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# Note on the existence and modulus of continuity of the $SLE_8$ curve

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**Abstract** We review one method for estimating the modulus of continuity of a Schramm-Loewner evolution (*SLE*) curve in terms of the inverse Loewner map. Then we prove estimates about the distribution of the inverse Loewner map, which underpin the difficulty in bounding the modulus of continuity of *SLE* for $\kappa = 8$. The main idea in the proof of these estimates is applying the Girsanov theorem to reduce the problem to estimates about one-dimensional Brownian motion.

**Keywords** Schamm-Loewner evolution · modulus of continuity

## 1 Introduction

The *chordal* or *half-plane Loewner equation* is

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{1}$$

where $U : [0, \infty) \to \mathbb{R}$ is a continuous function and $a > 0$. If $U_0 = 0$, then for each $z \in \mathbb{C} \setminus \{0\}$, the solution to (1) exists for $t < T_z \in (0, \infty]$, where $T_z$ can be defined as the supremum of times $t$ such that

$$\min\{|g_s(z) - U_s| : 0 \leq s \leq t\} > 0.$$

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(See [3] for basic facts on the Loewner equation.) Since $T_z = T_{\overline{z}}$ and $g_t(\overline{z}) = \overline{g_t(z)}$, we often restrict the equation to $z$ in the closure of the upper half-plane $\mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$. In this case, if $H_t$ is the set $\{z : T_z > t\}$, then $g_t$ is the unique conformal transformation of $H_t$ onto $\mathbb{H}$ with $g_t(z) - z = o(1)$ as $z \to \infty$. Moreover, as $z \to \infty$,

$$g_t(z) = z + \frac{at}{z} + O(|z|^{-2}).$$

We say that $g_t$ is *generated by the curve* $\gamma$ if $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a continuous function such that for each $t$, $H_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$. Although the maps $g_t$ exist for every continuous $U_t$, it is not always the case that they are generated by a curve. However, if $U_t$ is Hölder-(1/2), it is known [7] that for $a$ sufficiently large, the curve $\gamma$ exists and is simple with $\gamma(0, \infty) \subset \mathbb{H}$. On the other hand, there exist Hölder-(1/2) functions $U_t$ such that the solution to (1) is not generated by a curve for some values of $a$.

The chordal Schramm-Loewner evolution with parameter $\kappa$ ($SLE_\kappa$) is the solution to (1) with $a = 2/\kappa$ and $U_t$ a standard Brownian motion. This process was introduced by Oded Schramm [9] as a candidate for the continuum limit of two-dimensional models in statistical physics, and is now an important part of their study. Since a standard Brownian motion is not Hölder-(1/2) continuous, the deterministic results cannot be applied to show existence of the curve. Rohde and Schramm [8] proved that for $\kappa \neq 8$, the curve does exist. Moreover, they showed that there are three phases:

- For $\kappa \leq 4$, the curve is simple.
- For $4 < \kappa < 8$, the curve has double points but is not plane filling.
- For $\kappa > 8$, the curve is plane-filling; their argument also showed that the $SLE_8$ curve, if it exists, is plane-filling.

The existence of the curve for $\kappa = 8$, which is the most delicate and which we study in this paper, could not be resolved by their argument. Existence of the curve was proved indirectly in [5] by proving that $SLE_8$ is the scaling limit of a certain Peano curve arising in uniform spanning trees. This proof gave no estimate for the modulus of continuity.

The delicacy of the question at $\kappa = 8$ can be seen in the Hölder exponent of $\gamma$. For $\kappa \neq 8$, there exists $\alpha > 0$, depending on $\kappa$, such that $\gamma$ is Hölder-$\alpha$. However, for $\kappa = 8$, it is known that the curve is not Hölder-$\alpha$ for any $\alpha > 0$, see [1,6].

Our goal is to give a direct proof of the existence of the $SLE_8$ curve, and to estimate the modulus of continuity. Although we do not accomplish this goal in this paper, we give some partial results towards this goal, as well as some negative results that show that our goal is difficult to achieve. We state our main conjecture.

*Conjecture 1* There exists $\beta > 0$ such that if $\gamma$ is an $SLE_8$ curve, then with probability one,

$$\sup_{0 \leq s < t \leq 1} [1 + |\log(t - s)|]^\beta \, |\gamma(t) - \gamma(s)| < \infty. \tag{2}$$

The outline of the paper is as follows. In Section 2, we review how to show existence of a Loewner curve and show how estimates on the derivative of $g_t^{-1}$ can be used to bound the modulus of continuity. We then describe our main results in terms of the derivatives. In Section 3, we give most of the proofs. Here we use the reverse Loewner equation, and, as in [1, 2, 4], a particular martingale in a particular parametrization. This martingale takes on a simple form for $\kappa = 8$. The idea is to use the Girsanov theorem to weight the measure by the martingale, find probabilities of particular events in this new measure, and then use these to estimate probabilities in the original measure. Essentially, we boil down our problem to some estimates about one-dimensional standard Brownian motion. These are stated as lemmas in Section 3, and they are proved in Section 4.

## 2 Existence of the curve

In this section we review how to show existence of a Loewner curve and to establish bounds on the modulus of continuity. See the early sections of [1, 2] for more details. Suppose that $a > 0$, $U_t$ is a continuous real-valued function, and $g_t$ is defined by (1). Let $f_t(z) = g_t^{-1}(U_t + z)$. For each positive integer $n$, there is a curve $\gamma^{(n)}(t)$ defined by

$$\gamma^{(n)}(t) = f_t(2^{-n}i) = g_t^{-1}(U_t + 2^{-n}i).$$

For fixed $n$, $\gamma^{(n)}$ is a well-defined curve, and if $\gamma$ exists, we know that

$$\gamma(t) = \lim_{n\to\infty} \gamma^{(n)}(t) = \lim_{y\downarrow 0} f_t(yi).$$

Indeed, if we show the limit above exists and that $\gamma$ is a continuous function of $t$, then $g_t$ is generated by $\gamma$. Let us be more precise showing how one can obtain estimates of the modulus of continuity of $\gamma$. We will restrict our discussion to $0 \le t \le 1$.

