1 Definition

The Schramm-Loewner evolution (SLE) was defined by Oded Schramm as the only conformally invariant families of random curves that satisfy the domain Markov property. To make a precise definition we consider the implications of these assumptions on probability measures $\mathbb{P}$ on simple curves $\gamma : (0, \infty) \to \mathbb{H}$ with $\gamma(0+) = 0$. We will consider curves modulo (increasing) reparametrization, that is, we are only interested in the path that is traversed and not on how "quickly" one goes through the path. The assumptions on $\mathbb{P}$ are *scale invariance* and the *conformal Markov property*,

- **Scale invariance.** If $r > 0$ and $\mathbb{P}_r$ denotes the measure on curves obtained by considering $r\gamma$, then $\mathbb{P}_r = \mathbb{P}$. (Remember we are considering curves modulo reparametrization so, for example, the point mass on the straight line $\gamma(t) = it$ satisfies this property.)
- **Conformal Markov property**. Suppose the beginning segment $\gamma_t = \gamma[0, t]$ is observed and let $g : \mathbb{H} \setminus \gamma_t \to \mathbb{H}$ be a conformal transformation with $g(\gamma(t)) = 0, \hat{g}(\infty) = \infty$. Then the conditional distribution of $g[\gamma[t, \infty))$ given γ_t is $\mathbb{P}$.

The conformal transformation g is not unique, but any other such transformation is of the form rg for some $r > 0$; hence, scale invariance implies that the distribution of $g[\gamma[t, \infty))$ is independent of the choice of g.

Let us work towards the definition. Suppose $\mathbb{P}$ satisfies scale invariance and the conformal Markov property and is supported on simple curves γ with $\gamma(0, \infty) \subset \mathbb{H}$. Let $\gamma_t = \gamma[0, t]$ and let $g_t : \mathbb{H} \setminus \gamma_t \to \mathbb{H}$ with $g_t(z) = z + o(1)$ as $z \to \infty$. Theorem **??** implies that if we parametrize the curve so that $\mathrm{hcap}[\gamma_t] = 2t$, then

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z,$$

where U_t is a random function with continuous paths. Conformal invariance and the domain Markov property imply that U_t is a continuous process with stationary, independent increments. This implies that U_t must be a (one-dimensional) Brownian motion with drift m and variance parameter κ. Scale invariance can be used to see that $m = 0$. This leaves one parameter, κ and SLE_κ was defined as the solution of this equation with driving function equal to a driftless Brownian motion with variance parameter κ.

For ease, we will choose a slightly different parametrization. If g_t is as above and $\tilde{g}_t = g_{t/\kappa}$ then

$$\partial_t \, \tilde{g}_t = \frac{2/\kappa}{\tilde{g}_t(z) - \tilde{U}_t}, \quad g_0(z) = z,$$

where $\tilde{U}_t = U_{t/\kappa}$ which is a standard Brownian motion. Here we have parametrized so that $\mathrm{hcap}[\tilde{\gamma}_t] = at$ where $a = 2/\kappa$. At the moment, we can take all of this as motivation for the following definition.

Definition Suppose $a = 2/\kappa > 0$ and $U_t = -B_t$ is a standard one-dimensional Brownian motion. Let g_t denote the solution of the initial value problem

$$\dot{g}_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z. \tag{1}$$

Then g_t is called the *Schramm-Loewner evolution with parameter κ (SLE_κ) from 0 to ∞ in $\mathbb{H}$.*

We have started by defining SLE_κ as the random collection of maps $\{g_t\}$. We will also use the term for the curve induced by the maps, but it requires some work to show that the latter is well defined. We know from Section **??** that for all $z \in \mathbb{C} \setminus \{0\}$. the solution to (1) exists for all $t < T_z = \inf\{t : g_t(z) - U_t = 0\}$. If we write $Z_t(z) = g_t(z) - U_t$ we can write (1) as a stochastic differential equation

$$dZ_t(z) = \frac{a}{Z_t(z)}\,dt + dB_t, \quad Z_0(z) = z.$$

This is an example of a stochastic flow, that is, a family of process $Z_t(z)$ indexed by starting points $z \in \mathbb{C} \setminus \{0\}$ where the same Brownian motion B_t is used for all of the processes. One must take care in reading the equation. The quantity $Z_t(z)$ is complex-valued, but the Brownian motion B_t is real-valued.

If z is fixed and we write

$$Z_t = Z_t(z) = X_t + i\,Y_t, \tag{2}$$

then (2) becomes

$$dX_t = \frac{aX_t}{X_t^2 + Y_t^2}\,dt + dB_t = X_t\,\left[\frac{a}{X_t^2 + Y_t^2}\,dt + \frac{1}{X_t}\,dB_t\right], \quad X_0 = \mathrm{Re}[z]; \tag{3}$$

$$\partial_t Y_t = -\,Y_t\,\frac{a}{X_t^2 + Y_t^2}, \quad Y_0 = \mathrm{Im}[z]. \tag{4}$$

If $z = x \in \mathbb{R} \setminus \{0\}$, then $Y_t = 0$ for all t, and X_t satisfies the Bessel equation

$$dX_t = \frac{a}{X_t}\,dt + dB_t, \quad X_0 = x. \tag{5}$$

Let us review our logic. We started with trying to find probability measures on curves that satisfy scale invariance and the conformal Markov property. We conclude that the only possible candidates can be reparametrized so they are solutions to the Loewner equation whose driving function is a driftless Brownian motion. We then use the Loewner equation as the definition for SLE, but it remains to see if this is a measure on curves for all values of $\kappa > 0$. While this is true, we will see that it is not always a measure on *simple* curves.

Here and throughout, we will reserve partial derivative notation ∂_t for actual derivatives. Stochastic differentials will be denoted by d. As above, we will often drop the z dependence on Z_t, X_t, Y_t and similar quantities but it is important to remember that they depend on the initial point.

Our definition used a particular parametrization on the curves. This parametrization is very useful for analysis of the curve but is not always the most natural. We are really considering curves modulo reparametrization, so we should also think of SLE as a probability on random maps $\{g_t\}$ modulo reparametrization. We prove some easy facts about C^1 time changes.

Proposition 1.1. *Suppose g_t satisfies* (1), $\sigma : [0,\infty) \to [0,\infty)$ *is a C^1 function with $\dot\sigma > 0$, and $\tilde g_t(z) = g_{\sigma(t)}(z)$. Then*

$$\partial_t \tilde g_t(z) = \frac{a\,\dot\sigma(t)}{\tilde g_t(z) - \tilde U_t}, \quad \tilde U_t = U_{\sigma(t)}.$$

Proof. Immediate from the chain rule. □

A particularly important application of this is the scaling rule for SLE which shows that SLE in the upper half plane (in the capacity parametrization) satisfies Brownian (heat equation) scaling.

Proposition 1.2 (Scaling)**.** *Suppose g_t satisfies* (1)*, $r > 0$, and $\tilde{g}_t(z) = r^{-1}\, g_{r^2 t}(rz)$. Then $\tilde{g}_t$ has the distribution of SLE_κ.*

Proof. Using the chain rule, we see that

$$\partial_t \tilde{g}_t(z) = r\, \dot{g}_{r^2 t}(rz) = \frac{ar}{g_{r^2 t}(rz) - U_{r^2 t}} = \frac{a}{\tilde{g}_t(z) - \tilde{U}_t},$$

where $\tilde{U}_t = r^{-1}\, U_{r^2 t}$. The scaling property of Brownian motion implies that $\tilde{U}_t$ is a standard Brownian motion.

□

We write $H_t = \{z \in \mathbb{H} : T_z > 0\}$ and let $K_t = \mathbb{H} \setminus H_t$ be the corresponding hull. Under our parametrization, $\mathrm{hcap}(K_t) = at$. Although we will not prove it at the moment, the following theorem holds.

Theorem 1. *There exists a curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ such that for each t, H_t is the unbounded component of $\mathbb{H} \setminus \gamma_t$ where $\gamma_t = \gamma[0, t]$.*

We call γ the SLE_κ *curve (parametrized so that* $\mathrm{hcap}[\gamma_t] = at$*).*

Proof. We will not prove this at the moment. We would like to define

$$\gamma(t) = g_t^{-1}(U_t) = \lim_{y \downarrow 0} g_t^{-1}(U_t + iy).$$

However, there are Loewner chains for which the limit does not exists for some t, and other chains for which the limit exists but does not give a continuous function of t. For $\kappa \neq 8$, we show in Section 8 that with probability one, the conditions of Proposition **??** hold. The case $\kappa = 8$ is much more delicate and will not be proved in this book. □

Many of the statements we make can be phrased without reference to the curve, but it will be easier to assume this theorem now. In order to do this, it will be useful to define the set γ_t in a way that does not require γ to be a curve. We say that z is a *pioneer point* for the Loewner chain $\{g_s\}$ at time t if $z \in H_s$ for all $s < t$ and $z \in \partial H_t$. We let γ_t be the union of all pioneer points with $s \leq t$. Then we have

- If the chain is generated by a curve γ, then $\gamma_t = \gamma[0, t]$.
- H_t is the unbounded connected component of $\mathbb{H} \setminus \gamma_t$.

Note that $\partial H_t \subset \mathbb{R} \cup \gamma_t$. If $T_z < \infty$, then there are two possibilities: either $\gamma(T_z) = z$, or there exists a bounded connected component of $\mathbb{H} \setminus \gamma_{T_z}$ containing z. In the latter case, this will also be the connected component of $\mathbb{H} \setminus \gamma_t$ containing z for all $t > T_z$.

In the definition of SLE, the curve is parametrized by half-plane capacity. This is what allows stochastic calculus to be used to analyze the curve. However, this is not necessarily the most intrinsic parametrization for curves. For this reason, it is often more useful to view SLE as a measure on curves *modulo reparameterization.*

Definition Suppose γ is a random curve from 0 to ∞ in $\mathbb{H}$ and let

$$a(t) = \inf\{s : \mathrm{hcap}(\gamma_s) = (2/\kappa)\, t\}.$$

- We say that γ is an SLE_κ *path* if $\tilde{\gamma}(t) := \gamma(a(t))$ is an SLE_κ curve as above.
- We say that γ has a *capacity parametrization* if $a(t)$ is a strictly increasing C^1 function of t.

Using this definition and Proposition 1.2, we can see that if γ is an SLE_κ then so is $r\gamma$ for all r; moreover, $r\gamma$ has a capacity parametrization if and only if γ does.

Definition Suppose D is a simply connected domain and z, w are distinct points on ∂D. Let $F : D \to \mathbb{H}$ be a conformal transformation with $F(z) = 0, F(w) = \infty$. We say that a random curve γ is a *chordal* SLE_κ from z to w in D if $\tilde{\gamma}(t) := F(\gamma(t))$ is an SLE_κ from 0 to ∞ in $\mathbb{H}$. We say that γ has a capacity parametrization if $\tilde{\gamma}$ has a capacity parametrization.

The conformal transformation F in the last definition is not unique. However, if $\hat{F}$ is another such transformation, then $\hat{F} = rF$ for some $r > 0$. Using the discussion before the last definition, we can see the definition is independent of which F we choose.

We are using parametrization by capacity to mean parametrizing so that $a(t)$ is a linear function of t but "a capacity paramterization" if $a(t)$ is C^1. The main other cases of capacity parametrizations will be the radial parameterization (with respect to a particular interior point) and the imaginary part parametrization in $\mathbb{H}$.

---

2 The curve near a point $z \in \overline{\mathbb{H}} \setminus \{0\}$

The Schramm-Loewner evolution is unusual in that the curve is defined indirectly in terms of the conformal maps g_t. If the curve grows with a capacity parametrization, the effect of the curve on points away from the curve is described by a relatively simple SDE. The curve itself is very irregular, and one cannot describe its motion as an SDE. Indeed, the dynamics of the curve are very nonMarkovian. The basic strategy is to use the behavior "away from the curve" to determine facts about the curve.

Consider the questions.

- How "thick" is the curve γ, that is, what is its fractal dimension?
- Could it possibly be plane filling and hit every point?

Addressing these questions may seem daunting, especially since we have not proved that the curve exists, but let us start with some heuristics. Suppose the curve had fractal dimension d. If we consider the intersection of the curve with a closed disk of, say, radius 1, then the expected number of disks of radius ϵ needed to cover the curve would be of order ϵ^{-d}. Since the number of discs needed to cover the disc of radius 1 of order ϵ^{-2}, we see that for a fixed disc of radius ϵ, we would expect that the probability of hitting that disk is order ϵ^{2-d}.

Using this as motivation, we will studying the following questions in this section.

- For a fixed $z \in \overline{\mathbb{H}} \setminus \{0\}$, does the curve hit z?
- If not, what is the probability that it gets within distance ϵ of z?
- If it misses z does it go to the "left" or to the "right" of z?

The answers will depend on κ and the results will be slightly different for $z \in \mathbb{H}$ and for $z \in \mathbb{R} \setminus \{0\}$.

We can phrase these questions in terms of the domains H_t, which we know are well defined, rather than in terms of γ, so we do not need to prove existence of the curve before discussing them. For example, if $\mathrm{dist}(z, \gamma_t) < \mathrm{Im}(z)$, then $\mathrm{dist}(z, \gamma_t) = \mathrm{dist}(z, \partial H_t)$. The latter quantity is comparable to the conformal radius $\mathrm{crad}_{H_t}(z)$.

We now fix z and let $Z_t = Z_t(z) = X_t + iY_t$ as in (2)–(5). If $z = x \in \mathbb{R} \setminus \{0\}$, then $Y_t = 0$ for all x, and we often write X_t instead of Z_t. By differentiating (1) with respect to z, we see that

$$\partial_t g_t'(z) = -\frac{a\, g_t'(z)}{(g_t(z) - U_t)^2} = -\frac{a\, g_t'(z)}{Z_t^2},$$

$$\partial_t [\log g_t'(z)] = -\frac{a}{Z_t^2}.$$

Although $\log g_t'(z)$ is defined only up to an initial additive multiple of $2\pi i$, the derivative $\partial_t[\log g_t'(z)]$ is independent of the choice. The next proposition follows immediately.

Proposition 2.1. *If $z \in \mathbb{H}$ and u_t, v_t are defined by $\log g_t'(z) = u_t(z) + iv_t(z)$, then for $t < T_z$,*

$$\partial_t u_t(z) = \frac{a\,(Y_t^2 - X_t^2)}{(X_t^2 + Y_t^2)^2}, \quad \partial_t v_t(z) = \frac{2aX_tY_t}{(X_t^2 + Y_t^2)^2}.$$

In particular,

$$\partial_t |g_t'(z)| = |g_t'(z)|\, \frac{a\,(Y_t^2 - X_t^2)}{(X_t^2 + Y_t^2)^2}. \tag{6}$$

If $x \in \mathbb{R} \setminus \{0\}$,

$$\partial_t g_t'(x) = -a\, \frac{g_t'(x)}{X_t^2}. \tag{7}$$

As before, we write $\partial_t g_t$ or $\dot{g}_t$ for derivatives with respect to time and reserve the prime notation $'$ for spatial derivatives.

Since g_t is a conformal transformation of H_t onto $\mathbb{H}$ we can see that $g_t'(x) > 0$ for all $x \in \mathbb{R}$ with $t < T_x$. For this reason we do not need absolute values in (7).

For now, let us assume that $z \in \mathbb{H}$. It is standard to represent a point in $\mathbb{H}$ in either rectangular coordinates $x + iy$ or polar coordinates $re^{i\theta}$. We will choose a compromise by representing the point by (y, θ). Here θ is the argument and y is the imaginary part which can also be considered as one-half the conformal radius of $\mathbb{H}$ with respect to y. The representation $(\mathrm{crad}/2, \theta)$ works well with conformal transformations and we will use this. Let

$$\Theta_t = \Theta_t(z) = \arg Z_t(z), \quad \Upsilon_t = \Upsilon_t(z) = \frac{Y_t}{|g_t'(z)|} = \frac{1}{2}\,\mathrm{crad}_{H_t}(z).$$

The last equality is justified in the following proposition.

Proposition 2.2. *If $t < T_z$, then Υ_t is one-half times the conformal radius of H_t with respect to z. In particular,*

$$\frac{\operatorname{dist}(z, \partial H_t)}{2} \le \Upsilon_t \le 2 \operatorname{dist}(z, \partial H_t). \tag{8}$$

Proof. Recall that by definition, $\operatorname{crad}_D(z) = |f'(z)|^{-1}$ where $f : D \to \mathbb{D}$ is a conformal transformation with $f(z) = 0$. By using a conformal transformation of $\mathbb{H}$ onto $\mathbb{D}$ we see that $\operatorname{crad}_{\mathbb{H}}(x+iy) = 2y$. More generally, $\operatorname{crad}_{H_t}(z) = |g_t'(z)|^{-1} \operatorname{crad}_{\mathbb{H}}(g_t(z)) = 2Y_t/|g_t'(z)|$. The inequalities in (8) follow from the Koebe 1/4-theorem, see (**??**). □

If $T_z < \infty$, we define $\Upsilon_t = \Upsilon_{T_z} := \lim_{s \uparrow T_z} \Upsilon_s$ for $t \ge T_z$. This is either zero (if $\operatorname{dist}(\gamma_{T_z}, z) = 0$) or equals $\operatorname{crad}_{H_z}(z)/2$, where H_z is the component of $\mathbb{H} \setminus \gamma_{T_z}$ containing z. Similarly, we define

$$\Upsilon_\infty(z) = \lim_{t \to \infty} \Upsilon_t(z),$$

noting that $\Upsilon_\infty(z) = \Upsilon_{T_z}(z)$ if $T_z < \infty$. Using the Koebe 1/4-theorem as above, we see that

$$\Upsilon_\infty(z) \asymp_2 \operatorname{dist}(z, \gamma \cup \mathbb{R}).$$

(Here we write $f(x) \asymp_r g(x)$ if $r^{-1} f(x) \le g(x) \le r\, f(x)$.)

Let us consider what happens to the curve from the perspective of the point z up to the time T_z. There are three possibilities for a continuous curve going to infinity:

- The curve never reaches z and goes to infinity never trapping z. In this case $\Upsilon_t \downarrow \Upsilon_\infty > 0$ and as $z \to \infty$, $\Theta_z \to 0$ or $\Theta_z \to \pi$ depending on whether the curve goes to the "left" of z or to the "right" of z, respectively.
- The curve never reaches z and at the finite time T_z it disconnects z from infinity. Then $\Upsilon_{T_z} > 0$ and Θ_{T_z} equals 0 or π depending on whether the loop formed around z goes to the "left" of z or to the "right" of z, respectively.
- The curve reaches z in finite time in which case $\Upsilon_{T_z} = 0$ and we would expect Θ_t to fluctuate up to time T_z.

We will now investigate the quantities Υ_t, Θ_t using the Loewner equation and standard methods in stochastic calculus.

Proposition 2.3. *For $t < T_z$,*

$$\partial_t \Upsilon_t = -\Upsilon_t \frac{2aY_t^2}{(X_t^2 + Y_t^2)^2}. \tag{9}$$

$$d\Theta_t = \frac{(1-2a)\, X_t\, Y_t}{(X_t^2 + Y_t^2)^2}\, dt - \frac{Y_t}{X_t^2 + Y_t^2}\, dB_t. \tag{10}$$

Proof. Both Y_t and $|g_t'(z)|$ are differentiable with respect to t with

$$\partial Y_t = -\frac{a\, Y_t}{X_t^2 + Y_t^2} = -Y_t \frac{a(X_t^2 + Y_t^2)}{(X_t^2 + Y_t^2)^2},$$

so (9) follows from (6) and the product rule.

Let $L_t = \log Z_t$. Then Itô's formula shows that

$$dL_t = \frac{1}{Z_t}\, dZ_t - \frac{1}{2Z_t^2}\, d\langle Z\rangle_t = \frac{a-\frac{1}{2}}{Z_t^2}\, dt + \frac{1}{Z_t}\, dB_t$$

Since $\Theta_t = \mathrm{Im} L_t$, we see that

$$d\Theta_t = \mathrm{Im}\left[\frac{a-\frac{1}{2}}{Z_t^2}\right] dt + \mathrm{Im}\left[\frac{1}{Z_t}\right] dB_t = \frac{(1-2a)\, X_t\, Y_t}{(X_t^2+Y_t^2)^2}\, dt - \frac{Y_t}{X_t^2+Y_t^2}\, dB_t.$$

□

In the proof we used Itô's formula with a complex function and then took the imaginary part. The reader may wish to check that this was legitimate!

The equation (10) will become nicer if we change the parametrization. Since the conformal radius $\mathrm{crad}_{H_t}(z)$ is strictly decreasing, $\phi(\mathrm{crad}_{H_t}(z))$ will be strictly increasing if ϕ is strictly decreasing and can be used to reparametrize the curve. As the curve approaches z, the argument of z will tend to vary (roughly speaking) in a way that when one halves the distance to z the argument has a change of order one. With this intuition, we can see that will be useful to parametrize the curve so that the logarithm of the conformal radius decays linearly. This is an example of a *radial parametrization* (with respect to z). Let

$$\sigma(t) = \sigma(t,z) = \inf\{s : \log(\Upsilon_0/\Upsilon_s) = 2at\}, \tag{11}$$

and define

$$\hat{Z}_t = \hat{X}_t + i\hat{Y}_t = Z_{\sigma(t)}, \quad \hat{\Theta}_t = \Theta_{\sigma(t)}, \quad \hat{\Upsilon}_t = \Upsilon_{\sigma(t)} = e^{-2at}\,\Upsilon_0.$$

(The choice of $2a$ in the exponent was made so that the coefficient of the Brownian motion in (12) below is 1.) Since $\partial_t \hat{\Upsilon}_t = -2a\hat{\Upsilon}_t$ and the chain rule gives $\partial_t \hat{\Upsilon}_t = \dot{\Upsilon}_{\sigma(t)}\, \dot{\sigma}(t)$, we see from (9) that

$$\dot{\sigma}(t) = \frac{(\hat{X}_t^2+\hat{Y}_t^2)^2}{\hat{Y}_t^2}.$$

Using this, we see that (10) implies that

$$d\hat{\Theta}_t = (1-2a)\,\frac{\hat{X}_t}{\hat{Y}_t}\, dt + d\hat{B}_t = (1-2a)\,\cot\hat{\Theta}_t\, dt + d\hat{B}_t, \tag{12}$$

where $\hat{B}_t$ is a standard Brownian motion. This time change is very useful when viewing the flow from a particular starting point z. However, different starting points give different parametrizations so it is not as convenient when considering more than one initial point.

Here we have done a time change of an SDE. Since this will happen often, it is useful to discuss it in some detail. If

$$dK_t = R_t\, dt + A_t\, dB_t,$$

and σ is a C^1 time change, then $\hat{K}_t := K_{\sigma(t)}$ satisfies

$$d\hat{K}_t = R_{\sigma(t)}\, \dot{\sigma}(t)\, dt + \sqrt{\dot{\sigma}(t)}\, A_{\sigma(t)}\, d\hat{B}_t,$$

where $\hat{B}_t$ is a standard Brownian motion given by

$$\hat{B}_t = \int_0^{\sigma(t)} \sqrt{\dot{\eta}(s)}\, dB_s, \quad \eta = \sigma^{-1}. \tag{13}$$

Note that if $\hat{B}_t$ is defined as above, then

$$\langle \hat{B} \rangle_t = \int_0^{\sigma(t)} \dot{\eta}(s)\, ds = \eta(\sigma(t)) - \eta(\sigma(0)) = t,$$

from which we can see that $\hat{B}_t$ is a standard Brownian motion. In the example above,

$$K_t = \Theta_t, \quad R_t = \frac{(1-2a)\, X_t\, Y_t}{(X_t^2 + Y_t^2)^2}, \quad A_t = -\frac{Y_t}{X_t^2 + Y_t^2}, \quad \dot{\sigma}(t) = \frac{(X_t^2 + Y_t^2)^2}{Y_t^2},$$

$$R_{\sigma(t)}\, \dot{\sigma}(t) = \frac{(1-2a)\, \hat{X}_t}{\hat{Y}_t} = (1-2a)\, \cot \hat{\Theta}_t, \quad \sqrt{\dot{\sigma}(t)}\, A_{\sigma(t)} = -1.$$

Therefore,

$$d\hat{\Theta}_t = (1-2a)\, \cot \hat{\Theta}_t\, dt - d\hat{B}_t.$$

Since $-\hat{B}_t$ is also a standard Brownian motion, we can also write (12) where the $\hat{B}_t$ there is the negative of the $\hat{B}_t$ defined in (13).

The next theorem shows the three "phases" of SLE_κ from the perspective of a point $z \in \mathbb{H}$.

- If $\kappa \le 4$, then $z \in H_t$ for all t.
- If $4 < \kappa < 8$, then $T_z < \infty$ and $z \notin H_t$ for $t \ge T_z$. However, $\operatorname{dist}(z, \gamma(T_z)) > 0$. At time T_z the path makes a loop that disconnect z from ∞.
- If $\kappa \ge 8$, then the curve actually reaches z.

Theorem 2. *If $z \in \mathbb{H}$,*

$$\mathbb{P}\{T_z = \infty, \operatorname{dist}(z, \gamma) > 0\} = 1, \qquad \textit{if } 0 < \kappa \le 4,$$

$$\mathbb{P}\{T_z < \infty, \operatorname{dist}(z, \gamma) > 0\} = 1, \qquad \textit{if } 4 < \kappa < 8,$$

$$\mathbb{P}\{T_z < \infty, \operatorname{dist}(z, \gamma) = 0\} = 1, \qquad \textit{if } 8 \le \kappa < \infty.$$

Proof. Let $z = x + iy \in \mathbb{H}$. By scaling we may assume that $y = 1$. Note that $\mathrm{dist}(z, \gamma) > 0$ if and only if $\Upsilon_\infty(z) > 0$. To analyze $\Upsilon_\infty = \Upsilon_\infty(z)$ we parametrize as in (11) so that $\log \Upsilon_t$ decays linearly. Under this parametrization $\hat{\Theta}_t$ satisfies

$$d\hat{\Theta}_t = (1 - 2a) \cot \hat{\Theta}_t \, dt + d\hat{B}_t,$$

for a standard Brownian motion $\hat{B}_t$. This is the radial Bessel equation as discussed in Section **??**. As $\theta \downarrow 0$, $\cot \theta = \theta^{-1} [1 + O(\theta^2)]$, and using this and comparison with the Bessel equation we see that $\hat{\Theta}_t$ reaches the origin in finite time if and only if $1 - 2a < 1/2$, that is, if $\kappa < 8$. Therefore,

$$\mathbb{P}\{\Upsilon_\infty > 0\} = \left\{ \begin{array}{ll} 1 & \text{if } \kappa < 8 \\ 0 & \text{if } \kappa \geq 8 \end{array} \right. .$$

We now address if $T_z = \infty$. There are several ways to do it, but we will use the *imaginary part parametrization*.

If $Z_t = X_t + iY_t = g_t(z) - U_t$, then the Loewner equations (3) and (4) can be written as

$$dX_t = \frac{aX_t}{X_t^2 + Y_t^2} \, dt + dB_t, \quad \partial_t[\log Y_t] = -\frac{a}{X_t^2 + Y_t^2}. \tag{14}$$

In the imaginary part parametrization $\log Y_t$ decreases linearly. Let

$$\sigma(t) = \inf\{s : \log Y_s = -at\}, \quad \tilde{Y}_t = Y_{\sigma(t)} = e^{-at}, \quad \tilde{X}_t = X_{\sigma(t)}, \quad K_t = e^{at} \, \tilde{X}_t.$$

Note that $\dot{\sigma}(t) = \tilde{X}_t^2 + \tilde{Y}_t^2 = e^{-2at} \, (K_t^2 + 1)$. By viewing the SDE for X_t in (14) in the new parametrization, we see that

$$\begin{aligned} d\tilde{X}_t &= a \, \tilde{X}_t \, dt + \sqrt{\tilde{X}_t^2 + \tilde{Y}_t^2} \, dW_t \\ &= a \, e^{-at} \, K_t \, dt + e^{-at} \sqrt{K_t^2 + 1} \, dW_t, \end{aligned}$$

where W_t is a standard Brownian motion. Since

$$dK_t = a \, e^{at} \, \tilde{X}_t \, dt + e^{at} \, d\tilde{X}_t,$$

we get

$$dK_t = 2a \, K_t \, dt + \sqrt{K_t^2 + 1} \, dW_t.$$

We will do a change of variables that will change the coefficient of the Brownian term to one. Let $J_t = \sinh^{-1}[K_t]$, that is, $K_t = \sinh J_t$. Itô's formula using the function $\sinh^{-1}(x)$ shows that

$$dJ_t = \left(2a - \frac{1}{2}\right) \tanh J_t \, dt + dW_t. \tag{15}$$

One can check (15) directly, or one could start with (15) where W_t is a standard Brownian motion and check that if $K_t = \sinh J_t$, then

$$\begin{aligned} dK_t &= \cosh J_t \, dJ_t + \frac{1}{2} \sinh J_t \, d\langle J \rangle_t \\ &= 2a \, \sinh J_t \, dt + \cosh J_t \, dW_t \\ &= 2a \, K_t \, dt + \sqrt{K_t^2 + 1} \, dW_t. \end{aligned}$$

In the imaginary part parametrization it takes an infinite amount of time for Y_t to reach 0. Indeed, Since $|\tanh x| < 1$, the solutions to the SDE (15) exist for all time. However, in the original capacity parametrization the total amount of time spent is

$$\sigma(\infty) = \int_0^\infty \dot{\sigma}(t)\, dt = \int_0^\infty e^{-2at}\, (K_t^2 + 1)\, dt = \int_0^\infty e^{-2at}\, \cosh^2 J_t\, dt.$$

Thus $T_z < \infty$ corresponds to $\sigma(\infty) < \infty$. There are three cases to be considered.

