots, \omega_n\} = \emptyset.$$

Throughout this section, A will denote a nonempty finite subset of \mathbb{Z}^2 and

$$\partial A = \{z \in \mathbb{Z}^2 : \text{dist}(z, A) = 1\}.$$

2.1. Excursion measure

An excursion in A is a random walk that starts on ∂A , immediately goes into A , and ends when it reaches ∂A . To be more precise, a (*boundary*) *excursion* in A is a path $\omega = [\omega_0, \dots, \omega_n]$ with $n \geq 2$, $\omega_0, \omega_n \in \partial A$ and $\omega_j \in A$ for $1 \leq j \leq n-1$. Let \mathcal{E}_A denote the set of excursions in A and if $x, y \in \partial A$ let $\mathcal{E}_A(x, y)$ denote the set of excursions with $\omega_0 = x, \omega_n = y$. Let q_A denote q restricted to \mathcal{E}_A . Let

$$H_A(x, y) = q_A[\mathcal{E}_A(x, y)]$$

which is called the *boundary* or excursion Poisson kernel. Note that $q_A(\mathcal{E}_A) < \infty$. Indeed, if $x \in \partial A$,

$$\sum_{y \in \partial A} H_A(x, y) = r/4,$$

where r denotes the number of nearest neighbors of x in A .

2.2. Loop-erased measure

Let $\hat{\mathcal{E}}_A$ denote the set of excursions in A that are self-avoiding and define $\hat{\mathcal{E}}_A(x, y)$ similarly. The (*boundary*) *loop-erased measure* \hat{q}_A is the measure on $\hat{\mathcal{E}}_A$, where

$$\hat{q}_A(\eta) = q_A \{ \omega \in \mathcal{E}_A : L(\omega) = \eta \}.$$

Note that $\hat{q}_A[\hat{\mathcal{E}}_A(x, y)] = H_A(x, y)$. If $\eta \in \hat{\mathcal{E}}_A$, it is not obvious how to calculate $\hat{q}_A(\eta)$ which is the measure of the set of excursions $\omega \in \hat{\mathcal{E}}_A$ whose loop-erasure is η . It turns out that the calculation of this leads to some classical formulas involving the determinant of the Laplacian for the random walk in A . We will write

$$\hat{q}_A(\eta) = q_A(\eta) F_\eta(A),$$

where $F_\eta(A)$ will be a quantity that depends only on the sites in η and not on the order that they are traversed. In the next subsection we will compute $F_\eta(A)$ in terms of a measure on unrooted loops.

2.3. Random walk loop measure

If $\omega = [\omega_0, \dots, \omega_n]$ is a loop, we call $\omega_0 = \omega_n$ the *root* of the loop. We say that ω is in A and write $\omega \subset A$ if all the sites in ω are in A . Let $\mathcal{L}(A)$ denote the set of loops in A with length at least two, and $\mathcal{L}^x(A)$ the set of such loops rooted at x . We define the measure $m = m_A$ on $\mathcal{L}(A)$ by

$$m(\omega) = \frac{q(\omega)}{|\omega|}, \quad \omega \in \mathcal{L}(A). \quad (6)$$

This called the *rooted loop measure on A* .

◆ Throughout this section we will abuse notation so that ω can either refer to the path or to the set of sites visited by ω . For example in the previous paragraph, the expression $\omega \subset A$ uses ω to mean the set of sites while the expression $\omega \in \mathcal{L}(A)$ uses ω to mean the path. I hope this does not create confusion. Also, if x is a site, we also use x to denote the singleton set $\{x\}$.

An *unrooted loop* is a rooted loop where one forgets which point is the starting point. More precisely, it is an equivalence class of rooted loops under the equivalence

$$[\omega_0, \omega_1, \dots, \omega_n] \sim [\omega_j, \omega_{j+1}, \dots, \omega_n, \omega_1, \dots, \omega_{j-1}, \omega_j].$$

We write $\bar{\omega}$ for an unrooted loop and we write $\omega \sim \bar{\omega}$ if ω is a rooted loop in the equivalence class $\bar{\omega}$. We write $\bar{\mathcal{L}}(A)$ for the set of unrooted loops. The (*unrooted*) *loop measure* \bar{m} is the measure given by

$$\bar{m}(\bar{\omega}) = \sum_{\omega \sim \bar{\omega}} m(\omega).$$

◆ One thinks of the unrooted loop measure as assigning measure $(1/4)^n$ to each unrooted loop of length n . However, this is not exactly correct because an unrooted loop of length n may have fewer than n rooted representatives. For example, consider $\omega = [x, y, x, y, x]$. The length of the walk is four, but there are only two representatives of the unrooted loop.

Let

$$F(A) = \exp \left\{ \sum_{\bar{\omega} \in \bar{\mathcal{L}}(A)} \bar{m}(\bar{\omega}) \right\},$$

and if $B \subset A$, let $F_B(A)$ be the corresponding quantity restricted to loops that intersect B ,

$$F_B(A) = \exp \left\{ \sum_{\bar{\omega} \in \bar{\mathcal{L}}(A), \bar{\omega} \cap B \neq \emptyset} \bar{m}(\bar{\omega}) \right\},$$

Note that

$$F(A) = F_B(A) F(A \setminus B).$$

In particular, if $\eta = [\eta_0, \dots, \eta_k] \in \hat{\mathcal{E}}_A$,

$$F_\eta(A) = \prod_{j=1}^k F_{\eta_j}(A \setminus \{\eta_1, \dots, \eta_{j-1}\}).$$

In the expression on the right it is not so obvious that $F_\eta(A)$ is independent of the ordering of the vertices.

Proposition 1. *If $\eta \in \hat{\mathcal{E}}_A$, then*

$$\hat{q}(\eta) = q(\eta) F_\eta(A). \quad (7)$$

We will not prove the proposition, but we will state some other expressions for $F_\eta(A)$. Let $G(x, x; A)$ denote the usual Green's function for the random walk, that is, the expected number of visits to x , starting at x , before leaving A . Then [11, Lemma 9.3.2],

$$G(x, x; A) = F_x(A). \quad (8)$$

In particular,

$$F_\eta(A) = \prod_{j=1}^k G(\eta_j, \eta_j; A \setminus \{\eta_1, \dots, \eta_{j-1}\}).$$

The expression on the right-hand side is what one naturally gets if one tries to compute $\hat{q}(\eta)/q(\eta)$. The proposition above then requires the generating function identity (8). Another expression involves the determinant of the Laplacian. Let $Q = Q_A$ denote the $\#(A) \times \#(A)$ matrix $Q(x, y) = 1/4$ if $|x - y| = 1$ and $Q(x, y) = 0$ for other x . Then the (negative of the) Laplacian of the random walk on A is the matrix $I - Q$. Then [11, Proposition 9.3.3],

$$F(A) = \frac{1}{\det(I - Q)}.$$

2.4. Spanning trees and Wilson's algorithm

A spanning tree of a connected graph is a subgraph containing all the vertices that is also a tree. If A is a finite subset of \mathbb{Z}^2 , then a *wired spanning tree* of A is a spanning tree of the graph $A \cup \{\partial\}$ where a vertex $x \in A$ is adjacent to ∂ if $\text{dist}(x, \partial A) = 1$. *Wilson's algorithm* [24] for choosing a wired spanning tree is as follows.

- Choose a vertex in A and run a random walk until it reaches ∂A .
- Perform loop-erasure on the walk and add those edges to the tree.
- If there are vertices that are not on the tree, choose one and do a random walk until it reaches a vertex in the tree.
- Perform loop-erasure on this walk and add those new edges to the tree.
- Continue this procedure until all vertices have been added to the tree.

This is a very efficient way to choose a random tree, and the amazing fact is that it chooses a tree uniformly over the set of all wired spanning trees. This is called informally a *uniform spanning tree* where this means “a spanning tree chosen uniformly among all spanning trees”.

Proposition 2. *If \mathcal{T} is a wired spanning tree of A , then the probability that \mathcal{T} is chosen in Wilson's algorithm is $4^{-\#(A)} F(A)$. In particular, every spanning tree is equally likely to be chosen and the total number of spanning trees is*

$$\frac{4^{\#(A)}}{F(A)} = 4^{\#(A)} \det[I - Q] = \det[4(I - Q)].$$

While the algorithm is due to Wilson, the formula for the number of spanning trees is much older, going back to Kirchhoff. In graph theory the Laplacian is written as $4(I - Q)$ which makes this formula nicer. To prove the proposition, one just takes any tree and considers the probability that it is chosen, see [11, Proposition 9.7.1].

2.5. Laplacian random walk

If $x, y \in \partial A$ are distinct, then the loop-erased walk gives a measure $\hat{q}_{A,x,y}$ on $\hat{\mathcal{E}}_A(x, y)$ of total mass $H_A(x, y)$. We can normalize to make this a probability measure. Under this measure we have a stochastic process X_j with $X_0 = x$ and $X_T = y$ where T is a random stopping time. It is clearly nonMarkovian since, for example, we cannot visit any site that we have already visited. To give the transition probabilities, we need to give for every $\eta = [\eta_0 = a, \dots, \eta_j]$ and $z \in A \setminus \eta$,

$$\mathbf{P}\{X_{j+1} = z \mid [X_0, \dots, X_j] = \eta\}.$$

Using the definition, it is easy to check that this is given by

$$\frac{1}{4} H_{A \setminus \eta}(z, y).$$

Note that the function $z \mapsto H_{A \setminus \eta}(z, y)$ is the unique function that is discrete harmonic (satisfies the discrete Laplace equation) on $A \setminus \eta$ with boundary value δ_y on $\partial[A \setminus \eta]$. For this reason the loop-erased walk is often called the *Laplacian random walk (with exponent 1)*.

Using the transition probability, one can check that the stochastic process X_n satisfies the following *domain Markov property*.

- Given $[X_0, \dots, X_j] = \eta$, the distribution of the remainder of the path is the same as the loop-erased walk from X_j to w in $A \setminus \eta$.

◆The term “loop-erased random walk” has two different, but similar, meanings. Sometimes it is used for the probability measure and sometimes for the measure \hat{q}_A or $\hat{q}_{A,x,y}$ which is not a probability measure. Of course we can write

$$\hat{q}_{A,x,y} = H_A(x, y) \hat{q}_{A,x,y}^\#$$

where $\hat{q}_{A,x,y}^\#$ is a probability measure. The Laplacian random walk is a description of the probability measure.

2.6. Boundary perturbation

The boundary perturbation rule is the way to compare \hat{q}_A and $\hat{q}_{A'}$ for different sets A, A' . It is easier to state this property using the unnormalized measure $\hat{q}_{A,x,y}$. It follows immediately from (7).

- Suppose $A' \subset A$ and x, y are distinct points in $(\partial A') \cap (\partial A)$ with $H_{A'}(x, y) > 0$. Then for every $\eta \in \mathcal{E}_{A'}(x, y)$,

$$\frac{\hat{q}_{A',x,y}(\eta)}{\hat{q}_{A,x,y}(\eta)} = \frac{\hat{q}_{A'}(\eta)}{\hat{q}_A(\eta)} = \exp \left\{ - \sum \overline{m}(\overline{\omega}) \right\},$$

where the sum is over all loops $\overline{\omega} \subset A$ that intersect both η and $A \setminus A'$. In other words, over loops in A that intersect η but are not loops in A' .

2.7. Loop soup

The measure \hat{q} on $\hat{\mathcal{E}}_A(x, y)$ is obtained from the measure q on $\mathcal{E}_A(x, y)$ by the deterministic operation of erasing loops. The total mass of the measure does not change. This leads to asking if we can obtain q from \hat{q} by “adding on loops”. Here we show how to do this. The operation is not deterministic (there is no way it could be since many different simple paths give the same loop erasure). We will describe it by introducing independent randomness, the random walk loop soup, and then giving a deterministic algorithm using the soup and the SAW η to produce the path ω .

The *random walk loop soup* is a Poissonian realization of unrooted loops from the measure \overline{m} . To be more precise, for each unrooted loop $\overline{\omega}$, there is a Poisson process $N_t^{\overline{\omega}}$ with rate $\overline{m}(\overline{\omega})$. These processes are independent. A realization of these processes gives for each $\overline{\omega}$ an increasing sequence of times

$$0 < t(1, \overline{\omega}) < t(2, \overline{\omega}) < \dots$$

where the corresponding Poisson process jumps. With probability one, the times $t(j, \overline{\omega})$ over all $j, \overline{\omega}$ are distinct.