Let $\mathcal{D}_n$ denote the dyadic rationals in $[0, 1]$ with denominator $2^{2n}$, that is,

$$\mathcal{D}_n = \{k2^{-2n} : k = 0, 1, \ldots, 2^{2n}\}.$$

If $n$ is a positive integer, let

$$\Theta_n = \sup\left\{|\gamma^{(n)}(t+s) - \gamma^{(n)}(t)| : t \in \mathcal{D}_n, 0 \le s \le 2^{-2n}\right\},$$

$$\hat{\Lambda}_n = \sup\{|\gamma^{(n)}(t) - \gamma^{(m)}(t)| : n \le m, 0 \le t \le 1\}.$$

If $\hat{\Lambda}_n \downarrow 0$, then we can define

$$\gamma(t) = \lim_{n\to\infty} \gamma^{(n)}(t).$$

The triangle inequality implies that if $0 \le s \le t \le 1$ and $t - s \le 2^{-2n}$,

$$\begin{aligned} |\gamma(t) - \gamma(s)| &\le |\gamma^{(n)}(t) - \gamma(t)| + |\gamma^{(n)}(s) - \gamma(s)| + |\gamma^{(n)}(t) - \gamma^{(n)}(s)| \\ &\le 2\,\hat{\Lambda}_n + 3\,\Theta_n \end{aligned}$$

Hence, if also $\Theta_n \downarrow 0$, then $\gamma$ is continuous in $t$, and we have an estimate on its modulus of continuity.

Let

$$\hat{K}_{n,m} = \sup\left\{\sum_{j=n}^{m} 2^{-j}\,|f_t'(i2^{-j})| : 0 \leq t \leq 1\right\}.$$

Here and throughout this paper we use primes to denote derivatives with respect to $z$. For fixed $n$, this quantity is increasing in $m$, and we set $\hat{K}_n = \hat{K}_{n,\infty}$. The distortion theorem implies that there is a universal constant $c$ such that

$$|\gamma^{(n)}(t) - \gamma^{(m)}(t)| \leq \int_{2^{-m}}^{2^{-n}} |f_t'(iy)|\,dy \leq c\sum_{j=n}^{m} 2^{-j}\,|f_t'(i2^{-j})|,$$

and hence $\hat{\Lambda}_n \leq c\,\hat{K}_n$. We summarize in the following proposition.

**Proposition 1** *Suppose that $\Theta_n + \hat{K}_n \downarrow 0$. Then $g_t$ is generated by the curve $\gamma$. Moreover, if $u : (0,\infty) \to (0,\infty)$ is a continuous, increasing function with $u(0+) = 0$ and such that $u(2^{-2n-1}) \geq \Theta_n + \hat{K}_n$ for all $n$ sufficiently large, then*

$$\sup_{0\leq s<t\leq 1} \frac{|\gamma(t) - \gamma(s)|}{u(t-s)} < \infty.$$

In order to estimate $\Theta_n$ and $\hat{K}_n$, we need to consider the modulus of continuity of $U_t$. It is convenient to define the following quantity which measures the modulus of continuity in terms of Brownian scaling. Let

$$\Delta(j,n) = 1 + 2^n \sup\left\{|U_t - U_{j2^{-2n}}| : j2^{-2n} \leq t \leq (j+1)2^{-2n}\right\}.$$

**Lemma 1** *There exists $c_3$ such that if $t = j2^{-2n} \in \mathcal{D}_n, 0 \leq s \leq 2^{-2n}$, then*

$$\frac{|f_t'(i2^{-n})|}{c_3\,\Delta(j,n)^4} \leq |f_{t+s}'(i2^{-n})| \leq c_3\,\Delta(j,n)^4\,|f_t'(i2^{-n})|,$$

$$|\gamma^{(n)}(t+s) - \gamma^{(n)}(t)| \leq c_3\,\Delta(j,n)^4\,2^{-n}\,|f_t'(i2^{-n})|.$$

*Proof* See [2, Propositions 2.3 and 2.5]. The tools are the distortion theorem and the Loewner equation.

Let

$$D_n = \max\{|f_t'(i2^{-n})| : t \in \mathcal{D}_n\},$$
$$\hat{D}_n = \max\{|f_t'(i2^{-n})| : 0 \leq t \leq 1\},$$
$$D_n^* = \max\{\Delta(j,n)^4\,|f_t'(i2^{-n})| : t = j2^{-2n} \in \mathcal{D}_n\},$$

and note that we have shown

$$\hat{K}_n \leq \sum_{j=n}^{\infty} 2^{-j}\,\hat{D}_j, \quad \hat{D}_n \leq c_3\,D_n^*, \quad \Theta_n \leq c_3\,2^{-n}\,D_n^*.$$

### 2.1 Main results

We will phrase our main results using the notation of the previous section. In this section, we assume that $g_t$ is the solution to (1) with $a = 1/4$ and $U_t$ a standard Brownian motion.