- $\kappa < 4$ ($a > 1/2$). Since $\tanh x = \pm 1 - O(e^{-2|x|})$ as $x \to \pm\infty$, we can see by comparison with a Brownian motion with drift, any solution to (15) must satisfy

  $$\lim_{t\to\infty} \frac{|J_t|}{t} = 2a - \frac{1}{2}.$$

  Indeed, the process has the same asymptotics as a Brownian motion with drift $\pm\left(2a - \frac{1}{2}\right)$. Choose r with $a < r < 2a - \frac{1}{2}$. We see that for all t sufficiently large, $\cosh^2 J_t \geq e^{2rt}/4$, and hence $\sigma(\infty) = \infty$.

- $\kappa = 4$ ($a = 1/2$). Then J_t satisfies

  $$dJ_t = \frac{1}{2}\, \tanh J_t\, dt + dW_t.$$

  If instead of this equation, we had $dJ_t = (1/2)\, dt + dW_t$, then we would have $J_t - (t/2)$ is a standard Brownian motion and hence there exists infinite number of n with $J_t \geq n/2$ (and hence $\cosh^2 J_t \geq e^n/4$) for $n \leq t \leq n+1$. This would imply that $\sigma(\infty) = \infty$. We need to do a little work, but the estimate $\tanh |J_t| = 1 - O(e^{-2|J_t|})$ can be used to establish this. We omit the details.

- $\kappa > 4$ ($a < 1/2$). As above, we see that

  $$\lim_{t\to\infty} \frac{|J_t|}{t} = 2a - \frac{1}{2}.$$

  Choose r with $2a - \frac{1}{2} < r < a$. We see that for all t sufficiently large $\cosh^2 J_t \leq e^{2rt}$, and hence $\sigma(\infty) < \infty$.

□

We now prove the analogous theorem for $x \in \mathbb{R}$. We will state it for $x > 0$ but it obviously holds for $x < 0$ as well.

Theorem 3. *If $x > 0$,*

$$\mathbb{P}\{T_x = \infty, \operatorname{dist}(x, \gamma) > 0\} = 1, \qquad \textit{if } 0 < \kappa \leq 4,$$

$$\mathbb{P}\{T_x < \infty, \operatorname{dist}(x, \gamma) > 0\} = 1, \qquad \textit{if } 4 < \kappa < 8,$$

$$\mathbb{P}\{T_x < \infty, \operatorname{dist}(x, \gamma) = 0\} = 1, \qquad \textit{if } 8 \leq \kappa < \infty.$$

If $z \in \mathbb{H}$ we considered Υ_t which is a constant multiple times $\mathrm{crad}_{H_t}(z)$ and is comparable to the distance. This will not be useful for $x > 0$ because $\mathrm{crad}_{H_t}(x) > 0$; however, we can use a different quantity based on a reflected domain. Let $H_t^+ = H_t \cup \{z : \overline{z} \in H_t\} \cup (x_t^+, \infty)$, where

$$x_t^+ = \max\{y \in \mathbb{R} : T_y \le t\}.$$

Note that $H_0^+ = \mathbb{C} \setminus (-\infty, 0]$. If the curve γ exists, then H_t^+ is the unbounded connected component of $\mathbb{C} \setminus [\gamma_t \cup \overline{\gamma}_t \cup (-\infty, 0])$; here $\overline{\gamma}_t = \{\overline{\gamma(s)} : 0 \le s \le t\}$. Let $X_t = g_t(x) - U_t, O_t = g_t(x_t^+) - U_t$, and

$$K_t = X_t - O_t, \quad J_t = \frac{K_t}{X_t}, \quad 1 - J_t = \frac{O_t}{X_t}, \quad \Psi_t = \frac{K_t}{g_t'(x)} = \frac{J_t\, X_t}{g_t'(x)}. \tag{16}$$

Proposition 2.4. *If $x > 0$ and $t < T_x$, then*

$$\Psi_t = \frac{1}{4}\, \mathrm{crad}_{H_t^+}(x).$$

In particular,

$$\Psi_t \le \mathrm{dist}(x, \gamma) \le 4\, \Psi_t.$$

Proof. Using the Koebe function, we can see that $\mathrm{crad}_{\mathbb{H}\setminus(-\infty,0]}(x) = 4x$. By the scaling rule for conformal radius

$$\mathrm{crad}_{H_t^+}(x) = \frac{\mathrm{crad}_{\mathbb{R}\setminus(-\infty, g_t(x_t^+)]}(g_t(x))}{g_t'(x)} = \frac{4\,(X_t - O_t)}{g_t'(x)} = 4\,\Psi_t.$$

The second inequality follows from the Schwarz lemma and the Koebe (1/4)-theorem.

□

Proof of Theorem 3. Without loss of generality assume that $x > 0$ and let $X_t = g_t(x) - U_t$. Then X_t satisfies

$$dX_t = \frac{a}{X_t}\, dt + dB_t, \quad X_0 = x.$$

This is a Bessel SDE for which it is known (see Proposition **??**) that with probability one, the process reaches the origin if and only if $a < 1/2$, that is $\kappa > 4$. Hence

$$\mathbb{P}\{T_x < \infty\} = \begin{cases} 0, & \kappa \le 4 \\ 1, & \kappa > 4. \end{cases}$$

By monotonicity, we can conclude that if $y < x$, then $T_y \le T_x$. Hence, we can conclude the stronger fact

$$\mathbb{P}\{T_x = \infty \text{ for all } x > 0\} = 1, \quad 0 < \kappa \le 4,$$

$$\mathbb{P}\{T_x < \infty \text{ for all } x > 0\} = 1, \quad 4 < \kappa < \infty.$$

With more argument, see Proposition **??**, we can see that if $0 < x < y$, then

$$\mathbb{P}\{T_x < T_y\} \begin{cases} < 1, & 4 < \kappa < 8 \\ = 1, & \kappa \ge 8 \end{cases},$$

and, in particular, for $\kappa \geq 8$,

$$\mathbb{P}\{T_x < T_y \text{ for all } 0 < x < y\} = 1. \tag{17}$$

For the remainder of this proof we consider $x \in \mathbb{R}$, and by scaling we can assume that $x = 1$. If $\kappa \geq 8$, we can use (17) to see that with probability one $\mathbb{R} \subset \gamma$. Let us consider the case $\kappa < 8$, and look at the quantities in (16). Using the Loewner equation and (7), we see that

$$\partial_t K_t = \frac{a}{O_t} - \frac{a}{X_t} = -a\,\frac{K_t}{O_t\,X_t} = -a\,\frac{K_t}{(1-J_t)\,X_t^2}, \quad \partial_t\, g_t'(x)^{-1} = g_t'(x)^{-1}\,\frac{a}{X_t^2}$$

Using the chain rule, (7), and Itô's formula, we can compute

$$\partial_t\,\Psi_t = -a\,K_t\,g_t'(x)^{-1}\,\left[\frac{1}{(1-J_t)\,X_t^2} - \frac{1}{X_t^2}\right] = -a\,\Psi_t\,\frac{J_t}{X_t^2\,(1-J_t)},$$

$$\begin{aligned} dJ_t &= K_t\,d[1/X_t] + [1/X_t]\,dK_t \\ &= K_t\,[-X_t^{-2}\,(a\,X_t^{-1}\,dt + dB_t) + X_t^3\,dt] - a\,\frac{K_t}{(1-J_t)\,X_t^3}\,dt \\ &= \frac{J_t}{X_t^2}\left(1 - a - \frac{a}{1-J_t}\right)\,dt - \frac{J_t}{X_t}\,dB_t. \end{aligned}$$

As in the case of the Υ_t, this equation becomes nicer if we parametrize so that $\log \Psi_t$ decays linearly. Let

$$\sigma(t) = \inf\{s : \Psi_s = e^{-at}\}, \tag{18}$$

and if $\sigma(t) < \infty$,

$$\hat{\Psi}_t = \Psi_{\sigma(t)} = e^{-at}, \quad \hat{X}_t = X_{\sigma(t)}, \quad \hat{K}_t = K_{\sigma(t)}, \quad \hat{J}_t = J_{\sigma(t)}.$$

By the chain rule, we see that

$$-a\,\hat{\Psi}_t = \partial_t \hat{\Psi}_t = -a\,\hat{\Psi}_t\,\frac{\hat{J}_t}{\hat{X}_t^2\,(1-\hat{J}_t)}\,\dot{\sigma}(t),$$

and hence

$$\dot{\sigma}(t) = \frac{\hat{X}_t^2\,(1-\hat{J}_t)}{\hat{J}_t}.$$

Using this, we see that

$$d\hat{J}_t = \left[1 - 2a - (1-a)\hat{J}_t\right]\,dt + \sqrt{\hat{J}_t\,(1-\hat{J}_t)}\,dW_t, \tag{19}$$

for a standard Brownian motion W. This is nicer if we do another change of variables. Define Q_t by

$$\hat{J}_t = \frac{1-\cos Q_t}{2},$$

and note that we can write (19) as

$$d\hat{J}_t = \left[\frac{1-3a}{2} + \frac{1-a}{2}\cos Q_t\right] dt + \frac{1}{2}\sin Q_t \, dW_t.$$

Using Itô's formula, we can see that this holds if Q_t satisfies the equation

$$d\,Q_t = \left[\left(\frac{1}{2} - a\right)\cot Q_t + \frac{1-3a}{\sin Q_t}\right] dt + dW_t.$$

We must actually define this with reflection at 1. The question is whether or not this process reaches 0 in finite (in the new parametrization) time. This is an example of a reflected "'asymptotically Bessel" process as discussed in Section **??**. Near the origin

$$\left(\frac{1}{2} - a\right)\cot u + \frac{1-3a}{\sin u} = \frac{\frac{3}{2} - 4a}{u} + O(u).$$

By comparison with a Bessel process we see that Q_t reaches the origin in finite time if and only if $\frac{3}{2} - 4a < \frac{1}{2}$ or $a > \frac{1}{4}$. This corresponds to $\kappa < 8$.

□

The next proposition is a continuation of the description of the "phases" of SLE. In the proof we are assuming that SLE comes from a curve.

Proposition 2.5. *Let γ be an SLE_κ curve from 0 to ∞ in $\mathbb{H}$. Then the following holds with probability one.*

1. *If $\kappa \le 4$, then γ is a simple curve with $\gamma(0,\infty) \cap \mathbb{R} = \emptyset$.*
2. *If $4 < \kappa < 8$, then γ has self-intersections and $\gamma(0,\infty) \cap \mathbb{R} \neq \emptyset$. However, for almost every $z \in \mathbb{C}$, $z \notin \gamma$.*
3. *If $8 \le \kappa < \infty$, then γ is plane-filling, that is,*

$$\gamma[0,\infty) = \overline{\mathbb{H}}.$$

Proof.

1. If $0 < x < y$, we know that $T_x < T_y$. We also know that with probability one $T_x < \infty$ for all rational x. Therefore, with probability one $T_x = \infty$ for all $x \in \mathbb{R}$ and hence $\gamma(0,\infty) \cap \mathbb{R} = \emptyset$. If $s > 0$, let
$$\gamma^s(t) = g_s[\gamma(s+t)] - U_s.$$
For each s, the distribution of γ^s is that of SLE_κ. Hence, with probability one, for every rational s, $\gamma^s(0,\infty) \cap \mathbb{R} = \emptyset$. If there existed $0 \le t_1 < t_2 < \infty$ with $\gamma(t_2) = \gamma(t_1)$, then $\gamma^s(0,\infty) \cap \mathbb{R} \neq \emptyset$ for all rational $t_1 < s < t_2$.
2. We have seen that for each $z \in \mathbb{C} \setminus \{0\}$, $\mathbb{P}\{\mathrm{dist}(z,\gamma) > 0\} = 1$. Using Fubini's theorem, we see that with probabilty one the area of $\gamma[0,\infty)$ is zero.

3. Using Fubini's theorem, we know that with probability one for almost all z, $T_z < \infty$ and $z \in \gamma[0, T_z]$. Since γ is a continuous curve we can see that for such z $z = \gamma(T_z)$. This almost prove the result, and we delay the proof of the remainder.

□

The next proposition was one of the first calculations done for SLE reducing a geometric problem to the solution of a second-order ordinary differential equation.

Proposition 2.6. *Suppose γ is a chordal SLE_κ curve (from 0 to ∞) in $\mathbb{H}$ with $\kappa < 8$. Let $q(\theta)$ be the probability that the curve goes to the right of $re^{i\theta}$. Then*

$$q(\theta) = q_\kappa(\theta) = C \int_0^\theta \sin^{4a-2} u \, du, \qquad \text{where } C = \left[\int_0^\pi \sin^{4a-2} u \, du \right]^{-1}. \tag{20}$$

One may note that the condition $\kappa < 8$ is necessary for the integrals to be convergent. There are several interesting cases:

$$q_2(\theta) = \frac{\theta - \sin\theta \, \cos\theta}{\pi}, \quad q_{8/3}(\theta) = \frac{1 - \cos\theta}{2}, \quad q_4(\theta) = \frac{\theta}{\pi}.$$

The $\kappa = 4$ case is just the gambler's ruin estimate for Brownian motion.

Proof. To say that the curve goes to the right is to say that $\Theta_T(z) := \Theta_{T-} = \pi$ where $T = T_z$. By scaling, the probability depends only on θ. If we parametrize so that the log conformal radius decays linearly as in (12), we see that $q(\theta) = \mathbb{P}^\theta\{\Theta_T = \pi\}$ where Θ_t satisfies

$$d\Theta_t = (1 - 2a) \, \cot \Theta_t \, dt + dB_t, \quad \Theta_t = \theta,$$

and $T = \inf\{t : \sin \Theta_t = 0\}$. Note that $q(\Theta_{t \wedge T})$ is a continuous martingale. If we also assume that q is C^2, then Itô's formula gives

$$\begin{aligned} dq(\Theta_t) &= q'(\Theta_t) \, d\Theta_t + \frac{1}{2} q''(\Theta_t) \, d\langle \Theta \rangle_t \\ &= \left[(1 - 2a) \, q'(\Theta_t) \, \cot \Theta_t + \frac{q''(\Theta_t)}{2} \right] dt + q'(\Theta_t) \, d\Theta_t. \end{aligned}$$

In order to make this a martingale, we solve the simple second order ODE,

$$(1 - 2a) \, q'(\theta) \, \cot \theta + \frac{q''(\theta)}{2} = 0,$$

which is a first order ODE in q'. Separation of variables shows that $q'(\theta) = c \, [\sin \theta]^{4a-2}$, and then using the boundary conditions $q(0) = 0$ and $q(\pi) = 1$ we get (20). Although we assumed that q was C^2 to start with, we could go back with q as in (20) and show that $q(\Theta_{t \wedge T})$ is a bounded martingale and use the optional sampling theorem.

□

The function $q(\theta)$ is the solution of a linear second-order ODE and could also be written as a hypergeometric function.

The arguments in the proof of Theorem 2 can be extended to establish two important "one-point" estimates for SLE_κ for $\kappa < 8$. The first discusses the probability of getting near an interior point z and the second deals with the probability of getting near a boundary point x.

Proposition 2.7. *If $\kappa < 8$, there exists $\alpha > 0$ such that if γ is a chordal SLE_κ curve from 0 to ∞ in $\mathbb{H}$, $z \in \mathbb{H}$, and $\Upsilon = \Upsilon_\infty(z)$, then for $0 < r \le 1/2$,*

$$\mathbb{P}\{\Upsilon \le r\,\Upsilon_0\} = c_*\, r^{2-d}\,[\sin(\arg z)]^{4a-1}[1 + O(r^\alpha)],$$

where

$$d = 1 + \frac{1}{4a} = 1 + \frac{\kappa}{8}, \quad c_* = 2\left[\int_0^\pi \sin^{4a}\theta\, d\theta\right]^{-1}.$$

The statement of the theorem is shorthand for the following. There exists c, α such that for all z and all $r < 1/2$,

$$\left|\mathbb{P}\{\Upsilon \le r\,\Upsilon_0\} - c_*\, r^\beta\right| \le c\, r^{\beta+\alpha}.$$

The same fact holds for all $r < 1 - \epsilon$ for any $\epsilon > 0$, but the constant c depends on ϵ.

Proof. By scaling, we may assume that $\Upsilon_0 = 1$, that is, $z = x + i$ for some $x \in \mathbb{R}$. We will use the radial parametrization as in (11) so that $\Upsilon_{\sigma(t)} = e^{-2at}$, and let $\hat{\Theta}_t = \Theta_{\sigma(t)}$, that satisfies

$$d\hat{\Theta}_t = (1 - 2a)\,\cot\hat{\Theta}_t\, dt + dB_t,$$

for a standard Brownian motion B_t. This equation is valid until $T = \inf\{t : \sin\hat{\Theta}_t = 0\}$ at which time $\hat{\Upsilon}_T = e^{-2aT}$. Hence, if we write $r = e^{-2as}$, we can write

$$\mathbb{P}\{\Upsilon < r\} = \mathbb{P}\{T > s\}.$$

Let $S_t = \sin\hat{\Theta}_t$ and let

$$M_t = e^{t(2a-\frac{1}{2})}\, S_t^{4a-1}. \tag{21}$$

We claim that $M_{t\wedge T}$ is a martingale satisfying

$$dM_t = (4a - 1)\,[\cot\hat{\Theta}_t]\, M_t\, dB_t. \tag{22}$$

To check this, we use Itô's formula. Using $\cot^2\hat{\Theta}_t = S_t^{-2} - 1$, we see that for $t < T$,

$$\begin{aligned} dS_t &= \cos\hat{\Theta}_t\, d\hat{\Theta}_t - \frac{1}{2}\sin\hat{\Theta}_t\, dt \\ &= S_t\left[((1-2a)\,\cot^2\hat{\Theta}_t - \frac{1}{2})\, dt + \cot\hat{\Theta}_t\, dB_t\right] \\ &= S_t\left[(2a - \frac{3}{2} + \frac{1-2a}{S_t^2})\, dt + \cot\hat{\Theta}_t\, dB_t\right] \end{aligned}$$

$$\begin{aligned} dS_t^{4a-1} &= S_t^{4a-1}\left[\frac{4a-1}{S_t}\,dS_t + \frac{(2a-1)(4a-1)}{S_t^2}\,d\langle S\rangle_t\right] \\ &= (4a-1)\,S_t^{4a-1}\left[(2a-\frac{3}{2}+\frac{1-2a}{S_t^2})\,dt + (2a-1)\,[\frac{1}{S_t^2}-1]\,dt + \cot\hat{\Theta}_t\,dB_t\right] \\ &= S_t^{4a-1}\left[(\frac{1}{2}-2a)\,dt + (4a-1)\,\cot\hat{\Theta}_t\,dB_t\right] \end{aligned}$$

This verifies that (22) holds for $t < T$. It is easy to check that the martingale is continuous as $t \uparrow T$, and since the process is uniformly bounded on $[0, t_0]$ for all $t_0 < \infty$, we can see that $M_{t\wedge T}$ is, in fact, a continuous martingale.

Using the Girsanov theorem, we define a new probability measure $\mathbb{P}^*$ with expectations $\mathbb{E}^*$ by stating that if V is a nonnegative random variable measurable with respect to $B_s, 0 \le s \le t$, then

$$\mathbb{E}^*(V) = M_0^{-1}\,\mathbb{E}[V\,M_t] = S_0^{1-4a}\,\mathbb{E}[V\,M_t]\ .$$

By (22) and the Girsnaov theorem, we can see that

$$dB_t = (4a-1)\,\cot\hat{\Theta}_t\,dt + dW_t,$$

where W_t is a standard Brownian motion with respect to the probability measure $\mathbb{P}^*$. This equation only holds for $t < T$, but it is easy to see that $\mathbb{P}^*\{t < T\} = 1$. (Indeed, since $M_{t\wedge T} = 0$ on the event $T \le t$, this must be the case.) Therefore,

$$d\hat{\Theta}_t = 2a\,\cot\hat{\Theta}_t\,dt + dW_t. \tag{23}$$

We now write

$$\begin{aligned} \mathbb{P}\{T > t\} &= \mathbb{E}\left[1\{T > t\}\right] \\ &= \mathbb{E}\left[M_t\,M_t^{-1}; T > t\right] \\ &= e^{t(\frac{1}{2}-2a)}\,\mathbb{E}\left[M_t\,S_t^{1-4a}; T > t\right] \\ &= e^{t(\frac{1}{2}-2a)}\,S_0^{4a-1}\,\mathbb{E}^*[S_t^{1-4a}; T > t] \\ &= e^{t(\frac{1}{2}-2a)}\,S_0^{4a-1}\,\mathbb{E}^*[S_t^{1-4a}]. \end{aligned}$$

The last equality holds because $\mathbb{P}^*\{T > t\} = 1$. We are left with estimating $\mathbb{E}^*[S_t^{1-4a}]$ which is a problem about the radial Bessel equation (23). As in Proposition **??**, we see that the invariant density is

$$f_{4a}(\theta) = c_{4a}\,\sin^{4a}\theta, \qquad \text{where } c_r = \left[\int_0^\pi \sin^r\,y\,dy\right]^{-1},$$

and

$$\mathbb{E}^*[S_t^{1-4a}] = [1 + O(e^{-tu})]\int_0^\pi [\sin^{1-4a}\,\theta]\,f_{4a}(\theta)\,d\theta = 2\,c_{4a}\,[1 + O(e^{-tu})].$$

Therefore, if $r = e^{-2at}$,

$$\mathbb{P}\left\{\Upsilon < r\right\} = \mathbb{P}\{T > t\} = 2\,c_{4a}\,e^{t(\frac{1}{2}-2a)}\,S_0^{4a-1}\,[1 + O(e^{-tu})] = 2\,c_{4a}\,r^{1-\frac{1}{4a}}\,S_0^{4a-1}\,[1 + O(r^\alpha)],$$

where $\alpha = u/(2a)$.

□

It may seem that the martingale in (21) came out of the blue, but it arises very naturally. If we consider the radial Bessel equation

$$d\Theta_t = r \cot \Theta_t \, dt + dB_t,$$

with $r < 1/2$ and we want to compute $f(\theta, t) = \mathbb{P}\{T > t \mid \Theta_t = \theta\}$ we might hope that this would satisfy

$$f(\theta, t) \sim f(\theta) \, e^{-\lambda t} \qquad t \to \infty.$$

We try to find f, λ so that

$$M_t = e^{\lambda t} \, f(\Theta_t)$$

is a martingale. Using Itô's formula, we see that a sufficient condition to make this a martingale is

$$\frac{1}{2} \, f''(x) + r \, [\cot x] \, f'(x) + \lambda \, f(x) = 0.$$

This is an eigenvalue/eigenfuction problem: try to find the (unique) positive solution to this second order differential equation with $f(0) = f(\pi) = 0$. If one tries functions of the form $f(x) = [\sin x]^\beta$, then one gets

$$f'(x) = \beta \, f(x) \, \cot x,$$

$$f''(x) = \beta \, f(x) \, \left[\beta \, \cot^2 x - \frac{1}{\sin^2 x}\right] = \beta \, f(x) \, \left[(\beta - 1) \, \cot^2 x - 1\right],$$

which leads to the equation

$$[\beta(\beta - 1) + 2r\beta] \, \cot^2 x + (2\lambda - \beta) = 0.$$

Equating coefficients gives

$$\beta = 1 - 2r, \qquad \lambda = r - \frac{1}{2}.$$

We therefore get the martingale

$$M_t = e^{t(r - \frac{1}{2})} \, [\sin \Theta_t]^{1-2r},$$

which satisfies

$$dM_t = (1 - 2r) \, M_t \, \cot \Theta_t \, dB_t.$$

When we tilt by the martingale to give a new measure $\mathbb{P}^*$, we see that

$$dB_t = (1 - 2r) \, \cot \Theta_t \, dt + dW_t,$$

where W_t is a standard Brownian motion with respect to $\mathbb{P}^*$. Therefore,

$$d\Theta_t = (1 - r) \, \cot \Theta_t \, dt + dW_t.$$

Since $r < 1/2$, we see that $\mathbb{P}^*\{T < \infty\} = 0$. This diffusion has invariant density $g(\theta) = [\sin \theta]^{2-2r}$. The quasi-invariant density h is the limiting density (with respect to the original measure $\mathbb{P}$) as $t \to \infty$ of Θ_t given $T > t$. The density g, which is the density in the measure $\mathbb{P}^*$, can be written as $g(\theta) = c \, [\sin \theta]^{4a-1} \, h(\theta)$. In other words, $h(\theta) = c_1 \, \sin \theta = \sin \theta / 2$.

Corollary 2.8. *If $\kappa < 8$, there exists $\alpha > 0$ such that that the following holds. Suppose that $z, w \in \partial D$ are distinct, γ is a chordal SLE_κ path from z to w in D, and $\zeta \in D$. Let D' be the connected component of $D \setminus \gamma$ containing ζ. Then if $r \le 1/2$,*

$$\mathbb{P}\{\mathrm{crad}_{D'}(\zeta) \le r\,\mathrm{crad}_D(\zeta)\} = c_*\, r^{2-d}\, S_D(\zeta; z, w)^{4a-1}\,[1 + O(r^\alpha)],$$

where d, c_* are as in Proposition 2.7.

Corollary 2.9. *If $\kappa < 8$, there exists $\alpha > 0$ such that that the following holds. Suppose that $z, w \in \partial D$ are distinct, γ is a chordal SLE_κ path from z to w in D, and $\zeta \in D$. Let D' be the connected component of $D \setminus \gamma$ containing ζ. Then if $r \le 1/2$,*

$$c_1\, S_D(\zeta; z, w)^{4a-1}\, r^{2-d} \le \mathbb{P}\left\{\mathrm{dist}(\zeta, \gamma) \le r\,\mathrm{dist}(\zeta, \partial D)\right\} \le c_2\, S_D(\zeta; z, w)^{4a-1}\, r^{2-d}.$$

Proof. Most of this follows from the previous corollary that $\mathrm{dist}_{D'}(\zeta) \asymp_2 2\,\mathrm{dist}(z, \partial D)$. For the upper bound for larger values of r, we also need the following fact that is easy to prove: there exists $\rho < 1$, such that if $\zeta \in D_2 \subset D_1$, then

$$\mathrm{crad}_{D_2}(\zeta) \le \rho \mathrm{crad}_{D_1}(\zeta).$$

□

We will now prove the analogue of Proposition 2.7 where conformal radius is replaced with distance to the boundary. We will get as strong a result although we will not get an explicit form for the constant.