If we restrict loops to a finite set A , then

$$\sum_{\overline{\omega} \subset A} \overline{m}(\overline{\omega}) = \log F(A) < \infty,$$

from which we can conclude

$$\{t(j, \omega) : t(j, \omega) \leq 1, \overline{\omega} \subset A\},$$

is finite. Writing this differently, we can write down the loops that have appeared by time one,

$$\overline{\omega}_1, \overline{\omega}_2, \dots, \overline{\omega}_k$$

where k is finite (random) and the loops have been written in the order that they appeared. Some loops may appear more than once in this listing. Indeed, if $t(j, \overline{\omega}) < 1 < t(j + 1, \overline{\omega})$, then the loop $\overline{\omega}$ appears j times. We can now give the “loop addition” algorithm.

- Choose $\eta = [\eta_0, \dots, \eta_n] \in \mathcal{E}_A(x, y)$ using measure \hat{q}_A .
- Take an independent loop soup in A and consider the loops that have arrived by time one.
- Take the subset of these which correspond to loops that have at least one point in common with η . Let us write these loops (using the same order as before)

$$\bar{\omega}_1, \dots, \bar{\omega}_l.$$

- For each $\bar{\omega}_j$ let i be the smallest integer such that $\eta_i \in \bar{\omega}_j$. Choose a rooted representative of ω_j that is rooted at η_i . If there is more than one choice at this stage choose uniformly among the choices. This gives a finite sequence of rooted loops

$$\omega_1, \dots, \omega_l$$

whose roots are $\eta_{\sigma(1)}, \dots, \eta_{\sigma(l)}$ with $\sigma(i) \leq \sigma(j)$ if $i < j$. Moreover,

$$\omega_j \cap \{\eta_0, \dots, \eta_{\sigma(j)-1}\} = \emptyset.$$

- Consider the loops

$$\omega_1^*, \dots, \omega_{n-1}^*$$

where ω_i^* is the concatenation of all the loops ω_j with $\sigma(j) = i$. If there are no such loops, choose ω_i^* to be the trivial loop of zero steps at η_i .

- Form a nearest neighbor path by

$$\omega = [\eta_0, \eta_1] \oplus \omega_1^* \oplus [\eta_1, \eta_2] \oplus \dots \oplus [\eta_{n-2}, \eta_{n-1}] \oplus \omega_{n-1}^* \oplus [\eta_{n-1}, \eta_n].$$

Proposition 3. *If η is chosen according to the measure \hat{q}_A on $\hat{\mathcal{E}}_A$ and ω is formed using the above algorithm, then the induced measure on paths is q_A .*

See [11, 9.4-9.5] for a proof.

3. Brownian loop measure

We will consider the scaling limit of the random walk loop measure described in the last section. For any positive integer N , let $\mathcal{Z}_N = N^{-1}\mathbb{Z}^2$. If ω is a loop, we let ω^N denote the corresponding loop on \mathcal{Z} obtained from scaling. If $|\omega| = n$ we can also view ω as a continuous function $\omega : [0, n] \rightarrow \mathbb{C}$ obtained in the natural way by linear interpolation. For ω^N , we use Brownian scaling,

$$\omega^N(t) = N^{-1}\omega(tN^2), \quad 0 \leq t \leq n/N^2.$$

We consider the limit as $N \rightarrow \infty$ of the rooted random walk loop measure. What happens is that small loops shrink to points; we will ignore these. However, at each scaling level N there are “macroscopic” loops. The Brownian loop measure will be the limit of these. We will define the Brownian loop measure directly, and then return to the question of how the scaled random walk loop measure approaches this limit.

3.1. Brownian measures

Probabilists tend to view Brownian motion as a stochastic process, that is, a probability measure on paths. However, it is useful sometimes to consider “configurational” measures given by Brownian paths. This will be useful when studying the Brownian loop measure, and also helps in understanding other configurational measures such as that given by *SLE*. We will do this for two-dimensional Brownian motion. We write $|\mu|$ for the total mass of a measure μ , and if $0 < |\mu| < \infty$, we write $\mu^\# = \mu/|\mu|$ for the probability measure obtained by normalization. We will write $\mu_D(z, w; t)$ for measures on paths in D , starting at z , ending at w , of time duration t . When the t is omitted, we allow different time durations, but we will always be considering paths of finite duration. When $D = \mathbb{C}$ we will drop the subscript.

Let $\mu(z, w; t)$ be the measure corresponding to Brownian paths starting at z , ending at w , of time duration t . We can write this as

$$\mu(z, w; t) = p_t(w - z) \mu^\#(z, w; t),$$

where

$$p_t(z) = \frac{1}{2\pi t} \exp\left\{-\frac{|z|^2}{2t}\right\},$$

is the density for Brownian motion and $\mu^\#(z, w; t)$ is the corresponding probability measure often called the *Brownian bridge*. We let

$$\mu(z, w) = \int_0^\infty \mu(z, w; t) dt.$$

This is an example of an infinite measure. All of the infinite measures we deal with here will be σ -finite. We could take other integrals. For example the measure

$$\mu(z, \cdot; t) = \int_{\mathbb{C}} \mu(z, w; t) dA(w)$$

is a probability measure, exactly the measure on paths given by a Brownian motion starting at z stopped at time t . Here and throughout dA will represent integrals with respect to usual Lebesgue measure (area) on \mathbb{C} .

◆ One might be worried about what it means to integrate when the integrands are measures. In all our examples, we could put topologies on measures so that the corresponding functions are continuous and these integrals are Riemann integrals. We will not bother with these tedious details.

We write $\mu_D(z, w; t)$, $\mu_D(z, w)$, etc. for the corresponding measures obtained by restricting the measure to curves staying in D . By definition our measures

will have the *restriction property*: if $D_1 \subset D$, then $\mu_{D_1}(\cdots)$ is $\mu_D(\cdots)$ restricted to curves lying in D_1 .

◆ We emphasize that we are *restricting* rather than *conditioning*; in other words, we are not normalizing to have total mass one. When we restrict a measure we decrease the total mass.

If D is a bounded domain, or, more generally, a domain with sufficiently large boundary that Brownian motion will eventually exit the domain, then if $z \neq w$ the measure $\mu_D(z, w)$ is a finite measure whose total mass is the *Green's function*

$$G(z, w) = \int_0^\infty |\mu_D(z, w; t)| dt < \infty.$$

The measure $\mu_D(z, z)$ is well defined but is an infinite measure.

3.2. Conformal invariance

Brownian motion is a conformally invariant object. We will make a precise statement of this in terms of the measures above. Suppose $D \subset \mathbb{C}$ is a domain and $f : D \rightarrow f(D)$ is a conformal transformation. If $\gamma : [0, t_\gamma] \rightarrow D$ is a curve, we define the curve $f \circ \gamma$ as follows. Let

$$\sigma(t) = \int_0^t |f'(\gamma(s))|^2 ds.$$

Then $f \circ \gamma$ is the curve of time duration $\sigma(t_\gamma)$ given by

$$(f \circ \gamma)(\sigma(s)) = f(\gamma(s)), \quad 0 \leq s \leq t_\gamma.$$

If μ is a measure on curves in D , we define $f \circ \mu$ to be the measure on curves in $f(D)$ defined by

$$(f \circ \mu)(V) = \mu\{\gamma : f \circ \gamma \in V\}.$$

Proposition 4 (Conformal invariance). *Suppose $z, w \in D$ and $f : D \rightarrow f(D)$ is a conformal transformation. Then*

$$f \circ \mu_D(z, w) = \mu_{f(D)}(f(z), f(w)). \quad (9)$$

While this is not the usual way that conformal invariance is stated it is a useful formulation when considering configurational measures. A consequence of the statement is that the total masses of the measures are the same. Indeed, it is well known that the Green's function is conformally invariant

$$G_{f(D)}(f(z), f(w)) = G_D(z, w),$$

In studying the Brownian loop measure, we will use (9) with $z = w$ where the measures are infinite:

$$f \circ \mu_D(z, z) = \mu_{f(D)}(f(z), f(z)).$$

◆ There are no topological assumptions on the domain D . In particular, it can be multiply connected.

We will consider the infinite measure on loops

$$\mu_D = \int_D \mu_D(z, z) dA(z).$$

The measure μ_D is not a conformal invariant. Indeed,

$$\begin{aligned} f \circ \mu_D &= \int_D f \circ \mu_D(z, z) dA(z) \\ &= \int_D \mu_{f(D)}(f(z), f(z)) dA(z) \\ &= \int_D \mu_{f(D)}(f(z), f(z)) |f'(z)|^{-2} |f'(z)|^2 dA(z) \\ &= \int_{f(D)} \mu_{f(D)}(w, w) |g'(w)|^2 dA(w), \end{aligned} \tag{10}$$

where $g = f^{-1}$. We will write $\bar{\mu}_D$ for the measure μ_D considered as a measure on *unrooted loops* by forgetting the root. The measure μ_D is the scaling limit of the macroscopic loops in $D \cap \mathcal{Z}_N$ under the measure q . The measure we want should be the scaling limit under m and this leads to the following definition.

Definition

- The *rooted Brownian loop measure* in D is the measure on (rooted) loops ν_D defined by

$$\frac{d\nu_D}{d\mu_D}(\gamma) = \frac{1}{t_\gamma}.$$

- The *(unrooted) Brownian loop measure* in D is the measure on unrooted loops defined by

$$\frac{d\bar{\nu}_D}{d\bar{\mu}_D}(\gamma) = \frac{1}{t_\gamma}.$$

Equivalently, it is the measure obtained from ν_D by forgetting the root.

The factor $1/t_\gamma$ is analogous to the factor $1/|\omega|$ in (6). The phrase Brownian loop measure refers to the unrooted version which is most important because it turns out to be the conformal invariant as the next theorem shows. However, the rooted version is often more convenient for computations.

Theorem 5. *If $f : D \rightarrow f(D)$ is a conformal transformation, then*

$$f \circ \bar{\nu}_D = \bar{\nu}_{f(D)}.$$

◆ The family of measures $\{\mu_D\}$ satisfies both the restriction property and conformal invariance. One can easily see that any nontrivial family that satisfies both of these properties must be a family of infinite measures.

Sketch of proof. If γ is a loop of time duration t_γ we can view γ as a curve $\gamma : (-\infty, \infty) \rightarrow \mathbb{C}$ of period t_γ . If $s \in \mathbb{R}$, we write $\theta_s \gamma$ for the loop $\theta_s \gamma(t) = \gamma(t+s)$. Note that γ and $\theta_s \gamma$ generate the same unrooted loop. Suppose that $\tilde{\nu}_D$ is a measure that is absolutely continuous with respect to μ_D with Radon-Nikodym derivative ϕ . Note that ν_D is an example with $\phi(\gamma) = 1/t_\gamma$. Suppose that for every loop γ of time duration t_γ ,

$$\int_0^{t_\gamma} \phi(\theta_s \gamma) ds = 1.$$

Then $\tilde{\nu}_D$ viewed as a measure on unrooted loops is the same as $\bar{\nu}_D$. Let $g = f^{-1}$ and choose

$$\phi(\theta_s \gamma) = \frac{|f'(\theta_s \gamma(0))|^2}{t_{f \circ \gamma}} = \frac{|f'(\gamma(s))|^2}{t_{f \circ \gamma}}.$$

By the definition of the parametrization of $f \circ \gamma$ we know that

$$\int_0^{t_\gamma} \phi(\theta_s \gamma) ds = \frac{1}{t_{f \circ \gamma}} \int_0^{t_\gamma} |f'(\gamma(s))|^2 ds = 1.$$

We will now give an important property of the measure μ_D . Consider the measure on curves obtained by selecting γ according to μ_D ; choosing a number $s \in [0, t_\gamma]$ uniformly; and then outputting $\theta_s \gamma$. Then the new measure is the same as μ_D . More generally, suppose that Ψ is a positive, continuous function on D . Consider the measure μ_Ψ defined by:

- For each γ , we have a measure ρ_γ on $\{\theta_s \gamma : 0 \leq s \leq t_\gamma\}$ with density $\Psi(\gamma(s))$. The total mass of this measure is

$$\int_0^{t_\gamma} \Psi(\gamma(s)) ds.$$

- First choose γ according to μ_D and then choose s according to ρ_γ and output $\theta_s \gamma$.