**Theorem 1** *With probability one, if $\beta < 1/4$,*

$$\lim_{n\to\infty} n^\beta \,\Theta_n = 0.$$

**Theorem 2** *For every $\alpha < \infty$, there exists $c_\alpha < \infty$ such that with probability one for all $n$ sufficiently large,*

$$D_n^* \le c_\alpha \,[n^{-\alpha}\, 2^n + (\log n)^2 \, D_n].$$

These theorems suggest an approach for determining the existence of the curve. Theorem 1 implies that $\Theta_n \downarrow 0$ with probability one. Suppose that we could find a sequence $\epsilon_n$ with

$$\sum_{n=1}^{\infty} (\log n)^2 \, \epsilon_n < \infty$$

and such that with probability one

$$\sup_n \frac{D_n}{\epsilon_n \, 2^n} < \infty. \tag{3}$$

Then Theorem 2 would imply that $\hat{K}_n \downarrow 0$ with probability one. By the Borel-Cantelli lemma, a sufficient condition to conclude (3) is

$$\sum_{n=1}^{\infty} \mathbb{P}\{D_n \ge \epsilon_n \, 2^n\} < \infty. \tag{4}$$

Note that

$$\mathbb{P}\{D_n \ge \epsilon_n \, 2^n\} \le \sum_{t\in\mathcal{D}_n} \mathbb{P}\{|f_t'(i2^{-n})| \ge \epsilon_n \, 2^n\}.$$

Hence to conclude (4), it would suffice to find a summable sequence $\delta_n$ such that for each $t \in \mathcal{D}_n$,

$$\mathbb{P}\{|f_t'(i2^{-n})| \ge \epsilon_n \, 2^n\} \le 2^{-2n} \, \delta_n. \tag{5}$$

For $SLE_\kappa$ with $\kappa \ne 8$, that is, if $g_t$ satisfies (1) with $a \ne 1/4$ and $U_t$ a standard Brownian motion, this approach works to show the existence of the curve. In fact, there exists $\theta = \theta_\kappa > 0$ such that for all $n$ sufficiently large

$$\mathbb{P}\{|f_t'(i2^{-n})| \ge 2^{n(1-\theta)}\} \le 2^{-2n(1+\theta)}.$$

Our main negative results in this paper, Theorem 3 and Corollary 1, show that this approach will not work for $\kappa = 8$ in the sense that there is no choice of summable $\epsilon_n$ and $\delta_n$ for which (5) holds. We do not know if sequences exist satisfying (4).

**Theorem 3** *For every $0 < \beta < 2$, there exist $0 < c_1 < c_2 < \infty$ such that*

$$c_1 2^{-2n}\, n^{2\beta-\frac{3}{2}} \leq \mathbb{P}\{|f_1'(2^{-n}i)| \geq 2^n\, n^{-\beta}\} \leq c_2\, 2^{-2n}\, n^{2\beta-\frac{3}{2}}\, (\log n)^5.$$

The power of $\log n$ on the right-hand side is not optimal, but we will not improve it here. The scaling property for $SLE_\kappa$ [3, Proposition 6.5] implies that the distribution of $f_t'(z)$ is the same as the distribution of $f_1'(z/\sqrt{t})$. Using this, the following corollary follows.

**Corollary 1**

$$\sum_{n=1}^{\infty} \sum_{t \in \mathcal{D}_n} \mathbb{P}\{|f_t'(i2^{-n})| \geq 2^n\, n^{-\beta}\} \begin{cases} < \infty, \ \beta < 1/4 \\ = \infty, \ \beta \geq 1/4 \end{cases}$$

*In particular, if $\epsilon_n$ is a summable sequence, then*

$$\sum_{n=1}^{\infty} \sum_{t \in \mathcal{D}_n} \mathbb{P}\{|f_t'(i2^{-n})| \geq 2^n\, \epsilon_n\} = \infty.$$

### 2.2 Modulus of continuity of Brownian motion

We review some estimates about the modulus of continuity of Brownian motion. Let $U_t$ denote a standard Brownian motion, and let $\Delta(j,n)$ be as above. Using scaling and the reflection principle for Brownian motion, there exists $c_1 < \infty$ such that for all $r > 0$,

$$\mathbb{P}\{\Delta(j,n) \geq 1 + r\} \leq c_1\, e^{-r^2/2}. \tag{6}$$

Using this and the Borel-Cantelli lemma we get the following.

- If $n \geq 1, r > 0$,

$$\mathbb{P}\left\{\Delta(j,n) \geq 1 + \sqrt{2r \log n}\right\} \leq c_1\, n^{-r}. \tag{7}$$

- With probability one, for all $n$ sufficiently large,

$$\max\left\{\Delta(j,n) : j = 0, 1, \ldots, 2^{2n}\right\} \leq 2\sqrt{n}. \tag{8}$$

## 3 Using the reverse Loewner flow

As observed in [8], estimates for $f'$ can be done using the reverse Loewner flow. The first observation is the following, see [8, Lemma 3.1].

**Proposition 2** *Suppose that $h_t$ satisfies the reverse Loewner equation*

$$\partial_t h_t(z) = \frac{1}{4[U_t - h_t(z)]}, \quad h_0(z) = z, \tag{9}$$

*where $U_t$ is a standard Brownian motion. Then $h_t'(z)$ has the same distribution as $f_t'(z)$.*

Using this and scaling, we give a restatement of Theorem 3. For convenience we write $t = 4(\log 2)n$, so that $2^n = e^{t/4}$, and

$$R_t = |h'_{e^{t/2}}(i)|.$$

**Theorem 4** *(Restatement of Theorem 3) For every* $0 < \beta < 2$*, there exist* $0 < c_1 < c_2 < \infty$ *such that if* $t \geq 2$*,*

$$c_1 e^{-t/2}\, t^{2\beta - \frac{3}{2}} \leq \mathbb{P}\{R_t \geq e^{t/4}\, t^{-\beta}\} \leq c_2\, e^{-t/2}\, t^{2\beta - \frac{3}{2}}\, (\log t)^5.$$

As in [4], we analyze $h'_t(i)$ by considering an appropriate martingale and using the Girsanov theorem to study the process weighted by the martingale. Let

$$Z_t = h_t(i) - U_t, \quad Y_t = \mathrm{Im} Z_t, \quad \Theta_t = \arg Z_t, \quad \Phi_t = |h'_t(i)|.$$

Note that $R_t = \Phi_{e^{t/2}}$.