Theorem 4. *If $\kappa < 8$, D is a simply connected domain, and $z, w \in \partial D$ are distinct, there exists a function $\mathbb{G}_D(\cdot; z, w)$ on D such that if γ is a chordal SLE path from z to w in D the following holds.*

1. *If $\zeta \in D$ and $r \le \mathrm{dist}(\zeta, D)/2$,*

$$\mathbb{P}\{\mathrm{dist}(\zeta, \gamma) \le r\} = \mathbb{G}_D(\zeta; z, w)\, r^{2-d}\,[1 + O(r^\alpha)].$$

2. *There exists $\hat{c} = \hat{c}(\kappa)$ such that*

$$\mathbb{G}_D(\zeta; z, w) = \hat{c}\,\mathrm{crad}_D(\zeta)^{d-2}\, S_D(\zeta; z, w)^{4a-1}.$$

We call $\mathbb{G}_D(\zeta; z, w)$ the *chordal SLE_κ Green's function (from z to w in D)*. The last assertion can be written as a combination of

$$\mathbb{G}_{\mathbb{D}}(0; 1, e^{2i\theta}) = \hat{c}\,(\sin\theta)^{4a-1},$$

and the conformal covariance rule: if $f : D \to f(D)$ is a conformal transformation, then

$$\mathbb{G}_D(\zeta; z, w) = |f'(\zeta)|^{2-d}\,\mathbb{G}_{f(D)}(f(\zeta); f(z), f(w)).$$

The proof will not give the value of the κ-dependent constant $\hat{c}$.

Proof. We first consider $D = \mathbb{D}, \zeta = 0, z = 1, w = e^{2\theta i}$. We will write $r = e^{-s}$.

We let $\mathbb{P}_\theta$ denote the probabilities for SLE_κ from 1 to $e^{2\theta i}$ in $\mathbb{D}$. We will show that there exists $\hat{c}$ such that for all $0 < \theta < \pi$,

$$\mathbb{P}_\theta\{\mathrm{dist}(0,\gamma) \le r\} = \hat{c}\, r^{2-d}\, (\sin\theta)^{4a-1}\, [1 + O(r^\alpha)]. \tag{24}$$

For each $r_0 > 0$ and $r_0 \le r \le 1/2$, small, we can use Corollary 2.9 to see that

$$\mathbb{P}_\theta\{\mathrm{dist}(0,\gamma) \le r\} \asymp [\sin\theta]^{4a-1}.$$

Hence it suffices to find r_0 so that the relation holds for $r \le r_0$.

We will assume that γ has the radial parametrization as in (11), so that if D_t is the connected component of $\mathbb{D} \setminus \gamma_t$ containing the origin, then

$$\mathrm{crad}_{D_t}(0) = e^{-2at}.$$

We let $\tilde{g}_t : D_t \to \mathbb{D}$ be the conformal transformation with $\tilde{g}_t(0) = 0, \tilde{g}_t(\gamma(t)) = 1$ and define Θ_t by $\tilde{g}_t(e^{2i\theta}) = e^{2i\Theta_t}$. Then Θ_t satisfies

$$d\Theta_t = (1 - 2a)\, \cot\Theta_t\, dt + dB_t,$$

where B_t is a standard Brownian motion. We write g_t for $\tilde{g}_t$ followed by a rotation such that $g_t'(0) > 0$. The curve has finite lifetime T in this parametrization. As before, using the Koebe 1/4-theorem,

$$2\,\mathrm{dist}(0, \partial D_t) \asymp_2 e^{-2at}.$$

Let $\tau_s = \inf\{t : |\gamma(t)| = e^{-s}\}$, so that (24) can be rephrased as

$$\mathbb{P}_\theta\{T > \tau_s\} = \hat{c}\, e^{s(d-2)}\, (\sin\theta)^{4a-1}\, [1 + O(e^{-\alpha s})]. \tag{25}$$

We will first give a sketch for why (25) holds. Note that $\tau_s \ge \frac{s - \log 4}{2a} > q$ where $q = q_s = \frac{s-2}{2a}$. The relation (25) can be established for large s by showing there exists u, C, C' such that

$$\mathbb{P}_\theta\{T > q\} = C\, e^{(d-2)s}\, (\sin\theta)^{4a-1}\, [1 + O(e^{-us})] \tag{26}$$

$$\mathbb{P}_\theta\{T > \tau_s \mid T > q\} = C' + O(e^{-us}). \tag{27}$$

Theorem 2.7 gives us (26). For the second equality note that conformal invariance implies that

$$\mathbb{P}_\theta\{T > \tau_s \mid T > q\} = \mathbb{P}_{\Theta_q}\{\gamma \cap \tilde{g}_q(e^{-s}\,\mathbb{D}) \ne \emptyset\}.$$

Using the Koebe 1/4-theorem, we can see that there exist $0 < \rho_1 < \rho_2 < 1$ such that

$$\rho_1\,\mathbb{D} \subset \tilde{g}_q(e^{-s}\,\mathbb{D}) \subset \rho_2\,\mathbb{D}.$$

If we can show that the conditional distribution of $(\Theta_q, \tilde{g}_q(e^{-s}\,\mathbb{D}))$ given $T > q$ converges exponentially fast to some distribution, then we would have (27).

For a proof, we consider the *quasi-invariant* density corresponding to the density of Θ_t given $T > t$. Let $\mathbb{P}^*_\theta, \mathbb{E}^*_\theta$ denote the probability measures obtained by tilting by the martingale

$$M_t = e^{t(a-\frac{1}{2})}\, [\sin\Theta_t]^{4a-1}$$

as in the proof of Theorem 2.7 , if $S_t = \sin \Theta_t$, Then, if F is a continuous function,

$$\begin{aligned}
\mathbb{E}_\theta[F(\Theta_t); T > t] &= e^{t(\frac{1}{2}-a)}\, \mathbb{E}_\theta \left[M_t\, F(\Theta_t)\, S_t^{1-4a}; T > t \right] \\
&= e^{t(\frac{1}{2}-a)}\, [\sin\theta]^{4a-1}\, \mathbb{E}_\theta^* \left[F(\Theta_t)\, S_t^{1-4a}; \right] \\
&= c_{4a}\, e^{t(\frac{1}{2}-a)}\, [\sin\theta]^{4a-1}\, [1 + O(e^{-\alpha u})] \int_0^\pi F(x)\, [\sin x]^{4a}\, [\sin x]^{1-4a}\, dx \\
&= 2\, c_{4a}\, e^{t(\frac{1}{2}-a)}\, [\sin\theta]^{4a-1}\, [1 + O(e^{-\alpha u})] \int_0^\pi F(x)\, \frac{\sin x}{2}\, dx
\end{aligned}$$

In other words, $(\sin x)/2$ is the quasi-invariant density. Indeed, if we write $\mathbb{P}_\mu, \mathbb{E}_\mu$ for probabilities and expectations assuming that the initial θ has density $(\sin x)/2$, then

$$\mathbb{E}_\mu[F(\Theta_t); T > t] = e^{t(\frac{1}{2}-a)}\, \mathbb{E}_\mu[F(\Theta_0)] = \int_0^\pi F(x)\, \frac{\sin x}{2}\, dx.$$

Let

$$P(s) = e^{s(2-d)}\, \mathbb{P}_\mu\{T > s\}.$$

We will now show that there exists c_0, c such that

$$|P(s) - c_0| \le c\, e^{-s}. \tag{28}$$

Suppose $0 \le \delta \le 1$, and note that $g_\delta'(0) = e^{2a\delta}$. Distortion estimates imply that there is a universal c (we could find an exact value but will not bother) such that for s sufficiently large and $|z| \le e^{-s}$,

$$|g_\delta(z) - e^{2a\delta}\, z| \le c\, e^{-2s}.$$

In particular, there exists universal c_1, c_2 such that

$$(e^{-s} - c_1 e^{-2s})\, \mathbb{D} \subset g_\delta(e^{-(s+2a\delta)}\mathbb{D}) \subset (e^{-s} + c_1 e^{-2s})\, \mathbb{D}$$

$$\mathbb{P}_\mu\{T > \tau_{s+c_2 e^{-s}}\} \le \mathbb{P}_\mu\{T > \tau_{s+\delta} \mid T > \delta\} \le \mathbb{P}_\mu\{T > \tau_{s-c_2 e^{-s}}\}.$$

Using the fact that μ is a quasi-invariant distribution, we see that this implies that for $0 \le \delta \le 1$ and s sufficiently large,

$$|\log P(s) - \log P(s+\delta)| \le c\, e^{-s}.$$

Similarly, if k is a positive integer

$$|\log P(s) - \log P(s+k)| \le \sum_{j=1}^{k} |\log P(s+j-1) - \log P(s+j)| = \sum_{j=1}^{k} O(e^{-s-j}) = O(e^{-s}).$$

We therefore get

$$\sup_{\delta > 0} |\log P(s) - \log P(s+\delta)| \le c\, e^{-s},$$

which is another way to express (28).

If our initial condition is $\Theta_0 = \theta$, we write

$$\mathbb{P}_\theta\{T > \tau_{2s}\} = \mathbb{P}_\theta\left\{T > \frac{s}{2a}\right\} \mathbb{P}_\theta\left\{T > \tau_{2s} \mid T > \frac{s}{2a}\right\}.$$

Using distortion estimates and $g'_{s/2a}(0) = e^s$ as in the previous paragraph, we can see that there exists c_3 such that

$$\mathbb{P}_{\Theta_{s/2a}} \left\{ T > \tau_{s+c_3\, e^{-s}} \right\} \le \mathbb{P}_\theta \{ T > \tau_{2s} \mid T > \frac{s}{2a} \} \le \mathbb{P}_{\Theta_{s/2a}} \left\{ T > \tau_{s-c_3\, e^{-s}} \right\},$$

Combining this, with (28), we see that

$$\begin{aligned} \mathbb{P}_\theta \{ T > \tau_{2s} \mid T > \frac{s}{2a} \} &= [1 + O(e^{-us})]\, \mathbb{P}_\mu \{ T > \tau_{s+O(e^{-s})} \} \\ &= c_0\, e^{-s(2-d)}\, [1 + O(e^{-us})]. \end{aligned}$$

$$\mathbb{P}_\theta \{ T > s \} = c_*\, e^{-2as(2-d)}\, [\sin\theta]^{4a-1}\, [1 + O(e^{-us})].$$

Proposition 2.7 implies that

$$\mathbb{P}_\theta \left\{ T > \frac{s}{2a} \right\} = c_*\, e^{-s(2-d)}\, [1 + O(e^{-us})].$$

This finishes the result for $D = \mathbb{D}, \zeta = 0, z = 1, w = e^{2i\theta}$. For more general D, z, w, let $f : D \to \mathbb{D}$ be the unique conformal transformation $f : D \to \mathbb{D}$ with $f(\zeta) = 0, f(z) = 1$ and define θ by $f(w) = e^{2i\theta}$. Note that $\sin\theta = S_D(\zeta; z, w)$. By scaling, we may assume that $|f'(\zeta)| = 1$. Note that $\mathrm{crad}_D(\zeta)^{-1} = |f'(\zeta)| \asymp 1$. Let V_r denote the image under f of the open disk of radius r about ζ. By distortion estimates, there exists a universal c' such that if $r \le \mathrm{dist}(\zeta, \partial D)/2$, then

$$(r - c'\, r^2)\mathbb{D} \subset f(V_r) \subset (r + c'\, r^2)\, \mathbb{D}.$$

Hence, we can use the result for the disk.

□

The estimates above can be considered as "interior" or "bulk" estimates for the probability that the SLE curve gets close to an interior point. We will now give the estimates for getting near a boundary point. By scaling, we can consider the probability that an SLE_κ path from 0 to ∞ gets near 1. The interior estimate gives us a (correct) guess for the correct exponent of decay. Suppose $\zeta = 1 + i\epsilon$. Then one might expect that the probability of getting within distance ϵ of 1 is comparable to the probability of getting within distance $\epsilon/2$ of ζ. Since $S_{\mathbb{H}}(1 + \epsilon; 0, \infty) = \epsilon$, we see that the latter probability is comparable to ϵ^{4a-1}. Indeed, this is correct, and the next proposition gives a much finer estimate. We will use the alternative conformal radius Ψ_t that was introduced in the proof of Theorem 2. Recall that Ψ_t is one-fourth times the conformal radius of 1 in the unbounded component of

$$\mathbb{C} \setminus \left(\gamma_t \cup \{ \overline{\gamma(s)} : 0 \le s \le t \} \cup (-\infty, 0] \right).$$

Proposition 2.10. *There exists $\alpha > 0$ such that if $x > 0$,*

$$\mathbb{P}\{ \Psi_\infty(x) \le rx \} = \frac{\Gamma(6a)}{\Gamma(4a)\, \Gamma(2a+1)}\, r^{4a-1}\, [1 + O(r^\alpha)].$$

Proof. Without loss of generality we assume that $x = 1$. Let $X_t = X_t(1) = g_t(1) - U_t$ and

$$M_t = X_t^{1-4a}\, g_t'(1)^{4a-1}.$$

Since

$$\begin{aligned} dX_t^{1-4a} &= X_t^{1-4a} \left[\frac{1-4a}{X_t}\, dX_t + \frac{(1-4a)(-2a)}{X_t^2}\, d\langle X\rangle_t \right] \\ &= X_t^{1-4a} \left[\frac{-a(1-4a)}{X_t^2}\, dt + \frac{1-4a}{X_t}\, dB_t \right]. \end{aligned}$$

$$\partial_t g_t'(1)^{4a-1} = -g_t'(1)^{4a-1}\, \frac{a(4a-1)}{X_t^2},$$

we see that $M_t = g_t'(1)^{4a-1}\, X_t^{1-4a}$ is a local martingale satisfying

$$dM_t = \frac{1-4a}{X_t}\, M_t\, dB_t.$$

Let $\mathbb{P}^*, \mathbb{E}^*$ denote probabilities and expectations obtained by tilting by the local martingale M_t. Then the Girsanov theorem implies that

$$dB_t = \frac{1-4a}{X_t}\, dt + dB_t^*,$$

where B_t^* is a standard Brownian motion with respect to $\mathbb{P}^*$. Using the notation as in (16), we get

$$dJ_t = \frac{J_t}{X_t^2} \left(3a - \frac{a}{1-J_t} \right) dt - \frac{J_t}{X_t}\, dB_t^*.$$

If we reparametrize as in (18) so that $\Psi_{\sigma(t)} = e^{-at}$, we see that $\hat{J}_t := J_{\sigma(t)}$ satisfies

$$dJ_t = \left(2a - 3a\hat{J}_t \right) dt + \sqrt{\hat{J}_t\,(1-\hat{J}_t)}\, W_t, \tag{29}$$

where W_t is also a standard Brownian motion with respect to $\mathbb{P}^*$. In the new parametrization,

$$\hat{M}_t := \hat{\Psi}_t^{1-4a}\, \hat{J}_t^{4a-1} = M_{\sigma(t)} = e^{(4a-1)at}\, \hat{J}_t^{4a-1}.$$

As before, we define Q_t by

$$\hat{J}_t = \frac{1-\cos \hat{Q}_t}{2},$$

so that (29) can be written as

$$dJ_t = \left(\frac{a}{2} + \frac{3a}{2} \cos Q_t \right) dt + \frac{\sin Q_t}{2}\, dW_t.$$

Since

$$d\hat{Q}_t = \frac{1}{2}\, \sin Q_t\, dQ_t + \frac{1}{4}\, \cos Q_t\, d\langle Q\rangle_t,$$

we can see that

$$dQ_t = \left[\frac{a}{\sin Q_t} + \left(3a - \frac{1}{2}\right) \cot Q_t \right] dt + dW_t.$$

This is the SDE studied in Section **??** with

$$v = a, \quad u = 2a - \frac{1}{2}.$$

As in the Proof of Proposition 2.7,

$$\begin{aligned} \mathbb{P}\{\Psi_\infty < e^{-at}\} &= \mathbb{E}\left[1\{\Psi_\infty < e^{-at}\}\right] \\ &= e^{(1-4a)at}\, \mathbb{E}\left[\hat{M}_t\, \hat{J}_t^{1-4a}\ ;\ \Psi_\infty < e^{-at}\right] \\ &= e^{(1-4a)at}\, \mathbb{E}^*\left[\hat{J}_t^{1-4a}\right]. \end{aligned}$$

So it suffices to show there exists α such that

$$\mathbb{E}^*\left[\hat{J}_t^{1-4a}\right] = \frac{\Gamma(6a)}{\Gamma(4a)\,\Gamma(2a+1)} + O(e^{-t\alpha}).$$

This is obtained by plugging $v = a, u = 2a - \frac{1}{2}, k = 1 - 4a$ into (**??**).

□

It follows from the last proposition and the Koebe-1/4 theorem that for $0 < r \le 1$, and $x \in \mathbb{R}$,

$$\mathbb{P}\{\mathrm{dist}(x, \gamma) \le xr\} \asymp r^{4a-1}.$$

This can be improved using essentially the same proof as Theorem (4) so we omit it. As in that theorem, we are unable to determine the value of the constant c'.

Theorem 5. *There exist $c' = c'_\kappa$ and $\alpha > 0$ such that for $r \le 1$,*

$$\mathbb{P}\{\mathrm{dist}(x, \gamma) \le r|x|\} = c\, r^{4a-1}\, \left[1 + O(r^\alpha)\right].$$

For practical purposes, the following simple corollary of Proposition 2.10 usually suffices. It is useful to write it in terms of a conformally invariant quantity, excursion measure.

Proposition 2.11. *If $\kappa < 8$, there exists $c < \infty$ such that the following holds. Suppose D is a simply connected domain and z, w are distinct boundary points. Let ∂_- denote one of the two arcs of ∂D with endpoints z, w and let η be a crosscut of D whose endpoints are on $\partial D \setminus \partial_-$. Let D' be the connected component of $D \setminus \eta$ whose boundary contains z and w. If γ is an SLE_κ curve from z to w in D, then*

$$\mathbb{P}\{\gamma \cap \eta \ne \emptyset\} \le c\, \mathcal{E}_{D'}(\eta, \partial_-)^{4a-1}.$$

Proof. By conformal invariance, we may assume $D = \mathbb{H}, z = 0, w = \infty$ and $\partial_-$is the negative real axis. We can also assume that one of the endpoints of η is 1. In this case (see Lemma **??**),

$$\mathrm{cap}_{\mathbb{H}}(\eta) \ge c\, \mathrm{diam}(\eta),$$

and from this we see that $\mathcal{E}_{D'}(\eta, \partial_-) \ge c\, r$ where $r = 1 \wedge \mathrm{diam}(\eta)$.

$$\mathbb{P}\{\gamma \cap \eta \ne \emptyset\} \le \mathbb{P}\{\mathrm{dist}(\gamma, 1) \le r\} \le c\, r^{4a-1} \le c\, \mathcal{E}_{D'}(\eta, \partial_-)^{4a-1}.$$

□

Here we give a similar result which is a little easier to prove because we only need to consider a real SDE.

Proposition 2.12. *Suppose $4 < \kappa < 8$ and γ is an SLE_κ curve from 0 to ∞. Let*

$$T = \inf\{t : \gamma(t) \in [1, \infty)\}.$$

Then

$$\mathbb{P}\{\gamma(T) < 1 + x\} = \frac{\Gamma(2a)}{\Gamma(4a-1)\,\Gamma(1-2a)} \int_0^{\frac{x}{1+x}} \frac{du}{u^{2-4a}\,(1-u)^{2a}}.$$

Note in particular

$$\mathbb{P}\{\gamma(T) < 1 + x) \sim \frac{\Gamma(2a)}{\Gamma(4a-1)\,\Gamma(1-2a)}\, x^{4a-1}, \quad x \downarrow 0.$$

Proof. Let $X_t = g_t(x) - U_t, Y_t = g_t(1+x) - U_t$. Then $T = \inf\{t : X_t = 0\}$. Let $T_x = \inf\{t : Y_t = 0\}$. Then the event $\{\gamma(T) < 1 + x\}$ is the same as the event $\{T < T_x\}$. Let $R_t = [Y_t - X_t]/X_t$. Then this event can also be described as $\{R_{T-} = \infty\}$. Since

$$dX_t = \frac{a}{X_t}\,dt + dB_t, \quad dY_t = \frac{a}{Y_t}\,dt + dB_t,$$

an Itô's formula calculation shows that

$$dR_t = \left[\frac{(1-a)\,R_t}{X_t^2} - \frac{a\,R_t}{(R_t+1)\,X_t^2}\right] dt - \frac{R_t}{X_t}\,dB_t.$$

Under a suitable time change we get that $\hat{R}_t := R_{\sigma(t)}$ satisfies

$$\hat{R}_t = \left[\frac{1-2a}{\hat{R}_t} + \frac{a}{\hat{R}_t + 1}\right] dt + dW_t,$$

for a Brownian motion W_t.

If $\psi(x) = \mathbb{P}\{T_x > T\}$, then scaling shows that $\psi(\hat{R}_t)$ is a martingale and hence (assuming ψ is C^2),

$$\frac{1}{2}\,\psi''(x) + \left[\frac{1-2a}{x} + \frac{a}{x+1}\right] \psi'(x) = 0.$$

This is a first-order differential equation for ψ' that can be solved easily. Using the boundary conditions $\psi(0) = 0, \psi(\infty) = 1$, we get the formula.

□

3 Dimension and natural parametrization

The definition of SLE was given modulo reparametrization. When the curve is parametrized by capacity, the conformal transformation removing the curve evolves so that it is C^1 in time. However, the capacity parametrization has some major disadvantages. For example, if one takes a particular realization of the curve and changes the domain that the curve lies in, then the parametrization changes. Also, SLE curves are models for scaling limits of discrete lattice curves

that are parametrized so that edge is traversed at the same rate. For this reason, we would like to parametrize the SLE curve by length. If the curve γ were a C^1 curve, then we could use the *natural parametrization*, that is, parametrization by arclength. However, Theorem 4 implies that $SLE_\kappa, \kappa < 8$ paths are d-dimensional where

$$d = d_\kappa = 1 + \frac{\kappa}{8}.$$

In this section, we will assume $\kappa < 8$ and define the *natural parametrization* for SLE curves. We need a concept of "d-dimensional length" to replace arclength. One possibility is to consider Hausdorff d-measure of the curve, but it turns out that this is zero. The correct analogue is (*d-dimensional*) *Minkowski content.*We will state some theorems in this section but will delay the proof until ???. The main technical tool needed to prove these theorems is a "two-point" analogue of Theorem 4.

Definition If $V \subset \mathbb{C}$ is bounded and $1 < d < 2$, the *d-dimensional Minkowski content* of V is defined by

$$\mathrm{Cont}_d(V) = \lim_{\epsilon \downarrow 0} \epsilon^{d-2} \,\mathrm{Area}\{z : \mathrm{dist}(z, V) \le \epsilon\}, \tag{30}$$

provided that the limit exists.

One can also define the upper and lower Minkowski contents by taking limsups and liminfs instead of limits. It is easy to see that if the box dimension of V is strictly less than d, then $\mathrm{Cont}_d(V) = 0$. The limit on the right-hand side of (30) often does not exist, but for SLE_κ paths it exists and is nontrivial.

Theorem 6. *If $\kappa < 8$, and γ is an SLE_κ path from 0 to ∞ in $\mathbb{H}$, then with probability one, for all t, the Minkowski content $\mathrm{Cont}_d(\gamma_t)$ exists. Moreover, the function $t \mapsto \gamma(t)$ is continuous, strictly increasing, and satisfies the additivity rule: if $s < t$,*

$$\mathrm{Cont}_d(\gamma_t) = \mathrm{Cont}_d(\gamma_s) + \mathrm{Cont}_d(\gamma[s,t]).$$

The additivity rule is immediate (given the rest of the theorem) if $\kappa \le 4$ since the curve is simple. For $4 < \kappa < 8$, it requires a little more work but essentially follows from the fact that the double points of the paths have fractal dimension strictly less than d. Since the function $t \mapsto \gamma_t$ is continuous and strictly increasing, one can reparametrize the curve so that the content grows linearly.

Although we delay the full proof to later, let us discuss the construction. We will, in fact, construct the *occupation measure* ν^{occ} for SLE which is a nontrivial, positive measure supported on γ such that for each $s < t$, $0 < \nu^{\mathrm{occ}}(\gamma_t \setminus \gamma_s) < \infty$. We say that S is a (closed) dyadic square in $\mathbb{H}$ if it is of the form

$$S = \left\{ x + iy : \frac{j}{2^n} \le x \le \frac{j+1}{2^n}, \ \frac{k}{2^n} \le y \le \frac{k+1}{2^n} \right\}$$

where j, n are integers and k is a positive integer. Let

$$\nu^{\mathrm{occ}}_\epsilon(S) = \epsilon^{d-2} \int_S 1\{\mathrm{dist}(z, \gamma) \le \epsilon\} \, dA(z),$$

and note that Theorem 4 implies that

$$\lim_{\epsilon \downarrow 0} \mathbb{E}\left[\nu_\epsilon^{\rm occ}(S)\right] = \int_S \mathbb{G}(z)\, dA(z).$$

We will show that for each dyadic S, the limit $\nu^{\rm occ}(S) = \nu_{0+}^{\rm occ}(S)$ exists with probability one and in L^2. With a little more, we see that any subsequential limit ν of the measures $\nu_\epsilon^{\rm occ}$ must satisfy $\nu(\partial S) = 0$, from this see that any two limit measures agree on all open sets from which we conclude that the limit is unique and we can define

$$\nu^{\rm occ} = \lim_{\epsilon \downarrow 0} \nu_\epsilon^{\rm occ}.$$

Definition We say that an SLE_κ curve γ from 0 to ∞ in $\mathbb{H}$ has the *natural parametrization* if for each t,

$$\mathrm{Cont}_d[\gamma_t] = t, \quad d = d_\kappa = 1 + \frac{\kappa}{8}.$$

If $\kappa \geq 8$, we could say that SLE_κ has the natural parametrization if for each t, $\mathrm{Area}[\gamma_t] = t$. In this book we will only talk about natural parametrizaton for $\kappa < 8$.

We want to extend this definition to talk about SLE_κ from z to w in D in the natural parametrization. We will take a little care to avoid pathological domains.

Definition Let $\mathcal{Q}_{\rm sc}$ denote the collection of triples (D, z, w) where D is a simply connected domain; z, w are distinct points in ∂D; $z \neq \infty$; and satisfying

- If $w \neq \infty$, then
$$\mathbb{G}_D(D; z, w) := \int_D \mathbb{G}_D(\zeta; z, w)\, dA(\zeta) < \infty. \tag{31}$$
- If $w = \infty$, then for every bounded set V,
$$\mathbb{G}_D(V; z, \infty) := \int_V \mathbb{G}_D(\zeta; z, \infty)\, dA(\zeta) < \infty. \tag{32}$$

Recall that SLE_κ from z to w in D is defined as the image of SLE_κ from 0 to ∞ under a conformal map $f : D \to f(D)$. Let us define the curve $f \circ \gamma = f \circ_d \gamma$ as follows:

$$f \circ \gamma(t) = f[\gamma(\sigma(t))]$$

where $\sigma(t)$ is defined by

$$\int_0^{\sigma(t)} |f'(\gamma(s))|^d\, ds = t.$$

In other words, the time for $f \circ \gamma$ to traverse $f[\gamma_t]$ is

$$\int_0^t |f'(\gamma(s))|^d\, ds.$$

This definition assumes that D, z are sufficiently nice so that

$$\int_0^t |f'(\gamma(s))|^d \, ds < \infty,$$

for all $t < \infty$. This will be guaranteed, for example, if for all $t < \infty$,

$$\max_{0 \le s \le t} |f'(\gamma(s))| < \infty.$$

In order to keep the notation short, we have decided to write $f \circ \gamma$ rather than $f \circ_d \gamma$. However, it is important to remember that the notation implicitly assumes the dimension d. When studying SLE_κ paths, we fix one value of κ, and then d is implied by $d = \min\{1 + \frac{\kappa}{8}, 2\}$. If one is only interested in the curves modulo reparametrization, then one does need to worry about d.

Here we are doing an analogue of what is done in establishing the conformal invariance of two-dimensional Brownian motion. If B_t is a complex Brownian motion and f is a conformal map, then $f \circ B = f \circ_2 B$ is also a complex Brownian motion. The 2 represents the dimension of Brownian paths, and this gives a shorthand for the change in time parametrization needed to state conformal invariance.

