Fast convergence to an invariant measure for non-intersecting 3-dimensional Brownian paths

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Abstract

We consider pairs of 3-dimensional Brownian paths, started at the origin and conditioned to have no intersections after time zero. We show that there exists a unique measure on pairs of paths that is invariant under this conditioning, while improving the rate of convergence to stationarity from [7].

1 Introduction

Suppose B_t^1, B_t^2 are independent Brownian motions taking values in $\mathbb{R}^3$, starting at different points. It is well known [4] that

$$\lim_{t \to \infty} \mathbf{P}\{B^1[0,t] \cap B^2[0,t] = \emptyset\} = 0,$$

and, from this, one can conclude that the paths of the Brownian motions almost surely have double points. Using a subadditivity argument [2, 3], one can show that there exists a ξ, called the *(3-dimensional) Brownian intersection exponent*, such that

$$\mathbf{P}\{B^1[0,t^2] \cap B^2[0,t^2] = \emptyset\} \approx t^{-\xi}, \quad t \to \infty,$$

where $\approx$ indicates that the logarithms of both sides are asymptotic. The value of ξ is not known exactly. Rigorous estimates [2, 5] show that $.5 < \xi < 1$ and previous numerical simulations [3] suggest a value of approximately .58. If we define the set of cut points for B^1 to be

$$\{B_s^1 : B^1[0,s) \cap B^1[s,\infty) = \emptyset\},$$

then it was proved in [5] that, with probability one, the Hausdorff dimension of the set of cut points is $2 - \xi$.

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To understand the behavior of a Brownian path B near a typical cut point, one is led to study the distribution of B_t, when $0 \le t \le 2$, given that B_1 is a cut point. This conditioning is on an event of probability zero, and in order to make this conditioning precise, one needs to take a limit, e.g, one can condition on $B[0, 1-\epsilon] \cap B[1+\epsilon, 2] = \emptyset$ and then take the limit as $\epsilon \to 0$. Equivalently, by translating so that B_1 is the origin and using B^1, B^2 to denote the "past" and the "future" of the walk, we can consider the measure on pairs of paths (B_t^1, B_t^2), when $0 \le t \le 1$, conditioned so that $B_t^1[\epsilon, 1] \cap B_t^2[\epsilon, 1] = \emptyset$. A similar limit, where ϵ is replaced with the first visit to the sphere of radius ϵ, was studied in [7] for dimensions 2 and 3 and [6] for dimension 2, where it was shown that there exists a unique limit distribution which can be considered an invariant (or, as sometimes called, quasi-invariant) measure for the nonintersecting paths.

In this paper, we will reprove the result in [7], making an important improvement in the rate of convergence to the invariant measure. More precisely, our proof gives an exponential rate of convergence. The reason for establishing this result is not just to make an improvement of a result in the literature. We hope to extend these ideas to the more general intersection exponents. See Section 6 for a discussion of some goals for this program of research. The final section summarizes the results of some simulations we have done for the exponent.

2 Main result

2.1 Preliminaries

Throughout this paper, B_t, B_t^1, B_t^2 will denote standard Brownian motions taking values in $\mathbb{R}^3$. We write elements of $\mathbb{R}^3$ as $w, w_1, w_2, \ldots$ and we use $\mathbf{w} = (w_1, w_2)$ for ordered pairs of points in $\mathbb{R}^3$. Let $\mathcal{B}_n$ denote the open ball of radius e^n about the origin and let $\mathcal{B} = \mathcal{B}_0$. Although the notation n suggests integer values, unless specified otherwise, n can take on real values. We write $\partial\mathcal{B}^2$ for $(\partial\mathcal{B})^2$. Let

$$T_n = \inf\{t : B_t \in \partial\mathcal{B}_n\},$$

and define T_n^1, T_n^2 similarly.

We state, without proof, some standard facts about Brownian motion.

Lemma 2.1 (Gambler's ruin estimate). *Let $V_a = \{(x, y, z) \in \mathbb{R}^3 : x = a\}$ and suppose $w = (1, y, z)$. For $n \ge 1$, let τ_n be the first time t that a Brownian motion B_t visits $V_0 \cup V_n$. Then*

$$\mathbf{P}^w\{B_{\tau_n} \in V_n\} = 1/n.$$

Lemma 2.2 (Harnack inequality). *If $U \subset \mathbb{R}^3$ is open and connected and $K \subset U$ is compact, then there exists $c = c(K, U) < \infty$ such that if $f : U \to (0, \infty)$ is harmonic, then $f(w_1) \le c\, f(w_2)$ for all $w_1, w_2 \in K$.*

Lemma 2.3. *If $w \in \partial\mathcal{B}$ and $k > 0$, then*

$$\mathbf{P}^w\{B[0,\infty) \cap \partial\mathcal{B}_{-k} \neq \emptyset\} = e^{-k}. \tag{1}$$

Lemma 2.4 (Cone estimate)**.** *Suppose U is a (relatively) open subset of $\partial\mathcal{B}$ containing $w = (1,0,0)$. Let O denote the corresponding cone*

$$O = \{rw : r > 0, w \in U\}.$$

Then there exist $0 < c, \alpha < \infty$, depending on U, such that for all positive integers n

$$\mathbf{P}^w\{B[0,T_n] \subset O\} \geq ce^{-n\alpha}. \tag{2}$$

Remark One can further show that

$$\mathbf{P}^w\{B[0,T_n] \subset O\} \asymp e^{-\alpha n}$$

for some $\alpha < \infty$, where $\asymp$ means "within multiplicative constants of". One way to do this is to follow an argument similar to (but easier than) the argument in this paper. See [8]. We will not need this stronger result.

If B_t is started at $|w| < 1$, then the density of B_{T_0} with respect to surface measure is given by the Poisson kernel

$$H(w,z) = c\,\frac{1-|w|^2}{|w-z|^3}, \quad |w| < 1, |z| = 1.$$

Using this, we easily conclude the following.

Lemma 2.5. *There exists $c < \infty$ such that if $r \leq 1$ and $|w_1|, |w_2| \leq r$, then we can define standard Brownian motions B_t^1, B_t^2 on the same probability space such that $B_0^1 = w_1, B_0^2 = w_2$ and*

$$\mathbf{P}\left\{B^1_{T_0^1} = B^2_{T_0^2}\right\} \geq 1 - c\,r.$$

Slightly more generally, using maximal coupling (see [14]), we have the following result.