**Proposition 3**

$$\partial_t Y_t = \frac{\sin^2 \Theta_t}{4Y_t}, \quad d\Theta_t = \frac{3\cos\Theta_t\, \sin^3\Theta_t}{2Y_t^2}\, dt - \frac{\sin^2\Theta_t}{Y_t}\, dB_t, \tag{10}$$

$$\partial_t \Phi_t = \frac{\cos^2\Theta_t \sin^2\Theta_t - \sin^4\Theta_t}{4Y_t^2}\, \Phi_t. \tag{11}$$

*Proof* These are straightforward calculations using (9) and Itô's formula.

**Proposition 4** *If*

$$M_t = \frac{\Phi_t^2}{\sin\Theta_t},$$

*then* $M_t$ *is a martingale satisfying*

$$dM_t = \frac{\cos\Theta_t\, \sin\Theta_t}{Y_t}\, M_t\, dB_t. \tag{12}$$

*In particular,*

$$\mathbb{E}[\Phi_t^2] \leq \mathbb{E}[M_t] = \mathbb{E}[M_0] = 1.$$

*Proof* Again, (12) is a straightforward Itô's formula calculation. This shows that $M_t$ is a local martingale. Since $Y_t$ increases with $t$, $|\cos\Theta_t\, \sin\Theta_t / Y_t| \leq 1$; using this, one can check that $M_t$ is a martingale.

**Corollary 2** *For every* $r > 0, t \geq 0$*,*

$$\mathbb{P}\{|f'_t(iy)| \geq r\} \leq r^{-2}. \tag{13}$$

*Proof* The distribution of $f'_t(iy)$ is the same as that of $h'_t(iy)$ which is the same as $h'_s(i)$ where $s = t/y^2$. Hence,

$$\mathbb{P}\{|f'_t(iy)| \geq r\} = \mathbb{P}\{\Phi_s \geq r\} \leq r^{-2}\, \mathbb{E}[\Phi_s^2] \leq r^{-2}.$$

### 3.1 Proof of Theorem 2

We fix $\alpha$ and allow constants to depend on $\alpha$. Let $J(j,n)$ denote the event

$$J(j,n) = \left\{ |f'_{j2^{-2n}}(i2^{-n})| \leq 2^n\, n^{-\alpha-2} \right\}.$$

If we choose $n$ sufficiently large so that (8) holds, then on the event $J(j,n)$,

$$\Delta(j,n)^4\, |f'_{j2^{-2n}}(i2^{-n})| \leq 2^{n+4}\, n^{-\alpha}.$$

By (13), $\mathbb{P}[J(j,n)^c] \leq n^{2(\alpha+2)}\, 2^{-2n}$, and by (7), we can find $C < \infty$ such that $\mathbb{P}\{\Delta(j,n) \geq C\sqrt{\log n}\} \leq c\, n^{-2(\alpha+3)}$. Also, the events $J(j,n)^c$ and $\{\Delta(j,n) \geq C\sqrt{\log n}\}$ are independent. Therefore,

$$\mathbb{P}\left\{ |f'_{j2^{-2n}}(i2^{-n})| \geq 2^n\, n^{-\alpha-2}, \Delta(j,n) \geq C\sqrt{\log n} \right\} \leq c\, 2^{-2n}\, n^{-2}.$$

In particular,

$$\sum_{n=1}^{\infty} \sum_{j=0}^{2^{2n}} \mathbb{P}\left\{ |f'_{j2^{-2n}}(i2^{-n})| \geq 2^n\, n^{-\alpha-2}, \Delta(j,n) \geq C\sqrt{\log n} \right\} < \infty,$$

and hence the Borel-Cantelli implies the following: with probability one for all $n$ sufficiently large, on the event $J(j,n)^c$, we have $\Delta(j,n) \leq C\sqrt{\log n}$ and hence

$$\Delta(j,n)^4\, |f'_{j2^{-2n}}(i2^{-n})| \leq C^4\, (\log n)^2\, |f'_{j2^{-2n}}(i2^{-n})| \leq C^4\, (\log n)^2\, D_n.$$

### 3.2 Imaginary part parametrization

As in [4,8], we reparametrize the curve so that $\log Y_t$ grows linearly. Let

$$\sigma(t) = \inf\{s : Y_s = e^{t/4}\}, \quad \hat{\Theta}_t = \Theta_{\sigma(t)}, \quad K_t = \cot \hat{\Theta}_t, \quad \hat{\Phi}_t = \Phi_{\sigma(t)}. \tag{14}$$

Since $Y_{\sigma(t)} = e^{t/4}$, we can use (10) to see that

$$\partial_t \sigma(t) = \frac{e^{t/2}}{\sin^2 \hat{\Theta}_t} = (K_t^2 + 1)\, e^{t/2}, \qquad \sigma(t) = \int_0^t e^{s/2}\,(K_s^2+1)\, ds. \tag{15}$$

Let

$$\hat{M}_t = M_{\sigma(t)} = \frac{\hat{\Phi}_t^2}{\sin \hat{\Theta}_t} = \sqrt{K_t^2+1}\, \hat{\Phi}_t^2.$$

Using (15) and Proposition 3, we see that there is a standard Brownian motion $\hat{B}_t$ such that

$$d\hat{\Theta}_t = \frac{3\cos\hat{\Theta}_t\, \sin\hat{\Theta}_t}{2}\, dt - \sin\hat{\Theta}_t\, d\hat{B}_t,$$

$$d\hat{M}_t = \hat{M}_t \cos\hat{\Theta}_t \, d\hat{B}_t,$$

$$dK_t = \frac{1}{\sin\hat{\Theta}_t} \, d\hat{B}_t - \frac{1}{2} \cot\hat{\Theta}_t \, dt, \quad K_0 = 0.$$