Proposition 3.1. *Suppose γ is an $SLE_\kappa(\kappa < 8)$ path from 0 to ∞ in $\mathbb{H}$ with the natural parametrization. Suppose $f : \mathbb{H} \to D$ is a conformal transformation with $f(0) = z, f(\infty) = w$ satisfying* (31)*. Then with probability one, for every t,*

$$\mathrm{Cont}_d[f(\gamma_t)] = \int_0^t |f'(\gamma(s))|^d \, ds.$$

In other words, the curve $\tilde{\gamma} := f \circ \gamma$ has the natural parametrization in the sense

$$\mathrm{Cont}_d[\tilde{\gamma}_t] = t.$$

Proof. We will not give the details of the proof. If f' is constant, this is immediate from the definition of the Minkowski content. Otherwise, we can use the additivity of the content to partition time $t_0 = 0 < t_1 < t_2 < \cdots < t_n = t$ so that the derivative does not change much in each interval. The condition (31) is used to see that the contribution of the content for the curve near the boundary (for which the derivative can be very large) is negligible. □

The total "natural time duration" T of a path is $\mathrm{Cont}_d[\gamma]$. We can see that if (31) holds, then

$$\mathbb{E}[T] = \mathbb{G}_D(D; z, w) < \infty.$$

In the case $z = 0, w = \infty, D = \mathbb{H}$, $T = \infty$ with probability one. but if D is a bounded domain and z, w are analytic boundary points, then $\mathbb{E}[T] < \infty$.

One major advantage of the natural parametrization over the capacity parametrization is that the time to traverse a curve γ depends only on the curve γ and not on the domain it sits.

Definition Suppose $\kappa < 8$ and $(D, z, w) \in \mathcal{Q}_{\rm sc}$.

- We define $\mu_D^{\#}(z, w)$ to be the probability measure induced by SLE_κ from z to w with the curves curves parametrized naturally.
- If $f : D \to f(D)$ is a conformal transformation and μ is a measure on naturally parametrized d-dimensional curves from z to w in D, we define the measure $f \circ \mu$ by

$$f \circ \mu[V] = f \circ_d \mu[V] = \mu\{\gamma : f \circ \gamma \in V\}.$$

  Note that there is an implicit d in the definition.

Under this definition, the conformal invariance of SLE_κ can be stated as follows.

Proposition 3.2. *Suppose $(D, z, w), (D_1, z_1, w_1) \in \mathcal{Q}_{\rm sc}$, and $f : D \to D_1$ is a conformal transformation with $f(z) = z_1, f(w) = w_1$. Then*

$$f \circ \mu_D^{\#}(z, w) = \mu_{D_1}^{\#}(z_1, w_1).$$

Throughout this book we use superscripts of # to indicate probability measures. As we will see, it is very natural to consider measures on paths that are not probability measures.

The measure $\mu_D^{\#}(z, w)$ is just another way to define SLE_κ in the same way that Brownian motion can be defined equivalently as the Wiener measure on continuous paths. The Minkowski content gives the convenient way to parametrize the paths for the path measure formulation.

We have used d-dimensional Minkowski content to characterize the fractal dimension of the curve. One can also use Hausdorff dimension, and this agrees. This proof also needs the two-point estimates, and so we delay it.

Theorem 7. *If $\kappa < 8$ and γ is an SLE_κ curve from 0 to ∞ in $\mathbb{H}$, then with probability one the Hausdorff dimension of the curve is $d = 1 + \frac{\kappa}{8}$.*

Proof. Consider γ restricted to a compact subset K in $\mathbb{H}$. There exist $c = c_K$ such that for $0 < \epsilon \le 1$, $z \in K$,

$$c_K^{-1}\, \epsilon^{2-d} \le \mathbb{P}\{\mathrm{dist}(z, \gamma_\infty) \le \epsilon\} \le c_K\, \epsilon^{2-d}.$$

In Section 9, we prove a two-point estimate,

$$\mathbb{P}\{\mathrm{dist}(z, \gamma_\infty) \le \epsilon,\ \mathrm{dist}(w, \gamma_\infty) \le \epsilon\} \le c_K\, \epsilon^{2-d}\, |z - w|^{d-2}.$$

This is enough to show that the dimension is at most d with probability one, and that with positive probability it is at least d; see Section 10. To show that it is equal to d with probability one, we can use the fact that the dimension of $\gamma[0, 1]$ is the maximum of the dimensions of $\gamma[\frac{k-1}{n}, \frac{k}{n}], k = 1, \ldots, n$; we leave the details as the next exercise.

□

Exercise 1. *Complete the proof by showing that with probability one, for all $s < t$, $\dim_h(\gamma[s, t]) = d$.*

It is typical for random curves with Hausdorff dimension d to have zero d-Hausdorff dimension. Indeed, this true for SLE_κ curves for $\kappa < 8$ but we will not prove it here. This is different than the case of deterministic fractals such as the Koch snowflake curve. It is possible that there is a Hausdorff gauge function such that the measure with this gauge function is strictly between zero and infinity, but this is unknown.

4 Rate of escape for SLE

Let γ be an SLE_κ path from 0 to ∞ in $\mathbb{H}$. Let

$$\xi_s = \inf\{t : |\gamma(t)| = e^s\}.$$

Since $\mathrm{hcap}[e^s\,\overline{\mathbb{D}}_+] = e^{2s}$, we can see that $\xi_s \le e^{2s}/a$.

In this section we prove the following theorem.

Proposition 4.1. *For every $\kappa < 8$, there exists $c < \infty$ such that the following holds. Suppose $\eta : [0,1] \to \overline{\mathbb{H}}$ is a curve with $\eta(0+) = 1, \eta(1) = w \in \partial\mathbb{D}$ and $0 \le |\eta(t)| < 1$ for $0 < t < 1$. Let D denote the unbounded component of $\mathbb{H} \setminus \eta$. Let γ be an SLE_κ curve from w to ∞ in D. Then*

$$\mathbb{P}\{\gamma \cap \{|z| \le r\} \ne \emptyset\} \le c\, r^{4a-1}.$$

We will only give a complete proof for $\kappa \le 4$. For $4 < \kappa < 8$, we will sketch a proof (this second proof also works for $\kappa \le 4$) but we omit some details.

Corollary 4.2. *For every $\kappa < 8$ and $\beta > (4a-1)^{-1} = \kappa/(8-\kappa)$, with probability one, for all integers n sufficiently large,*

$$\gamma[\xi_n, \infty) \cap (n^{-\beta}\, e^n\, \overline{\mathbb{D}}_+) \ne \emptyset, \tag{33}$$

and for all t sufficiently large

$$|\gamma(t)| \ge t^{1/2} (\log t)^{-\beta}.$$

Proof. If E_n denotes the event in (33), then Proposition 4.1 applied to $D = H_{\xi_n}$ and scaling imply that $\mathbb{P}(E_n) = O(n^{-u})$ for some $u > 1$ and hence the first assertion follows from the Borel-Cantelli lemma. The second follows from $\xi_n \le e^{2n}/a$. □

Proof of Proposition 4.1 for $\kappa \le 4$. Let C denote the half-circle of radius r about the origin in $\mathbb{H}$. Let $g : D \to \mathbb{H}$ be a conformal transformation with $g(w) = 0, g(\infty) = \infty$. Note that $D \cap \partial\mathbb{D}$ consists of two circular crosscuts L_+, L_- where $g \circ L_+$ (resp., L_-) is a crosscut of $\mathbb{H}$ from 0 to a positive (resp., negative) number.

Note that $C \cap D$ consists of a finite or countably infinite number of circular crosscuts $\mathcal{L} = \{\ell_j\}$. In order for γ to hit $\{|z| \le r\}$ it must hit one of the crosscuts, and so

$$\mathbb{P}\{\gamma \cap \{|z| \le r\} \ne \emptyset\} \le \sum_{\ell \in \mathcal{L}} \mathbb{P}\{\gamma \cap \ell \ne \emptyset\}.$$

A curve from ℓ_j to infinity in D must first hit the unit circle at L_+ or L_-. We will restrict ourselves to the positive crosscuts $\mathcal{L}_+$, that is, those that go through L_+. (A similar argument will handle the negative crosscuts). Using Proposition 2.11, we have

$$\sum_{\ell\in\mathcal{L}_+} \mathbb{P}\{\gamma\cap\ell\neq\emptyset\} = \sum_{\ell\in\mathcal{L}_+} \mathbb{P}\{\tilde{\gamma}\cap(g\circ\ell)\neq\emptyset\} \le c\sum_{\ell\in\mathcal{L}_+} \mathcal{E}_{\mathbb{H}}(\mathbb{R}_-, g\circ\ell)^{4a-1} \le c\left[\sum_{\ell\in\mathcal{L}_+} \mathcal{E}_{\mathbb{H}}(\mathbb{R}_-, g\circ\ell)\right]^{4a-1}.$$

The last inequality uses $4a-1\ge 1$; this is where the assumption $\kappa\le 4$ is being used. Hence we have reduced to showing the estimate

$$\sum_{\ell\in\mathcal{L}_+} \mathcal{E}_{\mathbb{H}}(\mathbb{R}_-, g\circ\ell) \le c\,r.$$

Note that by continuity and monotonicity of excursion measure, $\mathcal{E}_{\mathbb{H}}(\mathbb{R}_-, g\circ\ell) \le \mathcal{E}_D(L_+,\ell)$. Note that

$$\mathcal{E}_D\left(L_+, \bigcup_{\ell\in\mathcal{L}_+}\ell\right) \le \mathcal{E}_{\mathbb{H}}(\partial\mathbb{D}, C) = O(r).$$

We claim (and this will finish the proof) that

$$\sum_{\ell\in\mathcal{L}_+} \mathcal{E}_{\mathbb{H}}(L_+,\ell) \le 2\,\mathcal{E}_D\left(L_+, \bigcup_{\ell\in\mathcal{L}_+}\ell\right).$$

If $z\in D$, let $f(z)$ be the probability that a Brownian motion starting at z hits $\cup_{\ell\in\mathcal{L}_+}\ell$ before leaving D or hitting L_+. Let $f_\ell(z)$ be the probability that a Brownian motion starting at z hits ℓ before leaving D or hitting L_+. Then,

$$\sum_{\ell\in\mathcal{L}_+} \mathcal{E}_{\mathbb{H}}(L_+,\ell) = \int_{L_+} \sum_{\ell\in\mathcal{L}_+} \partial_n\, f_\ell(z)\, d|z|,$$

$$\mathcal{E}_D\left(L_+, \bigcup_{\ell\in\mathcal{L}_+}\ell\right) = \int_{L_+} \partial_n\, f(z)\, d|z|.$$

where ∂_n denotes inward normal derivative. It suffices to show for each $z\in D$,

$$\sum_{\ell\in\mathcal{L}_+} f_\ell(z) \le 2f(z).$$

Note that the left-hand side gives the *expected number* of crosscuts hit by the Brownian motion. So it suffices to show that for every z, the expected number of crosscuts hit given that at least one is hit is at most two. In turn, to show this it suffices to show that if we start a Brownain motion on a crosscut ℓ, then the probability that it will reach any other crosscut before leaving D is at most $1/2$. This last fact can be seen by noting that if we have any curve starting on the crosscut and ending on a different crosscut, there is a dual path obtained by reflecting across the circle C. These are equally likely for the Brownian motion, but simple connectedness implies that at most one of these paths is in D. □

We will need another proof to handle $4 < \kappa < 8$. We start with the notation of the proof and let $h(z) = \log g(z)$ which is a conformal transformation of D onto the doubly infinite strip $S = \{x + iy : 0 < y < \pi\}$ sending 0 to $-\infty$ and ∞ to $+\infty$. Here $-\infty, \infty$ refer to the left and right boundary points of S, respectively. For $\ell \in \mathcal{L}_+$, $h \circ \ell$ is a crosscut of S with both endpoints on the real line. Let

$$K = \bigcup_{\ell \in \mathcal{L}_+} (h \circ \ell).$$

Proposition 2.11 and conformal invariance show that the probability that an SLE_κ path from $-\infty$ to ∞ in S hits one of the crosscuts is bounded above by a constant times

$$\sum_{\ell \in \mathcal{L}_+} [\mathrm{diam}(h \circ \ell)]^{4a-1}.$$

This estimate is not sufficient. However, we not that if $U_1, U_2, \ldots$ is any cover of K with balls centereed on the real line, then the probability of hitting K is bounded above by the probability of hitting $\bigcup U_n$ which in turn is bounded by a constant times

$$\sum_{n=1}^{\infty} \mathrm{diam}(U_n)^{4a-1}. \tag{34}$$

If we ease the restriction that the balls be centered on the real line, one can only reduce this quantity by a multiplicative constant and so we get that the probability of hitting K is bounded above by a constant times the infimum of the quantity in (34) where the infimum is over all covers of K. This infimum is called the $(4a-1)$-Hausdorff content.

The problem then reduces to an estimate for the content. We omit it here because we will not need the proposition for $4 < \kappa < 8$.

5 SLE in a smaller domain

5.1 Introduction

We will study chordal SLE_κ from 0 to ∞ in a subdomain of $\mathbb{H}$. Suppose D is a simply connected subdomain of $\mathbb{H}$ with $K := \mathbb{H} \setminus D$ bounded and $\mathrm{dist}(0, K) > 0$. Let $\Phi = \Phi_D : D \to \mathbb{H}$ denote the unique conformal transformation with $\Phi(z) = z + o(1)$ as $z \to \infty$. Let $F = \Phi^{-1}$ which takes $\mathbb{H}$ onto D. Then SLE_κ in D from 0 to infinity is defined as follows.

- Let γ be an SLE_κ from 0 to $\mathbb{H}$ and consider $F \circ \gamma$.

We can consider this as a probabiity measure on curves modulo reparametrization or for curves with the natural parametrization.

If $\kappa \leq 4$, we will see that SLE_κ in D, is absolutely continuous with respect to SLE_κ in $\mathbb{H}$. Let us see what this will imply. Since the measure is a probability measure on simple curves γ, we can parametrize the curves so that $\mathrm{hcap}[\gamma_t] = at$ and the conformal maps g_t satisfy the Loewner equation

$$\dot{g}_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{35}$$

where $U_t = g_t(\gamma(t))$ is the driving function. To say that this probability measure is absolutely continuous with respect to usual SLE_κ is to say that the probability distribution on the driving function is absolutely continuous with respect to a standard Brownian motion. The way to get such measures is to give a drift to the Brownian motion, that is if B_t is a standard Brownian motion, we can let

$$dU_t = A_t\, dt - dB_t, \tag{36}$$

where A_t is a continuous, adapted process. One way to find A_t is starting with an SLE_κ $\gamma^*(t)$ in $\mathbb{H}$ with driving function $U_t^* = -B_t^*$ and carefully using a version of Itô's formula to compute $dF(U_t^*)$. We will do essentially the equivalent, by finding which A_t to choose so that $\Phi(U_t)$ is a (time change of a) Brownian motion.

One way to put drifts onto Brownian motion is to find an appropriate local martingale and use Girsanov theorem. We will start with SLE_κ in $\mathbb{H}$ using driving function $-B_t$ and then tilt by a local martingale to get (36). Recall that if $x \in \mathbb{R} \setminus \overline{K}$, then $\Phi'(x)$ denotes the probability that a Brownian excursion from x to ∞ in $\mathbb{H}$ stays in D; in particular, $0 < \Phi'(x) \le 1$. We will "tilt" by $\Phi_t'(U_t)^b$ where $b = b_\kappa$ is an exponent that we will call the boundary scaling exponent. It will be chosen so the calculations work out nicely but the choice will help describe the SLE path. More precisely, we will consider a local martingale $M_t = C_t\, \Phi_t'(U_t)^b$ where C_t is a C^1 compensator. The calculations will be valid for all κ for t sufficiently small but for $\kappa \le 4$ it will be valid for all t and we will be able to write $\mathbb{E}[C_\infty = \mathbb{E}[M_\infty] = \mathbb{E}[M_0] = \Phi'(0)^b$.

The term C_∞ will be written in terms of integrals over time of spatial derivatives, but it can also be interpreted in terms of Brownian loops. Indeed, if $\kappa \le 4$,

$$M_\infty = \exp\left\{\frac{\mathbf{c}}{2}\,\Lambda(\gamma, K; \mathbb{H})\right\}\, 1\{\gamma(0,\infty) \subset D\}, \tag{37}$$

where $\Lambda(\gamma, K; \mathbb{H})$ denote the Brownian loop measure of loops in $\mathbb{H}$ that intersect both γ and K. Here $\mathbf{c} = \mathbf{c}_\kappa$ is an exponent that is also computed along the way.

The strongest results hold only for $\kappa \le 4$, but some things still hold for $\kappa > 4$. If $\kappa > 4$, then there is a positive probability that the SLE_κ path hits K; if $\kappa \ge 8$, this probability equals one. If the curve hits K, then SLE_κ in D will act differently than SLE_κ in H since the curve bounces off of K.

- If $\kappa \ge 8$, SLE_κ in D is singular with respect to SLE_κ in $\mathbb{H}$.
- If $4 < \kappa < 8$, SLE_κ in D has a singular part and an absolutely continuous part with respect to SLE_κ in $\mathbb{H}$. The absolutely continuous part corresponds to curves in $\mathbb{H}$ that do not hit K,

Since they will appear in the computations, we define two parameters now.

Definition

The *boundary scaling exponent* is

$$b = b_\kappa = \frac{6-\kappa}{2\kappa} = \frac{3a-1}{2}.$$

The *central charge* $\mathbf{c} = \mathbf{c}_\kappa$ is defined by

$$\mathbf{c} = \frac{2b(3-4a)}{a} = \frac{(3a-1)\,(3-4a)}{a} = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}.$$

We make several remarks.

- The value $b = 0$ for $\kappa = 6$ gives a property we call the *locality property.*
- We have $\mathbf{c} = 0$ for $\kappa = 8/3, 4$. The value $\kappa = 8/3$ gives the measure on simple curves such that the Brownian loop term in (37) is trivial. This is called the *restriction property.*
- We can invert the formula for central charge

$$a = \frac{(13-\mathbf{c}) \pm \sqrt{(13-\mathbf{c})^2 - 144}}{24}, \quad \kappa = \frac{(13-\mathbf{c}) \pm \sqrt{(13-\mathbf{c})^2 - 144}}{3}.$$

  Note that $\kappa \mapsto \mathbf{c}_\kappa$ is a two-to-one mapping from $(0,\infty)$ to $(-\infty, 1]$; more precisely it is two-to-one except that $\mathbf{c} = 1$ has a single preimage $\kappa = 4$.
- For any $\mathbf{c} < 1$, there are two corresponding values of κ, κ' with $\mathbf{c}_\kappa = \mathbf{c}_{\kappa'} = \mathbf{c}$. They satisfy $\kappa\,\kappa' = 16$ (as does the double root $\kappa = \kappa' = 4$ for $\mathbf{c} = 1$).
- The term central charge comes from conformal field theory. There are algebraic meanings of the term which we will not discuss here. The standard notation for central charge is c, but it would be too confusing to use this when c so often means an arbitrary constant. As a compromise, we use $\mathbf{c}$.

5.2 Computations and locality

Let γ be an SLE_κ curve from 0 to infinity in $\mathbb{H}$ satisfying (35) where $U_t = -B_t$ is a standard Brownian motion, and, as before, let H_t denote the unbounded component of $\mathbb{H} \setminus \gamma_t$. Let $T_\epsilon = \inf\{t : \mathrm{dist}(\gamma_t, \mathbb{H} \setminus D) \le \epsilon\}$ and $T = T_{0+} = \inf\{t : \gamma(t) \in \overline{\mathbb{H} \setminus D}\}$. For fixed $\epsilon > 0$, we can consider SLE_κ in $\mathbb{H}$ (from 0 to ∞) and chordal SLE_κ in D (from 0 to ∞). Note that

$$\mathbb{P}\{T = \infty\} > 0, \quad \kappa < 8,$$

$$\mathbb{P}\{T = \infty\} = 0, \quad \kappa \ge 8.$$

We will show that the two distributions are absolutely continuous if viewed as measures on curves up to time T_ϵ. Also, on the event $\{T = \infty\}$, the measures on the full curves are absolutely continuous.

We will start by doing some computations that are valid for all κ for $t < T$. We will use some deterministic results from Section **??** which we now recall. Let $D_t = g_t(D) = \mathbb{H} \setminus K_t$ where $K_t = g_t(K)$. Let $\Phi_t = \Phi_{D_t}$ which is the unique conformal transformation of D_t onto $\mathbb{H}$ with $\Phi_t(z) = z + o(1), z \to \infty$. Let $\gamma^*(t) = \Phi(\gamma(t))$ and $H_t^* = \Phi(H_t \setminus K)$. Let $g_t^* : H_t^* \to \mathbb{H}$ be the conformal transformation with $g_t^*(z) = z + o(1), z \to \infty$. Then (**??**) tells us that

$$\mathrm{hcap}[\gamma_t^*] = a \int_0^t \Phi_s'(U_s)^2\, ds, \quad \partial_t g_t^*(z) = \frac{a\,\Phi_t'(U_t)^2}{g_t^*(z) - U_t^*}, \tag{38}$$

where $U_t^* = \Phi_t(U_t)$. (Here we have replaced the factor 2 with the factor a because we are growing our curve so that the hcap grows at rate a.) We will use the following deterministic results which are restatements of (**??**) and (**??**), where they are stated for $a = 2$,

$$\dot\Phi_t(U_t) = \lim_{z \to U_t} \dot\Phi_t(z) = -\frac{3a}{2}\,\Phi_t''(U_t), \tag{39}$$

$$\dot{\Phi}_t'(U_t) = \frac{a}{4}\,\frac{\Phi_t''(U_t)^2}{\Phi_t'(U_t)} - \frac{2a}{3}\,\Phi_t'''(U_t). \tag{40}$$

We have mentioned before that the capacity parametrization of a curve depends on the domain. We have parametrized γ so that the half plane capacity of γ_t in $\mathbb{H}$ (from infinity) grows at rate a. We can interpret (38) as saying that the capactiy of γ_t viewed as a curve in D grows at rate $a\,\Phi_t'(U_t)^2$.

Theorem 8. *Suppose $\gamma(t), 0 \le t < T$ is a solution to the Loewner equation* (35) *where the driving function U_t satisfies*

$$dU_t = b\,\frac{\Phi_t''(U_t)}{\Phi_t'(U_t)}\,dt + dW_t, \quad t < T,$$

where W_t is a standard Brownian motion. Then γ has the distribution of SLE_κ in D up to time T. Here $b = (3a-1)/2$ is the boundary scaling exponent.

Proof. Itô's formula and (39) give

$$\begin{aligned} dU_t^* = d[\Phi_t(U_t)] &= \left[\dot{\Phi}_t(U_t) + \frac{1}{2}\,\Phi_t''(U_t)\right] dt + \Phi_t'(U_t)\,dU_t \\ &= -b\,\Phi_t''(U_t)\,dt + \Phi_t'(U_t)\,dU_t \\ &= \Phi_t'(U_t)\,dW_t. \end{aligned} \tag{41}$$

Let $\gamma^*(t) = \Phi[\gamma(t)]$ with corresponding conformal maps g_t^* as in (38). Let $\hat{\gamma}(t) = \gamma^*(\sigma(t))$ where the time change $\sigma(t)$ is chosen so that $\mathrm{hcap}[\hat{\gamma}_t] = at$, that is

$$\int_0^{\sigma(t)} \Phi_s'(U_s)^2\,ds = t.$$

In other words, U^* is a continuous local martingale with quadratic variation

$$\langle U^* \rangle_t = \int_0^t \Phi_s'(U_s)^2\,ds = \mathrm{hcap}[\gamma_t^*]$$

Using scaling (Proposition 1.1), we can see that γ^* is (a time change of) SLE_κ. □

Here we are using Itô's formula on a function $\Phi_t(\cdot)$ that is random but adapated to the Brownian motion. The results (**??**) and (**??**) show that these are C^1 in t and hence the usual proof of Itô's formula is valid. We will often write quantitites such as $\dot{\Phi}_t(U_t)$; this denotes the function $\dot{\Phi}_t(z)$ evaluated at the point $z = U_t$. The time derivative is not taken on the (in fact, not even differentiable) function U_t.

The following is an immediate corollary since $b = 0$ if $\kappa = 6$.

Theorem 9 (Locality for SLE_6)**.** *Suppose γ is an SLE_6 in $\mathbb{H}$ and T is defined as above. Then the distribution of γ_T is the same as the distribution of SLE_6 from 0 to ∞ in D stopped at the first visit to K.*

The two distributions are clearly different after time T since SLE_6 in D "bounces off" K which SLE_6 in $\mathbb{H}$ does not see K.

This is somewhat analogous to a reflected Brownian motion in a domain D. Such a Brownian motion does not "see" the boundary ∂D until it hits it, that is, the distribution of the path before the hitting time is the same whether or not the boundary is there. For other values of κ, the SLE_κ does feel the boundary before reaching it.

The discrete model that satisfies the locality property is the percolation exploration process and this is why $\kappa = 6$ is the natural choice for the scaling limit.

5.3 The fundamental local martingale

In the section we will do one of the early fundamental calculations about SLE. It was first done when the restriction property was conjectured for $SLE_{8/3}$; more precisely, it was conjectured that the scaling limit of self-avoiding walks would be a probability measure on simple curves and that the restriction propertry, which held trivially on the discrete level, would also hold for this process. So it was asked which valuse of κ give a process that satisfies restriction. If it did it could be shown that there must be a β such that the probability that SLE_κ does not hit K is $\Phi'(0)^\beta$. Using the conformal Markov property for SLE_κ that would imply that $M_t = \Phi'(U_t)^\beta$ was a martingale.

Using this as motivation, we will consider for all κ what happens when one tilts by $\Phi'(U_t)^\beta$. This is not always a local martingale but by choosing an approriate value of β we can also find the compensator to make this a martingale. Recall the exponents $b, \mathbf{c}$ defined above. The next proposition is an exercise in Itô's formula using the deterministic identity (40).

Proposition 5.1. *Suppose D is as above and g_t satisfies* (35) *where U_t is a standard Brownian motion. Let*

$$M_t = \Phi_t'(U_t)^b \exp\left\{-\frac{a\mathbf{c}}{12}\int_0^t S\Phi_s(U_s)\,ds\right\}, \quad t < T,$$

where b is the boundary scaling exponent, $\mathbf{c}$ is the central charge, and S denotes the Schwarzian derivative

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3f''(z)^2}{2f'(z)^2}.$$

Then M_t is a local martingale for $t < T$ satisfying

$$dM_t = b\,\frac{\Phi_t''(U_t)}{\Phi_t'(U_t)}\,M_t\,dU_t. \tag{42}$$

Proof. This is a straightforward calculation. Let $\psi_t(z) = \Phi_t'(z)^\beta$. Then, using (40) we see that

$$\dot{\psi}_t(U_t) = \beta\,\psi_t(U_t)\,\frac{\dot{\Phi}_t'(U_t)}{\Phi_t(U_t)} = \beta\;\psi_t(U_t)\,\left[\frac{a\,\Phi_t''(U_t)^2}{4\,\Phi_t'(U_t)^2} - \frac{2a\,\Phi_t'''(U_t)}{3\Phi_t'(U_t)}\right],$$

$$\psi_t'(z) = \beta\, \psi_t(z)\, \frac{\Phi_t''(z)}{\Phi_t'(z)},$$

$$\psi_t''(z) = \beta\, \psi_t(z)\, \left[(\beta-1)\, \frac{\Phi_t''(z)^2}{\Phi_t'(z)^2} + \frac{\Phi_t'''(z)}{\Phi_t'(z)}\right],$$

Therefore, Itô's formula gives

$$d\Phi_t'(U_t)^\beta = \beta\, \Phi_t'(U_t)^\beta\, \left(\left[\frac{a+2\beta-2}{4}\, \frac{\Phi_t''(U_t)^2}{\Phi_t'(U_t)^2} + \frac{3-4a}{6}\, \frac{\Phi_t'''(U_t)}{\Phi_t'(U_t)}\right] dt + \frac{\Phi_t''(U_t)}{\Phi_t'(U_t)}\, dU_t\right).$$

The term inside the square brackets can be written as

$$\frac{3-4a}{6}\, S\Phi_t(U_t) + \frac{2\beta+1-3a}{4}\, \frac{\Phi_t''(U_t)^2}{\Phi_t'(U_t)^2}.$$

If we choose $\beta = b = (3a-1)/2$, then the second term vanishes and we have

$$\begin{aligned} d\Phi_t'(U_t)^b &= \Phi_t'(U_t)^b\, \left[b\, \frac{3-4a}{6}\, S\Phi_t(U_t)\, dt + b\, \frac{\Phi_t''(U_t)}{\Phi_t'(U_t)}\, dU_t\right] \\ &= \Phi_t'(U_t)^b\, \left[\frac{a\mathbf{c}}{12}\, S\Phi_t(U_t)\, dt + b\, \frac{\Phi_t''(U_t)}{\Phi_t'(U_t)}\, dU_t\right]. \end{aligned}$$

Given this, the computation for dM_t follows from the product rule. □

One surprise that comes from the calculation is that choosing $\beta = b$ not only makes the compensator term nicer, it also gives the equation (42) which tells us what the drift will be if we tilt by M_t and use the Girsanov theorem.