Lemma 2.6 (Coupling)**.** *There exists $c < \infty$ such that the following holds. Suppose $w_1, w_2 \in \partial\mathcal{B}$. Then we can find a probability space on which we can define B_t^1, B_t^2, Brownian motions with $B_0^j = w_j$, such that for all $n \geq 0$,*

$$\mathbf{P}\left\{B^1_{t+T_n^1} = B^2_{t+T_n^2} \text{ for all } t \geq 0\right\} \geq 1 - c\,e^{-n}.$$

2.2 Intersection exponent

Suppose B_t^1, B_t^2 are independent Brownian motions. Let A_n denote the event that the paths do not intersect before reaching $\partial\mathcal{B}_n$,

$$A_n = \{B^1[0, T_n^1] \cap B^2[0, T_n^2] = \emptyset\}.$$

More generally, if K_1, K_2 are closed subsets of $\mathbb{R}^3$, let

$$A_n(K_1, K_2) = \{(B^1[0, T_n^1] \cup K_1) \cap (B^2[0, T_n^2] \cup K_2) = \emptyset \text{ or } \{0\}\}.$$

This event is trivial unless $K_1 \cap K_2 = \emptyset$ or $\{0\}$. Let $\mathcal{F}_n$ denote the σ-algebra generated by

$$\left\{B_s^1, B_t^2 : 0 \leq s \leq T_n^1, 0 \leq t \leq T_n^2\right\}.$$

We use $\mathbf{P}^{(w_1,w_2)}$ to denote probabilities assuming $B_0^1 = w_1, B_0^2 = w_2$; if the $\mathbf{w}$ does not appear, then the implicit assumption is $\mathbf{w} = (0, 0)$.

If $\mathbf{w} \in (w_1, w_2) \in \overline{\mathcal{B}}^2$, let

$$q_n(\mathbf{w}) = q_n(w_1, w_2) = \mathbf{P}^{\mathbf{w}}(A_n).$$

If $n \geq 0$, let

$$\overline{q}_n = \sup_{\mathbf{w}\in\partial\mathcal{B}^2} q_n(\mathbf{w}) = \sup_{\mathbf{w}\in\mathcal{B}^2} q_n(\mathbf{w}).$$

We conjecture that the supremum is taken on if $w_2 = -w_1$, but this has not been proved. However, it is not difficult to show that for fixed n, $q_n(\mathbf{w})$ is continuous in $\mathbf{w}$ and hence there exists $\mathbf{w} = \mathbf{w}(n) \in \partial\mathcal{B}^2$ at which the supremum is attained. Let q_n denote the probability assuming that the starting points are chosen uniformly and independently on $\partial\mathcal{B}$,

$$q_n = \mathbf{P}\{B^1[T_0^1, T_n^1] \cap B^2[T_0^2, T_n^2] = \emptyset\} = \int_{\partial\mathcal{B}} q_n(w_1, w_2)\, ds(w_2).$$

Here w_1 is any point on $\partial\mathcal{B}$ and s denotes surface measure on $\partial\mathcal{B}$ normalized to have total mass one. Rotational invariance implies that this quantity does not depend on the choice of w_1.

If $0 \leq m \leq n$, let

$$A_{m,n} = \{B^1[T_m^1, T_n^1] \cap B^2[T_m^2, T_n^2] = \emptyset\}.$$

The strong Markov property and Brownian scaling imply

$$q_{m+n}(\mathbf{w}) = \mathbf{P}^{\mathbf{w}}(A_{m+n}) \leq \mathbf{P}^{\mathbf{w}}(A_m \cap A_{m,m+n}) = \mathbf{P}^{\mathbf{w}}(A_m)\, \mathbf{P}^{\mathbf{w}}(A_{m,m+n} \mid A_m) \leq q_m(\mathbf{w})\, \overline{q}_n. \quad (3)$$

In particular, $\overline{q}_{m+n} \leq \overline{q}_m\, \overline{q}_n$. From the subadditivity of $\log \overline{q}_n$, we see that there exists $\xi > 0$ such that

$$\overline{q}_n \approx e^{-n\xi}, \quad \overline{q}_n \geq e^{-n\xi},$$

where $\approx$ means that the logarithms of both sides are asymptotic. Using Lemma 2.1, it is easy to check that $\xi \leq 2$. In fact, it can be shown that $1/2 < \xi < 1$, but we will not need this estimate in this paper. While the exact value of ξ is not known, simulations point to a value close to .57, as we shall see in Appendix 7.

Using the Harnack inequality one can see that there is a $c < \infty$ such that for all $\mathbf{w} \in \overline{\mathcal{B}}^2$,

$$q_{n+1}(\mathbf{w}) \leq \mathbf{P}^{\mathbf{w}}(A_{1,n+1}) \leq c\, \mathbf{P}^{\mathbf{0}}(A_{1,n+1}) = c\, q_n,$$

and hence

$$\overline{q}_{n+1} \leq c\, q_n. \tag{4}$$

The first major step in establishing the existence of the invariant measure is to prove that $\overline{q}_n \asymp e^{-n\xi}$, meaning $\overline{q}_n$ is within multiplicative constants of $e^{-n\xi}$. Note that this immediately implies $q_n \asymp e^{-n\xi}$.

Proposition 2.7. *There exists $c_* < \infty$ such that*

$$e^{-n\xi} \leq \overline{q}_n \leq c_*\, e^{-n\xi}. \tag{5}$$

Proof. Although this was essentially proved in [5], we give the proof in Section 3. We start by remarking that the second inequality follows from the super-multiplicativity inequality

$$\overline{q}_n\, \overline{q}_m \leq c\, \overline{q}_{n+m}, \tag{6}$$

which is what we will prove. $\square$

2.3 Markov process on path space

If B_t is a standard Brownian motion starting at the origin, then the path B_t, for $0 \leq t \leq T_n$, can be scaled to give a continuous path from 0 to ∂B. This gives a Markov process indexed by n on the path space. This process is not ergodic in a strict sense, since one never completely forgets the beginning of the path. However, if we only look at the path from the first time it reaches ∂B_{-k} to the first time it reaches ∂B, then it is ergodic. We set up the appropriate notation in this subsection.

Let $\mathcal{C}$ denote the set of continuous paths $\gamma : [0, t_\gamma] \to \overline{\mathcal{B}}$ with $\gamma(0) = 0, |\gamma(t_\gamma)| = 1$ and $0 < |\gamma(s)| < 1$ for $0 < s < t_\gamma$. If $\gamma \in \mathcal{C}$, for $k \geq 0$, let

$$s_k = s_k(\gamma) = \inf \left\{ t : |\gamma(t)| = e^{-k} \right\}$$

be the first visit of γ to $\mathcal{B}_{-k}$ and let $\pi_k\gamma$ denote the curve starting at $\gamma(s_k)$,

$$\pi_k\gamma : [0, t_\gamma - s_k] \to \overline{\mathcal{B}}, \quad \pi_k\gamma(t) = \gamma(t + s_k).$$

If $\gamma, \gamma' \in \mathcal{C}$, we write $\gamma =_k \gamma'$ if $\pi_k\gamma = \pi_k\gamma'$. We sometimes write just γ for the set $\gamma[0, t_\gamma]$.