Let $\mathbb{P}^*$ be the probability measure obtained by weighting by the martingale $\hat{M}_t$. (We note this is the same as the measure obtained by weighting by $M_t$. Making a time change in a martingale does not affect the probability.) Then

$$d\hat{B}_t = \cos\hat{\Theta}_t \, dt + dW_t,$$

where $W_t$ is a $\mathbb{P}^*$-Brownian motion, and

$$\begin{aligned} dK_t &= \cot\hat{\Theta}_t \, dt - \frac{1}{2} \cot\hat{\Theta}_t \, dt + \frac{1}{\sin\hat{\Theta}_t} \, dW_t \\ &= \frac{1}{2} K_t \, dt + \sqrt{K_t^2 + 1} \, dW_t. \end{aligned}$$

Using Itô's formula, we see that a solution to this equation is $K_t = \sinh W_t$, and choosing $W_0 = 0$, we see that

$$W_t = \sinh^{-1} K_t.$$

Combining this relation with (11) and (14), we see that

$$\hat{\Phi}_t = e^{t/4} \, e^{-I_t/2}, \quad \text{where} \quad I_t = \int_0^t \frac{ds}{\cosh^2 W_s}. \tag{16}$$

Note also the following:

$$\sigma(t) = \int_0^t e^{s/2} \cosh^2 W_s \, ds,$$

$$\hat{M}_t = e^{t/2} \, e^{-I_t} \cosh W_t,$$

$$\Phi_t = Y_t \, \exp\left\{-I_{4\log Y_t}/2\right\},$$

$$R_t \le Y_{e^{t/2}}.$$

3.3 Upper bound for Theorem 4

**Proposition 5** *Let $E_1 = E_{1,t}$ be the event*

$$E_1 = \{|W_{t-8\log t}| \geq 8\log t\}. \tag{17}$$

*If $0 < \beta < 2$, there exists $t_0 < \infty$ such that if $t \geq t_0$,*

$$\mathbb{P}[E_1 \cap \{R_t \geq e^{t/4}\, t^{-\beta}\}] \leq t^{2\beta}\, e^{-t/2}\, e^{-8(\log t)^2}.$$

*Proof* Let $E_2 = E_{2,t}$ be the event

$$E_2 = \{|W_r| \geq 4\, \log t \ \text{ for all } r \in [t - 8\log t, t - 8\log t + 1]\}.$$

Note that on the event $E_2$, if $t \geq 2$,

$$\begin{aligned} \sigma(t - 8\log t + 1) &\geq \int_{t-8\log t}^{t-8\log t+1} e^{s/2}\, \cosh^2 W_s\, ds \\ &\geq e^{(t-8\log t)/2}\, [e^{4\log t}/2]^2 \\ &\geq \frac{1}{4}\, e^{t/2}\, t^4 > e^{t/2}. \end{aligned}$$

Hence on the event $E_2$,

$$R_t \leq Y_{e^{t/2}} \leq e^{(t-8\log t+1)/4} \leq 2\, e^{t/4}\, t^{-2} < e^{t/4} t^{-\beta}$$

for $t$ sufficiently large. Thus,

$$\mathbb{P}\left[E_1 \cap E_2 \cap \{R_t \geq e^{t/4}\, t^{-\beta}\}\right] = 0.$$

Using the reflection principle and standard estimates, we see that for $t$ sufficiently large,

$$\mathbb{P}^*(E_1 \setminus E_2) \leq \mathbb{P}^*(E_2^c \mid E_1) \leq \mathbb{P}^*\left\{\max_{0 \leq s \leq 1} B_s \geq 4\, \log t\right\} \leq \exp\left\{-8(\log t)^2\right\},$$

and hence

$$\mathbb{E}\left[R_t^2\, 1_{E_1 \setminus E_2}\right] \leq \mathbb{E}\left[M_{e^{t/2}}\, 1_{\mathbb{E}_1 \setminus E_2}\right] = \mathbb{P}^*(E_1 \setminus E_2) \leq \exp\left\{-8(\log t)^2\right\},$$

and

$$\begin{aligned} \mathbb{P}\left[E_1 \cap \{R_t \geq e^{t/4}\, t^{-\beta}\}\right] &= \mathbb{P}[(E_1 \setminus E_2) \cap \{R_t \geq e^{t/4}\, t^{-\beta}\}] \\ &\leq t^{2\beta}\, e^{-t/2}\, \mathbb{E}\left[R_t^2\, 1_{E_1 \setminus E_2}\right] \\ &\leq t^{2\beta}\, e^{-t/2}\, \exp\left\{-8(\log t)^2\right\}. \end{aligned}$$

Therefore, to prove the upper bound of Theorem 4, it suffices to bound

$$\mathbb{P}\{R_t \geq e^{t/4}\, t^{-\beta}; |W_{t-8\log t}| \leq 8\log t\}.$$

If $\sigma(t - 8\log t) \geq e^{t/2}$, then

$$R_t \leq Y_{e^{t/2}} \leq t^{-2}\, e^{t/4} < t^{-\beta} e^{t/4}.$$

Hence, it suffices to give an upper bound for $\mathbb{P}(E_3)$ where

$$E_3 = \left\{ R_t \geq t^{-\beta}\, e^{t/4} \; ; \; \sigma(t - 8\log t) \leq e^{t/2} \; ; \; |W_{t-8\log t}| \leq 8\log t \right\}.$$

Let

$$L_t = \int_0^t 1\{|W_s| \leq 3\}\, ds, \qquad \lambda = [2\cosh^2 3]^{-1},$$

and note that $I_t \geq 2\lambda L_t$. Thus, if $\sigma(t - 8\log t) \leq e^{t/2}$ and $L_{t-8\log t} \geq (2/\lambda)\log t$, then

$$R_t \leq Y_{e^{t/2}}\, e^{-\lambda L_{t-8\log t}} \leq e^{t/4}\, t^{-2},$$

where we have used the fact that $Y_{e^{t/2}} \leq [1 + (e^{t/2}/2)]^{1/2}$, which for large $t$ is bounded above by $e^{t/4}$.