Then μ_Ψ is the same as

$$\int_D \Psi(z) \mu_D(z, z) dA(z).$$

In other words

$$\frac{d\mu_\Psi}{d\mu_D}(\gamma) = \Psi(\gamma(0)).$$

Computing as in (10), we see that

$$f \circ \mu_\Psi = \int_{f(D)} \Psi(g(w)) |g'(w)|^2 \mu_D(z, z) dA(z).$$

In particular, if $\Psi(z) = |f'(z)|^2$,

$$f \circ \mu_\Psi = \mu_{f(D)}.$$

Returning to ϕ as above, we see that if ν_ϕ is the measure on rooted loops with $d\nu_\phi/d\mu_D = \phi$,

$$f \circ \nu_\phi = \nu_{f(D)}.$$

Since ν_ϕ considered as a measure on unrooted loops is the same as $\bar{\nu}_D$, we get

$$f \circ \bar{\nu}_D = \bar{\nu}_{f(D)}.$$

□

3.3. Brownian loop soup

The *Brownian loop soup* is a Poissonian realization from the measure $\bar{\nu}_D$. Alternatively, we can take a rooted Brownian loop soup, that is, a realization of ν_D , and then forget the roots of the loops. We can think of the soup as growing a set of loops where \mathcal{A}_t denotes the set of loops that have been created at time t . By conformal invariance, the conformal image of a soup in D gives a soup in $f(D)$. Soups also satisfy the restriction property.

The Brownian loop soup is the limit of the macroscopic loops from the random walk loop soup. This can be made precise. We refer to [14] for the precise result, but we give a rough version here. Let D be a bounded domain. There exists $\delta > 0$ such that, except for an event of probability $O(N^{-\delta})$, a Brownian soup \mathcal{A}_t in D and a (scaled) random walk soup $\tilde{\mathcal{A}}_t$ in $D \cap \mathcal{Z}_N$ can be defined on the same probability space such that for $0 \leq t \leq 1$, there is a one-to-one correspondence between the loops in \mathcal{A}_t and those in $\tilde{\mathcal{A}}_t$ of time duration at least $N^{-\delta}$ such that if two loops are paired up, their time duration differs by at most N^{-2} and the parametrized curves lie within distance $O(N^{-1} \log N)$ of each other.

3.4. Excursion measure and boundary bubbles

Suppose D is a domain with piecewise smooth boundary. The *Poisson kernel* $H_D(z, w)$, $z \in D$, $w \in \partial D$ is the density of harmonic measure starting at z . In

other words, the probability that a Brownian motion starting at z exists D at $V \subset \partial D$ is

$$\text{hm}_D(z, V) := \int_V H_D(z, w) |dw|.$$

We can write the probability measure associated to Brownian motion starting at z stopped when it reaches ∂D as

$$\int_V \mu_D(z, w) |dw|$$

where $\mu_D(z, w)$ is a finite measure on paths starting at z ending at w . Indeed,

$$\mu_D(z, w) = H_D(z, w) \mu_D^\#(z, w),$$

where the probability measure $\mu_D^\#(z, w)$ corresponds to Brownian motion started at z conditioned to leave D at w . This is conditioning on an event of probability zero but one can make rigorous sense of this using a number of methods, e.g., the theory of h -processes.

◆We will use $\mu_D(z, w)$ for various Brownian measures. Before we used it with $z, w \in D$ and here we use it with $z \in D, w \in \partial D$. Below we will define a version with $z, w \in \partial D$. I hope that using the same notation will not be confusing.

Conformal invariance of Brownian motion implies that harmonic measure is conformally invariant:

$$\text{hm}_D(z, V) = \text{hm}_{f(D)}(f(z), f(V)).$$

If ∂D and $\partial f(D)$ are smooth near $w, f(w)$, this gives a conformal covariance rule for $\mu_D(z, w)$

$$f \circ \mu_D(z, w) = |f'(w)| \mu_{f(D)}(f(z), f(w)).$$

This is shorthand for the scaling rule on H :

$$H_D(z, w) = |f'(w)| H_{f(D)}(f(z), f(w)),$$

and the conformal invariance of the probability measures:

$$f \circ \mu_D^\#(z, w) = \mu_{f(D)}^\#(f(z), f(w)).$$

The scaling limit of the random walk excursion measure from Section 2.1 is the (*Brownian motion*) *excursion measure* that we now describe. Suppose ∂D

is smooth. If z, w are distinct points in ∂D , let $H_{\partial D}(z, w)$ be the *boundary or excursion Poisson kernel* defined by

$$H_{\partial D}(z, w) = \partial_{\mathbf{n}} H_D(z, w),$$

where the derivative is in the first component and $\mathbf{n} = \mathbf{n}_z$ is the inward unit normal at z . We get the same number if we take $\partial_{\mathbf{n}_w} H_D(w, z)$. There is a corresponding measure on paths that we denote by

$$\mu_D(z, w) = H_{\partial D}(z, w) \mu_D^\#(z, w).$$

Note that $\mu_D(z, w)$ satisfies the conformal covariance relation

$$f \circ \mu_D(z, w) = |f'(z)| |f'(w)| \mu_{f(D)}(f(z), f(w)). \quad (11)$$

A simple calculation shows that

$$H_{\partial \mathbb{H}}(0, x) = \frac{1}{\pi x^2}.$$

(Sometimes it is useful to multiply the Poisson kernel by π in order to avoid having a constant in this formula.)

◆ Combining these ideas we can say that the Brownian measures have a boundary scaling exponent of 1 and an interior scaling exponent of 0.

Excursion measure is the infinite measure on paths connecting boundary points of D given by

$$\mathcal{E}_D = \int_{\partial D} \int_{\partial D} \mu_D(z, w) |dz| |dw|.$$

Using (11) we can check that the excursion measure is conformally *invariant*:

$$f \circ \mathcal{E}_D = \mathcal{E}_{f(D)}.$$

In particular, it is well defined even if the boundaries are not smooth provided that we can map D conformally to a domain with piecewise smooth boundaries. (This is always possible for finitely connected domains.) The term excursion measure is sometimes used for the measure on $\partial D \times \partial D$ given by

$$\mathcal{E}_D(V_1, V_2) = \left| \int_{V_1} \int_{V_2} \mu_D(z, w) |dz| |dw| \right| = \int_{V_1} \int_{V_2} H_{\partial D}(z, w) |dz| |dw|.$$

This latter view of the excursion measure is a nice conformal invariant and can be used in the study of conformal mappings. It is related to, but not the same as, extremal distance or extremal length which is often used for such purposes.

As $w \rightarrow z$, the measures $\mu_D(z, w)$ approach an infinite measure called the *Brownian bubble measure*. There are a number of ways of writing this measure. For ease, assume that $D = \mathbb{H}$ and $z = 0$. Then

$$\mu_{\mathbb{H}}(0, 0) = \lim_{x \downarrow 0} \pi \mu_{\mathbb{H}}(0, x) = \lim_{\epsilon \downarrow 0} \pi \mu_{\mathbb{H}}(\epsilon i, 0),$$

provided that one interprets these limits correctly. For example, if $r > 0$ and $\mu_{\mathbb{H}}(0, 0; r), \mu_{\mathbb{H}}(0, x; r)$ denote these measures restricted to curves that do not lie in the disk of radius r about the origin, then these are finite measures and

$$\mu_{\mathbb{H}}(0, 0; r) = \lim_{x \downarrow 0} \pi \mu_{\mathbb{H}}(0, x; r).$$

(Convergence of finite measures means convergence of the total masses and convergence of the probability measures in some appropriate topology.) If $D \subset \mathbb{H}$ with $\text{dist}(0, \mathbb{H} \setminus D) > 0$, let $\Gamma(D)$ denote the $\mu_{\mathbb{H}}(0, 0)$ measure of loops that do not stay in $D \cap \{0\}$. The factor of π is put in the definition so that

$$\Gamma(\mathbb{D}_+) = 1, \quad \mathbb{D}_+ = \mathbb{H} \cap \mathbb{D}.$$

If D is simply connected and $f : D \rightarrow \mathbb{H}$ is a conformal transformation with $f(0) = 0$, then [7, Proposition 5.22]

$$\Gamma(D) = -\frac{Sf(0)}{6},$$

where S denotes the Schwarzian derivative

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3f''(z)^2}{2f'(z)^2}.$$

We know of no such nice formulas for multiply connected D . The Brownian bubble measure satisfies the conformal covariance rule

$$f \circ \mu_D(z, z) = |f'(z)|^2 \mu_{f(D)}(f(z), f(z)).$$

We will state one more result which will be important in the analysis of *SLE*. Suppose K is a bounded, relatively closed subset of \mathbb{H} . Then the half-plane capacity of K is defined by

$$\text{hcap}(K) = \lim_{y \rightarrow \infty} y \mathbf{E}^{iy}[\text{Im}(B_\tau)].$$

Here B is a standard complex Brownian motion, τ is the first time that that it hits $\mathbb{R} \cup K$, and \mathbf{E}^{iy} denotes expectations assuming that $B_0 = iy$. One can check that the limit exists. In fact

$$\text{hcap}(K) = \Gamma(D)$$

where D is the image of $\mathbb{H} \setminus K$ under the inversion $z \mapsto -1/z$.

Proposition 6. *Suppose $\gamma : (0, \infty) \rightarrow \overline{\mathbb{H}}$ is a curve with $\gamma(0) = 0$ and such that for each t ,*

$$\text{hcap}[\gamma(0, t)] = t.$$

Suppose $D \subset \mathbb{H}$ is a domain with $\text{dist}(0, \mathbb{H} \setminus D) > 0$. Let $m(t)$ denote the Brownian loop measure of loops in \mathbb{H} that intersect both $\gamma(0, t]$ and $\mathbb{H} \setminus D$. Then as $t \downarrow 0$,

$$m(t) = \Gamma(D) t [1 + o(1)]. \quad (12)$$

4. Complex variables and conformal mappings

In order to study *SLE*, one must know some basic facts about conformal maps. Some of this material appears in standard first courses in complex variables and a few are more specialized. We will discuss the main results here. For proofs and more details see, e.g., [5, 6, 7].

4.1. Review of complex analysis

Definition

- A *domain* in \mathbb{C} is an open, connected subset. Two main examples are the *unit disk*

$$\mathbb{D} = \{z : |z| < 1\}$$

and the *upper half plane*

$$\mathbb{H} = \{z = x + iy : y > 0\}.$$

- The *Riemann sphere* is the set $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ with the topology of the sphere which is to say that the open neighborhoods of ∞ are the complements of compact subsets of \mathbb{C} .
- A domain $D \subset \mathbb{C}$ is *simply connected* if its complement in \mathbb{C}^* is a connected subset of \mathbb{C}^* .
- A domain $D \subset \mathbb{C}$ is *finitely connected* if its complement in \mathbb{C}^* has a finite number of connected components.
- A function $f : D \rightarrow \mathbb{H}$ is *holomorphic* or *analytic* if the complex derivative $f'(z)$ exists at every point.

If $0 \in D$, and f is holomorphic on D then we can write

$$f(z) = \sum_{j=0}^{\infty} a_j z^j,$$

where the radius of convergence is at least $\text{dist}(0, \partial D)$. In particular, if $f(0) = 0$, then either f is identically zero, or there exists a nonnegative integer n such that

$$f(x) = z^n g(z),$$

where g is holomorphic in a neighborhood of 0 with $g(0) \neq 0$. In particular, if $f'(0) \neq 0$, then f is locally one-to-one, but if $f'(0) = 0$, it is not locally one-to-one.

If we write a holomorphic function $f = u + iv$, then the functions u, v are harmonic functions and satisfy the Cauchy-Riemann equations

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v.$$

Conversely, if u is a harmonic function on D and $z \in D$, then we can find a unique holomorphic function $f = u + iv$ with $v(z) = 0$ defined in a neighborhood of z by solving the Cauchy-Riemann equations. It is not always true that f can be extended to all of D , but there is no problem if D is simply connected.

Proposition 7. *Suppose $D \subset \mathbb{C}$ is simply connected.*

- *If u is a harmonic function on D and $z \in D$, there is a unique holomorphic function $f = u + iv$ on D with $v(z) = 0$.*
- *If f is a holomorphic function on D with $f(z) \neq 0$ for all z , then there exists a holomorphic function g on D with $e^g = f$. In particular, if $w \in \mathbb{C} \setminus \{0\}$ and $h = e^{g/w}$, then $h^w = f$.*

The *Cauchy integral formula* states that if f is holomorphic in a domain containing the closed unit disk, then

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(z) dz}{z^{n+1}}.$$

In particular,

$$|f^{(n)}(0)| \leq n! \|f\|_\infty.$$

Here $\|f\|_\infty$ denotes the maximum of $|f|$ on $\overline{\mathbb{D}}$ which (by the $n = 0$ case which is called the *maximum principle*) is the same as the maximum on $\partial\mathbb{D}$.

Proposition 8. *Suppose f is a holomorphic function on a domain D . Suppose $z \in D$, and let*

$$d_z = \text{dist}(z, \partial D), \quad M_z = \sup\{|f(w)| : |w - z| < d_z\}.$$

Then,

$$|f^{(n)}(z)| \leq n! d_z^{-n} M_z.$$

Proof. Consider $g(w) = f(z + d_z w)$. □

A similar estimate exists for harmonic functions in \mathbb{R}^d . It can be proved by representing a harmonic function in the unit ball in terms of the Poisson kernel.