If $\gamma \in \mathcal{C}$, we can consider a Brownian motion starting at $\gamma(t_\gamma)$ as a process in $\mathcal{C}$ with initial condition γ. To be more specific, let B be a Brownian motion starting at $\gamma(t_\gamma)$. For

$n \geq 1$, define $\tilde{\gamma}_n$ to be the path obtained by attaching the Brownian motion, stopped when it first reaches $\partial\mathcal{B}_n$. In other words, the path $\tilde{\gamma}_n$ has time duration $t_\gamma + T_n$ and

$$\tilde{\gamma}_n(t) = \begin{cases} \gamma(t), & 0 \leq t \leq t_\gamma \\ B_{t-t_\gamma}, & t_\gamma \leq t \leq t_\gamma + T_n. \end{cases}$$

Let γ_n be the curve in $\mathcal{C}$ obtained from $\tilde{\gamma}_n$ by Brownian scaling:

$$\gamma_n(t) = e^{-n}\,\tilde{\gamma}_n(te^{2n}), \quad 0 \leq t \leq e^{-2n}\,[t_\gamma + T_n].$$

Observe that the path γ_n is not continuous in n. For our purposes, we will only need to consider the process for integer times n.

Let $\mathcal{X}$ denote the set of ordered pairs $\overline{\gamma} = (\gamma^1, \gamma^2) \in \mathcal{C} \times \mathcal{C}$ with $\gamma^1 \cap \gamma^2 = \{0\}$. We write $\pi_k\overline{\gamma} = (\pi_k\gamma^1, \pi_k\gamma^2)$ and $\overline{\gamma} =_k \overline{\gamma}'$ if $\pi_k\overline{\gamma} = \pi_k\overline{\gamma}'$.

Suppose $\overline{\gamma} = (\gamma^1, \gamma^2) \in \mathcal{X}$ with endpoint $(w_1, w_2) \in \partial\mathcal{B}^2$. Let B^1, B^2 be independent Brownian motions starting at w_1, and w_2 respectively. Define γ_n^j as above, by attaching to γ^j the Brownian motion B^j stopped at $\partial\mathcal{B}_n$ and then scaling. Let $\overline{\gamma}_n = (\gamma_n^1, \gamma_n^2)$. Note that $\overline{\gamma}_n \in \mathcal{C} \times \mathcal{C}$, but it is possible that $\overline{\gamma}_n \notin \mathcal{X}$. If $\overline{\gamma}_n \notin \mathcal{X}$, then $\overline{\gamma}_m \notin \mathcal{X}$ for all $m \geq n$. Let $A_n(\overline{\gamma})$ denote the event $A_n(\gamma^1, \gamma^2)$ as in the previous section and note that we can write

$$A_n(\overline{\gamma}) = \left\{\gamma_n^1 \cap \gamma_n^2 = \{0\}\right\} = \{\overline{\gamma}_n \in \mathcal{X}\}.$$

Let

$$q_n(\overline{\gamma}) = \mathbf{P}\left[A_n(\overline{\gamma})\right].$$

Note that for every $w_1, w_2 \in \partial\mathcal{B}_1$,

$$q_n(w_1, w_2) = \sup q_n(\overline{\gamma}), \tag{7}$$

where the supremum on the right is over all $\overline{\gamma} = (\overline{\gamma}_1, \overline{\gamma}_2) \in \mathcal{X}$ whose terminal points are w_1, w_2, respectively. Indeed, it is clear from the definition that $q_n(\overline{\gamma}) \leq q_n(w_1, w_2)$ for each such $\overline{\gamma}$, and if we choose the curves to be straight lines from 0 to w_1, w_2, respectively, then $q_n(\overline{\gamma}) = q_n(w_1, w_2)$. Here we use the fact that Brownian motions in $\mathbb{R}^3$ do not hit lines. Similarly,

$$\overline{q}_n = \sup_{\overline{\gamma} \in \mathcal{X}} q_n(\overline{\gamma}). \tag{8}$$

Let $\mathcal{W}$ denote the Wiener measure on $\mathcal{C} \times \mathcal{C}$, that is to say the measure induced by taking two independent Brownian motions and stopping them when they reach $\partial\mathcal{B}$. More generally, if $\overline{\gamma} \in \mathcal{X} \times \mathcal{X}$, let $\mathcal{W}_n(\overline{\gamma})$ denote the probability measure induced by $\overline{\gamma}_n$ as above. If μ is a probability measure on $\mathcal{C} \times \mathcal{C}$, let $\pi_k\mu$ denote the measure generated from μ by the projection $\overline{\gamma} \mapsto \pi_k\overline{\gamma}$. Note that if $k < n$, then $\pi_k\mathcal{W}_n(\overline{\gamma})$ is mutually absolutely continuous with respect to $\pi_k\mathcal{W}$.

2.4 Results

Our main result discusses a measure on $\mathcal{X}$. In order to avoid talking about general measures, let us restrict to a family of measures, that we will call $\mathcal{W}$-probability measures on $\mathcal{X}$. We say that ν is a $\mathcal{W}$-probability measure on $\mathcal{X} \subset \mathcal{C} \times \mathcal{C}$ if, for each $0 \leq k < \infty$, $\pi_k \nu$ is absolutely continuous with respect to $\pi_k \mathcal{W}$. In order to specify such a probability measure, it suffices to specify the measures $\{\pi_k \nu\}$ and to show that the curves have finite time duration. To show the latter we need to show that the time durations under the measures $\pi_k \nu$ are tight.

If $\overline{\gamma} \in \mathcal{X}$, let $\mu_n(\overline{\gamma})$ denote the probability measure on $\mathcal{X}$ obtained as the distribution of $\overline{\gamma}_n$, given the event $A_n(\overline{\gamma})$. Note that $\mu_n(\overline{\gamma})$ is absolutely continuous with respect to $\mathcal{W}_n(\overline{\gamma})$.

Theorem 2.8. *There exists a $\mathcal{W}$-probability measure ν on $\mathcal{X}$, a function $Q : \mathcal{X} \to (0, \infty)$, and constants $\beta > 0, c < \infty$ such that if $\overline{\gamma} \in \mathcal{X}$ and $n \geq 1$.*

$$|e^{\xi n}\, q_n(\overline{\gamma}) - Q(\overline{\gamma})| \leq c\, e^{-\beta\, n},$$

$$\|\pi_{n/2} \mu_n(\overline{\gamma}_n) - \pi_{n/2}\, \nu\| \leq c\, e^{-\beta n},$$

where $\| \cdot \|$ denotes variation distance.

The proof uses a coupling argument and the main work is to prove the following.