Hence, it suffices to give an upper bound for $\mathbb{P}(E_4)$, where

$$E_4 = \left\{ R_t \geq t^{-\beta}\, e^{t/4} \; ; \; L_{t-8\log t} \leq (2/\lambda)\, \log t \; ; \; |W_{t-8\log t}| \leq 8\log t \right\}.$$

We do this by estimating the weighted probability $\mathbb{P}^*[E_4]$.

**Lemma 2** *For every $r < \infty$ there exists $c < \infty$ such that if $t \geq 2$,*

$$\mathbb{P}^*\left\{|W_t| \leq 8\log t; L_t \leq r\, \log t\right\} \leq c\, (\log t)^5\, t^{-3/2}.$$

*Proof* See Section 4.

In particular, $\mathbb{P}^*[E_4] \leq c\,(\log t)^5\, t^{-3/2}$. Therefore,

$$\mathbb{E}\left[R_t^2\, 1_{E_4}\right] \leq \mathbb{E}\left[M_{e^{t/2}}\, 1_{E_4}\right] = \mathbb{P}^*(E_4) \leq c\,(\log t)^5\, t^{-3/2}.$$

Chebyshev's inequality then gives

$$\mathbb{P}(E_4) \leq c\,(\log t)^5\, t^{2\beta - \frac{3}{2}}\, e^{-t/2}.$$

## 3.4 Lower bound for Theorem 4

Let $V = V(t, \rho, r)$ be the event that the following three estimates hold:

$$|W_s| \leq 3, \quad t \leq s \leq t + r,$$
$$\sigma(t) \leq e^{(t+r-1)/2},$$
$$\rho \leq I_t \leq \rho + r.$$

**Lemma 3** *There exists $r, \delta > 0$ such if $t \geq r$ and $0 \leq \rho \leq t^{1/8}$,*

$$\mathbb{P}^*(V) \geq \delta\, t^{-3/2}, \quad V = V(t, \rho, r).$$

*Proof* See Section 4.

Let $u = t + r$. Note that $t \leq \sigma^{-1}(e^{u/2}) \leq t + r$. Using only the Loewner equation, we see that there exist $c_3, c_4 > 0$ such that on the event $V$,

$$c_3\, e^{u/4}\, e^{-\rho/2} \leq R_u \leq c_4\, e^{u/4}\, e^{-\rho/2}.$$

We choose $\rho$ so that $c_3\, e^{-\rho/2} = u^{-\beta}$. Also, $|W_{\sigma^{-1}(e^{u/2})}| \leq 3$, and therefore

$$\mathbb{E}\left[R_u^2\, 1_V\right] \geq \mathbb{E}\left[\frac{M_{e^{u/2}}}{\cosh 3}\, 1_V\right].$$

From Lemma 3, we conclude that

$$\mathbb{E}\left[R_u^2\, 1_V\right] \geq \delta'\, u^{-3/2}$$

for some $\delta' > 0$. Since $R_u \leq c_4\, e^{u/4}\, e^{-\rho/2}$,

$$\mathbb{P}(V) \geq [c_4\, e^{u/4}\, e^{-\rho/2}]^{-2}\, \mathbb{E}\left[R_u^2\, 1_V\right] \geq c\, e^{\rho}\, u^{-3/2} e^{-u/2}.$$

Therefore,

$$\mathbb{P}\left\{R_u \geq u^{-\beta}\, e^{u/4}\right\} = \mathbb{P}\left\{R_u \geq c_3\, e^{u/4}\, e^{-\rho/2}\right\} \geq c\, e^{\rho}\, u^{-3/2}\, e^{-u/2}.$$

## 3.5 Proof of Theorem 1

Let $\beta < \alpha < 1/4$. By Corollary 1,

$$\sum_{n=1}^{\infty} \sum_{t \in \mathcal{D}_n} \mathbb{P}\{|f_t'(i2^{-n})| \geq\ (\log n)^{-3}\, n^{-\alpha}\, 2^n\} < \infty.$$

Hence, by the Borel-Cantelli Lemma and Theorem 2, we know that with probability one for all $n$ sufficiently large, $D_n^* \leq n^{-\alpha}\, 2^n$, and hence

$$\lim_{n \to \infty} n^{\beta}\, \Theta_n \leq c_3 \lim_{n \to \infty} n^{\beta}\, 2^{-n}\, D_n^* = 0.$$

## 4 Estimates for Brownian motion

In this section $W_t$ will be a standard Brownian motion with respect to the probability measure $\mathbb{P}$. We write $\mathbb{P}^x$ to denote expectations assuming that $W_0 = x$.

**Lemma 4** *There exists $c < \infty$ such that if $u, n \geq 1$, $T = T_u = \inf\{s : |W_s| = u\}$, and*

$$J_r = \int_0^r 1\{|W_s| < 1\}\, ds,$$

*then*

$$\mathbb{P}^x\{J_T \leq n\} \leq \frac{c\,(|x| + n)}{u}.$$

*Proof* The result is trivial if $u \leq 3$ so we assume $u \geq 3$. Let us first assume that $|x| \leq 1$. Let $\tau_0 = \inf\{s : |W_s| = 1\}$, and define recursively, $\sigma_k = \inf\{s \geq \tau_k : |W_s| = 2\}$, $\tau_k = \inf\{s \geq \sigma_{k-1} : |W_s| = 1\}$. Let

$$q = \mathbb{P}\{J_{\sigma_k} - J_{\tau_k} > 1 \mid W_s, 0 \leq s \leq \tau_k\} > 0.$$

Then by the strong Markov property and the gambler's ruin estimate,

$$\begin{aligned}\mathbb{P}^x\{J_T \leq 1; \sigma_k \leq T < \sigma_{k+1}\} &\leq \mathbb{P}^x\{J_{\sigma_K} \leq 1\}\, \mathbb{P}^x\{T < \sigma_{k+1} \mid T \geq \sigma_k\} \\ &\leq (1-q)^k\, 2/u.\end{aligned}$$