Proposition 9. *For all positive integers d, n , there exists $C(d, n) < \infty$ such that if u is a harmonic function on the unit ball $U = \{x \in \mathbb{R}^d : |x| < 1\}$ and D denotes an n th order mixed partial,*

$$|Du(0)| \leq C(d, n) \|u\|_\infty.$$

Derivative estimates allow one to show “equicontinuity” results. We write $f_n \implies f$ if for every compact $K \subset D$, f_n converges to f uniformly on K .

We state the following for holomorphic functions, but a similar result holds for harmonic functions.

Proposition 10. *Suppose D is a domain.*

- If f_n is a sequence of holomorphic functions on D , and $f_n \implies f$, then f is holomorphic.
- If f_n is a sequence of holomorphic functions on D that is locally bounded, then there exists a subsequence f_{n_j} and a (necessarily, holomorphic) function f such that $f_{n_j} \implies f$.

Proposition 11 (Schwarz lemma). *If $f : \mathbb{D} \rightarrow \mathbb{D}$ is holomorphic with $f(0) = 0$, then $|f(z)| \leq |z|$ for all z . If f is not a rotation, $|f'(0)| < 1$ and $|f(z)| < |z|$ for all $z \neq 0$.*

Proof. Let $g(z) = f(z)/z$ with $g(0) = f'(0)$ and use the maximum principle. \square

4.2. Conformal transformations

Definition

- A holomorphic function $f : D \rightarrow D_1$ is called a *conformal transformation* if it is one-to-one and onto.
- Two domains D, D_1 are *conformally equivalent* if there exists a conformal transformation $f : D \rightarrow D_1$.

It is easy to verify that “conformally equivalent” defines an equivalence relation. A necessary, but not sufficient, condition for a holomorphic function f to be a conformal transformation onto $f(D)$ is $f'(z) \neq 0$ for all z . Functions that satisfy this latter condition can be called *locally conformal*. The function $f(z) = e^z$ is a locally conformal transformation on \mathbb{C} that is not a conformal transformation. Proving global injectiveness can be tricky, but the following lemma gives a very useful criterion.

Proposition 12 (Hurwitz). *Suppose f_n is a sequence of one-to-one holomorphic functions on a domain D and $f_n \implies f$. Then either f is constant or f is one-to-one.*

This is the big theorem.

Theorem 13 (Riemann mapping theorem). *If $D \subset \mathbb{C}$ is a proper, simply connected domain, and $z \in D$, there exists a unique conformal transformation*

$$f : \mathbb{D} \rightarrow D$$

with $f(0) = z, f'(0) > 0$.

Proof. The hard part is existence. We will not discuss the details, but just list the major steps. The necessary ingredients to fill in the details are Propositions 7, 8, 10, 11, and 12. Consider the set \mathcal{A} of conformal transformations $g : D \rightarrow g(D)$ with $g(z) = 0, g'(z) > 0, g(D) \subset \mathbb{D}$. Then one shows:

- \mathcal{A} is nonempty.
- $M := \sup\{g'(z) : g \in \mathcal{A}\} < \infty$.
- There exists $\hat{g} \in \mathcal{A}$ with $\hat{g}'(z) = M$.
- $\hat{g}(D) = \mathbb{D}$

□

If D is a simply connected domain, then to specify the conformal transformation $f : \mathbb{D} \rightarrow D$ one needs to specify two quantities: $z = f(0)$ and the argument of $f'(0)$. We can think of this as “three real degrees of freedom”. Similarly, to specify the map it suffices to specify where three boundary points are sent.

The Riemann mapping theorem does not say anything about the limiting behavior of $f(z)$ as $|z| \rightarrow 1$. One needs to make more assumptions in order to obtain further results.

Proposition 14. *Suppose D is a simply connected domain and $f : \mathbb{D} \rightarrow D$ is a conformal transformation.*

- If $\mathbb{C} \setminus D$ is locally connected, then f extends to a continuous function on $\overline{\mathbb{D}}$.
- If ∂D is a Jordan curve (that is, homeomorphic to the unit circle), then f extends to a homeomorphism of $\overline{\mathbb{D}}$ onto \overline{D} .

4.3. Univalent functions

Definition

- A function f is *univalent* if f is holomorphic and one-to-one.
- A univalent function f on \mathbb{D} with $f(0) = 0, f'(0) = 1$ is called a *schlicht* function. Let \mathcal{S} denote the set of schlicht functions.

The Riemann mapping theorem implies that there is a one-to-one correspondence between proper simply connected domains D containing the origin and $(0, \infty) \times \mathcal{S}$. Any $f \in \mathcal{S}$ has a power series expansion at the origin

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m.$$

Much of the work of classical function theory of the twentieth century was focused on estimating the coefficients a_n of the schlicht functions. Three early results are:

- [Bieberbach] $|a_2| \leq 2$
- [Loewner] $|a_3| \leq 3$
- [Littlewood] For all $n, |a_n| < en$.

Bieberbach’s conjecture was that the coefficients were maximized when the simply connected domain D was as large as possible under the constraint $f'(0) = 1$. A good guess would be that a maximizing domain would be of the form

$$D = \mathbb{C} \setminus (-\infty, -x]$$

for some $x > 0$, where x is determined by the condition $f'(0) = 1$. It is not hard to show that the *Koebe function*

$$f_{\text{Koebe}}(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n,$$

maps \mathbb{D} conformally onto $\mathbb{C} \setminus (-\infty, -1/4]$. This led to Bieberbach's conjecture which was proved by de Branges.

Theorem 15 (de Branges). *If $f \in \mathcal{S}$, then $|a_n| \leq n$ for all n .*

To study *SLE*, it is not necessary to use as powerful a tool as de Branges' theorem. Indeed, the estimates of Bieberbach and Littlewood above suffice for most problems. A couple of other simpler results are very important for analysis of conformal maps. To motivate the first, suppose that $f : \mathbb{D} \rightarrow f(D)$ is a conformal transformation with $f(0) = 0$ and $z \in \mathbb{C} \setminus f(D)$ with $|z| = \text{dist}(0, \partial f(D))$. If one wanted to maximize $f'(0)$ under these constraints, then it seems that the best choice for $f(D)$ would be the complex plane with a radial line to infinity starting at z removed. This indeed is the case which shows that the Koebe function is a maximizer.

Theorem 16 (Koebe-1/4). *If $f \in \mathcal{S}$, then $f(\mathbb{D})$ contains the open disk of radius 1/4 about the origin.*

Uniform bounds on the coefficients a_n give uniform bounds on the rate of change of $|f'(z)|$. The optimal bounds are given in the next theorem.

Theorem 17 (Distortion). *If $f \in \mathcal{S}$ and $|z| = r < 1$,*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}.$$

The distortion theorem can be used to analyze conformal transformations of multiply connected domains since such domains are "locally simply connected". Suppose D is a domain, and $f : D \rightarrow f(D)$ is a conformal transformation. Let us write $z \sim w$ if

$$|z - w| < \frac{1}{2} \min \{ \text{dist}(z, \partial D), \text{dist}(w, \partial D) \}.$$

The distortion theorem implies that if $z \sim w$, then

$$\frac{|f'(z)|}{12} \leq |f'(w)| \leq 12 |f'(z)|.$$

The adjacency relation induces a metric ρ on D by $\rho(z, w) = n$ where n is the minimal length of a sequence

$$z = z_0, z_1, \dots, z_n = w,$$

with $z_j \sim z_{j-1}$, $j = 1, \dots, n$. We then get the inequality

$$12^{-\rho(z,w)} |f'(z)| \leq |f'(w)| \leq 12^{\rho(z,w)} |f'(z)|.$$

For simply connected domains, one can get better estimates than this using the distortion theorem directly. However, this kind of estimate applies to multiply connected domains (and also to estimates for positive harmonic functions in \mathbb{R}^d).

4.4. Harmonic measure and the Beurling estimate

Harmonic measure is the hitting measure by Brownian motion. (This is not how it is defined in the complex variables literature, but for probabilists it is the most direct definition.) If D is a domain, and B_t is a (standard) complex Brownian motion, let

$$\tau_D = \inf\{t \geq 0 : B_t \notin D\}.$$

Definition

- The harmonic measure (in D from $z \in D$) is the probability measure supported on ∂D given by

$$\text{hm}_D(z, V) = \mathbf{P}^z \{B_{\tau_D} \in V\}.$$

- The Poisson kernel, if it exists, is the function $H_D : D \times \partial D \rightarrow [0, \infty)$ such that

$$\text{hm}_D(z, V) = \int_V H_D(z, w) |dw|.$$

Conformal invariance of Brownian motion implies conformal invariance of harmonic measure and conformal covariance of the Poisson kernel. To be more specific, if $f : D \rightarrow f(D)$ is a conformal transformation,

$$\text{hm}_D(z, V) = \text{hm}_{f(D)}(f(z), f(V)),$$

$$H_D(z, w) = |f'(w)| H_{f(D)}(f(z), f(w)).$$

The latter equality requires some smoothness assumptions on the boundary; we will only need to use it when ∂D is analytic in a neighborhood of w , and hence (by Schwarz reflection) the map f can be analytically continued in a neighborhood of w . Two important examples are

$$H_{\mathbb{D}}(z, w) = \frac{1}{2\pi} \frac{1 - |z|^2}{|z - w|^2}, \quad H_{\mathbb{H}}(x + iy, \tilde{x}) = \frac{1}{\pi} \frac{y}{(x - \tilde{x})^2 + y^2}.$$

The Poisson kernel for any simply connected domain can be determined by conformal transformation and the scaling rule above. Finding explicit formulas for multiply connected domains can be difficult. Using the strong Markov property, we can see that if $D_1 \subset D$ and $w \in \partial D \cap \partial D_1$,

$$H_D(z, w) = H_{D_1}(z, w) + \int_{\partial D_1 \setminus \partial D} H_{D_1}(z, z_1) H_D(z_1, w) |dz_1|.$$

Example Let $D = \{z \in \mathbb{H} : |z| > 1\}$. Then

$$f(z) = z + \frac{1}{z}, \quad f'(z) = 1 - \frac{1}{z^2} = \frac{1}{z} \left[z - \frac{1}{z} \right],$$

is a conformal transformation of D onto \mathbb{H} . Therefore,

$$H_D(z, e^{i\theta}) = |f'(e^{i\theta})| H_{\mathbb{H}} \left(z + \frac{1}{z}, f(e^{i\theta}) \right) = 2 [\sin \theta] H_{\mathbb{H}} \left(z + \frac{1}{z}, f(e^{i\theta}) \right)$$

If we write $z = re^{i\psi}$, then we can see that for $r \geq 2$,

$$H_D(re^{i\psi}, e^{i\theta}) = \frac{2 \sin \psi \sin \theta}{\pi r} [1 + O(r^{-1})].$$

Theorem 18 (Beurling projection theorem). *Suppose $D \subset \mathbb{D}$ is a domain containing the origin and let*

$$V = \{0 < r < 1 : re^{i\theta} \notin D \text{ for some } 0 \leq \theta < 2\pi\}.$$

Then,

$$\text{hm}_D(0, \partial\mathbb{D}) \leq \text{hm}_{\mathbb{D} \setminus V}(0, \partial\mathbb{D}).$$

This theorem is particularly important when $\mathbb{D} \setminus D$ is a connected set connected to $\partial\mathbb{D}$. By conformal invariance, one can show that

$$\text{hm}_{\mathbb{D} \setminus [r, 1)}(0, \partial D) \asymp r^{-1/2}.$$

This leads to the following corollary.

Proposition 19 (Beurling estimate). *There exists $c < \infty$ such that if D is a simply connected domain containing the origin, B_t is a standard Brownian motion starting at the origin, and $r = \text{dist}(0, \partial D)$, then*

$$\mathbf{P} \{B[0, \tau_D] \not\subset \mathbb{D}\} \leq cr^{1/2}.$$

In particular,

$$\text{hm}_D(0, \partial D \setminus \mathbb{D}) \leq cr^{1/2}.$$

4.5. Multiply connected domains

Since conformal transformations are also topological homeomorphisms, topological properties must be preserved. In particular, multiply connected domains are not conformally equivalent to simply connected domains. In fact, for multiply connected domains, topological equivalence is not sufficient for conformal equivalence. When considering domains, isolated points in the complement are not interesting because they can be added to the domain. Let \mathcal{R} denote the set of domains that are proper subsets of the Riemann sphere and whose boundary contains no isolated points. (In particular, \mathbb{C} is not in \mathcal{R} because its complement is an isolated point in the sphere; if we add this point to the domain, then the domain is no longer a proper subset.)