Theorem 2.9. *There exist constants $\beta > 0, c < \infty$ such that if $\overline{\gamma}, \overline{\gamma}' \in \mathcal{X}$ and $n \geq 1$,*

$$\|\pi_{n/2} \mu_n(\overline{\gamma}_n) - \pi_{n/2} \mu_n(\overline{\gamma}'_n)\| \leq c\, e^{-\beta n}.$$

The rest of the paper is organized as follows. In Section 3, we prove Proposition 2.7. The coupling result (Theorem 2.9) is proved in Section 4 and convergence to an invariant measure and the proof of Theorem 2.8 are done in Section 5.

3 Up-to-constants estimates

3.1 Separation lemma

The key technical lemma that allows the argument to work is the separation lemma. The statement is very believable — two paths that are conditioned not to intersect are likely to be not very close at their endpoints. The separation lemma gives a stronger statement that, no matter how close paths are at when then reach $\partial \mathcal{B}_n$, those that reach $\partial \mathcal{B}_{n+1}$ have a good chance of having separated. More precisely, it asserts that there is a uniform estimate for the conditional probability of separation of the paths at times (T^1_{n+1}, T^2_{n+1}), uniform over all possible configurations up to time (T^1_n, T^2_n). It is an analogue of the boundary Harnack principle.

There are many ways to define the "separation" event; we will make one arbitrary choice. Let

$$I(r) = \{(x, y, z) \in \mathbb{R}^3 : x \geq e^r\},$$

and let Sep denote the set of $\overline{\gamma} = (\gamma^1, \gamma^2) \in \mathcal{X}$ such that for all $0 \leq r \leq 1/2$,

$$\gamma^1[s_r^1, t^1] \subset I\left(-r - \frac{1}{8}\right), \qquad \gamma^2[s_r^2, t^2] \subset -I\left(-r - \frac{1}{8}\right),$$

$$\gamma^1(s_r^1) \in I\left(-r - \frac{1}{16}\right), \qquad \gamma^2(s_r^2) \in -I\left(-r - \frac{1}{16}\right).$$

Here $t^j = t_{\gamma^j}$ and $s_r^j = \inf\{t : |\gamma^j(t)| = e^{-r}\}$. Let J_n denote the event

$$J_n = \{\overline{\gamma}_n \in \mathsf{Sep}\}.$$

A typical pair $\overline{\gamma} \in \mathsf{Sep}$ is pictured below, viewed as projected on the xz-plane. The inner and outer balls have radii $e^{-1/2}$ and 1, respectively.

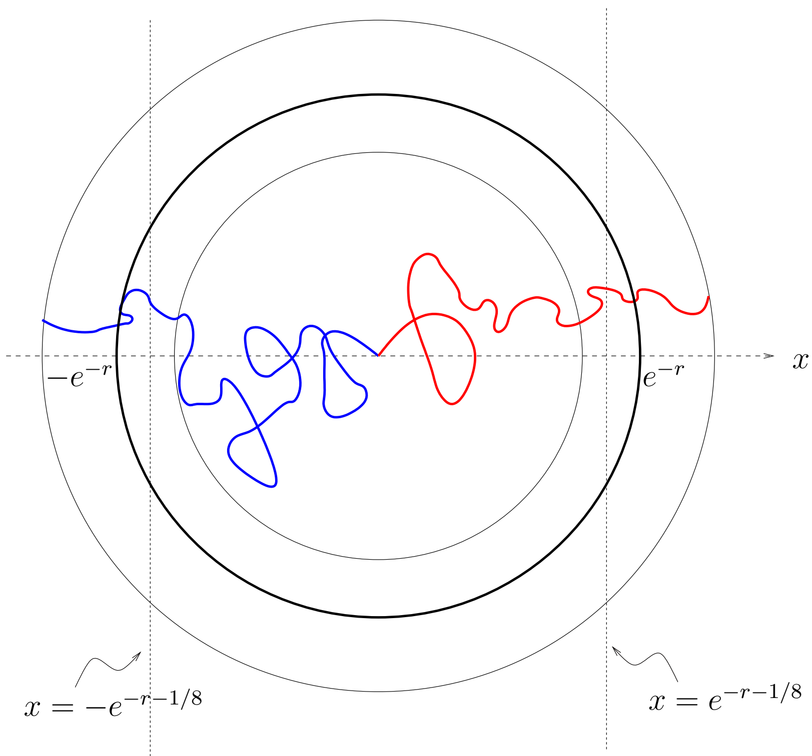

Figure 1: A separation event.

Lemma 3.1 (Separation lemma)**.** *There exists $\rho_1 > 0$ such that if $\overline{\gamma} \in \mathcal{X}$ and $n \geq 1$,*

$$\mathbf{P}\left\{\overline{\gamma}_n \in \mathsf{Sep} \mid A_n(\overline{\gamma})\right\} \geq \rho_1. \tag{9}$$

We first note that it suffices to prove (9) for $n = 1$; the general case can be deduced by applying this case to $\overline{\gamma}_{n-1}$. More generally, we can see that for all $n \geq 1$,

$$\mathbf{P}\left[A_n(\overline{\gamma}) \cap \{\overline{\gamma}_n \in \mathsf{Sep}\} | \mathcal{F}_{n-1}\right] \geq \rho_1 \, \mathbf{P}\left[A_n(\overline{\gamma}) | \mathcal{F}_{n-1}\right].$$

We will prove this slightly stronger form of the lemma for $n = 1$.

Lemma 3.2 (Separation lemma, alternative form)**.** *There exists $\rho_1 > 0$ such that if K_1, K_2 are closed subsets of $\overline{\mathcal{B}}$ and $\mathbf{w} = (w_1, w_2) \in \partial\mathcal{B}^2$ with $K_j \cap \partial\mathcal{B} = \{w_j\}$, then*

$$\mathbf{P}^{\mathbf{w}}(A_1(K_1, K_2) \cap J_1) \geq \rho_1 \, \mathbf{P}^{\mathbf{w}}(A_1(K_1, K_2)).$$

Proof. Let

$$D = D(K_1, K_2, w_1, w_2) = \min\left\{\operatorname{dist}(w_1, K_2), \operatorname{dist}(w_2, K_1)\right\}.$$

Let

$$u_n = \sum_{j=n}^{\infty} j^2 \, 2^{-j}.$$

Let $J(r_1, r_2)$ be the event that the following facts hold for $r_1 \leq s \leq r_2$:

$$B^1[T_s^1, T_{r_2}^1] \subset I\left(s - \frac{1}{8}\right), \qquad B^2[T_s^2, T_{r_2}^2] \subset -I\left(s - \frac{1}{8}\right),$$

$$B^1(T_s^1) \in I\left(s - \frac{1}{16}\right), \qquad B^2(T_s^2) \in -I\left(s - \frac{1}{16}\right).$$

Using this notation, we observe that $J_1 = J(1/2, 1)$.