By summing over $k$, we see that $\mathbb{P}\{J_T \leq 1\} \leq 2/(uq)$. Using the strong Markov property again, we see that $\mathbb{P}^x\{n \leq J_T \leq n+1\} \leq \mathbb{P}^x\{J_T \leq n+1 \mid J_T \geq n\} \leq 2/(uq)$. Summing over $n$ gives $\mathbb{P}^x\{J_T \leq n\} \leq 2n/(uq)$. If $|x| > 1$, let $\rho = \inf\{t : |W_t| = 1\}$. Then the gambler's ruin estimate implies that

$$\mathbb{P}^x\{J_T \leq n\} \leq \mathbb{P}^x\{\rho > T\} + \mathbb{P}^x\{J_T \leq n \mid \rho < T\} \leq \frac{|x|-1}{u-1} + \frac{2n}{uq} \leq \frac{c\,(|x|+n)}{u}.$$

**Lemma 5** *There exists $c < \infty$ such that if $n \geq 1, t \geq 2$, then*

$$\mathbb{P}^x\{J_t \leq n\} \leq \frac{c\,(|x| + n)\, \log t}{\sqrt{t}}.$$

This inequality is true even without the factor of $\log t$ on the right-hand side, but we will only need this weaker result which is easier to prove.

*Proof* Let $u = \delta\sqrt{t}/\log t$. By standard estimates, there exists $\delta > 0$ such that $\mathbb{P}^x\{T_u \geq t\} \leq t^{-1}$. If $|x| \leq u$,

$$\mathbb{P}^x\{J_t \leq n\} \leq t^{-1} + \mathbb{P}^x\{I_{T_u} \leq n\} \leq \frac{c\,(|x|+n)\, \log t}{\sqrt{t}}.$$

The inequality is immediate for $|x| \geq u$.

It will be convenient to construct a Brownian motion as follows. Suppose that $W_s^j, 1 \leq j \leq 3$, are independent Brownian motions and

$$W_s = W_s^1, \quad 0 \leq s \leq \frac{t}{3},$$

$$W_s = W_{t/3}^1 + W_{s-(t/3)}^2, \quad \frac{t}{3} \leq s \leq \frac{2t}{3},$$

$$W_s = W_{t/3}^1 + W_{t/3}^2 + [W_{t-s}^3 - W_{t/3}^3], \quad \frac{2t}{3} \leq s \leq t.$$

Then it is easy to see that $W_s, 0 \leq s \leq 3t$ is a standard Brownian motion, and $W_t = W_{t/3}^2 + (W_{t/3}^1 - W_{t/3}^3)$. Let $\mathcal{F}_t$ denote the $\sigma$-algebra generated by $\{W_s^1, W_s^3 : 0 \leq s \leq t/3\}$. Since $W_{t/3}^2 \sim N(0, t/3)$, if $E \in \mathcal{F}_t$ with $\mathbb{P}(E) > 0$, then

$$\mathbb{P}\{a \leq W_t \leq b \mid E\} \leq \sqrt{\frac{3}{2\pi t}}\,(b-a). \tag{18}$$

In the opposite direction, we note that if $\epsilon > 0$, and $Y_s, 0 \leq s \leq 1$, is a Brownian bridge, then $\mathbb{P}\{\max_{0 \leq s \leq 1} |Y_s| \leq \epsilon\} > 0$. Using this and scaling we can see the following.

– There exists $c > 0$ such that if $E \in \mathcal{F}_t$ with $\mathbb{P}(E) > 0$ such that on $E$,

$$\frac{\sqrt{t}}{2} \leq W_{t/3}^1, W_{t/3}^3 \leq 2\sqrt{t},$$

then

$$\mathbb{P}\left\{|W_t| \leq 1\,;\; \frac{\sqrt{t}}{4} \leq W_s \leq 3\sqrt{t} \;\text{ for }\; \frac{t}{3} \leq s \leq \frac{2t}{3} \mid E\right\} \geq c\,t^{-1/2}. \tag{19}$$

*Proof (Proof of Lemma 2)* We fix $r$ and allow constants to depend on $r$. For each $n \in \mathbb{Z}$, let $V = V(n,t)$ be the event

$$V = \{n \leq W_t < n+1,\; L_t \leq r\,\log t\}.$$

It suffices to show that if $|n| \leq 9 \log t$,

$$\mathbb{P}(V) \leq c\,(\log t)^4\,t^{-3/2}.$$

Note that $V \subset E$ where $E = E(n,t)$ is the event that the following two inequalities hold.

$$\int_0^{t/3} 1\{|W_s^1| \leq 1\}\,ds \leq r\,\log t,$$

$$\int_0^{t/3} 1\{|W_s^3 - n| \leq 1\}\,ds \leq r\,\log t.$$

Using Lemma 5 and the independence of $W^1$ and $W^3$, we see that $\mathbb{P}(E) = O((\log t)^4/t)$. Hence by (18),

$$\mathbb{P}(V) \leq \mathbb{P}[E \cap \{n \leq |W_t| < n+1\}] \leq \mathbb{P}(E)\,\mathbb{P}\{n \leq |W_t| < n+1 \mid E\} \leq \frac{c\,(\log t)^4}{t^{3/2}}.$$

*Proof (Proof of Lemma 3)* Let $V^1 = V^1(t, \rho, r)$ be the event that

$$|W_t| \le 1, \quad \sigma(t) \le e^{(t+r-1)/2}, \quad \rho \le I_t \le \rho + r.$$

Then $\mathbb{P}(V) \ge \mathbb{P}(V^1)\, \mathbb{P}\{|W_s - W_t| \le 2, t \le s \le t + r\}$. Hence, it suffices to find $c, r$, such that $\mathbb{P}(V^1) \ge c\, t^{-3/2}$.