Definition Let \mathcal{R}_k denote the set of k -connected domains of the Riemann sphere whose complement consists of $k + 1$ connected components.

The Riemann mapping theorem states that all domains in \mathcal{R}_0 are conformally equivalent. Domains in \mathcal{R}_1 are called *conformal annuli*. One example of such a domain is

$$A_r = \{z \in \mathbb{C} : r < |z| < 1\}, \quad 0 < r < 1.$$

The next theorem states that there is a one-parameter family of equivalence classes of 1-connected domains.

Theorem 20. *If $0 < r_1 < r_2 < 1$, then A_{r_1} and A_{r_2} are not conformally equivalent. If $D \in \mathcal{R}_1$, then there exists a (necessarily unique) r such that D is conformally equivalent to A_r .*

The next theorem shows that the equivalence classes for \mathcal{D}_k are parametrized by $3k - 2$ variables. Let \mathcal{R}_k^* denote the set of domains D in \mathcal{R}_k of the form

$$D = \mathbb{H} \setminus \bigcup_{j=1}^n I_j, \quad I_j = [x_j^- + iy_j, x_j^+ + iy_j].$$

Here $x_j^-, x_j^+ \in \mathbb{R}, y_j > 0$, and we assume that the I_j are disjoint. The set \mathcal{R}_k^* is parametrized by $3k$ variables. However, if $D \in \mathcal{R}_k^*, x \in \mathbb{R}$ and $r > 0$, then $x + D$ and rD are clearly conformally equivalent to D .

Theorem 21. *If $D_1, D_2 \in \mathcal{R}_k^*$ and $f : D_1 \rightarrow D_2$ is a conformal transformation with $f(\infty) = \infty$, then $f(z) = rz + x$ for some $r > 0, x \in \mathbb{R}$. Moreover, every k -connected domain is conformally equivalent to a domain in \mathcal{R}_k^* .*

5. The Loewner differential equation

5.1. Half-plane capacity

Definition

- Let \mathcal{D} denote the set of simply connected subdomains D of \mathbb{H} such that $K = \mathbb{H} \setminus D$ is bounded.
- We call $K = \mathbb{H} \setminus D$ a (compact) \mathbb{H} -hull
- Let \mathcal{D}_0 denote the set of $D \in \mathcal{D}$ with $\text{dist}(0, K) > 0$.

If $D \in \mathcal{D}$, let g_D denote a conformal transformation of D onto \mathbb{H} . Such a transformation is not unique; indeed, if f is a conformal transformation of \mathbb{H} onto \mathbb{H} , then $f \circ g_D$ is also a transformation. In order to specify g_D uniquely we specify the following condition:

$$\lim_{z \rightarrow \infty} [g_D(z) - z] = 0.$$

If we do this, then we can expand g_D at infinity as

$$g_D(z) = z + \frac{a(D)}{z} + O(|z|^{-2}). \tag{13}$$

Definition If K is a \mathbb{H} -hull, the half-plane capacity of K , denoted $\text{hcap}(K)$, is defined by

$$\text{hcap}(K) = a(D),$$

where $a(D)$ is the coefficient in (13).

Examples

- $K = \{z \in \mathbb{H} : |z| \leq 1\}$, $g_D(z) = z + \frac{1}{z}$, $\text{hcap}(K) = 1$.
- $K = (0, i]$, $g_D(z) = \sqrt{z^2 + 1} = z + \frac{1}{2z} + \dots$, $\text{hcap}(K) = \frac{1}{2}$.

Half-plane capacity satisfies the scaling rule

$$\text{hcap}(rK) = r^2 \text{hcap}(K).$$

The next proposition gives a characterization of hcap in terms of Brownian motion killed when it leaves D .

Proposition 22. *If K is a \mathbb{H} -hull then*

$$\text{Im}g_D(z) = \text{Im}(z) - \mathbf{E}^z[\text{Im}(B_{\tau_D})],$$

$$\text{hcap}(K) = \lim_{y \rightarrow \infty} y \mathbf{E}^{iy}[\text{Im}(B_{\tau_D})]. \quad (14)$$

Here B_t is a complex Brownian motion and

$$\tau_D = \inf\{t : B_t \notin D\}.$$

◆ We could use (14) as the definition of $\text{hcap}(K)$. This requires K to be bounded, but it is not necessary for $\mathbb{H} \setminus K$ to be simply connected. One can use this formulation to show that this definition of hcap is the same as that given in Section 3.4.

Proof. Write $g_D = u + iv$. Then v is a positive harmonic function on D that vanishes on ∂D and satisfies

$$v(x + iy) = y - a(D) \frac{y}{|z|^2} + O(|z|^{-2}), \quad z \rightarrow \infty. \quad (15)$$

In particular, $\text{Im}(z) - v(z)$ is a bounded harmonic function on D , and the optional sampling theorem implies that

$$\text{Im}(z) - v(z) = \mathbf{E}^z[\text{Im}(B_{\tau_D}) - v(B_{\tau_D})] = \mathbf{E}^z[\text{Im}(B_{\tau_D})].$$

This gives the first equality, and the second follows from (15). \square

Definition The *radius (with respect to zero)* of a set is

$$\text{rad}(K) = \sup\{|w| : w \in K\}.$$

More generally, $\text{rad}(K, z) = \sup\{|w - z| : z \in K\}$.

Proposition 23. *Suppose K is an \mathbb{H} -hull and $|z| \geq 2\text{rad}(K)$. Then*

$$\mathbf{E}^z [\text{Im}(B_{\tau_D})] = \text{hcap}(K) [\pi H_{\mathbb{H}}(z, 0)] \left[1 + O\left(\frac{\text{rad}(K)}{|z|}\right) \right].$$

Sketch. By scaling, we may assume that $\text{rad}(K) = 1$. Let $\tilde{D} = \mathbb{H} \setminus \overline{\mathbb{D}}_+$, $\xi = \tau_{\tilde{D}} = \inf\{t : B_t \in \mathbb{R} \text{ or } |B_t| = 1\}$. Note that $\xi \leq \tau_D$; indeed, any path from z that exits D at K must first visit a point in $\partial\mathbb{D}_+$. By the strong Markov property,

$$\mathbf{E}^z [\text{Im}(B_{\tau_D})] = \int_0^\pi H_{\tilde{D}}(z, e^{i\theta}) \mathbf{E}^{e^{i\theta}} [\text{Im}(B_{\tau_D})] d\theta.$$

The Poisson kernel $H_{\tilde{D}}(z, e^{i\theta})$ can be computed exactly by finding an appropriate conformal map. For our purposes we need only the estimate

$$H_{\tilde{D}}(z, e^{i\theta}) = 2 H_{\mathbb{H}}(z, 0) \sin \theta [1 + O(|z|^{-1})].$$

Therefore,

$$\frac{H_{\tilde{D}}(z, e^{i\theta})}{\pi H_{\mathbb{H}}(z, 0)} = [1 + O(|z|^{-1})] \int_0^\pi \mathbf{E}^{e^{i\theta}} [\text{Im}(B_{\tau_D})] \frac{2}{\pi} \sin \theta d\theta.$$

Using the Poisson kernel $H_{\tilde{D}}(iy, e^{i\theta})$, we see that

$$\text{hcap}(K) = \lim_{y \rightarrow \infty} y \mathbf{E}^{iy} [\text{Im}(B_\tau)] = \int_0^\pi \mathbf{E}^{e^{i\theta}} [\text{Im}(B_{\tau_D})] \frac{2}{\pi} \sin \theta d\theta.$$

□

5.2. Loewner differential equation in \mathbb{H}

Definition A *curve* is a continuous function of time. It is *simple* if it is one-to-one.

Suppose $\gamma : (0, \infty) \rightarrow \mathbb{H}$ is a simple curve with $\gamma(0+) = 0$. Let

$$K_t = \gamma(0, t], \quad D_t = \mathbb{H} \setminus K_t, \quad g_t = g_{D_t}, \quad a(t) = \text{hcap}(K_t).$$

It is not hard to show that $t \mapsto a(t)$ is a continuous, strictly increasing, function. We will also assume that $a(\infty) = \infty$.

Theorem 24 (Loewner differential equation). *Suppose γ is a simple curve as above such that a is continuous differentiable. Then for $z \in \mathbb{H}$, $g_t(z)$ satisfies the differential equation*

$$\partial_t g_t(z) = \frac{\dot{a}(t)}{g_t(z) - U_t}, \quad g_0(z) = z,$$

where $U_t = g_t(\gamma(t)) = \lim_{w \rightarrow \gamma(t)} g_t(w)$. Moreover, the function $t \mapsto U_t$ is continuous. If $z \notin \gamma(0, \infty)$, then the equation is valid for all t . If $z = \gamma(s)$, then the equation is valid for $t < s$.

Although we do not give the details, let us show where the equation arises by computing the right-derivative at time $t = 0$. Let $r_t = \text{rad}(K_t)$. If we write $g_t = u_t + iv_t$, then Proposition (23) implies

$$v_t(z) - \text{Im}(z) = -\mathbf{E}^z[\text{Im}(B_{\tau_{D_t}})] = -a(t) [\pi H_{\mathbb{H}}(z, 0)] [1 + O(r_t/|z|)].$$

Note that

$$\text{Im}[1/z] = -\pi H_{\mathbb{H}}(z, 0).$$

Hence (with a little care on the real part) we can show that

$$g_t(z) - z = a(t) [1/z] [1 + O(r_t/|z|)],$$

which implies

$$\lim_{t \downarrow 0} \frac{g_t(z) - z}{t} = \frac{\dot{a}(t)}{z}.$$

We see that it is convenient to parametrize the curve γ so that $a(t)$ is differentiable, and, in fact, and if we are going to go through the effort, we might as well make $a(t)$ linear.

Definition The curve γ is *parametrized by (half-plane) capacity with rate $a > 0$* if $a(t) = at$.

The usual choice is $a = 2$. In this case, if the curve is parametrized by capacity, then the Loewner equation becomes

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}. \quad g_0(z) = z. \quad (16)$$

Example Suppose $U_t = 0$ for all t . Then the Loewner equation becomes

$$\partial_t g_t(z) = \frac{2}{g_t(z)}, \quad g_0(z) = z,$$

which has the solution

$$g_t(z) = \sqrt{z^2 + 4t}, \quad K_t = (0, i2\sqrt{t}].$$

If we start with a simple curve, then the function U_t which is called the *driving function* is continuous. Let us go in the other direction. Suppose $t \mapsto U_t$ is a continuous, real-valued function. For each $z \in \mathbb{H}$ we define $g_t(z)$ as the solution to (16). Standard methods in differential equations show that the solution exists up to some time $T_z \in (0, \infty]$. We define

$$D_t = \{z : T_z > t\}.$$

Then it can be shown that g_t is a conformal transformation of D_t onto \mathbb{H} with $g_t(z) - z = o(1)$ as $z \rightarrow \infty$. We would like to define a curve γ by

$$\gamma(t) = g_t^{-1}(U_t) = \lim_{y \downarrow 0} g_t^{-1}(U_t + iy). \quad (17)$$

The quantity $g_t^{-1}(U_t + iy)$ always makes sense for $y > 0$, but it is not true that the limit can be taken for every continuous U_t . The “problem” functions U_t have the property that they move faster along the real line than the hull is growing. From the simple example above, we see that if the driving function remains constant, then in time $O(t)$ the hull grows at rate $O(\sqrt{t})$. If $U_t = o(\sqrt{t})$ for small t , then we are fine. In fact, the following holds.

Theorem 25. [20]

- There exists $c_0 > 0$ such that if U_t satisfies

$$|U_{t+s} - U_t| \leq c_0 \sqrt{s},$$

for all s sufficiently small, then the curve γ exists and is a simple curve.

- There exists $c_1 < \infty$ and a function U_t satisfying

$$|U_{t+s} - U_t| \leq c_1 \sqrt{s},$$

for all t, s for which the limit (17) does not exist for some t .

Definition Suppose $t \rightarrow U_t$ is a driving function. We say that U_t generates the curve $\gamma : [0, \infty) \rightarrow \overline{\mathbb{H}}$ if for each t , D_t is the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$.

5.3. Radial parametrization

The half-plane capacity parametrization is convenient for curves going from one boundary point to another (∞ is a boundary point of \mathbb{H}). When considering paths going from a boundary point to an interior point, it is convenient to consider the *radial parametrization* which is another kind of capacity parametrization. Suppose D is a simply connected domain and $z \in D$.

Definition If D is a simply connected domain and $z \in D$, then the *conformal radius* of z in D is defined to be $|f'(0)|$ where $f : \mathbb{D} \rightarrow D$ is a conformal transformation with $f(0) = z$. We let $\Upsilon_D(z)$ denote one-half times the conformal radius.