Proposition 5.2. *Under the assumptions of the previous theorem, if γ is an SLE_κ with driving function $U_t = -B_t$, and if we tilt by the local martingale $M_t, t < T$, then*

$$dU_t = b\, \frac{\Phi_t''(U_t)}{\Phi_t'(U_t)}\, dt + dW_t,$$

where W_t is a standard Brownian motion in the new measure. In particular, in the new measure, γ is SLE_κ in D stopped at time T.

Proof. The equation for dU_t follows immediately from the Girsanov theorem and the last comment follows by comparison with (8). □

The proof of Proposition 5.1 was a straightforward, if a bit tedious, calculation. We will give a more geometric and probabilistic interpretation to the compenstor term in the martingale. We start by recalling theh relationship among $S\Phi$, half-plane capacity and "Brownian bubbles". If $D = \mathbb{H} \setminus K$ is a subdomain of $\mathbb{H}$ (not necessarily simply connected) with $\mathrm{dist}(0, \mathbb{H} \setminus D) > 0$, we define

$$\Gamma(0, D) = \mathrm{hcap}[-1/K] = \lim_{y \downarrow 0} y^{-1}\, \mathbb{E}^{iy}\, [\mathrm{Im}(-1/B_\tau)]\,,$$

where B_t is a standard complex Brownian motion, τ_D is the first exit time from D. Recall that $\mathrm{Im}(-1/z) = H_{\mathbb{H}}(z, 0)$ where H denotes π times the Poisson kernel. This can be interpreted as the measure of Brownian bubbles at 0 in $\mathbb{H}$ that do not stay in D. We recall (**??**).

- Suppose D is a simply connected subdomain of $\mathbb{H}$ with $\mathrm{dist}(0, \partial D) > 0$. Let $\Phi : D \to \mathbb{H}$ be a conformal transformation with $\Phi(0) = 0$. Let $D' = \{-1/z : z \in D\}$ and $K' = \mathbb{H} \setminus D'$. Then

$$\Gamma(0, D) = \mathrm{hcap}(K') = -\frac{1}{6}\, S\Phi(0).$$

We can write the martingale in terms of the Brownian loop measure discussed in Section **??**. By (**??**), we can see that

$$a \int_0^t \Gamma(U_t, D_t)\, ds = \Lambda(\mathbb{H}; \mathbb{H} \setminus D, \gamma_t),$$

where the right-hand side is the Brownian loop measure of loops in $\mathbb{H}$ that intersect both $\mathbb{H} \setminus D$ and γ_t. This representation focuses on the first point of the curve γ hit by a loop. (The factor a is there because we have parametrized our curve so that the capacity grows at rate a.) For this reason, we will write our local martingale as

$$M_t = \Phi_t'(U_t)^b \exp\left\{\frac{\mathbf{c}}{2}\, \Lambda(\mathbb{H}; \mathbb{H} \setminus D, \gamma_t)\right\}, \quad t < T,$$

where T is the first time that γ hits K. One advantage of this representation is that looks the same for all parametrizations of the SLE curve. This local martingale is valid for all $\kappa > 0$, but for $\kappa \le 4$ it is a martingale.

Theorem 10. *Suppose $0 < \kappa \le 4$ and $D = \mathbb{H} \setminus K$ is a simply connected domain with K bounded and $\mathrm{dist}(0, K) > 0$. Let $\Phi_D : D \to \mathbb{H}$ be the conformal transformation with $\Phi_D(z) = z + o(1), z \to \infty$. Let γ denote an SLE_κ curve from 0 to ∞ in $\mathbb{H}$. Let*

$$M_\infty = \exp\left\{\frac{\mathbf{c}}{2}\, \Lambda(\mathbb{H}; K, \gamma)\right\} 1\{\gamma \cap K = \emptyset\}.$$

Then,

$$\mathbb{E}[M_\infty] = \Phi_D'(0)^b.$$

Moreover, if Q denotes the probability measure

$$\frac{dQ}{d\mathbb{P}} = \frac{M_\infty}{M_0} = \frac{M_\infty}{\Phi_D'(0)^b},$$

then with respect to Q, γ has the distribution of SLE_κ from 0 to ∞ in D.

Proof. We first make a remark about SLE_κ in D. Let $\tilde{\eta}$ denote an SLE_κ from 0 to ∞ in $\mathbb{H}$ and $\eta = \Phi_D^{-1} \circ \eta$. Then η has the distribution of chordal SLE_κ from 0 to infinity in D. It follows from Corollary 4.2 that with probability one, $\mathrm{dist}(\eta, K) > 0$ and $\eta_t \to \infty$ as $t \to \infty$. In particular, $\Gamma(\mathbb{H}; K, \eta) < \infty$.

We now prove the theorem. Recall that $\Phi_t'(U_t)$ denotes the probability that a Brownian excursion from U_t to infinity in $\mathbb{H}$ stays in the domain D_t. In particular, $0 < \Phi_t'(U_t) \le 1$. If $\mathbf{c} \le 0$ and $b > 0$ (that is, if $0 < \kappa \le 8/3$), then M_t is a bounded local martingale. However, for $8/3 < \kappa \le 4$, we have $\mathbf{c} > 0$ and hence M_t is not uniformly bounded. However, even if this case, we can find a sequence of stopping times T_n with $T_n \uparrow T$ such that $M_{t \wedge T_n}$ is bounded. For example, we could choose

$$T_n = n \wedge \inf\{t : M_t \ge n \text{ or } \mathrm{dist}(\gamma_t, \mathbb{H} \setminus D) \le 1/n\}.$$

For each n, there is a probability measure Q_n on paths $\gamma_t, 0 \le t \le T_n$ given by

$$dQ_n = \frac{M_{T_n}}{M_0}\, d\mathbb{P}.$$

Using the Girsanov theorem and (42), we see that if

$$W_t = U_t - \int_0^t b\, \frac{\Phi_s''(U_s)}{\Phi_s'(U_s)}\, ds,$$

Then $W_t, 0 \le t \le T_n$ is a standard Brownian motion with respect to the measure Q_n. In other words,

$$dU_t = b\, [\log \Phi_t(U_t)]'\, dt + dW_t.$$

Note that T_n does not appear in this equation so we can consider Q as a measure on paths $\gamma(s), 0 \le s < T$ satisfying the above SDE. This is exactly the equation one gets in Proposition 8. This shows that up to time T_n, the Q-distribution of γ is that of SLE_κ in D. But, as we remarked in the first paragraph of this proof, with probability one in this new measure, the path stays a positivie distance away from K and goes to infinity. When this happens, we see that $M_n \to M_\infty$ where M_∞ is as above.

To be a bit more precise, the martingale convergence theorem tells us that with $\mathbb{P}$-probability one, the limit M_{T-} exists. Also, we see that M_t is tight with respect to the measure Q; indeed we have $M_t \le 1$ for $\kappa \le 8/3$. For $8/3 < \kappa \le 4$ with $\mathbf{c} > 0$, we have with Q-probability one $T = \infty$ and

$$M_t \le \exp\left\{\frac{\mathbf{c}}{2}\Lambda(\mathbb{H}; K, \gamma)\right\}.$$

Therefore with Q-probability one the limit $\tilde{M}_\infty = \lim_{t\to\infty} M_t$ exists. We will have $\tilde{M}_\infty = M_\infty$ as above provided that $\Phi'(U_t) \to 1$. Since we know we have convergence and $\exp\left\{\frac{\mathbf{c}}{2}\Lambda(\mathbb{H}; K, \gamma_t)\right\}$ is increasing in t, it suffices to show that $\limsup \Phi'(U_t) = 1$ and we can do a deterministic estimate taking a conveninent increasing sequence of stopping times, e.g., the first time that curve reaches the circle of radius R.

□

Theorem 11. *(Restriction property) Under the assumptions of the previous theorem, the only value of $\kappa \le 4$ such that SLE_κ from 0 to ∞ in D is the same as SLE_κ from 0 to ∞ conditioned to avoid K is $\kappa = 8/3$. If γ is an $SLE_{8/3}$ path, then*

$$\mathbb{P}\{\gamma \cap K \ne \emptyset\} = \Phi_D'(0)^{5/8}.$$

For $\kappa \le 4$ with $\kappa \ne 8/3$, there is simple formula for $\mathbb{P}\{\gamma \cap K \ne \emptyset\}$. This is the primary reason that $SLE_{8/3}$ is the conjectured scaling limit for self-avoiding walks.

Here are the ideas of stochastic calculus used in the last proof.

Suppose X_t satisfies

$$dX_t = m_t\, dt + dB_t,$$

where B_t is a standard Brownian motion and m_t is adapted. Suppose Ψ_t is a positive function and T is a stopping time such that for $t < T$,

$$d\Psi_t = \Psi_t \, [A_t \, dB_t + R_t \, dt] \, .$$

Then the measure Q obtained by "tilting by Ψ" will mean the measure obtained by tilting by the local martingale

$$M_t = \Psi_t \, \exp \left\{ - \int_0^t R_s \, ds \right\}$$

which satisfies

$$dM_t = A_t \, M_t \, dB_t.$$

If T is not given explicitly, we let $T = \lim_{\epsilon \downarrow 0} T_\epsilon$ where

$$T_\epsilon = \min\{t : M_t \leq \epsilon \text{ or } M_t \geq 1/\epsilon\}.$$

By the Girsanov theorem,

$$dB_t = A_t \, dt + dW_t, \quad t < T,$$

where W_t is a standard Brownian motion with respect to Q. In particular,

$$dX_t = [m_t + A_t] \, dt + dW_t, \quad t < T$$

The martingale convergence theorem implies that with $\mathbb{P}$-probability 1, the limit

$$M_T = \lim_{t \uparrow T} M_t,$$

exists. If we also have tightness in the measure Q, that is, if $M^* = \sup_t M_t$,

$$\lim_{\epsilon \downarrow 0} Q\{T_\epsilon < \infty\} = 0,$$

then we also have that convergence to M_∞ with probability one with respect to Q and

$$M_\infty = \frac{dQ}{d\mathbb{P}}.$$

5.4 The chordal partition function (in simply connected domains)

The term partition function in statistical mechanics is ofteen used for the total mass of the measure for some weight on discrete models. It often depeends on a parameter such as inverse time and perhaps the lattice spacing. We will use the term here for the total mass of a nonprobabilty measure; in some cases it is conjected to be the normalized limit of a lattice partition function. We will define it for all κ but the meaning will be clearest for $\kappa \leq 4$ as we will see in the next subsection. For now we will define it only for simply connected domains, and some smoothness assumptions are needed on the boundary. We start with the assumptions.

Definition Suppose $(D, z, w) \in \mathcal{Q}_{\rm sc}$.

- If $w \neq \infty$, and ∂D is locally analytic in neighborhoods of z, w, then $(D, z, w) \in \mathcal{Q}_{\rm a}$.

- If $D \subset \mathbb{H}$, $\mathbb{H} \setminus D$ is bounded, and ∂D is locally analytic in a neighborhood of z, then $(D, z, \infty) \in \mathcal{Q}_{\rm a}$.

Definition Suppose $0 < \kappa \le 4$ and $(D, z, w) \in \mathcal{Q}_{\rm a}$. We define the SLE_κ *partition function* by

- If $w \neq \infty$, then
$$\Psi(D; z, w) = H_D(z, w)^b, \tag{43}$$
where $H_D(z, w)$ denotes the Poisson kernel normalized so that $H_{\mathbb{H}}(0, x) = x^{-2}$

- If $w = \infty$ and $\Phi_D : D \to \mathbb{H}$ is the conformal transformation with $\Phi_D(\zeta) = \zeta + o(1)$ as $\zeta \to \infty$, then
$$\Psi(D; z, \infty) = \Psi_\kappa(D; z, \infty) = |\Phi_D'(z)|^b.$$

We make some remarks.

- Although $\Psi(D; z, \infty)$, depends on κ, we will usually write them without the κ.

- It may seem to be a waste of notation to use a letter Ψ to denote the right-hand side of (43). However, when we extend the definition of $\Psi(D; z, w)$ to multiply connected domains we will see that (43) does not hold.

- If $D_1 \subset D$ and D_1, D agree in neighborhoods of z, w, then
$$H_{D_1|D}(z, w) := \frac{H_{D_1}(z, w)}{H_D(z, w)}$$
gives the probability that a Brownian excursion from z to w in D does not hit $D \setminus D_1$. Hence it is possible to define
$$\Psi(D_1 \mid D; z, w) := \frac{\Psi(D_1; z, w)}{\Psi(D; z, w)} = \left[\frac{H_{D_1}(z, w)}{H_D(z, w)}\right]^b \tag{44}$$
even if $D \in \mathcal{Q}_{\rm sc} \setminus \mathcal{Q}_{\rm a}$ (again, assuming that D, D_1 agree in neighborhoods of z, w).

- If $(D, z, w), (D_1.z_1, w_1) \in \mathcal{Q}_{\rm a}$ and $f : D \to D_1$ is a conformal transformation with $f(z) = z_1, f(w) = w_1$, then
$$\Psi(D; z, w) = |f'(z)|^b \, |f'(w)|^b \, \Psi(D_1; z_1, w_1). \tag{45}$$
If $w, w_1 \neq \infty$, this follows from the corresponding scaling property of the boundary Poisson kernel. Although the map f is not unique, the product $|f'(z)| \, |f'(w)|$ does not depend on the choice. One can choose a unique f by requiring that $|f'(w)| = 1$; for this particular, choice we get $\Psi(D; z, w) = |f'(z)|^b \, \Psi(D_1; z_1, w_1)$. This formula also holds, if either or both of w, w_1 equal infinity provided we choose the correct analogue of the derivative. It is easiest to choose the unique f such that "$|f'(w)| = 1$". If $w = w_1 = \infty$, then this becomes the familiar condition
$$f(\zeta) = \zeta + O(1), \quad \zeta \to \infty.$$
If $w \neq \infty, w_1 = \infty$ and $\mathbf{n}$ denotes the normal derivative at w, the condition becomes
$$f(w + \epsilon \, \mathbf{n}) = \frac{i}{\epsilon} + O(1),$$
and if $w = \infty, w_1 \neq \infty$, we get a similar condition on f^{-1}. We will write (45) with this understanding about derivatives at infinity.

- For example if $f(z) = z/(1-z)$, which is the Möbius transformation of $\mathbb{H}$ with $f(0) = 0$ and
$$f(1+\epsilon i) = \frac{i}{\epsilon},$$
then $f'(0) = 1$ and
$$1 = H_{\mathbb{H}}(0,1)^b = \Psi_{\mathbb{H}}(0,1) = |f'(0)|^b\, \Psi_{\mathbb{H}}(0,\infty) = \Psi_{\mathbb{H}}(0,\infty).$$

We will see an interpretation of the partition function as a total mass of a measure in the next subsection. Even when this interpretation is not available, one can use the partition function as a way to write the (local) martingale used to tilt paths to go from one type of SLE_κ to another SLE_κ. The case we have seen is comparing SLE_κ from 0 to ∞ in D to that in $\mathbb{H}$. The local martingale M_t can be written as
$$M_t = C_t \frac{\Psi(D_t; \gamma(t), \infty)}{\Psi(\mathbb{H}; \gamma(t), \infty)}, \tag{46}$$
where C_t is C^1 in t. In other word,

- **Different versions of SLE_κ (with the same κ) tend to be "locally absolutely continuous". One type of SLE_κ is obtained from another type by tilting locally by the ratio of the partition functions.**

Sometimes a compensator C_t will be needed but not always.

5.5 The SLE_κ measure for $\kappa \le 4$

While the partition function is defined for all κ, we will see that for $\kappa \le 4$ it can be used to redefine SLE_κ as a nonprobablity measure whose total mass is given by the partiition function. For this section we will assume that $0 < \kappa \le 4$, and as before we will write $\mu_D^{\#}(z,w)$ for the probability measure associated to chordal SLE_κ in D from z to w with the curves parametrized by the Minkowski content (naturally). This assumes that $(D,z,w) \in \mathcal{Q}_{\rm sc}$. If z, w are nice boundary points, then it is convenient to consider a nonprobability measure on paths. We first define nice.

Definition Suppose $0 < \kappa \le 4$ and $(D,z,w) \in \mathcal{Q}_{\rm a}$. The SLE_κ measure $\mu_D(z,w)$ is defined by
$$\mu_D(z,w) = \mu_{D,\kappa}(z,w) = \Psi(D,z,w)\, \mu_D^{\#}(z,w).$$

We make some remarks.

- As in the previous subsection, it is possible to define the ratio
$$\frac{\mu_{D_1}(z,w)[V]}{\mu_D(z,w)[V]},$$
even if the two measures $\mu_{D_1}(z,w), \mu_D(z,w)$ are not well defined. Indeed,
$$\mu_{D_1|D}(z,w) := \frac{\mu_{D_1}(z,w)[V]}{\mu_D(z,w)[V]} = \frac{\mu_{D_1}^{\#}(z,w)[V]}{\mu_D^{\#}(z,w)[V]}\, \frac{\Psi(D_1;z,w)}{\Psi(D;z,w)}, \tag{47}$$
where the right-hand side makes sense using (44).

- If $(D, z, w), (D_1.z_1, w_1) \in \mathcal{Q}_{\rm a}$ and $f : D \to D_1$ is a conformal transformation with $f(z) = z_1, f(w) = w_1$,
$$f \circ \mu_D(z, w) = |f'(z)|^b \, |f'(w)|^b \, \mu_{f(D)}(f(z), f(w)),$$
where the derivatives are interpreted as above.

We now restate the results of the last section in terms of the measures $\mu_D(z, w)$. As a slight abuse of notation, we will write $\gamma \subset D$ to mean that the curve lies in D except perhaps the endpoints. For example, since $\kappa \le 4$, the measure $\mu_D(z, w)$ is carried on curves γ with $\{\gamma \subset D\}$.

Theorem 12. *Suppose $0 < \kappa \le 4$, $(D, z, w) \in \mathcal{Q}_{\rm a}$ and $D' \subset D$ is a simply connected domain that agrees with D in neighborhoods of z, w. Then $\mu_{D'}(z, w) \ll \mu_D(z, w)$ with*
$$\frac{d\mu_{D'}(z, w)}{d\mu_D(z, w)}(\gamma) = 1\{\gamma \subset D'\} \, \exp\left\{\frac{\mathbf{c}}{2} \, \Lambda(D; \gamma, D \setminus D')\right\}.$$
In particular, if $\kappa = 8/3$,
$$\frac{d\mu_{D'}(z, w)}{d\mu_D(z, w)}(\gamma) = 1\{\gamma \subset D'\}.$$

The last part of the theorem is the most succinct way of stating the *restriction property* for $SLE_{8/3}$: if $D' \subset D$, then $\mu_{D'}(z, w)$ is the restriction of $\mu_D(z, w)$ to curves that lie in D'.

We now return to the martingale in the previous section. Suppose that $D = \mathbb{H} \setminus K$ is a simply connected domain with K bounded and $\mathrm{dist}(0, K) > 0$. We can write the martingale as
$$M_t = \Phi_t'(U_t)^b \, \exp\left\{\frac{\mathbf{c}}{2} \, \Lambda(\mathbb{H}; \gamma_t, K)\right\}, \quad t < T.$$
As before, we write $\mathbb{P}$ for the original probability measure and Q for the measure obtained by tilting by M_t. Since $\kappa \le 4$, we know that with Q-probability one, $\mathrm{dist}(\gamma_\infty, K)0.$ and $\Lambda(\mathbb{H}; \gamma_\infty, K) < \infty$. In particular, with $\mathbb{P}$-probability one
$$\lim_{t \uparrow T} M_t = 0,$$
on the event $\{T < \infty\}$. For this reason we can write
$$M_t = \Phi_t'(U_t)^b \, \exp\left\{\frac{\mathbf{c}}{2} \, \Lambda(\mathbb{H}; \gamma_t, K)\right\} \, 1\{\gamma_t \subset D\},$$
and stop the martingale at time T knowing that $M_{t \wedge T}$ is a continuous martingale. Since time changes of martingale are martingales (under suitable conditions), it does not matter how we parametrize γ_t. Let us assume that γ has the natural parametrization so that, in particular, the parametrization is the same whether we consider γ as an SLE path in $\mathbb{H}$ or an SLE in D. Note that if $t < T$, then
$$\Phi_t, (U_t)^b = \Psi(D_t; U_t, \infty) = \frac{\Psi(D_t; U_t, \infty)}{\Psi(\mathbb{H}, U_t, \infty)} = \frac{\Psi(D \setminus \gamma_t; \gamma(t), \infty)}{\Psi(\mathbb{H} \setminus \gamma_t; \gamma(t), \infty)}.$$
The numerator and the denominator of the expression on the right do not technically make sense, but the ratio is well defined by (44). Formally, we can write
$$\frac{\Psi(D \setminus \gamma_t; \gamma(t), \infty)}{\Psi(\mathbb{H} \setminus \gamma_t; \gamma(t), \infty)} = \frac{|g_t'(\gamma(t))|^b \, |g_t'(\infty)|^b \, \Psi(D_t; U_t; \infty)}{|g_t'(\gamma(t))|^b \, |g_t'(\infty)|^b \, \Psi_(\mathbb{H}; U_t.\infty)} = \Psi(D_t; U_t, \infty),$$
if we are willing to cancel the $|g_t'(\gamma(t))|^b$ even though this derivative does not exist. We will reiterate this as a proposition, replacing $(\mathbb{H}, D)$ with a more general (D, D_1).

Proposition 5.3. *Suppose $(D, z, w) \in \mathcal{Q}_{\rm a}$ and D_1 is a simply connected subdomain of D that agrees with D in neighborhoods of z, w. Suppose S is a stopping time and $\mu_{D_1}(z, w), \mu_D(z, w)$ are viewed as measures on the stopped paths $\gamma(t), 0 \le t \le S$. Then*

$$\frac{d\mu_{D_1|D}(z,w)}{d\mu_D(z,w)}\,(\gamma_S) = \Psi[D_1 \setminus \gamma_S \mid D \setminus \gamma_S; \gamma(S), w]\,\exp\left\{\frac{\mathbf{c}}{2}\,\Lambda(D; \gamma_S, D \setminus D_1)\right\}\,1\{\gamma_S \subset D_1\}.$$

This proposition does not care how the curves were originally parametrized since it refers to the stopping time S. The assumption $(D, z, w) \in \mathcal{Q}_{\rm a}$ can be replaced with $(D, z, w) \in \mathcal{Q}_{\rm sc}$ provided that we interpret the left-hand side as in (47).

Proposition 5.4. *Suppose $(D, z, w), (D_1, z, w_1) \in \mathcal{Q}_{\rm a}$ with $D_1 \subset D$ and such that D and D_1 agree in neighborhoods of z, w. Suppose S is a stopping time and $\mu_{D_1}(z, w_1), \mu_D(z, w)$ are viewed as measures on the stopped paths $\gamma(t), 0 \le t \le S$. Then*

$$\frac{d\mu_{D_1}(z,w_1)}{d\mu_D(z,w)}\,(\gamma_S) = \frac{\Psi_{D_1\setminus\gamma_S}(\gamma(S), w_1)}{\Psi_{D\setminus\gamma_S}(\gamma(S), w)}\,\exp\left\{\frac{\mathbf{c}}{2}\,\Lambda(D; \gamma_S, D \setminus D_1)\right\}\,1\{\gamma_S \subset D_1\}.$$

5.6 SLE in $\mathbb{H}$ connecting two real points

In this subsection we will view SLE_κ in $\mathbb{H}$ from 0 to $x > 0$ as SLE_κ from 0 to ∞ weighted by the partition function. Here we will take any $\kappa > 0$ and let $T = \inf\{t > 0 : \gamma(t) \in [x, \infty)\}$. For $t < T$, we let H_t be the unbounded component of $\mathbb{H} \setminus \gamma_t$ and we consider the local martingale

$$M_t = \frac{\Psi_{H_t}(\gamma(t), x)}{\Psi_{H_t}(\gamma(t), \infty)}, \quad t < T$$

which is in the form similar to (46). It may not be obvious that this is a local martingale and no compensator is needed, or even that it is well defined. To justify this, we first do the formal calculation

$$\frac{\Psi_{H_t}(\gamma(t), x)}{\Psi_{H_t}(\gamma(t), \infty)} = \frac{|g_t'(\gamma(t))|^b\, g_t'(x)^b\, \Psi_{H_t}(U_t, g_t(x))}{|g_t'(\gamma(t))|^b\, g_t'(\infty)\, \Psi_{\mathbb{H}}(U_t, \infty)} = g_t'(x)^b\, X_t^{-2b} = g_t'(x)^b\, X_t^{1-3a},$$

where $X_t = g_t(x) - U_t$. This shows that M_t is well defined and this allows us to use Itô's formula to show it is a local martingale.

Proposition 5.5. *If $M_t = g_t'(x)^b\, X_t^{1-3a}$, then M_t is a local martingale for $t < T$ satisfying*

$$dM_t = \frac{1-3a}{X_t}\, M_t\, dB_t, \quad t < T.$$

Proof. Itô's formula and (7) give

$$\begin{aligned} dX_t^{1-3a} &= X_t^{1-3a}\left[\frac{1-3a}{X_t}\, dX_t + \frac{3a(3a-1)}{2\,X_t^2}\, dt\right] \\ &= X_t^{1-3a}\left[\frac{ab}{X_t^2}\, dt + \frac{1-3a}{X_t}\, dB_t\right] \end{aligned}$$

$$\partial_t g_t'(x)^b = -\frac{ab}{X_t^2}\, dt,$$

and hence the proposition follows from the product rule. □

If we use the Girsanov theorem to tilt by M_t, we see that

$$dB_t = \frac{1-3a}{X_t}\,dt + dW_t, \quad dX_t = \frac{1-2a}{X_t}\,dt + dW_t, \tag{48}$$

where W_t is a standard Brownian motion in the new measure. We emphasize that this description is valid only up to time T. Although we will not prove it now, we explain what this means in the three regimes. When we talk about "time" we mean in the usual capacity parametrization such that $\mathrm{hcap}[\gamma_t] = at$.

- $0 < \kappa \le 4, a \ge 1/2$. This process stops in finite time and $\gamma(T) = x$. The amount of time to reach x is finite.
- $4 < \kappa < 8, 1/4 < a < 1/2$. The process stops in finite time but γ "overshoots" x, $\gamma(T) > x$.
- $\kappa \ge 8, a \le 1/4$. Here the process survives for infinite time. This can be understood in the plane filling nature of the path. By the time SLE_κ going to x reaches x it has to visit all other points in the plane and hence has infinite capacity.

We summarize as follows.