For n sufficiently large so that $u_n \leq 1/4$, let h_n be the infimum of

$$\frac{\mathbf{P}^{\mathbf{w}}(A_{1-r}(K_1, K_2) \cap J(\frac{1}{2} - r, 1 - r))}{\mathbf{P}^{\mathbf{w}}(A_{1-r}(K_1, K_2))},$$

where the infimum is over $0 \leq r \leq u_n$; all closed K_1, K_2 in $\overline{\mathcal{B}}$; and all $\mathbf{w} = (w_1, w_2) \in \partial\mathcal{B}^2$ such that $D(K_1, K_2, w_1, w_2) \geq 2^{-n}$. The lemma will follow if we prove that $\inf_n h_n > 0$ and then letting $n \to \infty$. For this, it suffices to show that $h_n > 0$, for each n, and that there exists a summable sequence $\delta_n < 1$ such that

$$h_{n+1} \geq h_n \, [1 - \delta_n]. \tag{10}$$

We claim that there exist c_1, α such that for all K_1, K_2, w_1, w_2 as above,

$$\mathbf{P}^{\mathbf{w}}(A_2(K_1, K_2) \cap J(1/4, 5/4)) \geq c_1 \, D^{\alpha}. \tag{11}$$

To see this, we find infinite cones O_1, O_2 as in Lemma 2.4 and vertices z_1, z_2 such that the following hold:

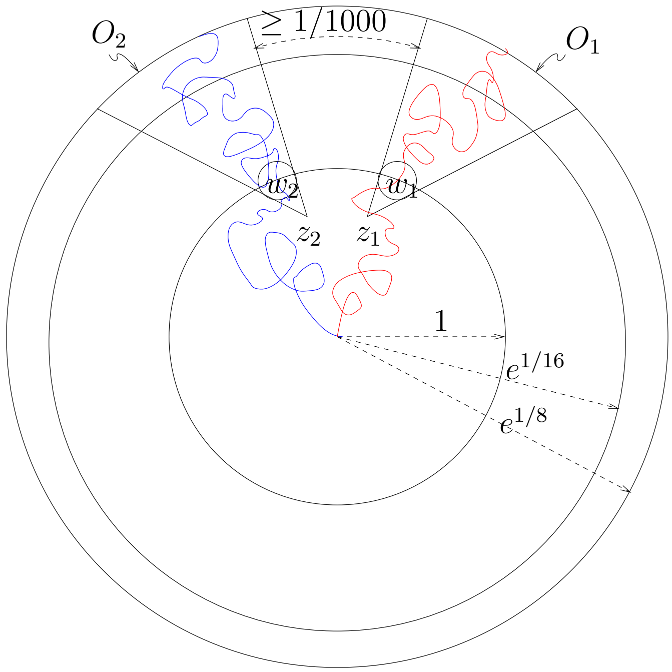

Figure 2: Separation into cones

- $D/100 < |z_j - w_j| < D/20$.
- $w_j \in O_j + z_j$ and $D/100 < \mathrm{dist}(w_j, \partial O_j) < D/20$.
- The intersection of $O_j + z_j$ with $\overline{B}$ is contained in the ball of radius $D/10$ about w_j.
- If $V_j = (O_j + z_j) \cap (\mathbb{R}^3 \setminus \mathcal{B}_{1/16})$, then $\mathrm{dist}(V_1, V_2) \geq 1/1000$.

We leave it to the reader to see that such cones can be found. Moreover, we can choose the same O_1, O_2, up to a rotation, for each value of D. Given this, Lemma 2.4 and Brownian scaling imply that there exist c, α such that with probability at least $c\, D^{-\alpha}$, $B[0, T^j_{1/8}] \subset O_j + z_j$ for $j = 1, 2$. Note that on this event the paths do not intersect and are somewhat "separated". It is not hard to convince oneself that given this event, there is a positive probability that the extended paths do not have an intersection and are in $J(1/4, 5/4)$. This establishes (11), and from this we see that $h_n > 0$ for each n with $u_n \leq 1/4$. Furthermore, from (11), we get that for all n with $u_n \leq 1/4$,

$$h_n \geq c_1 2^{-n\alpha}.$$

Let

$$K_j(s) = e^{-s}\, \left(K_j \cup B^j[0, T^j_s]\right),$$

$$D_s = D\left(K_1(s), K_2(s), e^{-s}\, B^1(T^1_s), e^{-s}\, B^2(T^2_s)\right),$$

$$\tilde{\tau}_n = \min\left\{s : D_s \geq 2^{-n}\right\}, \quad \tau_n = (n^2 2^{-n}) \wedge \tilde{\tau}_n.$$

It is easy to see that there is a $p > 0$ such that given $\mathcal{F}_0$, the probability that $D_{4 \cdot 2^{-n}} \geq 2^{-n+1}$ is at least p. Iterating this, we see that there exists c_2, β such that

$$\mathbf{P}\{\tau_n = n^2\, 2^{-n}\} \leq c_2\, 2^{-\beta\, n^2}. \tag{12}$$

Assume $0 \leq r \leq u_{n+1}$ and hence $0 \leq r + \tau_n \leq u_n$. On the event $\{\tau_n < n^2 2^{-n}\}$, we have $D_{\tau_n} \geq 2^{-n}$ and using the definition of h_n, we get

$$\begin{aligned} \mathbf{P}\left(A_{1-r} \cap J(\frac{1}{2} - r, 1-r)\right) &\geq \mathbf{P}\left(A_{1-r} \cap J(\frac{1}{2} - r, 1-r); \tau_n < n^2\, 2^{-n}\right) \\ &\geq h_n\, \mathbf{P}(A_{1-r}; \tau_n < n^2\, 2^{-n}). \end{aligned}$$

However, (11) and (12) imply that

$$\mathbf{P}(A_{1-r}; \tau_n < n^2\, 2^{-n}) \geq \mathbf{P}(A_{1-r}) - c_2\, 2^{-\beta n^2} \geq \mathbf{P}(A_{1-r}) \left[1 - \frac{c_2}{c_1}\, 2^{n\alpha - n^2\beta}\right],$$

from which (10) follows with $\delta_n = c_2/c_1\, 2^{n\alpha - n^2\beta}$.

$\square$

The lemma implies that there exists $\rho_2 > 0$ such that for all $n \geq 0$,

$$\overline{q}_{n+1} \geq \rho_2\, \overline{q}_n. \tag{13}$$

Indeed, it is not difficult to see that there exists $c > 0$ such that

$$\mathbf{P}(A_{n+1} \mid A_n, \overline{\gamma}_n \in \mathsf{Sep}) \geq c,$$

which together with Lemma 3.1 establish (13) for $n \geq 1$. It is also easy to see that $\overline{q}_1 \geq \tilde{c}\, \overline{q}_0$.

Remark A similar argument as above can prove boundary Harnack inequalities for many domains. The basic idea is that if a process is distance 2^{-n} from the boundary then, except for an event of small probability, in a short amount of time it must either hit the boundary or increase its distance to 2^{-n+1}. (This requires some assumptions about the boundary.) It is important that we have assumed that K_1, K_2 are subsets of $\overline{\mathcal{B}}$ and that $w_1, w_2 \in \partial\mathcal{B}$. This guarantees that the paths with $D = 2^{-n}$ have a positive probability of separating to $D = 2^{-n+1}$, without intersecting by the time they reach radius $1 + O(2^{-n})$.