By definition $\Upsilon_{\mathbb{D}}(0) = 1/2$ and a straightforward calculation shows that $\Upsilon_{\mathbb{H}}(z) = \text{Im}(z)$. (We put the factor $1/2$ in the definition of $\Upsilon_D(z)$ so that the latter equation holds.)

Suppose $\gamma : (0, \infty) \rightarrow D \setminus \{z\}$ is a simple curve with $\gamma(0+) = w, \gamma(\infty) = z$, and let $D_t = D \setminus \gamma(0, t]$.

Definition The curve γ has a *radial parametrization* (with respect to z) if $\log \Upsilon_{D_t}(z) = -at + r$ for some a, r .

5.4. Radial Loewner differential equation in \mathbb{D}

Suppose γ is a simple curve as in the last subsection with $D = \mathbb{D}, |w| = 1$, and $z = 0$. For each t , let g_t be the unique conformal transformation of D_t onto \mathbb{D} with $g_t(0) = 0, g'_t(0) > 0$. We assume that the curve has the radial parametrization with $\log[2\Upsilon_{D_t}(0)] = -2at$. In other words, $g'_t(0) = e^{2at}$. The following is proved similarly to Theorem 24.

Theorem 26 (Loewner differential equation). *Suppose γ is a simple curve as above. Then for $z \in \mathbb{D}$, $g_t(z)$ satisfies the differential equation*

$$\partial_t g_t(z) = 2a g_t(z) \frac{e^{i2U_t} + g_t(z)}{e^{i2U_t} - g_t(z)}, \quad g_0(z) = z,$$

where $e^{i2U_t} = g_t(\gamma(t)) = \lim_{w' \rightarrow \gamma(t)} g_t(w')$. Moreover, the function $t \mapsto U_t$ is continuous. If $z \notin \gamma(0, \infty)$, then the equation is valid for all t . If $z = \gamma(s)$, then the equation is valid for $t < s$.

For ease, let us assume $a = 1, t = 0, U_t = 0$, for which the equation becomes

$$\partial_t g_t(z)|_{t=0} = z \frac{1+z}{1-z},$$

or

$$\partial_t [\log g_t(z)] = \frac{1+z}{1-z}.$$

The function on the right hand side is (a multiple of) the complex form of the Poisson kernel $H_{\mathbb{D}}(z, 1)$. In \mathbb{H} we considered separately the real and imaginary parts; in \mathbb{D} we consider separately the radial and angular parts, or equivalently, the real and imaginary parts of the logarithm.

When analyzing the radial equation, it is useful to consider the function $h_t(z)$ defined by

$$g_t(e^{2iz}) = e^{2ih_t(z)}.$$

Then the Loewner equation becomes

$$\partial_t h_t(z) = a \cot(h_t(z) - U_t).$$

If $h_t(z) - U_t$ is near zero, then

$$\cot(h_t(z) - U_t) \sim \frac{1}{h_t(z) - U_t},$$

and hence this can be approximated by the chordal Loewner equation.

6. Schramm-Loewner evolution (SLE_κ)

6.1. Chordal SLE_κ

Suppose we have a family of probability measures $\mu_D^\#(z, w)$ on simple curves (modulo reparametrization) connecting distinct boundary points in simply connected domains satisfying the Domain Markov and conformal covariance properties as in Section 1.5. By considering the measure on curves in \mathbb{H} from 0 to ∞ , we see that this induces a probability measure on driving functions U_t . This measure satisfies:

- For every $s < t$, the random variable $U_t - U_s$ is independent of $\{U_r : 0 \leq r \leq s\}$ and has the same distribution as U_{t-s} .

Since U_t is also a continuous process, a well known result in probability tells us that U_t must be a one-dimensional Brownian motion. Using the fact that the measure is invariant under dilations, we can see that the drift of the Brownian motion must equal zero. This leaves one parameter κ , the variance parameter of the Brownian motion. This gives Schramm's definition.

Definition The *chordal Schramm-Loewner evolution with parameter κ (SLE_κ)* (from 0 to ∞ in \mathbb{H}) is the solution to the Loewner evolution

$$\partial_t \tilde{g}_t(z) = \frac{2}{\tilde{g}_t(z) - \tilde{U}_t}, \quad \tilde{g}_0(z) = z,$$

where \tilde{U}_t is a one-dimensional Brownian motion with variance parameter κ .

◆Schramm used the term stochastic Loewner evolution and this is still used by some in the literature. It was renamed the Schramm-Loewner evolution by others.

As before, we let \tilde{D}_t be the domain of \tilde{g}_t and $\tilde{K}_t = \mathbb{H} \setminus \tilde{D}_t$. Under this definition, $\text{hcap}(\tilde{K}_t) = 2t$. For computational purposes, it is useful to consider $g_t = \tilde{g}_{t/\kappa}$ which satisfies

$$\partial_t g_t(z) = \frac{(2/\kappa)}{g_t(z) - \tilde{U}_{t/\kappa}}.$$

Since $\tilde{U}_{t/\kappa}$ is a standard Brownian motion, we get an alternative definition. This is the definition we will use; it is a linear time change of Schramm's original definition. Throughout these notes we will write $a = 2/\kappa$.

Definition The *chordal Schramm-Loewner evolution with parameter κ (SLE_κ)* (from 0 to ∞ in \mathbb{H}) is the solution to the Loewner evolution

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{18}$$

where $U_t = -B_t$ is a standard one-dimensional Brownian motion and $a = 2/\kappa$. Under this parametrization, $\text{hcap}(K_t) = at$.

A Brownian motion path is Hölder continuous of all orders less than $1/2$, but is not Hölder continuous of order $1/2$. Hence, we cannot use the criterion of Theorem 25 to assert that this gives a random measure on curves. However, this is the case and we state this as a theorem. In many of the statements below we leave out the phrase “with probability one”.

Theorem 27. *Chordal SLE_κ is generated by a (random) curve.*

This was proved in [21] for $\kappa \neq 8$. The $\kappa = 8$ case is more delicate, and the only known proof involves showing that the measure is obtained as a limit of measures on discrete curves, see [12]. For $\kappa \neq 8$, the curve γ is Hölder continuous of some order (depending on κ), but for $\kappa = 8$ it is not Hölder continuous of any order $\alpha > 0$. When we speak of Hölder continuity here, we mean *with respect to the capacity parametrization*. It turns out that this is not the parametrization that gives the optimal modulus of continuity. The next theorem describes the phases of SLE .

Theorem 28.

- If $\kappa \leq 4$, then γ is a simple curve with $\gamma(0, \infty) \subset \mathbb{H}$.
- If $4 < \kappa < 8$, then γ has double points and $\gamma(0, \infty) \cap \mathbb{R} \neq \emptyset$. The curve is not plane-filling; in fact, for each $z \in \mathbb{H}$, $\mathbf{P}^z\{z \in \gamma(0, \infty)\} = 0$.
- If $\kappa \geq 8$, then the curve is space-filling, that is, $\gamma[0, \infty) = \overline{\mathbb{H}}$.

Sketch of proof. If $\gamma(s) = \gamma(t)$ for some $s < t$, and $s < r < t$, then the image of the curve $\gamma[r, t]$ under g_r has the property that it hits the real line. Hence, whether or not there are double points is equivalent to whether or not the real line is hit. Let T be the first time that $\gamma(t) \in [x, \infty)$ where $x > 0$. Let

$$X_t = X_t(x) = g_t(x) - U_t.$$

If $T < \infty$, then $X_T = 0$. By (18), we get

$$dX_t = \frac{a}{X_t} dt + dB_t, \quad X_0 = x.$$

This is a Bessel equation and is the same equation satisfied by the absolute value of a d -dimensional Brownian motion where $a = (d-1)/2$. It is well known that solutions of this equation avoid the origin if and only if $a \geq 1/2$ which corresponds to $\kappa \leq 4$.

To see if the curve is plane-filling, let us fix $z \in \mathbb{H}$ and ask if $\gamma(t) = z$ for some t . Let

$$Z_t = Z_t(z) = g_t(z) - U_t,$$

which satisfies

$$dZ_t = \frac{a}{Z_t} dt + dB_t.$$

(A little care must be taken in reading this equation — Z_t is complex but B_t is a real-valued Brownian motion.) We reparametrize the curve using the radial parametrization with respect to z ,

$$\hat{Z}_t = Z_{\sigma(t)}.$$

In this new parametrization, the lifetime will be finite if the curve stays away from z (for then the conformal radius does not go to zero), and the lifetime will be infinite if the curve goes to z . For ease assume $\text{Im}[z] = 1$, and let $\Theta_t = \arg[\hat{Z}_t]$. Then with the aid of some standard stochastic calculus, we can see that Θ_t satisfies the equation

$$d\Theta_t = (1 - 2a) \cot \Theta_t dt + dW_t, \quad \Theta_0 = \arg z, \quad (19)$$

where W_t is a standard Brownian motion. Whether or not the lifetime is finite in the radial parametrization boils down to whether or not a process satisfying (19) ever reaches $\{0, \pi\}$. Recalling that $\cot \theta \sim 1/\theta$ for small θ , by comparison with the Bessel process we find that the process avoids the origin if and only if $1 - 2a \geq 1/2$ which corresponds to $\kappa \geq 8$. \square

Two other interesting facts can be derived from (19).

- The process $\arg Z_t$ is a martingale if and only if $\kappa = 4$ ($1 - 2a = 0$). Note that it does not matter which parametrization we use when we want to see if a process is a martingale. The case $\kappa = 4$ is related to the harmonic explorer and the Gaussian free field.
- If $\kappa < 8$, and

$$\phi(\theta) = \mathbf{P}\{\Theta_T = \pi \mid \Theta_0 = \theta\},$$

then $\phi(\Theta_{t \wedge T})$ is a martingale, and hence by Itô's formula, F satisfies

$$\frac{1}{2} \phi''(\theta) + (1 - 2a) \phi'(\theta) = 0.$$

Solving this equation, with appropriate boundary conditions gives

$$\phi(\theta) = c \int_0^\theta \sin^{4a-2} x dx, \quad c = \left[\int_0^\pi \sin^{4a-2} x dx \right]^{-1}.$$

Note that $\kappa < 8$ implies $4a - 2 > -1$, and hence the integral is finite. One can check that $\phi(\arg z)$ represents the probability that the curve γ goes to the right of z .

◆The formula for ϕ was first given by Schramm [23] and written in terms of hypergeometric functions. Many of the functions from SLE (and conformal field theory) are solutions of second order differential equations which is why hypergeometric functions arise. Often one can give alternative forms of the solution. An important case historically was the formula for percolation crossing which was first given by Cardy [4] in the upper half plane terms of hypergeometric functions but, as first noted by Carleson, is much simpler in an equilateral triangle.

6.2. Dimension of the path

If $\kappa \geq 8$, the *SLE* curve is plane-filling and hence has dimension two. For this section, we assume $\kappa < 8$. Roughly speaking, the fractal dimension of the curve $\gamma[s, t]$ is given by d , where the number of balls of radius ϵ needed to cover $\gamma[s, t]$ grows like ϵ^{-d} as $\epsilon \rightarrow 0$. For fixed z , let $p(z, \epsilon)$ denote the probability that the curve gets within distance ϵ of z . Then a “back of the envelope” calculation shows that the *expected* number of balls of radius ϵ needed to cover, say,

$$\gamma(0, \infty) \cap \{z \in \mathbb{H} : |z - i| \leq 1/2\}$$

should decay like $\epsilon^{-2} p(i, \epsilon)$, or we would expect that $p(i, \epsilon) \approx \epsilon^{2-d}$.

Let us be more precise. Let $\Upsilon_t(z) = \Upsilon_{D_t}(z)$ where, as before, $\Upsilon_D(z)$ equals one-half times the conformal radius. Let $\Upsilon(z) = \lim_{t \rightarrow \infty} \Upsilon_t(z)$. Then we might expect that there is a function $\tilde{G}(z)$ and a dimension d such that

$$\mathbf{P}\{\Upsilon(z) \leq \epsilon\} \sim \tilde{G}(z)\epsilon^{2-d}, \quad \epsilon \rightarrow 0.$$

Again, let $Z_t = Z_t(z) - g_t(z)$. Then, if such a function existed, one can show that

$$|g'_t(z)|^{2-d} \hat{G}(Z_t),$$

would have to be a local martingale. Using Itô’s formula, we can show that this implies that $\hat{G} = cG$ where G is the *chordal SLE $_{\kappa}$ Green’s function*,

$$G(z) = \text{Im}(z)^{2-d} [\arg z]^{4a-1}, \quad d = 1 + \frac{1}{4a} = 1 + \frac{\kappa}{8}.$$

The proof of the following is discussed in [8].