Theorem 13. *Suppose $0 < \kappa < 8$ and $x > 0$. There are three different ways to construct SLE_κ from 0 to x in $\mathbb{H}$ up to time $T = T_x$.*

1. *The conformal image of SLE_κ in $\mathbb{H}$ from 0 to x under the transformation Φ^{-1} where*
$$\Phi(z) = \frac{xz}{z-x}.$$
2. *The solution of the Loewner equation (1) where $X_t := g_t(x) - U_t$ satisfies*
$$dX_t = -\frac{2b}{X_t}\,dt + dB_t = \frac{1-3a}{X_t}\,dt + dB_t,$$
and B_t is a Brownian motion. Equivalently, U_t satisfies
$$dU_t = \frac{2a-1}{X_t}\,dt - dB_t.$$
3. *Take SLE_κ from 0 to ∞ in $\mathbb{H}$ and tilt by $H_{\mathbb{H}}(0, X_t)^b$.*

Proof. Note that Φ is the conformal transformation of $\mathbb{H}$ onto $\mathbb{H}$ with $\Phi(0) = 0, \Phi(\infty) = -x, \Phi(x) = \infty$. Let $F = \Phi^{-1}$. Let γ be an SLE_κ path from 0 to ∞ and as in Section **??**, let $\gamma^*(t) = F(\gamma(t))$. As before, for $t < T$, let g_t^* be the conformal transformation of the unbounded component of $\mathbb{H} \setminus \gamma_t^*$ satisfying $g_t^*(z) = z + o(1), z \to \infty$, and let $F_t = g_t^* \circ F \circ g_t^{-1}$, $U_t^* = F_t(U_t)$. Then, Itô's formula as in (41) shows that

$$dU_t^* = -b\,F_t''(U_t)\,dt + F_t'(U_t)\,dU_t.$$

If we reparametrize, setting $\hat\gamma(t) = \gamma^*(\sigma(t))$ so that $\mathrm{hcap}[\hat\gamma_t] = at$, then we get

$$d\hat U_t = -b\,\frac{F''_{\sigma(t)}(\hat U_{\sigma(t)})}{F'_{\sigma(t)}(\hat U_{\sigma(t)})^2}\,dt + dW_t = b\,\frac{\hat\Phi_t''(\hat U_t)}{\hat\Phi_t'(\hat U_t)}\,dt + dW_t,$$

where $\hat{\Phi}_t = F^{-1}_{\sigma(t)}$ and W_t is a standard Brownian motion. Note that $\hat{\Phi}_t$ is a conformal transformation of $\mathbb{H}$ with $\Phi_t(\hat{g}_t(x)) = \infty$ where $\hat{g}_t = g^*_{\sigma(t)}$. There is a two-parameter family of such maps, but for each one,

$$\frac{\hat{\Phi}''_t(\hat{U}_t)}{\hat{\Phi}'_t(\hat{U}_t)} = \frac{2}{\hat{g}_t(x) - \hat{U}_t}.$$

If we let $X_t = \hat{g}_t(x) - \hat{U}_t$, then we get

$$dX_t = -\frac{2b}{X_t}\,dt + dB_t,$$

where $B_t = -W_t$. The upshot of this is that if we let g_t be the solution of the Loewner equation (1) with U_t satisfying

$$dU_t = \frac{2b}{X_t}\,dt - dB_t = \frac{3a-1}{X_t}\,dt - dB_t,$$

for a standard Brownian motion, then the distribution is the same as the image of SLE_κ in $\mathbb{H}$ from 0 to ∞ under F. In other words, this gives the distribution of SLE_κ from 0 to x up to time $T = \inf\{t : X_t = 0\}$. To see that this is the same as tilting by M_t, see (48).

This shows the equivalence of the first two definitions and the third follows from $H_{\mathbb{H}}(0,x) = x^{-2}$ and Itô's formula which shows that

$$dX_t^{-2b} = X_t^{-2b}\left[-\frac{2b}{X_t}\,dB_t + \frac{2b(2b+1)}{2X_t^2}\,dt\right].$$

□

6 Radial SLE

Chordal SLE_κ is a measure on curves connecting two boundary points of a domain. Radial SLE_κ is a similar construction, but in this case connecting a boundary point to an interior point. There are several ways to construct the measure. We will start with the original definition in terms of a process from the boundary of the unit disk to the origin and then we will give another construction that shows that radial and chordal SLE are locally absolutely continuous. Recall the discussion of the radial Loewner equation in Section **??**.

Definition If $\kappa = 2/a > 0$, then *radial* SLE_κ is the solution to the Loewner equation

$$\dot{g}_t(z) = 2a\, g_t(z)\,\frac{e^{2iU_t} + g_t(z)}{e^{2iU_t} - g_t(z)}, \tag{49}$$

where $U_t = -B_t$ is a standard Brownian motion.

There exists a curve $\gamma : (0,\infty) \to \mathbb{D}$ with $\gamma(0+) = e^{2iU_t}$ such if D_t is the connected component of $\gamma \setminus \gamma_t$ containing the origin, then g_t is the unique conformal transformation of D_t on $\mathbb{D}$ with $g'_t(0) = 0, g'_t(0) > 0$. The curve has been parametrized so the $g'_t(0) = e^{2at}$. An equivalent (up to time change) definition, and the original one due to Schramm, is parametrize the curve so that $g'_t(0) = e^t$ and then to let g_t be the solution of

$$\dot{g}_t(z) = g_t(z)\,\frac{w_t + g_t(z)}{w_t - g_t(z)},$$

where w_t is a Brownian motion on the unit circle with variance parameter κ. We will use (49).

Recall that if h_t is defined by

$$g_t(e^{2iz}) = \exp\{2ih_t(z)\},$$

then h satisfies the equation

$$\dot{h}_t(z) = a\,\cot(h_t(z) - U_t),$$

that is, if $X_t = X_t(z) = h_t(z) - U_t$, then X_t satisfies the radial Bessel equation

$$dX_t = a\,\cot X_t\,dt + dB_t. \tag{50}$$

Note that if $\theta \in \mathbb{R}$, then

$$\dot{h}_t'(\theta) = -\frac{a\,h_t'(\theta)}{\sin^2(h_t(\theta) - U_t)}, \qquad h_t'(\theta) = \exp\left\{-a\int_0^t \frac{ds}{\sin^2(h_t(\theta) - U_t)}\right\},$$

and since $|g_t'(e^{2i\theta})| = h_t'(\theta)$,

$$|g_t'(e^{2i\theta})| = \exp\left\{-a\int_0^t \frac{ds}{\sin^2(h_s(\theta) - U_s)}\right\}. \tag{51}$$

In Section 5.6, two SLE_κ paths in $\mathbb{H}$ with different boundary "target points" were compared. The measures on paths were locally absolutely continuous and could be described using a local martingale obtained by the ratio of the partition functions. We will do the same thing here with one of the target points being an interior point. Along the way we will have to determine the radial partition function.

Suppose that $U_0 = 0$ and let $w_0 = e^{2i\theta_0}$ with $0 < \theta_0 < \pi$. We will compare radial SLE_κ from 0 to ∞ with chordal SLE_κ from 0 to w_0. Suppose γ is chordal SLE_κ from 0 to w_0 and we parametrized the curve using a radial parametrization with respect to 0. In other words we parametrize so that the conformal radius at time t of $\mathbb{D} \setminus \gamma_t$ with respect to the origin is e^{-2at}. Let D_t be the connected component of $\mathbb{D} \setminus \gamma_t$ containing the origin and $\hat{g}_t$ the unique conformal transformation of D_t onto $\mathbb{D}$ with $\hat{g}_t(0) = 0$ and define Θ_t by $\hat{g}_t(w_0) = e^{2i\Theta_t}$. Let T be the first time t that γ_t disconnects w_0 from the oirign We have seen in (12) that Θ_t satisfies

$$d\Theta_t = (1 - 2a)\,\cot\Theta_t\,dt + dW_t, \quad t < T$$

where W_t is a standard Brownian motion. However, if it were radial SLE, we see from (50) that is should satisfy

$$d\Theta_t = a\,\cot\Theta_t\,dt + dB_t. \tag{52}$$

To get from one to the other, we use the Girsanov theorem. We want to get a drift of $(1-3a)\,\cot\Theta_t$ and this leads us to tilt by $\sin^{3a-1}\Theta_t$. We omit the proof of the next proposition which is an exercise in Itô's formula. It uses another scaling exponent so we will define it first.

Definition The *interior scaling exponent* $\tilde{b}$ is defined by

$$\tilde{b} = \frac{1-a}{2a}\,b = \frac{\kappa - 2}{4}\,b.$$

Proposition 6.1. *If*

$$M_t = e^{2a\hat{b}t} \exp\left\{ ab \int_0^t \frac{ds}{\sin^2 \Theta_s} \right\} [\sin \Theta_t]^{3a-1} = e^{2a\hat{b}t} \, |g_t'(w_0)|^{-b} \, [\sin \Theta_t]^{3a-1}, \tag{53}$$

where $\hat{b} = b(\kappa - 4)/2$ is the interior scaling exponent, then M_t is a local martingale satisfying

$$dM_t = [\cot \Theta_t]^{3a-1} \, M_t \, dB_t.$$

Proof. The first equality is an exercise in Itô's formula which we omit. The second follows from (51). □

We can think of this proposition as the computation of the interior exponent $\tilde{b}$; it is the unique number such that M_t as defined in (53) is a local martingale.

Girsanov's theorem now shows that radial SLE_κ can be obtained from chordal SLE_κ up to a stopping time by tilting by M_t.

Definition Suppose D is a simply connected domain, z a locally analytic point on ∂D; and $\zeta \in D$. The *radial partition function* $\Psi(D; z, \zeta)$ is defined by $\Psi(\mathbb{D}; 1, 0) = 1$ and the scaling rule: if $f : D \to f(D)$ is a conformal transformation,

$$\Psi_D(z, \zeta) = \Psi(D; z, \zeta) = |f'(z)|^b \, |f'(\zeta)|^{\tilde{b}} \, \Psi(f(D); f(z), f(\zeta)).$$

We will give another interpretation which helps understand how to view this local martingale. Recall that the partition function for chordal SLE_κ from 1 to w_0 is $H_{\mathbb{D}}(1, w_0)^b$ which (see Section **??**) is a constant times $[\sin \theta]^{-2b} = \sin^{1-3a} \theta$. Hence we can write

$$\frac{M_t}{M_0} = \frac{|g_t'(0)|^{\tilde{b}} \, \Psi_{\mathbb{D}}(1, w_0)}{|g_t'(w_0)|^b \, \Psi_{\mathbb{D}}(1, w_t)} = \frac{\Psi_{D_t}(\gamma(t), 0)}{\Psi_{D_t}(\gamma(t), w_0)}.$$

Proposition 6.2. *Suppose γ is a chordal SLE_κ curve with distribution $\mu^{\#}_{\mathbb{D}}(1, w)$ where $w = e^{2i\Theta_0}, 0 < \Theta_0 < \pi$. Let $\epsilon > 0$ and let T be a stopping time such that* $\mathrm{dist}(0, \gamma_t) \geq \epsilon$ *and w is on the boundary of the connected component of $\mathbb{D} \setminus \gamma_T$ containing the origin. Then radial SLE_κ from 1 to 0 is $\mathbb{D}$, viewed as a measure on paths $\gamma(t), 0 \leq t \leq T$, is absolutely continuous with respect to $\mu^{\#}_{\mathbb{D}}(1, w)$ with Radon-Nikodym derivative*

$$\frac{M_T}{M_0} = |g_t'(0)|^{\tilde{b}} \, |g_t'(w)|^{-b} \, \frac{\Psi_{\mathbb{D}}(1, w)}{\Psi_{\mathbb{D}}(1, e^{2i\Theta_T})} = e^{2a\hat{b}T} \exp\left\{ ab \int_0^T \frac{ds}{\sin^2 \Theta_s} \right\} \left[\frac{\sin \Theta_T}{\sin \Theta_0} \right]^{3a-1}.$$

Recalling that $a = 1/3, b = \hat{b} = 0$ for $\kappa = 6$, we get the following corollary.

Corollary 6.3. *Let T be the first time t that w is not on the boundary of the connected component of $\mathbb{D} \setminus D_t$ containing the origin. Then the distribution of chordal SLE_6 and radial SLE_6 up to time T is the same.*

After time T the distributions are different because chordal SLE_6 goes toward w while radial SLE_6 goes toward the origin.

For $\kappa > 4$, we can compare radial SLE to chordal SLE only up to the point that the curve separates the boundary target point from the origin. However, at this time, we can choose a new boundary point in the domain D_T and compare to that, stopping when the curve close the new point. Alternatively, we can go until we almost separate the boundary point from the interior point and at that time change to a new target point. The key point is that the path is absolutely continuous with respect to some chordal SLE_κ path and almost sure facts about chordal paths also hold for radial paths. For example, radial SLE_κ paths have fractal dimension $d = 1 + \frac{\kappa}{8}$ and can be parametrized naturally (by d-dimensional Minkowski content). The following can be proved but we omit the proofs.

Proposition 6.4.

1. *The radial Green's function*

$$\mathbb{G}_{\mathbb{D}}(\zeta; 1, 0) = \lim_{r \downarrow 0} r^{d-2}\, \mathbb{P}\{\mathrm{dist}(\zeta, \gamma) \le r\}$$

exists for $\zeta \in \mathbb{D} \setminus \{0\}$. Moreover,

$$\int_{\mathbb{D}} \mathbb{G}_{\mathbb{D}}(\zeta; 1, 0)\, dA(\zeta) < \infty.$$

2. *With probability one, the path is continuous at the origin.*

Definition For $\kappa \le 4$, we define the radial SLE_κ measure $\mu_D(z, w)$ where $z \in \partial D, w \in D$, by

$$\mu_D(z, w) = \Psi_D(z, w)\, \mu_D^{\#}(z, w).$$

Here $\Psi_D(z, w)$ is the radial partition function as above and $\mu_D^{\#}(z, w)$ is the corresponding probability measure. Then we can write

$$f \circ \mu_D^{\#}(z, w) = |f'(z)|^b\, |f'(w)|^{\hat{b}}\, \mu_{f(D)}(f(z), f(w)),$$

$$f \circ \mu_D(z, w) = \mu_{f(D)}^{\#}(f(z), f(w)),$$

where the first relation requires that the boundaries be nice at $z, f(z)$.

6.1 Fundamental local martingale and radial restriction

We are going to consider the fundamental local martingale and the restriction property for radial SLE_κ. While this could be done directly using stochastic calculus we will take a short cut that allows us to use the local martingales that we have already defined.

Suppose $D = \mathbb{D} \setminus K$ is a simply connected subdomain of the unit disk including the origin, with $\mathrm{dist}(1, K) > 0$. Let $\Phi = \Phi_D : D \to \mathbb{D}$ be the conformal transformation with $\Phi(0) = 0, \Phi'(0) > 0$. Let $\gamma(t)$ be a radial SLE_κ path from 1 to 0 in $\mathbb{D}$ and let $T = \inf\{t : \mathrm{dist}(\gamma(t), K) = 0\}$. It will not be important for use the exact parametrization; it can be by capacity or by Minkowski content.

Let $D_t = g_t(D)$. Let $\gamma^*(t) = \Phi(\gamma(t))$ and $\Phi_t = g_t^* \circ \Phi \circ g_t^{-1}$. Note that $\Phi_t : D_t \to \mathbb{D}$ is the unique conformal transformation with $\Phi_t(0) = 0, \Phi_t'(0) > 0$.

As in the chordal case, we will consider the ratio of partition functions,

$$\frac{\Psi_{D\setminus\gamma_t}(\gamma(t),0)}{\Psi_{\mathbb{D}\setminus\gamma_t}(\gamma(t),0)} = \frac{|g_t'(\gamma_t)|^b\,|g_t'(0)|^{\tilde{b}}\,\Psi_{D_t}(z_t,0)}{|g_t'(\gamma_t)|^b\,|g_t'(0)|^{\tilde{b}}\Psi_{\mathbb{D}}(z_t,0)} = \Psi_{D_t}(z_t,0) = |\Phi_t'(z_t)|^b\,\Phi_t'(0)^{\tilde{b}}.$$

where $z_t = g_t(\gamma(t))$. We can also, at least for small t, take another boundary point w_0, let $w_t = g_t(w_0)$ and write

$$\frac{\Psi_{D\setminus\gamma_t}(\gamma(t),0)}{\Psi_{\mathbb{D}\setminus\gamma_t}(\gamma(t),0)} = N_t\,R_t\,\tilde{N}_t,$$

where

$$N_t = \frac{\Psi_{D\setminus\gamma_t}(\gamma(t),0)}{\Psi_{D\setminus\gamma_t}(\gamma(t),w_t)}, \quad R_t = \frac{\Psi_{D\setminus\gamma_t}(\gamma(t),w_t)}{\Psi_{\mathbb{D}\setminus\gamma_t}(\gamma(t),w_t)}, \quad \tilde{N}_t = \frac{\Psi_{\mathbb{D}\setminus\gamma_t}(\gamma(t),w_t)}{\Psi_{\mathbb{D}\setminus\gamma_t}(\gamma(t),0)}.$$

We then do a series of tilting starting with the probability measure for radial SLE_κ from 1 to 0 in $\mathbb{D}$.

- In this measure, $\tilde{N}_t$ is a local martingale, and if we tilt by this we get chordal SLE_κ from 1 to w_0.
- If we are in the new measure, and

  $$\tilde{M}_t = 1\{\gamma_t \subset D\}\,\exp\left\{\frac{\mathbf{c}}{2}\,\lambda_{\mathbb{D}}(\gamma_t,K)\right\},$$

  , then $\tilde{M}_t$ is a local martingale in the new measure and tilting by $\tilde{M}_t$ gives a new measure that is chordal SLE_κ from 1 to w in $\mathbb{D}$.
- Finally in this newer measure, N_t is a local martingale, and tilting by N_t yields SLE_κ from 1 to 0 in D.

Combining this we see that if

$$M_t = N_t\,\tilde{M}_t\,\tilde{N}_t = \{\gamma_t \subset D\}\,\exp\left\{\frac{\mathbf{c}}{2}\,\lambda_{\mathbb{D}}(\gamma_t,K)\right\}\;|\Phi_t'(z_t)|^b\,\Phi_t'(0^{\tilde{b}},$$

then M_t is a local martingale and tilting by M_t gives SLE_κ in D from 1 to 0. We summarize here. We state the theorem for radial SLE_κ from 1 to 0 in $\mathbb{D}$ but it clearly applies to other domains by conformal invariance. We have stated the proposition so that the parametrization of the curve is not important.

Proposition 6.5. . *Suppose D is a simply connected subdomain of $\mathbb{D}$ with $0 \in D$ and* $\mathrm{dist}(1, \mathbb{D}\setminus D) > 0$. *Let $\gamma(t)$ be a radial SLE_κ path from 1 to 0 and let T be the first time that γ hits $\overline{\mathbb{D}\setminus D}$. Let $D_t = g_t(D), z_t = g_t(\gamma(t))$, and let Φ_t denote the conformal transformation of D_t onto $\mathbb{D}$ with $\Phi_t(0) = 0, \Phi_t'(0) > 0$. Then,*

$$M_t = \{\gamma_t \subset D\}\,\exp\left\{\frac{\mathbf{c}}{2}\,\lambda_{\mathbb{D}}(\gamma_t,K)\right\}\;|\Phi_t'(z_t)|^b\,\Phi_t'(0)^{\tilde{b}},$$

is a local martingale for $t < T$. If one tilts by M_t, then in the new measure $\gamma(t), 0 \le t < T$ is an SLE_κ from 1 to 0 in D.

If $\kappa \le 4$, then $T = \infty$ in the new measure and

$$\mathbb{E}[M_\infty] = \mathbb{E}\left[\{\gamma_t \subset D\}\,\exp\left\{\frac{\mathbf{c}}{2}\,\lambda_{\mathbb{D}}(\gamma,K)\right\}\right] = \Phi_0'(0)^{\tilde{b}}\,\Phi_0'(1)^b.$$

7 Computing an expectation

Here we use stochastic calculus to compute an expectation for chordal SLE with $\kappa < 8$.

Theorem 14. *If $\kappa < 8$, $0 < x < y$, $\lambda \geq -(a - \frac{1}{2})^2/2a$, and γ denotes an SLE_κ path from 0 to ∞ in $\mathbb{H}$, then*

$$\mathbb{E}\left[\left(\frac{H_{\mathbb{H}\setminus\gamma}(x,y)}{H_{\mathbb{H}}(x,y)}\right)^{\lambda}; H_{\mathbb{H}\setminus\gamma(x,y)} > 0\right] = \frac{\Gamma(2a)\,\Gamma(4a+2r-1)}{\Gamma(2r+2a)\,\Gamma(4a-1)}\,(x/y)^r\,F(2r, 1-2a, 2r+2a; x/y),$$

where

$$r = \frac{1}{2} - a + \sqrt{\left(a - \frac{1}{2}\right)^2 + 2\lambda a},$$

is the larger root of the equation $r(r-1) + 2ar - 2a\lambda = 0$ and $F = {}_2F_1$ denotes the hypergeometric function.

An important case is $\lambda = b = (3a-1)/2$ for which $r = a$ and

$$\mathbb{E}\left[\left(\frac{H_{\mathbb{H}\setminus\gamma}(x,y)}{H_{\mathbb{H}}(x,y)}\right)^{b}; H_{\mathbb{H}\setminus\gamma(x,y)} > 0\right] = \frac{\Gamma(2a)\,\Gamma(6a-1)}{\Gamma(4a)\,\Gamma(4a-1)}\,(x/y)^a\,F(2a, 1-2a, 4a; x/y).$$

It is instructive to see what happens as $\kappa \to 0$, that is, as $a \to \infty$. In this case, if $u = a - \frac{1}{2}$,

$$r = -u + u\sqrt{1 + \frac{2\lambda(u+\frac{1}{2})}{u^2}} = \lambda + o(1).$$

As $\kappa \to 0$, we expect that $\mathbb{H} \setminus \gamma$ has a "limit" of D, the northeast quadrant, since γ has a limit of the imaginary axis. Using conformal invariance, one can see that

$$H_D(x,1) \sim x\,H_{\mathbb{H}}(x,1), \quad x \downarrow 0.$$

Proof. Note that $H_{\mathbb{H}\setminus\gamma}(x,y)/H_{\mathbb{H}}(x,y)$ denotes the probability that a Brownian excursion from x to y in $\mathbb{H}$ stays in $\mathbb{H} \setminus \gamma$. Scaling shows that its distribution depends only on the ratio x/y so without loss of generality, we will assume that $y = 1$. We will use

$$r + a \geq \frac{1}{2}.$$

Let $X_t = g_t(x) - U_t, Y_t = g_t(1) - U_t$. Let $K_t = K_t(x,1)$ be the probability that a Brownian excursion from x to 1 in $\mathbb{H}$ stays in $\mathbb{H} \setminus \gamma_t$, that is,

$$K_t(x,y) = \frac{H_{\mathbb{H}\setminus\gamma_t}(x,y)}{H_{\mathbb{H}}(x,y)} = (y-x)^2\,H_{\mathbb{H}\setminus\gamma_t}(x,y) = K_0(x,y)\,\frac{g_t'(x)\,g_t'(y)}{(Y_t - X_t)^2}.$$

We need to compute $\phi(x) = \phi_{\lambda,\kappa}(x) = \mathbb{E}\left[K_\infty^\lambda; K_\infty > 0\right]$.

The Loewner equations give

$$d[\log X_t] = \frac{a-\frac{1}{2}}{X_t^2} + \frac{1}{X_t}\,dB_t, \quad \partial_t g_t'(x)^\lambda = -\frac{a\lambda g_t'(x)}{X_t^2},$$

and similarly for Y_t. If $Q_t = X_t/Y_t, L_t = \log Q_t$, we have

$$\begin{aligned} dL_t &= \left[\frac{a-\frac{1}{2}}{X_t^2} - \frac{a-\frac{1}{2}}{Y_t^2}\right] dt + \frac{Y_t - X_t}{X_t Y_t}\,dB_t. \\ &= \frac{a-\frac{1}{2}}{Y_t^2}\,\frac{1-Q_t^2}{Q_t^2}\,dt + \frac{1}{Y_t}\,\frac{1-Q_t}{Q_t}\,dB_t \\ &= \frac{a-\frac{1}{2}}{Y_t^2}\left[\frac{1-Q_t}{Q_t}\right]^2 \frac{1+Q_t}{1-Q_t}\,dt + \frac{1}{Y_t}\,\frac{1-Q_t}{Q_t}\,dB_t, \end{aligned}$$

and,

$$\begin{aligned} \partial_t K_t^\lambda &= a\lambda K_t\left[\frac{2}{X_t Y_t} - \frac{1}{X_t^2} - \frac{1}{Y_t^2}\right] \\ &= -\frac{a\lambda(1-Q_t)^2}{Y_t^2\,Q_t^2}. \end{aligned}$$

This suggests a time change. Let $\hat{X}_t = X_{s(t)}$, and $\hat{Y}_t, \hat{Q}_t, \hat{L}_t$ similarly where

$$\dot{s}(t) = \left[\frac{\hat{Q}_t\,\hat{Y}_t}{1-\hat{Q}_t}\right]^2.$$

Then $\hat{K}_t^\lambda = e^{-a\lambda t}$ and

$$d\hat{L}_t = (a - \frac{1}{2})\,\frac{1+\hat{Q}_t}{1-\hat{Q}_t}\,dt + dW_t,$$

for a standard Brownian motion W_t. Since $\hat{Q}_t = e^{\hat{L}_t}$, Itô's formula gives

$$\begin{aligned} d\hat{Q}_t = \hat{Q}_t\left[d\hat{L}_t + \frac{1}{2}dt\right] &= \hat{Q}_t\left[[(a-\frac{1}{2})\frac{1+\hat{Q}_t}{1-\hat{Q}_t} + \frac{1}{2}]\,dt + dW_t\right] \\ &= \hat{Q}_t\left[\left(a + \frac{(2a-1)\hat{Q}_t}{1-\hat{Q}_t}\right)dt + dW_t\right] \end{aligned}$$

We now use standard methods. Note that

$$\hat{M}_t := \mathbb{E}\left[\hat{K}_\infty^\lambda\,1\{\hat{K}_\infty > 0\} \mid \gamma_{\sigma(t)}\right] = \hat{K}_t^\lambda\,1\{\hat{K}_t > \infty\}\,\phi(\hat{Q}_t).$$

The left-hand side is a martingale and hence the dt term on the right-hand side must vanish giving the differential equation

$$x^2\,\phi''(x) + 2x\left(a + \frac{(2a-1)x}{1-x}\right)\phi'(x) - 2a\lambda\,\phi(x) = 0. \tag{54}$$

To solve this equation, we first consider only the largest order terms near the origin in (54) to get the equation

$$x^2 f'' + 2axf' - 2a\lambda f = 0. \tag{55}$$

Since $r(r-1) + 2ar - 2a\lambda = 0$, we see that x^r is a solution to (55). This observation leads us to try solutions of the form $\phi(x) = x^r\, h(x)$ for (54). Since

$$\phi'(x) = x^r\, h'(x) + rx^{r-1} h(x),$$

$$\phi''(x) = x^r\, h''(x) + 2rx^{r-1} h'(x) + r(r-1)x^{r-2}\, h(x),$$

we can plug in to get

$$h''(x) + 2\left[\frac{r+a}{x} + \frac{2a-1}{1-x}\right] h'(x) - \frac{r(2-4a)}{x(1-x)}\, h(x) = 0,$$

$$x(1-x)\, h''(x) + [2(r+a) - (2r-2a+2)x]\, h'(x) - 2r(1-2a)\, h(x) = 0.$$

This is a hypergeometric equation with $\alpha = 2r, \beta = 1-2a, \gamma = 2r+2a$. Note that $\gamma - \alpha - \beta = 4a - 1 > 0$. One solution to this equation is $F(x) := F(2r, 1-2a, 2r+2a; x)$. Two important fact about this function (which use $\gamma > \alpha + \beta$) are the following.

- As $x \uparrow 1$,
$$F(1) = \lim_{x\uparrow 1} F(x) = \frac{\Gamma(\gamma)\,\Gamma(\gamma - (\alpha+\beta))}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} = \frac{\Gamma(2r+2a), \Gamma(4a-1)}{\Gamma(2a)\,\Gamma(2r+4a-1)}.$$
See [Lebedev, (9.3.1)].

- There exists $u < 1$ such that as $x \uparrow 1$,
$$F'(x) = O((1-x)^{-u}).$$
If $\gamma > \beta > 0$, this follows from the integral representation [Lebedev, (9.14)]
$$F(z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}\,(1-t)^{\gamma-\beta-1}\,(1-tx)^{-\alpha}\, dx.$$
For other values, of (α, β, γ) with $\gamma \notin \{0, -1, -2, \ldots\}$ and $\gamma > \alpha + \beta$ it can be deduced using the recursion formula [Lebedev, (9.1.6)]
$$\gamma(\gamma+1)\, F(\alpha, \beta, \gamma; x) = \gamma(\gamma - \alpha + 1) F(\alpha, \beta+1; \gamma_2); x)$$
$$+\alpha[\gamma - (\gamma-\beta)x] F(\alpha+1, \beta+1, \gamma+2; x).$$

At this point, we step back and define $\phi(x) = x^r\, F(x)/F(1)$ and our goal is to show that $\phi(x) = \mathbb{E}[K_\infty^\lambda; K_\infty > 0]$. We know that ϕ is C^2 on $(0,1)$; is continuous on $[0,1]$; Also, as $x \to 0$,

$$[\log \phi(x)]' = \frac{r}{x} + O(1), \quad [\log \phi(1-x)]' = O((1-x)^{-u}),$$

for some $u < 1$. It is also true that $\phi(x) > 0$ for all $0 < x \le 1$; we will use this fact now but we will show later how this could actually be deduced separately. Let $\tau = \inf\{t : K_t = 0\}$. Itô's formula shows that $\phi(Q_t)\, K_t^\lambda$ is a local martingale for $0 \le t < \tau$.