3.2 Proof of Proposition 2.7

The separation lemma was the hard work. The results in this subsection are not as difficult. The main goal is to prove the following lemma.

Lemma 3.3. *There exists $\rho_3 > 0$ such that if $\overline{\gamma} \in \mathsf{Sep}$ and $m \geq 0$,*

$$q_m(\overline{\gamma}) \geq \rho_3 \, \overline{q}_m.$$

By combining Lemmas 3.1 and 3.3, we see that for all $n \geq 0$, $m \geq 0$,

$$\overline{q}_{n+m} \geq \overline{q}_{n+1+m} \geq \rho_1 \, \rho_3 \, \overline{q}_{n+1} \, \overline{q}_m \geq \rho_1 \, \rho_2 \, \rho_3 \, \overline{q}_n \, \overline{q}_m.$$

Hence this establishes (5). By combining the lemma with (5) and (9) we get the following corollary.

Corollary 3.4. *If $\overline{\gamma} \in \mathcal{X}$ and $m \geq 1$,*

$$\rho_1 \, \rho_3 \, q_1(\overline{\gamma}) \, e^{-\xi(m-1)} \leq q_m(\overline{\gamma}) \leq c_* \, q_1(\overline{\gamma}) \, e^{-\xi(m-1)} \tag{14}$$

We now proceed with the proof of Lemma 3.3. Recall (7) and (8).

Lemma 3.5. *There exists $C_3 < \infty$ such that if $w_1, w_2 \in \partial\mathcal{B}$ and $n \geq 1$,*

$$q_n(w_1, w_2) \leq C_3 \, |w_1 - w_2|^{\xi/2} \, \overline{q}_n.$$

In particular, there exists $C_4 > 0$ such that for each n, there exists $\mathbf{w} = (w_1, w_2) \in \overline{\mathcal{B}}^2$ with $|w_1 - w_2| \geq C_4$ and

$$q_n(w_1, w_2) = \overline{q}_n.$$

Proof. If $|w_1 - w_2| \geq 1$, the inequality follows trivially. So let us write $|w_1 - w_2| = e^{-s}$. Using (13),

$$q_n(w_1, w_2) \leq q_1(w_1, w_2) \, \overline{q}_{n-1} \leq \rho_2^{-1} \, q_1(w_1, w_2) \, \overline{q}_n.$$

Since the ball of radius 1 about w_1 is contained in $\mathcal{B}_1$, we can see by scaling that

$$q_1(w_1, w_2) \leq \overline{q}_s \leq c \, e^{-s\xi/2} = c \, |w_1 - w_2|^{\xi/2},$$

where the second inequality follows from the relation $\overline{q}_n \approx e^{-n\xi}$. To prove the last assertion in the lemma, choose C_4 such that it satisfies $C_3 \, C_4^{\xi/2} < 1$ and note that existence of a pair $(w_1, w_2) \in \partial\mathcal{B}^2$ which maximizes q_n was already proved in the introduction. □

Lemma 3.6. *Let E_n^j be the event $\{B^j[0, T_n^j] \cap \overline{\mathcal{B}}_{-1} = \emptyset\}$ and $E_n = E_n^1 \cap E_n^2$. Then for every n, there exists $\mathbf{w} = (w_1, w_2) \in \partial\mathcal{B}^2$ with $|w_1 - w_2| \geq C_4$ and*

$$\mathbf{P}^{\mathbf{w}}(A_n \cap E_n) \geq (1 - 2e^{-1}) \, \overline{q}_n.$$

Proof. Choose (w_1, w_2) with $|w_1 - w_2| \geq C_4$ and $q_n(w_1, w_2) = \overline{q}_n$ as in Lemma 3.5. Using (1), we see that if $w_j \in \partial\mathcal{B}$,

$$\mathbf{P}^{w_j}[(E_n^j)^c] \leq \mathbf{P}^{w_j}\{B^j[0, \infty) \cap \overline{\mathcal{B}}_{-1} \neq \emptyset\} = e^{-1}.$$

Let ρ be the first time that B^1 visits $\overline{\mathcal{B}}_{-1}$ and σ the first time greater than ρ that B^1 is on $\partial\mathcal{B}$. Then,

$$\mathbf{P}^{\mathbf{w}}(A_n \cap (E_n^1)^c) = \mathbf{P}^{\mathbf{w}}\{\rho < T_n^1\}\, \mathbf{P}^{\mathbf{w}}\{B^1[\sigma, T_n^1] \cap B^2[0, T_n^2] = \emptyset \mid \rho < T_n^1\} \leq e^{-1}\,\overline{q}_n.$$

The same holds for E_n^2 and hence for this choice of $\mathbf{w} = (w_1, w_2) \in \overline{\mathcal{B}}^2$,

$$\mathbf{P}^{\mathbf{w}}(A_n \cap E_n) \geq (1 - 2e^{-1})\overline{q}_n.$$

$\square$

If $w \in \partial\mathcal{B}$, let

$$L_\epsilon(w) = \left\{ z \in \mathbb{R}^3 : |z| \leq e, \quad \left| \frac{z}{|z|} - w \right| \leq \epsilon \right\}.$$

In other words, $L_\epsilon(w)$ is a cone centered around the line segment from 0 to ew. Three-dimensional Brownian motions do not hit line segments. Using this fact, the next lemma and corollary are almost immediate; we omit the proofs.

Lemma 3.7. *For every $\delta > 0$, there exists $\epsilon > 0$ such that if $\mathbf{w} = (w_1, w_2) \in \partial\mathcal{B}^2$ with $|w_1 - w_2| \geq C_4$, then*

$$\mathbf{P}^{w_2}\left\{B^2[0, \infty) \cap L_\epsilon(w_1) \neq \emptyset\right\} \leq \delta.$$

Corollary 3.8. *There exists $\epsilon_1 > 0$ such that the following is true. Let $U_n = U_{n,\epsilon_1}$ be the event that*

$$B^j[0, T_n^j] \cap L_{\epsilon_1}(B_0^{3-j}) = \emptyset, \quad j = 1, 2.$$

Then for every n, there exists $\mathbf{w} = (w_1, w_2) \in \partial\mathcal{B}^2$ with $|w_1 - w_2| \geq C_4$ such that

$$\mathbf{P}^{\mathbf{w}}(A_n \cap E_n \cap U_n) \geq \frac{1 - 2e^{-1}}{2}\,\overline{q}_n.$$