Theorem 29. *If $\kappa < 8$ and $z \in \mathbb{H}$,*

$$\lim_{\epsilon \rightarrow 0^+} \epsilon^{d-2} \mathbf{P}\{\Upsilon(z) \leq \epsilon\} = c_* G(z), \quad c_* = \left[\int_0^\pi \sin^{4a} x \, dx \right]^{-1}.$$

This “one-point” estimate is not good enough to compute the Hausdorff dimension of the path. A more difficult “two-point” estimate of the form

$$\mathbf{P}\{\Upsilon(z) \leq \epsilon, \Upsilon(w) \leq \epsilon\} \asymp \epsilon^{2(2-d)} |z - w|^{d-2},$$

was proved by Beffara [3] (see also [18]) from which he concluded the following.

Theorem 30. *If $\kappa < 8$, the *SLE $_{\kappa}$* paths have Hausdorff dimension*

$$d = 1 + \frac{\kappa}{8}.$$

In particular, for every $1 < d < 2$, there exists a unique κ that produces paths of dimension d .

6.3. SLE in simply connected domains

The starting assumption for chordal SLE_κ was that it was a conformally invariant family of probability measures connecting distinct boundary points. Using this, Schramm derived that there was only a one-parameter family of possible measures which he defined as SLE_κ . The definition was given in the upper half-plane, but one can then *define* SLE_κ connecting boundary points w_1, w_2 in a simply connected domain as the image of SLE_κ in \mathbb{H} under a conformal transformation $F : \mathbb{H} \rightarrow D$ with $F(0) = w_1, F(\infty) = w_2$.

Remarks

- This definition is really a measure on curves *modulo reparametrization*.
- The map F is not unique but if \tilde{F} is another such map one can show that $\tilde{F}(z) = F(rz)$ for some $r > 0$. Using scale invariance (modulo reparametrization) of SLE_κ , which follows from a scaling rule for Brownian motion, we see that the definition is independent of the choice of F .

In the calculation below, two important parameters will appear.

Definition

- The *central charge* $\mathbf{c} = \mathbf{c}(\kappa)$ is defined by

$$\mathbf{c} = \frac{(6 - \kappa)(3\kappa - 8)}{2\kappa} = \frac{(3a - 1)(3 - 4a)}{a}$$

- The *boundary scaling exponent* $b = b(\kappa)$ is defined by

$$b = \frac{6 - \kappa}{2\kappa} = \frac{3a - 1}{2}.$$

- For any κ , its *dual value* κ' is defined by $\kappa \kappa' = 16$.

We note that $\mathbf{c} \in (-\infty, 1]$ and $\mathbf{c}(\kappa) = \mathbf{c}(\kappa')$. The relationship $\kappa \leftrightarrow \mathbf{c}(\kappa)$ is two-to-one with a double root at $\kappa = 4, \mathbf{c} = 1$. As κ increases from 0 to ∞ , $b(\kappa)$ decreases from ∞ to $-1/2$.

6.4. Subdomains of \mathbb{H}

Suppose that $D = \mathbb{H} \setminus K \in \mathcal{D}$ with $\text{dist}(0, K) > 0$. Let $F = F_D$ be the conformal transformation of \mathbb{H} onto D with $F(0) = 0, F(\infty) = \infty, F'(\infty) = 1$, and let $\Phi = F^{-1}$. If γ is an SLE_κ curve in \mathbb{H} , then $\gamma^*(t) = F \circ \gamma(t)$ is (a time change of) SLE_κ in D from $F(0)$ to infinity. Let us write $\gamma_t = \gamma(0, t], \gamma_t^* = \gamma_t^*(0, t]$. Since γ_t^* is a curve in \mathbb{H} , we can write its Loewner equation,

$$\partial_t g_t^*(z) = \frac{\dot{a}^*(t)}{g_t^*(z) - U_t^*},$$

where $a^*(t) = \text{hcap}[\gamma_t^*]$. Let $F_t = g_t^* \circ F \circ g_t^{-1}$, and note that $F_t(U_t) = U_t^*$ and that F_t is the corresponding conformal transformation of \mathbb{H} onto $g_t^*(D)$

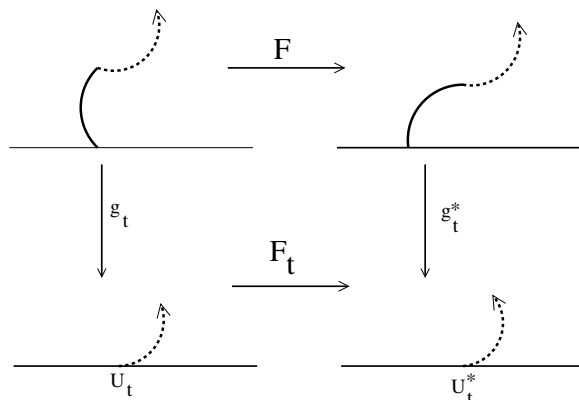


FIG 10. The maps.

with $F_t(U_t) = U_t^*$, $F_t(\infty) = \infty$, $F_t'(\infty) = 1$. By using the scaling rule for the half-plane capacity, we can show that

$$\partial_t a^*(t) = a F'(U_t)^2,$$

and hence

$$\partial_t g_t^*(z) = \frac{a F'(U_t)^2}{g_t^*(z) - U_t^*}.$$

With the aid of some careful chain rule and stochastic calculus computations, one can find the stochastic differential equation satisfied by the driving function U_t^* . It turns out to be nicer if one reparametrizes so that the half-plane capacity of γ^* grows linearly at rate a . Let $\hat{U}_t = U_{\sigma(t)}^*$ denote the driving function in the time change and let $\Phi_t = F_{\sigma(t)}^{-1}$.

Proposition 31. *Under the time change, the driving function \hat{U}_t satisfies*

$$d\hat{U}_t = b \frac{\Phi_t''(\hat{U}_t)}{\Phi_t'(\hat{U}_t)} dt + dW_t,$$

where W_t is a standard Brownian motion. This is valid up to the first time that the curve hits $K = \mathbb{H} \setminus D$.

This proposition implies that there is another way to define SLE_κ in D . Consider a solution to the Loewner equation (18) where U_t satisfies the SDE

$$dU_t = b \frac{\Phi_t''(U_t)}{\Phi_t'(U_t)} dt + dW_t.$$

Here Φ_t is a conformal transformation of $g_t(D)$ onto \mathbb{H} with $\Phi_t(\infty) = \infty$, $\Phi_t'(\infty) = 1$. Let γ denote the corresponding curve and let

$$\tau = \inf\{t : \gamma(t) \in K\}.$$

Then this gives the distribution of SLE_κ in D up to time τ .

The drift term is nontrivial unless $b = 0$ which corresponds to $\kappa = 6$. This particular property of SLE_6 is called *locality*. Note that the percolation exploration process satisfies a discrete analogue of the locality property, and this is one way to see that $\kappa = 6$ should correspond to percolation.

Proposition 32 (Locality). *If γ is an SLE_6 curve in D , then the distribution of γ is the same as that of SLE_6 in \mathbb{H} up to the first time that the curve hits $\mathbb{H} \setminus D$.*

6.5. Fundamental local martingale

We will start this section by stating an important computation which first appeared in [13]. We will not motivate it now, but we will discuss implication below. Suppose γ is an SLE_κ curve in \mathbb{H} with driving function $U_t = -B_t$, and D is a domain as above. Let Φ_t, τ be defined as in the previous subsection, and recall the values of b, c . Let S denote the *Schwarzian derivative*,

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3f''(z)^2}{2f'(z)^2}.$$

For $t < \tau$, we define

$$M_t = \Phi'_t(U_t)^b \exp \left\{ -\frac{ac}{12} \int_0^t S\Phi_s(U_s) ds \right\}.$$

Note that we can write M_t as

$$M_t = C_t \Phi'_t(U_t)^b$$

where C_t has paths that are continuously differentiable in t .

Proposition 33. *M_t is a local martingale satisfying*

$$dM_t = b \frac{\Phi''_t(U_t)}{\Phi'_t(U_t)} M_t dU_t, \quad t < \tau.$$

It is not hard to show that $\Phi'_t(U_t) \leq 1$ and $S\Phi_t(U_t) \leq 0$, and hence for $\kappa \leq 8/3$, M_t is a bounded martingale.

An important tool in understanding the curve is *Giranol's Theorem*. The form that we will need it is the following. Suppose M_t is a nonnegative martingale satisfying

$$dM_t = M_t J_t dU_t.$$

Then we can define a new probability measure by stating that if E is an event depending only on $\{U_s : 0 \leq s \leq t\}$, then

$$Q(E) = M_0^{-1} \mathbf{E}[M_t 1_E].$$

The theorem states that

$$W_t = U_t - \int_0^t J_s ds,$$

is a standard Brownian motion with respect to Q , or equivalently,

$$dU_t = J_t dt + dW_t.$$

This is a theorem about martingales, but we can apply this theorem to positive *local* martingales by choosing an increasing collection of stopping times ξ_n such that $M_{t \wedge \xi_n}$ are bounded martingales. Comparing this with Proposition 31 gives us the following.

Proposition 34. *SLE $_{\kappa}$ weighted (in the sense of the Girsanov theorem) by M_t has the same distribution as SLE $_{\kappa}$ in D (up to the first time that the curve hits $\mathbb{H} \setminus D$).*

Recall that for $\kappa \leq 4$, the SLE $_{\kappa}$ curve in \mathbb{H} never hits the real line. This implies that the SLE $_{\kappa}$ curve in D never hits $\mathbb{H} \setminus D$ and the last proposition is valid for all time. With a little more argument, one can let $t \rightarrow \infty$ and prove the following. As $t \rightarrow \infty$, $g_t(D)$ looks more and more like \mathbb{H} and from this one gets $\Phi'_t(U_t) \rightarrow 1$.

Theorem 35. *If $\kappa \leq 4$,*

$$\Phi'(0)^b = \mathbf{E}[M_0] = \mathbf{E}[M_{\infty}] = \mathbf{E} \left[1_{\{\gamma(0, \infty) \subset D\}} \exp \left\{ -\frac{ac}{12} \int_0^{\infty} S\Phi_t(U_t) dt \right\} \right].$$

6.6. Brownian loop measure

The “compensator” term

$$\exp \left\{ -\frac{ac}{12} \int_0^t S\Phi_s(U_s) ds \right\}$$

from last subsection comes from a direct calculation. However, there is a nice interpretation of this quantity in terms of loops of the Brownian loop measure. We give a slightly different definition here, but one can easily check that this is the same definition as given in Section 3.

A (*rooted*) *loop* ω is a continuous curve $\omega : [0, t_{\omega}] \rightarrow \mathbb{C}$ with $\omega(0) = \omega(t_{\omega})$. Such a loop can also be considered as a periodic function of period t_{ω} . An *unrooted loop* is a rooted loop with the root forgotten. To specify a rooted loop, one can give a triple $(z, t, \tilde{\omega})$ where $z \in \mathbb{C}$ is the root, t is the time duration, and $\tilde{\omega}$ is a loop of time duration one. To obtain ω from the triple, one uses Brownian scaling to convert $\tilde{\omega}$ to a loop of time duration t and then translates the loop so it has root z .

Definition

- The *rooted Brownian loop measure* $\tilde{\nu} = \tilde{\nu}_{\mathbb{C}}$ is the measure on rooted loops $(z, t, \tilde{\omega})$ given by

$$\text{area} \times \frac{dt}{2\pi t^2} \times \text{Brownian bridge},$$

where Brownian bridge denotes the probability measure associated to complex Brownian motions $B_t, 0 \leq t \leq 1$ conditioned so that $B_0 = B_1$.

- The (unrooted) Brownian loop measure in \mathbb{C} , $\nu = \nu_{\mathbb{C}}$, is the measure on unrooted loops obtained from $\tilde{\nu}_{\mathbb{C}}$ by forgetting the roots.
- If $D \subset \mathbb{C}$, the measures $\tilde{\nu}_D$ and ν_D are obtained by restricting $\tilde{\nu}$ and ν , respectively, to loops staying in D .

By definition the Brownian loop measure satisfies the “restriction property”. It is an infinite measure since small loops have large measure. Its importance comes from the fact that it also satisfies conformal invariance. The following theorem only holds for the measure on unrooted loops.

Theorem 36. *If $f : D \rightarrow f(D)$ is a conformal transformation, then*

$$f \circ \nu_D = \nu_{f(D)}.$$

The relationship between the Brownian loop measure and the compensator is as follows. If D is a domain and $V_1, V_2 \subset D$, let $\Lambda(D; V_1, V_2)$ denote the Brownian loop measure of the set of loops in D that intersect both V_1 and V_2 . Then, by analyzing the Brownian loop measure infinitesimally (see (12)), we have the following. Here and below, we write γ_t for $\gamma(0, t]$.