Let us do another time change so that $R_t = \hat{Q}_{\sigma(t)}$ satisfies

$$dR_t = \left(\frac{a}{R_t} + \frac{2a-1}{1-R_t} \right) dt + dB_t,$$

for a standard Brownian motion B_t. The usual chain rule implies that

$$K_{\sigma(t)} = \exp\left\{ -a \int_0^t \frac{ds}{R_s^2} \right\}.$$

Then, $M_t := K_{\sigma(t)}^{\lambda} \, \phi(R_t)$ is a local martingale for $0 < t < T$ where $T = \inf\{t : R_t \in \{0,1\}\}$. It satisfies

$$dM_t = \frac{\phi'(R_t)}{\phi(R_t)} \, M_t \, dW_t, \quad 0 \le t < T.$$

If we use Girsnaov and tilt by the local martingale M_t, then

$$dB_t = \frac{\phi'(R_t)}{\phi(R_t)} \, dt + dB_t^*$$

where B_t^* is a Brownian motion in the new measure. In other words,

$$dR_t = \left(\frac{a+r}{R_t} + \frac{2a-1}{1-R_t} + \delta(R_t) \right) dt + dB_t^*,$$

where

$$\delta(x) = O(1), \quad \delta(1-x) = O((1-x)^{-u}) \quad x \downarrow 0.$$

Since $a + r \ge \frac{1}{2}$, then solutions to this equation never reach the origin and since $a > 1/4$ they reach 1 in finite time so that $K_\infty = K_{\sigma(T)} > 0$. Hence M_t is actually a martingale and

$$\phi(x) = \mathbb{E}^x[M_0] = \mathbb{E}^x[M_\infty] = \mathbb{E}^x[\phi(1) \, K_\infty^{\lambda}; K_\infty > 0].$$

This assumed that $\phi > 0$ on $(0,1]$ which could be derived from facts about hypergeometric functions. If we do not want to use this fact, we can first note that it is immediate that $\phi(x) > 0$ for $0 < x \le \delta$. Suppose y were the smallest positive number with $\phi(y) = 0$. Then we could do the same argument with $T = \inf\{t : R_t \in \{0, y\}\}$. Then we would have

$$\phi(x) = \mathbb{E}^x \left[\phi(R_T) \, K_{\sigma(T)}^{\lambda} \right] = 0,$$

which would be a contradiction since in the tilted measure $K_{\sigma(T)} > 0$.

□

Suppose X_t satisfies

$$dX_t = \left(\frac{1}{2X_t} - r \right) dt + dB_t,$$

where r is a constant. Then X_t never reaches the origin. One way to see this is to write

$$dX_t = \frac{1}{2X_t} \, dt + dW_t,$$

where W_t is a Brownian motion with drift. But on every compact time interval B and W are absolutely continuous.

7.1 Multiple SLE paths

We will define the measure on n-tuples of $SLE_\kappa, \kappa \le 4$ paths connecting different boundary points in a domain D. The measure will lie of paths that do not intersect, but the measure is more complicated then just restricting to paths that are noninterecting.

It will be useful to set up notation. We will write

$$\mathbf{z} = (z_1, \dots, z_n), \quad \mathbf{w} = (w_1, \dots, w_n)$$

for $2n$ distinct points on ∂D, and we write

$$\boldsymbol{\gamma} = (\gamma^1, \dots, \gamma^n)$$

for an n-tuple of curves with γ^j connecting z_j to w_j in D. We let $I(\boldsymbol{\gamma})$ be the indicator function that the curves are nonintersecting, that is, $\gamma^j \cap \gamma^k = \emptyset$ for $j \ne k$. We define the measure

$$\mu_D(\mathbf{z}, \mathbf{w}) = \Psi_D(\mathbf{z}, \mathbf{w})\, \mu_D^{\#}(\mathbf{z}, \mathbf{w})$$

to be the measure absolutely continuous with respect to the product measure $d[\mu_D(z_1, w_1) \times \cdots \times \mu_D(z_n, w_n)]$ with Radon-Nikodym derivative

$$\frac{d\mu_D(\mathbf{z}, \mathbf{w})}{d[\mu_D(z_1, w_1) \times \cdots \times \mu_D(z_n, w_n)]}(\boldsymbol{\gamma}) = I(\boldsymbol{\gamma}) \exp\left\{ \frac{\mathbf{c}}{2} \sum_{j=2}^{n} \Lambda(D; \gamma^j, \gamma^1 \cup \cdots \cup \gamma^{j-1}) \right\}.$$

This formulation assume that D is locally analytic at the points z_j, w_j; however, the right-hand side does not require any smoothness. In particular, we have

$$\frac{\Psi_D(\mathbf{z}, \mathbf{w})}{\prod_{j=1}^n \Psi_D(z_j, w_j)} = \mathbb{E}\left[I(\boldsymbol{\gamma}) \exp\left\{ \frac{\mathbf{c}}{2} \sum_{j=2}^{n} \Lambda(D; \gamma^j, \gamma^1 \cup \cdots \cup \gamma^{j-1}) \right\} \right],$$

where the expectation on the right is with respect to the product measure $\mu_D^{\#}(z_1, w_1) \times \cdots \times \mu_D^{\#}(z_n, w_n)$. The left-hand side is a conformal invariant and the measures satisfy the scaling rules

$$f \circ \mu_D(\mathbf{z}, \mathbf{w}) = |f(\mathbf{z})|^b\, |f'(\mathbf{w})|^b\, \mu_{f(D)}(f(\mathbf{z}), f(\mathbf{w})), \quad f \circ \mu_D^{\#}(\mathbf{z}, \mathbf{w}) = \mu_{f(D)}^{\#}(f(\mathbf{z}), f(\mathbf{w})),$$

where $f'(\mathbf{z})$ is shorthand for $f'(z_1) \cdots f'(z_n)$.

Proposition 7.1. *Let $\hat{\mu}$ denote the marginal distribution on γ^1 in $\mu_D(\mathbf{z}, \mathbf{w})$. Then*

$$\frac{d\hat{\mu}}{\mu_D(z_1, w_1)}(\gamma^1) = \Psi_{D \setminus \gamma^1}(\hat{\mathbf{z}}, \hat{\mathbf{w}}).$$

Moreover, given γ^1, the conditional distribution on $\hat{\boldsymbol{\gamma}}$ is that of $\mu_{D \setminus \gamma^1}^{\#}(\hat{\mathbf{z}}, \hat{\mathbf{w}})$. Here

$$\hat{\mathbf{z}} = (z_2, \dots, z_n), \quad \hat{\mathbf{w}} = (w_2, \dots, w_n), \quad \hat{\boldsymbol{\gamma}} = (\gamma^2, \dots, \gamma^n).$$

The set $D \setminus \gamma^1$ has two components, but we can define $\mu_{D\setminus\gamma^1}(\hat{\mathbf{z}}, \hat{\mathbf{w}})$ in the obvious way.

We will compute a crossing exponent. Let $\mathcal{R}_L$ denote the rectangle

$$\mathcal{R}_L = \{x + iy : 0 < x < L, 0 < y < \pi\}.$$

Using conformal invariance, we can see that

$$H_{\mathcal{R}_L}(iy, L + iy') \asymp e^{-L} (\sin y)(\sin y).$$

We define the crossing exponents $\xi_n = \xi_{n,\kappa}$ by $\xi_1 = b$ and the relation

$$\xi_{n+1} = b + \xi_n + r_{\xi_n} = b + \xi_n + \frac{1}{2} - a + \sqrt{\left(a - \frac{1}{2}\right)^2 + 2a\xi_n}.$$

A simple induction argument shows that

$$\xi_n = \frac{an^2 + (2a-1)n}{2} = bn + \frac{an(n-1)}{2} = bn + \frac{n(n-1)}{\kappa}.$$

The following proposition can be derived from Theorem 14.

Proposition 7.2. *For every n, there exists $C = C_n < \infty$ such that if*

$$\mathbf{z} = (iy_1, iy_2, \dots, iy_n), \quad \mathbf{w} = (L + i\tilde{y}_1, \dots, L + i\tilde{y}_n),$$

with $y_0 = 0 < y_1 < \cdots < y_n < y_{n+1} = \pi, \tilde{y}_0 = 0 < \tilde{y}_1 < \cdots < \tilde{h}_n < \tilde{y}_{n+1} = \pi$, then for all $L \geq 1$,

$$\Psi_{\mathcal{R}_L}(\mathbf{z}, \mathbf{w}) \leq C\, e^{-\xi_n L}.$$

Moreover, for every $\epsilon > 0$, there exists $c_\epsilon > 0$ such that if $y_j - y_{j-1} \geq \epsilon$ and $\tilde{y}_j - \tilde{y}_{j-1} \geq \epsilon$ for $j = 1, 2, \dots, n-1$, then fpr $L \geq 1$,

$$\Psi_{\mathcal{R}_L}(\mathbf{z}, \mathbf{w}) \geq c_\epsilon\, e^{-\xi_n L}.$$

The last inequality can be written as

$$\Psi_{\mathcal{R}_L}(\mathbf{z}, \mathbf{w}) \asymp e^{-\zeta_n L} \prod_{j=1}^{n} \Psi_{\mathcal{R}_L}(z_j, w_j), \qquad \zeta_n = \frac{an(n-1)}{2} = \frac{n(n-1)}{\kappa}.$$

8 Reverse SLE_κ chains

Here we derive some of the facts about the reverse Loewner chains including the important estimate to establish the existence of the SLE_κ curve for $\kappa \neq 8$.

We will consider a solution to the reverse Loewner flow

$$\partial_t h_t(z) = -\frac{a}{h_t(z) - U_t}, \tag{56}$$

where $U_t = -B_t$ is a standard Brownian motion. We will call such solutions *reverse SLE_κ chains*. For each t, $h_t : \mathbb{H} \to h_t(\mathbb{H})$ is a conformal transformation with

$$h_t(z) = z - \frac{at}{z} + O(|z|^{-2}), \quad z \to \infty.$$

It is related to the usual (forward) SLE_κ with maps g_t and driving function V_t as follows.

Proposition 8.1. *For fixed $t > 0$, the distribution of h_t is the same as the distribution of the function $\hat{f}_t$ defined by*

$$\hat{f}_t(z) = g_t^{-1}(z + V_t) - V_t.$$

In particular, $\hat{f}_t'$ and $\hat{h}'$ have the same distribution.

Proof. For fixed t, the maps $h_s, 0 \le s \le t$, are exactly the reverse flow for the forward maps $\tilde{g}_s, 0 \le s \le t$ where $\tilde{g}$ has driving function $\tilde{V}_s = U_{t-s}$. We let g_t be the corresponding maps with driving function $V_s = \tilde{V}_s - \tilde{V}_0 = U_{t-s} - U_t$. □

The proposition shows that for a fixed t, h_t' and $\hat{f}_t'$ has the same distribution. However, it is not true that that the joint distribution (h_s', h_t') is the same as $(\hat{f}_s', \hat{f}_t')$.

The properties of reverse SLE_κ chains are similar but not identical to those of the forward flow. We start with the scaling relation whose proof is identical.

Proposition 8.2. *Suppose h_t is a reverse SLE_κ chain and $r > 0$.*

- *If $\tilde{h}_t(z) = r^{-1} h_{r^2 t}(rz)$, then $\tilde{h}_t$ is a reverse SLE_κ chain.*
- *The distribution of $h_{r^2 t}'(rz)$ is the same as that of $h_t'(z)$.*

Proof.

$$\partial_t \tilde{h}_t(z) = \frac{ar}{h_{t^2}(rz) - U_{t^2}} = \frac{a}{\tilde{h}_t(z) - \tilde{U}_t},$$

where $\tilde{U}_t = r^{-1} U_{r^2 t}$ is a standard Brownian motion. This gives the first assertion and the second follows from $\tilde{h}_t'(z) = h_{r^2 t}'(rz)$.

□

As in the forward case, if $z \in \mathbb{H}$, we let $Z_t = Z_t(z) = h_t(z) - U_t$ which in this case satisfies

$$dZ_t = -\frac{a}{Z_t}\, dt + dB_t.$$

If $Z_t = X_t + iY_t$, then

$$dX_t = -\frac{a\, X_t}{X_t^2 + Y_t^2}\, dt + dB_t, \quad \partial_t Y_t = \frac{a\, Y_t}{X_t^2 + Y_t^2}.$$

Note that the imaginary part increases, so this is valid for all times. Since

$$\partial_t Y_t^2 = \frac{2aY_t^2}{X_t^2 + Y_t^2} \le 2a,$$

we get

$$Y_t^2 \le 2at + \mathrm{Im}(z).$$

By differentiating (56) with respect to z, and then taking real parts, we see that

$$\partial_t \log |h_t'(z)| = \mathrm{Re}\left[\frac{a}{Z_t^2}\right] = \frac{a\,(X_t^2 - Y_t^2)}{(X_t^2 + Y_t^2)^2}.$$

We will do a change of time in which $\log Y_t$ grows linearly. Let $\sigma(t) = \inf\{s : Y_s = e^{at}\}$, and let $\hat{X}_t = X_{\sigma(t)}$, etc. Then

$$a\,Y_{\sigma(t)} = \partial_t\, Y_{\sigma(t)} = \frac{a\,Y_{\sigma(t)}}{X_{\sigma(t)}^2 + Y_{\sigma(t)}^2}\,\dot{\sigma}(t).$$

which implies

$$\dot{\sigma}(t) = \hat{X}_t^2 + e^{2at}.$$

$$d\hat{X}_t = -a\hat{X}_t\,dt + \sqrt{\hat{X}_t^2 + e^{2at}}\,dW_t.$$

Define K_t by $\hat{X}_t = e^{at}\,K_t$ and let $L_t = \sqrt{K_t^2 + 1}$. Then

$$\dot{\sigma}(t) = e^{2at}\,L_t^2,$$

$$dK_t = e^{-at}\,d\hat{X}_t - a\,K_t\,dt = -2a\,K_t\,dt + L_t\,dW_t.$$

$$\partial_t \log |\hat{h}'(z)| = \frac{a\,(\hat{X}_t^2 - \hat{Y}_t^2)}{\hat{X}_t^2 + \hat{Y}_t^2} = a\,\left(1 - 2\,L_t^{-2}\right).$$

We do a change of variables using the following easy Itô's formula calculation.

Lemma 8.3. *If J_t satisfies*

$$dJ_t = -\left(\frac{1}{2} + 2a\right)\,\tanh J_t\,dt + dW_t,$$

where W_t is a standard Brownian motion, and $K_t = \sinh J_t, L_t = \cosh J_t$, then

$$dK_t = -2a\,K_t\,dt + L_t\,dW_t,$$

$$dL_t = \left[\frac{L_t}{2} - \left(\frac{1}{2} + 2a\right)\frac{K_t^2}{L_t}\right]\,dt + K_t\,dW_t.$$

This gives us another way to construct the Loewner flow starting at z. For ease, let us assume $\mathrm{Im}(z) = 1$, so that $\sigma(0) = 0$. Let W_t be a standard Brownian motion, and define J_t, K_t, L_t as above. We define

$$\sigma(t) = \int_0^t \dot{\sigma}(s)\,ds = \int_0^t e^{as}\,L_s^2\,ds.$$

$$\tau = \sigma^{-1}, \quad Y_s = e^{a\tau(s)}, \quad X_s = K_{\tau(s)}\,Y_s.$$

As the analogue of $\log |\hat{h}'(z)|$ we can just define

$$\Delta_t = \exp\left\{a\int_0^t \left(1 - 2\,L_s^{-2}\right)\,ds\right\} = \exp\left\{a\int_0^t (2\tanh^2 J_s - 1)\,ds\right\}.$$

This gives a one parameter family of martingales.

Proposition 8.4. *If $r \in \mathbb{R}$ and*

$$\lambda = r\left(1+\frac{1}{2a}\right) - \frac{r^2}{4a}, \quad \zeta = \lambda - \frac{r}{2a}, \tag{57}$$

then

$$M_t := \Delta_t^\lambda \left[\cosh J_t\right]^r e^{a\zeta t},$$

is a martingale satisfying

$$dM_t = r\left[\tanh J_t\right] M_t\, dW_t. \tag{58}$$

We can also write the martingale in the other parametrization.

$$N_t = M_{\tau(t)} = |h_t'(i)|^\lambda\, Y_t^\zeta \left[\frac{Y_t}{\sqrt{X_t^2+Y_t^2}}\right]^{-r} = |h_t'(i)|^\lambda\, Y_t^\zeta \left[\sin\Theta_t\right]^{-r},$$

where $\Theta_t = \arg Z_t$.

Proof. Showing that M_t is a local martingale satisfying (58) is a straightforward exercise using

$$\partial_t \Delta_t^\lambda = a\lambda\left[2\tanh^2 J_t - 1\right]\Delta_t^\lambda, \qquad \partial_t e^{a\zeta t} = a\zeta\, e^{a\zeta t},$$

$$\begin{aligned} dL_t^r &= dL_t^r\left[\frac{r}{L_t}\, dL_t + \frac{r(r-1)}{2L_t^2}\, d\langle L\rangle_t\right] \\ &= dL_t^r\left[\left(\left[\frac{r(r-1)}{2} - \frac{r}{2} - 2ar\right]\tanh^2 J_t + \frac{r}{2}\right) dt + r\,\tanh J_t\, dW_t\right] \end{aligned}$$

To see that it is actually a martingale, we consider the measure obtained by tilting by M_t. Note that

$$dJ_t = -q\,\tanh J_t\, dt + dB_t, \quad q = \frac{1}{2} + 2a - r,$$

where B_t is a Brownian motion in the new measure. This is well defined if we use stopping times, but since $|\tanh x| \le 1$, we can see from this equation that there is no blow-up or other bad behavior in finite time with respect to the new measure, and hence it is actually a martingale. □

Corollary 8.5. *If $r \ge 0$ and λ, ζ are defined as in (57), and $z = i$, then*

$$\mathbb{E}\left[|h_t'(i)|^\lambda\, Y_t^\zeta\right] \le 1. \tag{59}$$

Proof.

$$\mathbb{E}\left[|h_t'(i)|^\lambda\, Y_t^\zeta\right] = \mathbb{E}\left[\Delta_{\tau(t)}^\lambda\, Y_{\tau(t)}^\zeta\right] \le \mathbb{E}\left[\Delta_{\tau(t)}^\lambda\, e^{at\zeta}\left[\cosh J_t\right]^r\right] \le 1.$$

□

Lemma 8.6. *If $a \ne 1/4$ ($\kappa \ne 8$), then we can find r satisfying*

$$0 < r < \frac{1}{2} + 2a, \quad \lambda + \zeta > 2,$$

and

$$\zeta > 0 \ \text{ if } \ a > 1/4, \qquad \text{and} \qquad \zeta < 0 \ \text{ if } a < 1/4.$$

Proof. Note that

$$\lambda'(r) = 1 + \frac{1}{2a} - \frac{r}{2a}, \quad \zeta'(r) = 1 - \frac{r}{2a}.$$

- If $a > 1/4$, we use $\lambda(1) + \zeta(1) = 2$, $\zeta(1) > 0$, and

$$\lambda'(1) + \zeta'(1) = 2 - \frac{1}{2a} > 0$$

  to see that we can find r slightly larger than 1 with $\lambda(r) + \zeta(r) > 2$ and $\zeta(r) > 0$.

- If $a < 1/4$, we use $\lambda(4a) + \zeta(4a) = 2$, $\zeta(4a) = 0$, and

$$\lambda'(4a) + \zeta'(4a) = \frac{1}{2a} - 2 > 0$$

  to see that we find r slightly larger than $4a$ with $\lambda(r) + \zeta(r) > 2$. Since $\zeta'(4a) = -1$, we have $\zeta(r) < 0$.

□

Proposition 8.7. *If $\kappa \neq 8$, then there exists $\delta > 0$ such that that the following holds with probability one for SLE_κ. For every $t_0 < \infty$ and all n sufficiently large, if $t = k2^{-2n}$ is a dyadic time with $t \leq t_0$, then*

$$|f_t'(U_t + i2^{-n})| \leq 2^{(1-\delta)n}.$$

Proof. By scaling it suffices to prove the result for $t_0 = 1$. By the Borel-Cantelli lemma it suffices to find $\delta, \epsilon > 0, C < \infty$ such that for all n sufficiently large and all $t \leq 1$,

$$\mathbb{P}\left\{|f_t'(U_t + i2^{-n})| \geq 2^{(1-\delta)n}\right\} \leq C\, 2^{-(2+\epsilon)n},$$

and by Proposition 8.1 this is equivalent to

$$\mathbb{P}\left\{|h_t'(i2^{-n})| \geq 2^{(1-\delta)n}\right\} \leq C\, 2^{-(2+\epsilon)n}.$$

By scaling (see Proposition 8.2), this will follow if we show for all t sufficiently large,

$$\mathbb{P}\left\{|h_{t^2}'(i)| \geq t^{1-\delta}\right\} \leq t^{-(2+\epsilon)}.$$

We choose r, λ, ζ as in the previous lemma and use (59). Recall that $Y_{t^2} \leq \sqrt{2at^2 + 1} \leq ct$ (for $t \geq 1$ say). The Koebe 1/4-theorem implies that image under h_{t^2} of the disk of radius 1 about i has radius at least $|h_{t^2}'(i)|/4$; hence $Y_{t^2} \geq |h_{t^2}'(i)|/4$. If $a \geq 1/4$, $\zeta > 0$ and hence

$$\mathbb{E}\left[|h_{t^2}'(i)|^{\lambda+\zeta}\right] \leq 4^\zeta\, \mathbb{E}\left[|h_{t^2}'(i)|^\lambda\, Y_{t^2}^\zeta\right] \leq 4^\zeta, \qquad \text{if } \zeta \geq 0.$$

If $a < 1/4$, $\zeta < 0$ and $Y_{t^2} \leq t$ for t large enough, hence for t large enough

$$\mathbb{E}\left[|h_{t^2}'(i)|^{\lambda+\zeta}\ ;\ |h_{t^2}'(i)| \geq t^{1-\delta}\right] \leq t^{-\delta\zeta}\, \mathbb{E}\left[|h_{t^2}'(i)|^\lambda\, Y_{t^2}^\zeta\right] \leq t^{-\delta\zeta}.$$

Combining these estimates, we see there exists a $C < \infty$ such that

$$\mathbb{P}\left\{|h'_{t^2}(i)| \geq t^{1-\delta}\right\} \leq C\, t^{-v}, \quad v = (1-\delta)(\lambda+\zeta) + \delta(0 \wedge \zeta).$$

We now choose δ sufficiently small so that $(1-\delta)(\lambda+\zeta) + \delta\,(0 \wedge \zeta) = 2 + 2\epsilon$ with $\epsilon > 0$. Then for all t sufficiently large

$$\mathbb{P}\left\{|h'_{t^2}(i)| \geq t^{1-\delta}\right\} \leq t^{-(2+\epsilon)}.$$

□

Proposition 8.8. *If $\kappa \neq 8$, then there exists $\delta > 0$ such that that the following holds with probability one for SLE_κ. For every $t_0 < \infty$ and all y sufficiently small*

$$\max_{0 \leq t \leq t_0} |f'_t(U_t + iy)| \leq y^{\delta - 1}.$$

Proof. By scaling, it suffices to prove this for $t_0 = 1$. Using the distortion estimate, it suffices to establish this for dyadic $y = 2^{-n}$. Let

$$I_n = \max\left\{|f'_t(U_t + i2^{-2n})| : t = k2^{-2n}, k = 0, 1, \ldots, 2^{2n}\right\}.$$

$$I^*_n = \max\left\{|f'_t(U_t + i2^{-2n})| : 0 \leq t \leq 1\right\}.$$

If $t = k2^{-2n} \leq s \leq t + 2^{-2n}$, then distortion estimates imply that

$$|f'_s(U_s + i2^{-n})| \leq c\, \left[2^n|U_t - U_s| + 1\right]^4\, |f'_s(U_t + i2^{-n})|.$$

By the inverse Loewner equation (see Lemma **??**), we have

$$|f'_s(U_t + i2^{-n})| \leq c\, |f'_t(U_t + i2^{-n})|.$$

Using the modulus of continuity for Brownian motion we know that for n sufficiently large

$$|U_t - U_s| \leq C\, \sqrt{n}\, 2^{-n}.$$

Combining this with the distortion theorem we can see that for all n sufficiently large

$$I^*_n \leq C\, n^2\, I_n.$$

Hence the result follows from the previous proposition. □

9 Two-point estimate

An important, but technically a bit tricky, estimate used to establish fractal properties of an SLE curve is the two-point estimate for the Green's function and related probabilities. We state a version here. Throughout this section we assume that $\kappa < 8$ and allow all constants to depend on κ. Let γ denote an SLE_κ path from 0 to infinity in $\mathbb{H}$.

If $z \in \mathbb{H}$ let

$$\xi^z_r = \min\{t : |\gamma(t) - z| = e^{-r}\, \mathrm{Im}(z)\}.$$

Theorem 15. *There exists c such that the following is true. If $w, z \in \mathbb{H}$, then for all $r, s \geq 0$,*

$$\mathbb{P}\{\xi_r^z < \infty, \xi_s^w < \infty\} \leq c\, e^{(d-2)(r+s)} \left(\frac{|w-z|}{\mathrm{Im}(z) + \mathrm{Im}(w)} \wedge 1 \right)^{d-2}. \tag{60}$$

By scaling, it suffices to prove this when $\mathrm{Im}(w) \leq \mathrm{Im}(z) = 1$ and in this case the right-hand side is comparable to

$$\mathbb{P}\{\xi_r^z < \infty\}\, \mathbb{P}\{\xi_s^w < \infty\}\, |w-z|^{d-2}.$$

In particular, if $|w-z| \asymp 1$, then the events $\{\xi_r^z < \infty\}$ and $\{\xi_s^w < \infty\}$ are "independent up to a multiplicative constant". This will allow second moments to be bounded. The hardest work is needed to establish this when $|w-z| \asymp 1$; we state this as a proposition.

Proposition 9.1. *For every $\delta > 0$, there exist $c < \infty$, such that the following is true. If $w, z \in \mathbb{H}$ with $\mathrm{Im}(w) \leq \mathrm{Im}(z) = 1$ and $|w-z| \geq \delta$. then for all $r, s \geq 0$,*

$$\mathbb{P}\{\xi_r^z < \infty, \xi_s^w < \infty\} \leq c\, e^{(d-2)(r+s)}.$$

Proof of Theorem 15 given Proposition 9.1. By scaling, we may assume that $\mathrm{Im}(w) < \mathrm{Im}(z) \leq 1$ and define u by $|w-z| = e^{-u-4}$. If $u \leq 0$, then the result follows immediately from Proposition 9.1. If $s \leq u+4$ then the right-hand side of (60) is greater than $O(e^{(d-2)r})$ and hence (60) follows from

$$\mathbb{P}\{\xi_r^z < \infty, \xi_s^w < \infty\} \leq \mathbb{P}\{\xi_r^z < \infty\} \leq c\, e^{(d-2)r}.$$

Similarly, if $r \leq u+4$, the one-point estimate applied to ξ_s^w gives the result.

Hence without loss of generality we assume $r, s \geq u+4 \geq 4$. We then write

$$\begin{aligned} \mathbb{P}\{\xi_r^z < \infty, \xi_s^w < \infty\} &= \mathbb{P}\{\xi_u^z < \infty\}\, \mathbb{P}\{\xi_r^z < \infty, \xi_s^w < \infty \mid \xi_u^z < \infty\} \\ &\leq c\, e^{(d-2)u}\, \mathbb{P}\{\xi_r^z < \infty, \xi_s^w < \infty \mid \xi_u^z < \infty\}. \end{aligned}$$

On the event $\{\xi_u^z < \infty\}$, we consider the image under the map $g_{\xi_u^z}$. Using distortion estimates (details omitted) and Proposition 9.1, we get

$$\mathbb{P}\{\xi_r^z < \infty, \xi_s^w < \infty \mid \xi_u^z < \infty\} \leq c\, e^{(d-2)(r-u)}\, e^{(d-2)(s-u)},$$

and hence

$$\mathbb{P}\{\xi_r^z < \infty, \xi_s^w < \infty\} \leq c\, e^{(d-2)r}\, e^{(d-2)s}\, e^{(2-d)u}.$$

□

We will now discuss the proof of Proposition 9.1. This next lemma is important. We start with some notation. Let $w, z \in \mathbb{H}$. Let H_t denote the unbounded component of $\mathbb{H} \setminus \gamma_t$ and assume that t is a time with $z, w \in H_t$ and $|z-w| \max\{\mathrm{dist}(w, \partial H_t), \mathrm{dist}(z, \partial H_t)\}$. Let $\mathcal{B}_t^w$ denote the open disk of radius $\mathrm{dist}(w, \gamma_t)$ about w with boundary circle C_t^w. There is a unique crosscut ℓ_t^w of H_t that lies in C_t^w and such that z and w are in different components of $H_t \setminus \ell_t^w$. Let V_t^w denote the connected component of $H_t \setminus \ell_t^w$ containing w, and note that $z \notin V_t^w$. We define $\mathcal{B}_t^z, \ell_t^z, V_t^z$ analogously. A key observation is that if $\gamma(t) \in \partial V_t^w$, then the distance from γ to z cannot decrease without hitting the crosscut ℓ_t^z.