Let $T = \inf\{s : W_s^3 = \sqrt{t}\}$, and let $\hat{E}_3$ denote the event that

$$T \le \frac{t}{3},$$
$$W_s^3 \ge -1, \quad 0 \le s \le T,$$
$$W_s^3 \ge 5, \quad 1 \le s \le T.$$

Using the gambler's ruin estimate, it is straightforward to see that there exists $c_1 > 0$ with $\mathbb{P}(\hat{E}^3) \ge 2\, c_1/\sqrt{t}$. For every integer $k \ge 5$ and $C < \infty$, let $\hat{E}^3(k, C)$ denote the intersection of $\hat{E}^3$ with the event

$$\int_0^T 1\{k \le W_s^3 \le k+1\}\, ds \ge C\, k \log k.$$

We claim that there exists $C < \infty$ such that for all $k \ge 5$,

$$\mathbb{P}[\hat{E}^3(k, C) \mid \hat{E}^3] \le \frac{1}{(2k)^2}. \tag{20}$$

To see this, we use the gambler's ruin estimate to see that

$$\mathbb{P}^{k+1}\{W_s^3 \le k \text{ for some } s \le T \mid \hat{E}^3\} \le \frac{k}{k+1} = 1 - \frac{1}{k+1}.$$

Let $\theta = \mathbb{P}\{W_1^3 \ge W_0^3 + 1\} > 0$. Then we can see that

$$\mathbb{P}\left\{\int_0^T 1\{k \le W_s^3 \le k+1\}\, ds \le 1\right\} \ge \frac{\rho}{k+1}.$$

Using the strong Markov property, for each $n$,

$$\mathbb{P}\left\{\int_0^T 1\{k \le W_s^3 \le k+1\}\, ds \ge n \;\middle|\; \hat{E}^3\right\} \le \left[1 - \frac{\rho}{k+1}\right]^n.$$

If we choose $n = C\, k \log k$ for $C$ large enough, the right-hand side is less than $(2k)^{-2}$. This gives (20) and by summing we see that

$$\mathbb{P}\left[\bigcup_{k=5}^{\infty} \hat{E}^3(k, C) \mid \hat{E}^3\right] \le \sum_{k=5}^{\infty} \frac{1}{(2k)^2} < 1.$$

We fix such a $C$ and we let $\tilde{E}^3$ be the intersection of $\hat{E}^3$ with the event

$$\int_0^T 1\{k \le |W_s^3| \le k+1\}\, ds \le C\, k \log k, \quad k = 5, 6, \ldots.$$

We have shown that there exists $c'$ such that $\mathbb{P}(\hat{E}^3) \geq c'/\sqrt{t}$. We now let $E^3$ be the intersection of $\tilde{E}^3$ with the event

$$\left\{ |W_s^3 - W_T^3| \leq \frac{\sqrt{t}}{2}, \quad T \leq s \leq T + \frac{t}{3} \right\}.$$

Then we can see that $\mathbb{P}(E^3) \geq c/\sqrt{t}$.

Let

$$\tau = \tau_\rho = \inf \left\{ s : \int_0^s \frac{dr}{\cosh^2 W_r^1} = \rho \right\},$$

and let $\tilde{W}_s^1 = W_{s+\tau}^1 - W_\tau^1$. Let $E^1$ denote the event

$$\tau \leq t^{1/4}, \quad W_\tau^1 \geq 0,$$

and that $\tilde{W}_s^1$ satisfies the same conditions as the event $\tilde{E}^3$ defined in the previous paragraph. We can find $c_2$ such that $\mathbb{P}(E^1) \geq c_2/\sqrt{t}$. Let

$$E = E^3 \cap E_1 \cap \left\{ |W_t| \leq 1, \quad \frac{\sqrt{t}}{4} \leq W_s \leq 2\sqrt{t}, \quad \frac{t}{3} \leq s \leq \frac{2t}{3} \right\}.$$

Using (19), we can see that $\mathbb{P}(\tilde{E}) \geq c\,\mathbb{P}(E^3 \cap E^1)/t^{1/2} \geq c/t^{3/2}$.

By the definition, we can see that there exists $r < \infty$ such that on the event $E$, $\rho \leq I_t \leq \rho + r$.

If $0 \leq s \leq t/3$, then

$$|W_{t-s}| \leq 1 + |W_s^3| \leq 1 + C\sqrt{s+2}\,\log(s+2) \leq C'(s+1)^{2/3}.$$

Combining this with $|W_s| \leq s^{2/3}$, we see that on the event $E$

$$\sigma(t) = \int_0^t e^{s/2} \cosh^2 W_s\, ds = e^{t/2} \int_0^t e^{-s/2} \cosh^2 W_{t-s}\, ds \leq c\, e^{t/2}.$$

Therefore, there exists $r$ such that $E \subset V^1(t,\rho,r)$.

## References


1. F. Johansson Viklund, G. Lawler (2011). Optimal Hölder exponent for the SLE path, Duke Math. J. **159**, 353–383.
2. F. Johansson Viklund, G. Lawler (2012). Almost sure multifractal spectrum for the tip of an SLE curve, Acta Math. **209**, 265–322.
3. G. Lawler (2005). *Conformally Invariant Processes in the Plane*, American Mathematical Society, Providence, RI.
4. G. Lawler (2009). Multifractal analysis of the reverse flow for the Schramm-Loewner evolution, in *Fractal Geometry and Stochastics IV*, Birkhäuser, Boston, 73–107.
5. G. Lawler, O. Schramm, W. Werner (2004). Conformal invariance of planar loop-erased random walks and uniform spanning trees, Ann. Probab. **32**, 939–995.
6. J. Lind (2008). Hölder regularity of the SLE trace, Trans. Amer. Math. Soc. **360**, 3557–3578.
7. D. Marshall, S. Rohde (2005). The Loewner differential equation and slit mappings, J. Amer. Math. Soc. **18**, 763–778.
8. S. Rohde, O. Schramm (2005). Basic properties of SLE, Ann. of Math. (2) **161**, 883–924.
9. O. Schramm (2000). Scaling limits of loop-erased random walks and uniform spanning trees, Israel J. Math. **118**, 221–288.

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