Proposition 37. *Suppose γ is a curve in \mathbb{H} parametrized so that $\text{hcap}(\gamma_t) = at$ and let $D = \mathbb{H} \setminus K \in \mathcal{D}$. Then if*

$$t < \tau := \inf\{s : \gamma(s) \in K\},$$

we have

$$-\frac{a}{6} \int_0^t S\Phi_s(U_s) ds = \Lambda(\mathbb{H}; \gamma_t, K).$$

Therefore, the fundamental local martingale can be written as

$$M_t = \Phi'_t(U_t)^b \exp\left\{\frac{\mathbf{c}}{2} \Lambda(\mathbb{H}; \gamma_t, K)\right\},$$

and if $\kappa \leq 4$, we have a limiting value

$$M_\infty = 1\{\gamma \subset D\} \exp\left\{\frac{\mathbf{c}}{2} \Lambda(\mathbb{H}; \gamma, K)\right\},$$

where $\gamma = \gamma_\infty$. Combining this we get the main theorem comparing SLE_κ in D to SLE_κ in \mathbb{H} .

Theorem 38. *Suppose $D = \mathbb{H} \setminus K \in \mathcal{D}$ with $\text{dist}(0, K) > 0$ and $\kappa \leq 4$. Let γ be an SLE_κ curve from 0 to ∞ in \mathbb{H} defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and let*

$$M_\infty = 1\{\gamma \subset D\} \exp\left\{\frac{\mathbf{c}}{2} \Lambda(\mathbb{H}; \gamma, K)\right\}.$$

Then,

$$\mathbf{E}[M_\infty] = \Phi'_D(0)^b.$$

Moreover, if Q is defined by

$$dQ = \frac{M_\infty}{\Phi'_D(0)^b} d\mathbf{P},$$

then the distribution of γ with respect to Q is that of SLE_κ in D from 0 to ∞ .

◆We review what we have done here. We used Itô's formula to derive a local martingale with a compensator term that had a Schwarzian derivative. We then interpreted this infinitesimal calculation in terms of a more global quantity, the Brownian loop measure. Some of these ideas can be extended to multiply connected domains, but the substitute for the Schwarzian derivative is the Brownian bubble quantity $\Gamma(D)$ from Section 3.4.

6.7. SLE_κ as a nonprobability measure

For this subsection, we assume that $\kappa \leq 4$ so that SLE_κ is supported on simple paths. If D is a simply connected domain, we say that $z, w \in \partial D$ are *smooth* (boundary points of D) if ∂D is locally analytic near z, w . For any such D , there is a unique conformal transformation

$$F = F_D : \mathbb{H} \rightarrow D$$

with $F(0) = z, F(\infty) = w, |F'(\infty)| = 1$. Here we abuse notation somewhat, to write $F(\infty) = w, |F'(\infty)| = 1$ to denote that as $w' \rightarrow \infty$,

$$F(w') = w - \frac{1}{w'} \mathbf{n},$$

where \mathbf{n} denotes the inward unit normal in D at w . We also call $0, \infty$ smooth boundary points for domains $D \in \mathcal{D}$. In this case, the map F is the same as the F defined in Section 6.4.

Definition Suppose D is a simply connected domain and z, w are smooth. If $\kappa \leq 4$, the (unparametrized) SLE_κ measure $\mu_D(z, w)$ is defined by

$$\mu_D(z, w) = C_D(z, w) \mu_D^\#(z, w),$$

where:

- $\mu_D^\#(z, w)$ is the probability measure on paths (modulo reparametrization) obtained as the image of SLE_κ in \mathbb{H} under F .
- $C_D(z, w) = |F'(z)|^b$.

In particular, we have normalized the measure so that $C_{\mathbb{H}}(0, \infty) = 1$. In fact, using the scaling rule for the Poisson kernel, we can see that for all simply

connected domains

$$C_D(z, w) = [\pi H_{\partial D}(z, w)]^b,$$

where $H_D(z, w)$ is the excursion or boundary Poisson kernel defined as the normal derivative of $H_D(\cdot, w)$ at z . In particular,

$$C_{\mathbb{H}}(0, x) = |x|^{-2b}.$$

We can summarize a number of the results in this section as follows.

Theorem 39. *Assume $\kappa \leq 4$, D, D' are simply connected domains, and z, w are smooth boundary points.*

- **Conformal covariance.** *If $f : D \rightarrow f(D)$ is a conformal transformation and $f(z), f(w)$ are smooth boundary points of $f(D)$, then*

$$f \circ \mu_D(z, w) = |f'(z)|^b |f'(w)|^b \mu_{f(D)}(f(z), f(w)).$$

- **Boundary perturbation.** *If $D_1 \subset D$ and $\partial D, \partial D_1$, then $\mu_{D_1}(z, w)$ is absolutely continuous with respect to $\mu_D(z, w)$ with*

$$\frac{d\mu_{D_1}(z, w)}{d\mu_D(z, w)}(\gamma) = Y(\gamma) \tag{20}$$

where

$$Y(\gamma) = Y(D_1, D; z, w)(\gamma) = 1\{\gamma \subset D_1\} \exp\left\{\frac{c}{2} \Lambda(D; \gamma, D \setminus D_1)\right\}.$$

◆The above characterization gives a natural way to extend the definition of SLE_κ to multiply connected domains at least for $\kappa \leq 4$. See [9, 10]

We can also state (20) in terms of probability measures,

$$\frac{d\mu_{D_1}^\#(z, w)}{d\mu_D^\#(z, w)}(\gamma) = \frac{Y(\gamma)}{E(Y)}$$

where E denotes expectation with respect to $\mu_D^\#(z, w)$. This formulation has the advantage that it does not require z, w to be smooth boundary points. Note that if $f : D \rightarrow f(D)$ is a conformal transformation, then

$$Y(f(D_1), f(D); f(z), f(w))(f \circ \gamma) = Y(D_1, D; z, w)(\gamma).$$

The definition of $\mu_D(z, w)$ very much used the fact that the curve was going from z to w . However, many of the discrete examples indicate that the measure should be reversible, that is, the measure $\mu_D(w, z)$ can be obtained from $\mu_D(z, w)$ by reversing the paths. This is not easy to show from the definition of SLE_κ , but fortunately, Zhan [25] has given a proof.

Theorem 40. *For $\kappa \leq 4$, $\mu_D(w, z)$ can be obtained from $\mu_D(z, w)$ by reversing the paths.*

6.8. Natural parametrization (length)

Everything we have done so far has considered SLE_κ in the capacity parametrization or “up to time change”. The *natural parametrization or length* should be a d -dimensional measure where d is the Hausdorff dimension of the path. Here we show how to define it. Let γ denote an SLE_κ curve in \mathbb{H} from 0 to ∞ and let Θ_t denote the amount of time to traverse $\gamma[0, t]$ in the natural parametrization. If $\kappa \geq 8$, the path is space filling and we define

$$\Theta_t = \text{area}(\gamma_t).$$

For the remainder of this section, assume that $\kappa < 8$.

The starting point for the definition is the belief that the expected amount of time spent in a bounded domain V should be (up to an arbitrary constant multiple) equal to

$$\int_V G(z) dA(z),$$

where G denotes the Green’s function and dA denotes integration with respect to area. More generally, the expected amount of time spent in V after time t , given the path $\gamma(0, t]$ should be given by

$$\Psi_t(V) := \int_V G_{D_t}(z; \gamma(t), \infty) dA(z).$$

For each z , the process $M_t(z) = G_{D_t}(z; \gamma(t), \infty)$ is a positive local martingale and hence is a supermartingale. Using this, we see that $\Psi_t(V)$ is a supermartingale. This leads to the following definition which comes from the Doob-Meyer decomposition of a supermartingale.

Definition The natural parametrization $\Theta_t(V)$ is the unique increasing process $\Theta_t(V)$ such that

$$\Theta_t(V) + \Psi_t(V)$$

is a martingale. The natural parametrization is given by

$$\Theta_t = \lim_{n \rightarrow \infty} \Theta_t(V_n),$$

where V_n is an increasing sequence of sets whose union is \mathbb{H} .

Work needs to be done to show that this is well defined and nontrivial [17, 19]. Indeed, if $\Psi_t(V)$ were a local martingale, then we would not have a nontrivial process.

This defines the natural length in \mathbb{H} ; there are two ways to define in subdomains. Suppose $D \in \mathcal{D}$. Then for each curve γ lying in D , we can consider as a curve in \mathbb{H} and compute its length, or we could use the conformal covariance rule (4). Fortunately, they give the same answer [16].

It may be surprising at first, but the capacity parametrization and the natural parametrization are singular with respect to each other.

References

- [1] A. BELAVIN, A. POLYAKOV, A. ZAMOŁODCHIKOV (1984). Infinite conformal symmetry of critical fluctuations in two dimensions, *J. Stat. Phys.* **34**, 763–774. [MR0751712](#)
- [2] A. BELAVIN, A. POLYAKOV, A. ZAMOŁODCHIKOV (1984). Infinite conformal symmetry in two-dimensional quantum field theory, *Nuclear Phys B* **241**, 333–380. [MR0757857](#)
- [3] V. BEFFARA (2008). The dimension of the SLE curves. *Ann. Probab.* **36**(4), 1421–1452. [MR2435854](#)
- [4] J. CARDY (1992). Critical percolation in finite geometries, *J. Phys. A* **25**, L201–L206. [MR1151081](#)
- [5] J. CONWAY (1978, 1995). *Functions of One Complex Variable I and II*, Springer-Verlag. [MR0503901](#)
- [6] P. DUREN (2001). *Univalent Functions*. Springer-Verlag. [MR0708494](#)
- [7] G. LAWLER (2005). *Conformally Invariant Processes in the Plane*, Amer. Math. Soc. [MR2129588](#)
- [8] G. LAWLER (2009). Schramm-Loewner evolution, in *Statistical Mechanics*, S. Sheffield and T. Spencer, ed., IAS/Park City Mathematical Series, AMS, 231–295. [MR2523461](#)
- [9] G. LAWLER (2009). Partition functions, loop measure, and versions of SLE, *J. Stat. Phys.* **134**, 813–837. [MR2518970](#)
- [10] G. LAWLER, Defining SLE in multiply connected domains with the loop measure, in preparation.
- [11] G. LAWLER, V. LIMIC (2010), *Random Walk: A Modern Introduction*, Cambridge Univ. Press. [MR2677157](#)
- [12] G. LAWLER, O. SCHRAMM, W. WERNER (2004), Conformal invariance of planar loop-erased random walks and uniform spanning trees, *Annals of Probab.* **32**, 939–995. [MR2044671](#)
- [13] G. LAWLER, O. SCHRAMM. AND W. WERNER (2003). Conformal restriction: the chordal case. *J. Amer. Math. Soc.* **16**, 917–955. [MR1992830](#)
- [14] G. LAWLER, J. A. TRUJILLO FERRERAS (2007). Random walk loop soup. *Trans. Amer. Math. Soc.* **359**, 767–787. [MR2255196](#)
- [15] G. LAWLER, W. WERNER (2004), The Brownian loop soup, *Probab. Theory Related Fields* **128**, 565–588. [MR2045953](#)
- [16] G. LAWLER, M. REZAEI, in preparation.
- [17] G. LAWLER, S. SHEFFIELD (2011). A natural parametrization for the Schramm-Loewner evolution, *Annals of Probab.* **39**, 1896–1937.
- [18] G. LAWLER, B. WERNESSE, Multi-point Green’s functions for SLE and an estimate of Beffara, to appear in *Annals of Probab.*
- [19] G. LAWLER, W. ZHOU, SLE curves and natural parametrization, preprint.
- [20] D. MARSHALL, S. ROHDE (2005), The Loewner differential equation and slit mappings. *J. Amer. Math. Soc.* **18**, 763–778. [MR2163382](#)
- [21] S. ROHDE, O. SCHRAMM (2005), Basic properties of SLE. *Ann. of Math.* **161**, 883–924. [MR2153402](#)

- [22] O. SCHRAMM (2000). Scaling limits of loop-erased random walk and uniform spanning trees, *Israel J. Math.* **118**, 221-288. [MR1776084](#)
- [23] O. SCHRAMM (2001). A percolation formula. *Electron Comm. Probab.* **6**, 115–120. [MR1871700](#)
- [24] D. WILSON (1996). Generating random spanning trees more quickly than the cover time, *Proc. STOC96*, 296–303. [MR1427525](#)
- [25] D. ZHAN (2007), Reversibility of chordal SLE, *Annals of Prob.* **36**, 1472–1494. [MR2435856](#)