We will write

$$\xi_r^w = \inf\{t : \mathrm{dist}(w, \gamma_t) = e^{-r}\}$$

(this is slightly different than defined earlier). Note that if $\xi_r^w > 0$, then $\ell_{\xi_r^w}$ is the entire circle of radius e^{-r} about w (with a single endpoint $\gamma(t)$).

Lemma 9.2. *If $0 < \kappa < 8$, there exist $c < \infty$ such that the following holds. Suppose γ is an SLE_κ path from 0 to ∞ in $\mathbb{H}$, $|w - z| \geq 1$. Suppose ρ is a stopping time with $w, z \in H_\rho$ satisfying the following:*

- $|\gamma(\rho) - w| = e^{-s} = \mathrm{dist}(\gamma_\rho, w)$,
- $\mathrm{dist}(\gamma_\rho, z) = e^{-r}$,

and let V denote the connected component of $H_\rho \cap \mathcal{B}_r^z$ containing z. Let $\tau_r = \inf\{t \geq \rho : \gamma(t) \in \ell_\rho^z$ and for positive integer j, let $\tau_{r+j} = \inf\{t \geq \rho : |\gamma(t) - z| = e^{-(r+j)}\}$. Let $\tilde{\rho}_j = \inf\{t \geq \tau_{r+j} : \gamma(t) \in \ell_{\tau_{r+j}}^w\}$ Then,

$$\mathbb{P}\{\tilde{\rho}_j < \infty \mid \gamma_\rho\} \leq c\, e^{-\beta(r+s)}\, e^{-(2-d)j}, \quad \beta = \frac{4a-1}{4}.$$

With more work one can prove this result with $\beta = (4a-1)/2$ but we will not need it so we will just prove this proposition.

The basic idea of the proof is the following. If the SLE curve is near w and it has also gotten near z then at one of the following is true: it will be unlikely starting near w to get closer to z, or, if we succeed, then it will be unlikely to return close to w after we have gotten close to z. Which of these is the case depends on the location of w and z. For example, if w is near the origin and z is far away, then it is not difficult to get near z but returning to w is difficult. The opposite is true if z is near the origin and w is far away.

Proof. Let $g = g_\rho - U_\rho$ and let $\hat{\ell}^w = g(\ell_\rho^w), \hat{\ell}^z = g(\ell_\rho^z)$. Note that $\hat{\ell}^w, \hat{\ell}^z$ are disjoint crosscuts of $\mathbb{H}$ with one of the endpoints of $\hat{\ell}^w$ being the origin. For ease we will assume that ℓ^z has at least one endpoint on the positive real axis $\mathbb{R}_+$ (there is an identical argument if both endpoints are on the negative axis). By conformal invariance and the Beurling estimate,

$$\mathcal{E}_{\mathbb{H}}(\hat{\ell}^w, \hat{\ell}^z) = \mathcal{E}_{H_\rho}(\ell_\rho^w, \ell_\rho^z) \leq c\, e^{-(r+s)/2}.$$

The second inequality uses only the facts that H_ρ is simply connected, its boundary intersects C_r^w and C_r^z, and $|w - z| \geq 1$.

We split into four cases. In two of the cases we will show that

$$\mathbb{P}\{\tau_r < \infty \mid \gamma_\rho\} \leq c\, e^{-\beta(r+s)}, \tag{61}$$

It then follows from the interior estimate that

$$\mathbb{P}\{\tilde{\rho}_j < \infty \mid \gamma_\rho\} \leq \mathbb{P}\{\tau_r < \infty \mid \gamma_\rho\}\, \mathbb{P}\{\tau_{r+j} < \infty \mid \tau_r < \infty\} \leq c\, e^{-\beta(r+s)}\, e^{-(2-d)j}.$$

In the other two cases, we will show that

$$\mathbb{P}\{\tilde{\rho}_j < \infty \mid \tau_{r+j} < \infty\} \leq c\, e^{-\beta(r+s)}. \tag{62}$$

It then follows from the interior estimate that

$$\mathbb{P}\{\tilde{\rho}_j < \infty \mid \gamma_\rho\} \leq \mathbb{P}\{\tau_{r+j} < \infty \mid \gamma_\rho\}\, \mathbb{P}\{\tilde{\rho}_j < \infty \mid \tau_{r+j} < \infty\} \leq c\, e^{-(2-d)j}\, e^{-\beta(r+s)}.$$

- **Case I:** $\hat{\ell}^z$ is in the bounded component of $\mathbb{H} \setminus \hat{\ell}^w$. In this case $\hat{\ell}^w$ separates $\hat{\ell}^z$ from $\mathbb{R}_-$ and

$$\mathcal{E}_{\mathbb{H}}(\hat{\ell}^z, \mathbb{R}_-) \le \mathcal{E}_{\mathbb{H}}(\hat{\ell}^w, \hat{\ell}^z) \le c\, e^{-(r+s)/2}.$$

  Hence from the boundary estimate, we get (61).

- **Case II:** $\hat{\ell}^w$ is in the bounded component of $\mathbb{H} \setminus \hat{\ell}^z$; let us denote this component by U. Assume $\tau_{r+j} < \infty$, let $\tilde{H} = H_{\tau_{r+j}}$. We assume $w \in \tilde{H}$; otherwise, the result is trivial. Let V be the component of $U \cap \tilde{H}$ containing w; V is clearly bounded. Let $\tilde{\ell}$ denote the unique crosscut of $\tilde{H}$ contained in $\hat{\ell}^z$ that is also contained in ∂V. If $j = 0$, then $\gamma(\tau_r)$ is an endpoint of $\tilde{\ell}$. If $j > 0$, then $\gamma(\tau_r)$ is not on the boundary of V. In either case, we see that $\tilde{\ell}$ separates $\ell^w_{\tau_{r+j}}$ and the probability that the SLE hits $\ell^w_{\tau_{r+j}}$ after time τ_{r+j} is bounded above by

$$\mathcal{E}_{\tilde{H}}(\ell^w_{\tau_{r+j}}, \tilde{\ell}) \le \mathcal{E}_{H_\rho}(\ell^w_\rho, \ell^z_\rho) \le e^{-(r+s)/2}.$$

  Hence, using the boundary exponent, we get (62).

For the remaining cases we assume implicitly that neither I nor II holds. Let $\phi^z = \mathrm{diam}(\hat{\ell}^z)/\mathrm{diam}(\hat{\ell}^w)$, $\delta = \mathrm{dist}(\hat{\ell}^w, \hat{\ell}^z)/\mathrm{diam}(\hat{\ell}^w)$. If $\delta \le 1 \wedge \phi_z$, then $\mathcal{E}_{\mathbb{H}}(\hat{\ell}^w, \hat{\ell}^z) \ge c$ which is impossible for $r+s$ sufficiently large. Hence we may assume that $\delta \ge 1 \wedge \phi_z$.

- **Case III:** Neither I nor II and $\phi_z \le 1$. If $\delta \le 2$, then

$$\mathcal{E}_{\mathbb{H}}(\hat{\ell}^w, \hat{\ell}^z) \ge c\, \mathcal{E}_{\mathbb{H}}(R_-, \hat{\ell}^z).$$

  If $\delta \ge 2$, then

$$\mathcal{E}_{\mathbb{H}}(R_-, \hat{\ell}^z) \asymp \phi_z/\delta, \quad \mathcal{E}_{\mathbb{H}}(\hat{\ell}^w, \hat{\ell}^z) \asymp \frac{\phi_z}{\delta^2}.$$

  Then

$$\mathcal{E}_{\mathbb{H}}(\hat{\ell}^w, \hat{\ell}^z) \ge c\,(\phi_z/\delta)^2 \asymp \mathcal{E}_{\mathbb{H}}(R_-, \hat{\ell}^z)^2.$$

  Therefore, by the boundary estimate, (61) holds.

- **Case IV:** Neither I nor II and $\phi_z > 1$. Note that $\mathcal{E}_{\mathbb{H}}(\hat{\ell}^w, \hat{\ell}^z) \ge c/\delta^2$ and hence there exists c_1 such that

$$\delta \ge 2c_1\, e^{(r+s)/4}.$$

  Let C be the half-circle of radius $c_1 e^{(r+s)/4}$ about the origin and let $\eta = g^{-1}(C)$. Then η is a crosscut of H_ρ with

$$\mathcal{E}_{H_\rho}(\ell^w_\rho, \eta) = \mathcal{E}_{\mathbb{H}}(\hat{\ell}^w, C) \asymp e^{(r+s)/4}.$$

  Let $\tau = \tau_{r+j}$ and let $\tilde{H} = H_\tau$. Let U be the component of $\tilde{H} \setminus \eta$ that contains w (if $w \notin \tilde{H}$, the result is trivial). There is a unique crosscut l of $\tilde{H}$ that is a subset of η and is contained in ∂U. Note that l disconnects w from ℓ^z_ρ and that

$$\mathcal{E}_{\tilde{H}}(\ell^w_\tau, l) \le \mathcal{E}_{H_\rho}(\ell^w_\rho, \eta) \le e^{(r+s)/4}.$$

  Using the boundary exponent, we see that the probability that $\gamma[\tau, \infty)$ hits ℓ^w_τ is $O(e^{(4a-1)(r+s)/4})$ and hence (62) holds.

□

Proof of Theorem 15. We will consider paths that go to w first and then z; the same argument works for paths that get to z first and then w. We will consider "excursions" between points near w and near z.

Let $\rho_{s,r} = \xi_s^w \vee \xi_r^z$. We will be considering the case where s, r are positive integers (this clearly suffices for our result). We think of $\rho_{s,r}$ as first stopping at time ξ_s^w, and then, if necessary, continuing until time ξ_r^z. Using this viewpoint, on the event $\{\rho_{s,r} < \infty\}$ define the finite sequence $\mathbf{j} = (j_1, \ldots, j_m), \mathbf{k} = (k_1, \ldots, k_m)$ by the following,

- Each integer is strictly positive except perhaps k_m which can equal zero.
- $k_m = 0$ if and only if $\xi_r^z < \xi_s^w$.
- $j_1 + \cdots + j_m = s,\ k_1 + \cdots + k_m = r$
- For $q = 1, \ldots, m-1$,

$$\xi^w_{j_1+\cdots+j_q} < \xi^z_{(k_1+\cdots+k_q)\wedge r} < \xi^w_{j_1+\cdots+j_q+1},$$
$$\xi^z_{k_1+\cdots+k_{q-1}} < \xi^w_{j_1+\cdots+j_q} < \xi^z_{k_1+\cdots+k_{q-1}+1},$$

We can now write

$$\mathbb{P}\{\xi_s^w \vee \xi_r^z < \infty; \xi_1^w < \xi_1^z\} = \sum_{m=1}^{\infty} \sum_{\mathbf{j}=(j_1,\ldots,j_m),\mathbf{k}=(k_1,\ldots,k_m)} \mathbb{P}\{\xi_s^w \vee \xi_r^z < \infty; \mathbf{j}, \mathbf{k}\}$$

where the sum is over all finite sequences $\mathbf{j} = (j_1, \ldots, j_m), \mathbf{k} = (k_1, \ldots, k_m)$ of nonnegative integers all of which are positive except perhaps k_m.

Using the proposition, we can see that there exist c, β such that for a given $m, \mathbf{j}, \mathbf{k}$,

$$e^{(2-d)(r+s)}\mathbb{P}\{\xi_s^w \vee \xi_r^z < \infty; \mathbf{j}, \mathbf{k}\} \le c^m \prod_{i=1}^{m-1} \exp\left\{-\beta(j_1 + \cdots + j_i + k_1 + \cdots + k_{i-1})\right\}.$$

Let

$$S_m = \sum c^m \prod_{i=1}^{m-1} \exp\left\{-\beta(j_1 + \cdots + j_i + k_1 + \cdots + k_{i-1})\right\}$$

where the sum is over all finite sequences $(j_1, \ldots, j_m)$, $(k_1, \ldots, k_{m-1})$ of finite integers. By focusing on the possible values of $(j_1, k_1) = (j, k)$ we get the inequality

$$S_{m+1} \le c \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} e^{-\beta(j+k)(m-1)} S_m.$$

Using this, we see that for m sufficiently large $S_{m+1} \le S_m/2$ and hence

$$\sum_{m=1}^{\infty} S_m < \infty.$$

□

10 Notes about fractal dimension

In this section we will prove some general facts about how to compute dimensions of random subsets. Since we do not need to use planarity, we will state our results for $\mathbb{R}^d$. Throughout this section we will make the following assumption.

- There is a fixed compact $K \subset \mathbb{R}^d$ and V is a random compact subset of K such that with probability one $\mathrm{dist}(V, \partial K) \geq 1$.

This may be more restrictive than one wants. If V is not bounded, we can use these results for the intersection of V with compact sets. We fix $0 < \alpha < d$, set $\zeta = d - \alpha$ and allow all constants to depend implicitly on α, ζ, d, K. We will consider ways to rigorously state that "V has fractal dimension α."

Let $V^\epsilon = \{x : \mathrm{dist}(x, V) \leq \epsilon\}$ denote the closed ϵ-"sausage" around V, and $\ell(V^\epsilon)$ the Lebesgue measure of V^ϵ. The sets V^ϵ increase with ϵ and $V^1 \subset K$. If V is α-dimensional, then we expect that $\ell(V^\epsilon) \approx \epsilon^{d-\alpha}$ as $\epsilon \to 0$. The upper and lower *box or Minkowski dimensions* of V are defined by

$$\underline{\dim}_b(V) = \liminf_{\epsilon \downarrow 0} \frac{\log[\epsilon^{-d}\ell(V_\epsilon)]}{\log(1/\epsilon)} \leq \limsup_{\epsilon \downarrow 0} \frac{\log[\epsilon^{-d}\ell(V_\epsilon)]}{\log(1/\epsilon)} = \overline{\dim}_b(V)$$

If the liminf and limsup agree, we call the common value $\dim_b(V)$ the *box or Minkowski dimension.* Note that we would get the same definition if we took the limit just along a geometric sequence, $\{\rho^n\}$ where $0 < \rho < 1$.

The box dimension α is often defined by saying that if N_ϵ is the minimum number of balls of radius ϵ need to cover V, then $N_\epsilon \approx \epsilon^{-\alpha}$ (with corresponding liminf and limsup versions). It is a straightfoward exercise to check that this agrees with our definition.

To define the Hausdorff dimension, we start by defining

$$\mathcal{H}^\alpha_\epsilon(V) = \inf \sum_{j=1}^{\infty} [\mathrm{diam}(U_j)]^\alpha,$$

where the infimum is over all covers of V by sets $U_1, U_2, \ldots,$ with $\mathrm{diam}(U_j) \leq \epsilon$. Since $\mathcal{H}^\alpha_\epsilon(V)$ is decreasing in ϵ, we can define the Hausdorff measure $\mathcal{H}^\alpha(V) = \mathcal{H}^\alpha_{0+}(V)$. The *Hausdorff dimension* $\dim_h(V)$ is the unique α such that

$$\mathcal{H}^\beta(V) = \begin{cases} 0, & \beta > \alpha \\ \infty, & \beta < \alpha. \end{cases}$$

This is well defined for all V.

The major difference between box and Hausdorff dimension is that the former uses cover by sets of size ϵ while the latter uses covers by sets of size *at most* ϵ. The former is more natural for numerical approximations and for comparison to lattice models, but the latter has the advantage of being defined for all sets and producing a mathematically nicer object.

It is not hard to see that $\dim_h(V) \le \underline{\dim}_b(V)$ but the inequality can be strict. We can give upper bounds for Hausdorff dimension by estimating the box dimension. Lower bounds are trickier. We will use the following fact that states that if a set supports an "α-dimensional measure", then it must be at least α-dimensional. If μ is a positive measure on $\mathbb{R}^d$ we define the β-energy by

$$\mathcal{E}_\beta(\mu) = \int\int \frac{\mu(dx)\,\mu(dy)}{|x-y|^\beta}.$$

Proposition 10.1. *Suppose V is a compact set and there exists a probability measure μ supported on V with $\mathcal{E}_\beta(\mu) < \infty$, then $\dim_h(V) \ge \beta$.*

For each $0 < \epsilon \le 1$, define

$$I_\epsilon(x) = \epsilon^{-\zeta}\, 1_{V^\epsilon}(x) = \epsilon^{-\zeta}\, 1\{\mathrm{dist}(x,V) \le \epsilon\},$$

and let μ_ϵ be the (random) measure whose Radon-Nikodym derivative with respect to Lebesgue measure is I_ϵ. Let

$$J_\epsilon = \mu_\epsilon(\mathbb{R}^d) = \int_K I_\epsilon(x)\,dx = \epsilon^{-\zeta}\,\ell(V^\epsilon).$$

Proposition 10.2. *Suppose V is a random subset of a compact set $K \subset \mathbb{R}^d$ and $0 < \zeta < d, \alpha = d-\zeta$. Suppose that there exist $0 < c_1 < c_2 < \infty$ and $\epsilon_0 > 0$ such that for all x, y and all $0 < \epsilon < \epsilon_0$,*

$$\mathbb{E}[I_\epsilon(x)] \le c_2, \quad \mathbb{E}[J_\epsilon] \ge c_1, \quad \mathbb{E}[I_\epsilon(x)\,I_\epsilon(y)] \le c_2\,|x-y|^{-\zeta}. \tag{63}$$

Then, there exists $\delta = \delta(c_1, c_2, K, d, \zeta) > 0$ such that

$$\delta \le \mathbb{P}\{\dim_h(V) = \alpha\} \le \mathbb{P}\{\overline{\dim}_b(V) \le \alpha\} = 1.$$

Proof. Note that the first inequality in (63) implies that $\mathbb{E}[J_\epsilon] \le c_2\,\ell(K)$. Also, using the third inequality,

$$\begin{aligned}
\mathbb{E}[J_\epsilon^2] &\le \mathbb{E}\left[\left(\int_K I_\epsilon(x)\,dx\right)\left(\int_K I_\epsilon(y)\,dy\right)\right] \\
&= \int_K\int_K \mathbb{E}[I_\epsilon(x)\,I_\epsilon(y)]\,dy\,dx \\
&\le c_2\int_K\int_K |x-y|^{-\zeta}\,dy\,dz := c_3 < \infty.
\end{aligned}$$

For any nonnegative random variable X, we have

$$\mathbb{P}\left\{X \ge \frac{\mathbb{E}(X)}{2}\right\} \ge \frac{(\mathbb{E}[X])^2}{4\mathbb{E}[X^2]}.$$

Therefore, using the second inequality,

$$\mathbb{P}\left\{J_\epsilon \ge \frac{c_1}{2}\right\} \ge \frac{c_1^2}{4c_3} > 0. \tag{64}$$

If $\beta < \alpha$, let

$$\mathcal{E}_\beta(\mu_\epsilon) = \int_K \int_K \frac{\mu_\epsilon(dx)\,\mu_\epsilon(dy)}{|x-y|^\beta},$$

and note that the third inequality gives

$$\mathbb{E}\left[\mathcal{E}_\beta(\mu_\epsilon)\right] \le c_2 \int_K \int_K \frac{dx\,dy}{|x-y|^\zeta\,|x-y|^\beta} := C_\beta < \infty.$$

Therefore,

$$\mathbb{P}\left\{\mathcal{E}_\beta(\mu_\epsilon) \ge \frac{8c_3\,C_\beta}{c_1^2}\right\} \le \frac{c_1^2}{8c_3}$$

Combining this with (64), we see that

$$\mathbb{P}\left\{\mu_\epsilon(K) \ge \frac{c_1}{2}, \mathcal{E}_\beta(\mu_\epsilon) \le \frac{8c_3\,C_\beta}{c_1^2}\right\} \ge \delta := \frac{c_1^2}{8c_3} > 0. \tag{65}$$

- **Upper bound**. Since $\mathbb{E}[J_{2^{-n}}]$, is uniformly bounded, we can see from the Markov inequality that

  $$\sum_{n=1}^{\infty} \mathbb{P}\{J_{2^{-n}} \ge n^2\} < \infty,$$

  and hence from the Borel-Cantelli lemma that with probability one for all n sufficiently large

  $$2^{nd}\,\ell(V^{2^{-n}}) = 2^{n\alpha}\,J_{2^{-n}} \le n^2\,2^{n\alpha}.$$

- **Lower bound.** It suffices to show for every $\beta < \alpha$, $\mathbb{P}\{\dim_h(V) \ge \beta\} \ge \delta$. Let $E = E_\beta$ be the event that for infinitely many n,

  $$\mu_\epsilon(K) \ge \frac{c_1}{2}, \quad \mathcal{E}_\beta(\mu_\epsilon) \le \frac{8c_3\,C_\beta}{c_1^2}. \tag{66}$$

  Using (65), we see that $\mathbb{P}(E) \ge \delta$. On the event E, find a subsequence that satisfies (66), and then take a further subsequence that converges to a measure μ. Using compactness of K and Fatou's lemma we see that

  $$\mu(K) \ge \frac{c_1}{2}, \quad \mathcal{E}_\beta(\mu) \le \frac{8c_3\,C_\beta}{c_1^2},$$

  and since μ_ϵ is supported on V^ϵ, we can see that μ is supported on V. Using Proposition 10.1, on the event E we have $\dim_h(V) \ge \beta$.

□

We let $\mathcal{S}_n$ denote the set of dyadic cubes of side length 2^{-n}, that is, the set of closed cubes of the form

$$S = [(k_1-1)2^{-n}, k_1 2^n] \times \cdots \times [(k_d-1)2^{-n}, k_d 2^n],$$

where $k_1, \ldots, k_d$ are integers. Let $\mathcal{S} = \cup_n \mathcal{S}_n$ and

$$J_\epsilon(S) = \mu_\epsilon(S) = \int_S I_\epsilon(x)\,dx.$$

Definition We say that the Minkowski content exists if (63) hold, and with probability one, for every $S \in \mathcal{S}$, the limit

$$J(S) = \lim_{\epsilon \downarrow 0} J_\epsilon(S) \tag{67}$$

exists.

Lemma 10.3. *Suppose the Minkowski content exists. If $S \in \mathcal{S}_m$ and $n \geq m$, let S^n denote the union of all squares in $\mathcal{S}_n$ that intersect ∂S. Then, with probability one, for all n sufficiently large,*

$$J(S^n) \leq 2^{-dm}\, 2^{(m-n)/2}. \tag{68}$$

In particular, $\lim_{n\to\infty} J(S^n) = 0$.

Proof. Since $\mathbb{E}[J(S^n)] \leq c\,\ell[J(S^n)] \approx 2^{-dm}\, 2^{m-n}$ we have $\mathbb{P}\{J(S^n) \geq 2^{-dm}\, 2^{(m-n)/2}\} \leq c\, 2^{(m-n)/2}$, and the result follows by Borel-Cantelli. □

Proposition 10.4. *Suppose that a random subset V satisfies the hypothesis of Proposition 10.2, A sufficient condition for the existence of the limit* (67) *for $S \in \mathcal{S}$ is to show that for every $\rho < 1$ there exists $u > 0$ such that for all n sufficiently large,*

$$\mathbb{E}[(Y_{n+1} - Y_n)^2] \leq n^{-(3+2u)} \qquad \textit{where } Y_n = J_{\rho^n}(S). \tag{69}$$

The hard work in establishing the existence of Minkowski content is to show (69). In practice, one often shows a stronger estimate, $\mathbb{E}[(Y_{n+1} - Y_n)^2] = O(e^{-\beta n})$ for some $\beta = \beta_\rho > 0$.

Proof. We fix S and write $J_\epsilon = J_\epsilon(S)$. For a fixed ρ, (69) immediately shows that $\{Y_n\}$ is a Cauchy sequence in L^2, and hence there exists $Y_\infty = Y_{\infty,\rho}$ such that $Y_n \to Y_\infty$ in L^2. Using Markov's inequality,

$$\sum_{n=1}^{\infty} \mathbb{P}\{(Y_{n+1} - Y_n)^2 \geq n^{-(2+u)}\} < \infty,$$

and hence with probability one $|Y_{n+1} - Y_n| \leq n^{-(1+\frac{u}{2})}$ for all n sufficiently large. On this event $Y_n \to Y_\infty$.

We will choose $\rho_{k+1} = \sqrt{\rho_k}$, that is $\rho_k = 2^{-2^{-k}}$. For fixed k, the condition (69) implies that there exists a random variable Y_∞ such that $Y_n \to Y_\infty$ in L^2 and with probability one. Since $\{\rho_k^n : n \geq 1\} \subset \{\rho_{k+1}^n : n \geq 1\}$, we see that the limit Y_∞ must be the same for all k. If $\rho_k^{n+1} \leq \epsilon \leq \rho_k^n$, then

$$J_{\rho_k^{n+1}} \leq \rho_k^\zeta\, J_\epsilon \leq \rho_k^{2\zeta}\, J_{\rho_k^n}.$$

Using this and the uniform bound on $\mathbb{E}[J_\epsilon^2]$, we see that $J_\epsilon \to Y_\infty$ in L^2 and with probability one. □

If U is an open set, we can write U uniquely as

$$U = \bigcup_{S \in \mathcal{S}_U} S, \tag{70}$$

where $\mathcal{S}_U$ is the set of $S \in \mathcal{S}$ with $S \subset U$ but whose "parent" cube is not contained in U. Note that $\{\mathrm{int(S)} : S \in \mathcal{S}_U\}$ are disjoint

Proposition 10.5. *If the Minkowski content exists, then with probability one the measure* $\mu = \lim_{\epsilon \downarrow 0} \mu_\epsilon$ *exists. Moreover, if* U *is an open set written as in* (70),

$$\mu(U) = \sum_{S \in \mathcal{S}_U} J(S). \tag{71}$$

Proof. We will consider the event of probability one on which (67) and (68) holds for all $S \in \mathcal{S}$. By choosing $S \in \mathcal{S}$ containing K in its interior, we can see that with probability one, $\{\mu_\epsilon\}$ is a collection of measures whose support is contained in S with $\mu_\epsilon(S) \to J(S)$. Hence the collection is precompact and every subsequence contains a convergent subsequence. To prove that the limit exists, it suffices to show that every subsequential limit is the same. Since these are finite Borel measures, it suffices to show that all subsequential limits agree on every open set U. Hence, it suffices to show that any subsequential limit satisfies (71).

Let μ be a subsequential limit. We claim that $\mu(\partial S) = 0$ for all $S \in \mathcal{S}$. To see this we use Lemma 10.3. Suppose $S \in \mathcal{S}_m$ and let S^n be defined as in that lemma. Then for all n sufficiently large, $J(S^n) \leq 2^{-dm}\, 2^{(m-n)/2}$. If we now fix n (so that there are only a finite number of cubes from $\mathcal{S}_n$ in S^n), we can see that there exists ϵ' such that for $\epsilon < \epsilon'$, $\mu_\epsilon(S') = J_\epsilon(S') \leq 2 \cdot 2^{-dm}\, 2^{(m-n)/2}$. From this we conclude that $\mu(\partial S) \leq 2 \cdot 2^{-dm}\, 2^{(m-n)/2}$. If we now let $n \to \infty$, we get $\mu(\partial S) = 0$. With the claim, we now see that

$$\mu(U) = \mu\left[\bigcup_{S \in \mathcal{S}_U} \operatorname{int}(S)\right] = \sum_{S \in \mathcal{S}_U} \mu(\operatorname{int}(S)) = \sum_{S \in \mathcal{S}_U} \mu(S) = \sum_{S \in \mathcal{S}_U} J(S).$$

□