Proposition 3.9. *For every $\epsilon > 0$ there is a $c_\epsilon > 0$ such that the following is true. Suppose $\mathbf{w} = (w_1, w_2) \in \partial\mathcal{B}^2$ with $|w_1 - w_2| \geq \epsilon$. Let $\Lambda_n = \Lambda_{n,\epsilon}$ denote the event*

$$\Lambda_n = \left\{B^j[0, T_n^j] \cap \mathcal{B}_1 \subset L_\epsilon(B_0^j) \setminus \mathcal{B}_{-\epsilon}\right\}.$$

Then

$$\mathbf{P}^{\mathbf{w}}(A_n \cap \Lambda_n) \geq c_\epsilon\,\overline{q}_n.$$

Proof. We will not discuss the entire proof. First we will prove the result for $n + 4$. Start with w_1, w_2 and consider the line segments to $e^2 w_1, e^2 w_2$. Let z_1, z_2 be maximizers for n for Corollary 3.8 and take line segments from $e^2 w_1$ to $e^4 z_1$ and $e^2 w_2$ to $e^4 z_2$. (If these intersect or get very close, interchange z_1 and z_2.) We now consider the event that Brownian motions start at w_1, w_2 and follow these lines very closely until they reach $e^4 z_1, e^4 z_2$. After this we attach paths as in Corollary 3.8. We leave the details to the reader. $\square$

Proof of Lemma 3.3. Choose $\epsilon = 1/100$ (or any other sufficiently small number) in the previous proposition and note that if $\overline{\gamma} \in \mathsf{Sep}$, then $A_n \cap \Lambda_n \subset A_n(\overline{\gamma})$. We choose $\rho_3 = c_{1/100}$. $\square$

4 Proof of Theorem 2.9

It suffices to prove Theorem 2.9 for integers n. We will use upper case N rather than n for the index in the statement of the theorem. We restate the result in terms of coupling.

Theorem 4.1 (Equivalent form of Theorem 2.9)**.** *There exist* $0 < c, \beta < \infty$ *such that for all positive integers* N *and all* $\overline{\gamma}, \overline{\gamma}' \in \mathcal{X}$*, we can define* $\overline{\gamma}_N, \overline{\gamma}'_N$ *on the same probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ *such that* $\overline{\gamma}_N$ *has the distribution* $\mu_N(\overline{\gamma})$*,* $\overline{\gamma}'_N$ *has the distribution* $\mu_N(\overline{\gamma}')$*, and*

$$\mathbf{P}\left\{\overline{\gamma}_N =_{N/2} \overline{\gamma}'_N\right\} \geq 1 - ce^{-\beta N}.$$

4.1 Preliminary estimates

Let $\mathcal{W}_N(\overline{\gamma})$ denote the measure on $\mathcal{C} \times \mathcal{C}$ induced from $\overline{\gamma}$ using Wiener measure as in Section 2.3. Note that this is not a measure on $\mathcal{X}$ since it gives nonzero measure to paths $\overline{\gamma}_n = (\gamma_n^1, \gamma_n^2)$ with $\gamma_n^1 \cap \gamma_n^2 \neq \{0\}$. If $n \leq N$, let $\mu_{n,N} = \mu_{n,N}(\overline{\gamma})$ be the probability measure on $\mathcal{X}$ induced by $\overline{\gamma}_n$ conditioned on the event $A_N(\overline{\gamma})$. Note that $\mu_{n,N}$ is supported on $\mathcal{X}$ and is absolutely continuous with respect to $\mathcal{W}_N(\overline{\gamma})$ (which is essentially the same as $\mathcal{W}_n(\overline{\gamma})$ if we only view the curves up to the time they first reach $\partial \mathcal{B}_n$) with Radon-Nikodym derivative

$$\frac{d\mu_{n,N}}{d\mathcal{W}_N}(\overline{\gamma}_n) = \frac{q_{N-n}(\overline{\gamma}_n)}{q_N(\overline{\gamma})}\, 1\{\overline{\gamma}_n \in \mathcal{X}\}.$$

If we write

$$\mu_N(\overline{\gamma}_1|\overline{\gamma}) = \frac{d\mu_{1,N}}{d\mathcal{W}_N}(\overline{\gamma}_1) = \frac{q_{N-1}(\overline{\gamma}_1)}{q_N(\overline{\gamma})}\, 1\{\overline{\gamma}_1 \in \mathcal{X}\},$$

then for positive integer $n \leq N$,

$$\frac{d\mu_{n,N}}{d\mathcal{W}_N}(\overline{\gamma}_n) = 1\{\overline{\gamma}_n \in \mathcal{X}\} \prod_{j=0}^{n-1} \mu_{N-j}(\overline{\gamma}_{j+1} \mid \overline{\gamma}_j).$$

If $\overline{\gamma}$ and $\overline{\gamma}'$ have the same endpoints, then $\mathcal{W}_N(\overline{\gamma})$ is the same as $\mathcal{W}_N(\overline{\gamma}')$, and we can define $\overline{\gamma}_1, \overline{\gamma}'_1$ by attaching the same Brownian motion. If the paths $\overline{\gamma}, \overline{\gamma}'$ agree except near the origin, it is reasonable to believe that $\dfrac{\mu_N(\overline{\gamma}_1|\overline{\gamma})}{\mu_N(\overline{\gamma}'_1|\overline{\gamma}')}$ is close to 1. Although we do not know if there exists a uniform estimate that holds for all paths, there is a uniform estimate if we restrict to a good set of paths. Let

$$\mathsf{Good}_k = \{\overline{\gamma} \in \mathcal{X} : q_1(\overline{\gamma}) \geq e^{-k/2}\}.$$

Note that $\cup_k \mathsf{Good}_k = \mathcal{X}$, and (14) implies that if $n \geq 1$, then

$$q_n(\overline{\gamma}) \geq \rho_1 \rho_3\, e^{-k/2}\, e^{-(n-1)\xi}, \quad \overline{\gamma} \in \mathsf{Good}_k, \tag{15}$$

$$q_n(\overline{\gamma}) \leq c_*\, e^{-k/2}\, e^{-(n-1)\xi}, \quad \overline{\gamma} \in \mathcal{X} \setminus \mathsf{Good}_k, \tag{16}$$

Let

$$\mathsf{Nice}_{k,m} := \{\overline{\gamma} \in \mathcal{X} : \pi_m \overline{\gamma} \cap \mathcal{B}_{-k-m} = \emptyset\}.$$

In other words, $\mathsf{Nice}_{k,m}$ is the set of ordered pairs of paths that do not enter $\mathcal{B}_{-k-m}$ after the first visit to $\mathcal{B}_{-m}$. Note that if $\overline{\gamma} \in \mathsf{Nice}_{k,m}$ and $\overline{\gamma} =_m \overline{\gamma}'$, then $\overline{\gamma}' \in \mathsf{Nice}_{k,m}$. Most paths $\overline{\gamma}$ which have a positive chance of non-intersection are Nice and Good. More precisely, we have the following lemma:

Lemma 4.2. *There exists $c_0 < \infty$ such that if k, m, n are positive integers with $m \leq n$, then for all $\overline{\gamma} \in \mathsf{Good}_k$,*

$$|\mathbf{P}\left[A_n(\overline{\gamma}) \cap \{\overline{\gamma}_m \in \mathsf{Nice}_{k,m}\}\right] - q_n(\overline{\gamma})| \leq c_0\, e^{-k/2}\, q_n(\overline{\gamma}),$$

$$|\mathbf{P}\left[A_{n+1}(\overline{\gamma}) \cap \{\overline{\gamma}_m \in \mathsf{Nice}_{k,m} \cap \mathsf{Good}_k\}\right] - q_{n+1}(\overline{\gamma})| \leq c_0\, e^{-k/2}\, q_{n+1}(\overline{\gamma}).$$

Proof. Let k, m, n be given and let (B^1, B^2) denote Brownian motions starting at the endpoints of (γ^1, γ^2). Let

$$E^j_{m,k} = \{B^j[0, T^j_m] \cap \partial \mathcal{B}_{-k} = \emptyset\}, \quad E_{m,k} = E^1_{m,k} \cap E^2_{m,k}.$$

Using (1), for all $|w_j| = 1$ we have

$$\mathbf{P}^{w_j}[(E^j_{m,k})^c] \leq \mathbf{P}^{w_j}\left\{\sup_{0 \leq t < \infty} |B_t| \leq e^{-k}\right\} = e^{-k}.$$

Using the strong Markov property and (5), we can see that

$$\mathbf{P}[A_n(\overline{\gamma}) \mid (E^j_{m,k})^c] \leq \overline{q}_n \leq c_*\, e^{-n\xi}.$$

Therefore,

$$\mathbf{P}\left[A_n(\overline{\gamma}) \cap \{\overline{\gamma}_m \notin \mathsf{Nice}_{k,m}\}\right] = \mathbf{P}[A_n(\overline{\gamma}) \cap E_{m,k}] \leq 2\, e^{-k}\, c_*\, e^{-n\xi}. \tag{17}$$

Now the first inequality in the lemma follows from (15).

For the second inequality, for all $\overline{\gamma} \in \mathcal{X}$, using (14) and (16), we get

$$\begin{aligned}
\mathbf{P}\left[A_{n+1}(\overline{\gamma}) \cap \{\overline{\gamma}_m \notin \mathsf{Good}_k\}\right] &\leq \mathbf{P}[A_m(\overline{\gamma})]\, \mathbf{P}\left[A_{n+1}(\overline{\gamma}) \mid A_m(\overline{\gamma}), \overline{\gamma}_m \notin \mathsf{Good}_k\right] \\
&\leq \left[c_* q_1(\overline{\gamma}) e^{-(m-1)\xi}\right] \left[c_*\, e^{-(n-m)\xi}\, e^{-k/2}\right] \\
&\leq c\, q_1(\overline{\gamma})\, e^{-n\xi}\, e^{-k/2} \\
&\leq c'\, q_{n+1}(\overline{\gamma})\, e^{-k/2}.
\end{aligned}$$

The inequality follows from this, together with the first part of the lemma.

□

Lemma 4.3. *There exists $c_0' < \infty$ such that if n, k are positive integers, $\overline{\gamma} \in \mathsf{Good}_k, \overline{\gamma}' \in \mathcal{X}$ and $\overline{\gamma} =_k \overline{\gamma}'$, then*

$$|q_n(\overline{\gamma}) - q_n(\overline{\gamma}')| \leq c_0'\, e^{-k/2}\, q_n(\overline{\gamma}). \tag{18}$$

Proof. Using the notation of the previous lemma, we see that if $\overline{\gamma} =_k \overline{\gamma}'$ and we attach the same Brownian motions to $\overline{\gamma}$ and $\overline{\gamma}'$, and if additionally the attached Brownian motions do not enter $\mathcal{B}_{-k}$ before reaching $\partial\mathcal{B}_n$, then non-intersection probabilities for the pairs starting with $\overline{\gamma}$ and $\overline{\gamma}'$, respectively, are equal. Formally,

$$\mathbf{P}[A_n(\overline{\gamma}) \cap E_{n,k}] = \mathbf{P}[A_n(\overline{\gamma}') \cap E_{n,k}].$$

Using (17), we see that

$$|q_n(\overline{\gamma}) - q_n(\overline{\gamma}')| \leq \mathbf{P}[A_n(\overline{\gamma}) \cap (E_{n,k})^c] + \mathbf{P}[A_n(\overline{\gamma}') \cap (E_{n,k})^c] \leq c\, e^{-k}\, e^{-n\xi}. \tag{19}$$

But since $\overline{\gamma} \in \mathsf{Good}_k$, (15) implies that $q_n(\overline{\gamma}) \geq c'\, e^{-k/2}\, e^{-n\xi}$ and the lemma follows. $\square$

4.2 Coupling

Fix a large integer N and assume $\overline{\gamma}, \overline{\gamma}' \in \mathcal{X}$. In order to show that $\mu_N(\overline{\gamma})$ and $\mu_N(\overline{\gamma}')$ are close, we will define a coupling. If, for k large enough, $\overline{\gamma} =_k \overline{\gamma}'$, then the paths stay coupled with high probability, depending only on k. However, if k is not large, or even if $\overline{\gamma}$ and $\overline{\gamma}'$ do not have the same endpoints, the coupling can be started, with positive probability. We prove these facts in the next two propositions.

Proposition 4.4. *There exists C_0 such that the following holds. Suppose k, m, N are positive integers with $m \leq N$, $\overline{\gamma} \in \mathsf{Good}_k$, and $\overline{\gamma} =_k \overline{\gamma}'$. Then we can define $\overline{\gamma}_m, \overline{\gamma}'_m$ on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that $\overline{\gamma}_m$ has distribution $\mu_{m,N}(\overline{\gamma})$, $\overline{\gamma}'_m$ has distribution $\mu_{m,N}(\overline{\gamma}')$, and*

$$\mathbf{P}\left\{\overline{\gamma}_m =_{k+m} \overline{\gamma}'_m\right\} \geq 1 - C_0 e^{-k/2}.$$

Moreover, if $N \geq m+1$,

$$\mathbf{P}\left\{\overline{\gamma}_m \in \mathsf{Good}_k\right\} \geq 1 - C_0 e^{-k/2}.$$

Proof. Using maximal coupling (see [14]), the estimate on the coupling rate follows directly from estimates on the total variation distance between $\mu_{m,N}(\overline{\gamma}_m)$ and $\mu_{m,N}(\overline{\gamma}'_m)$.

Suppose we attach Brownian motions that result in $\overline{\gamma}_m \in \mathsf{Nice}_{k,m} \cap \mathsf{Good}_k$. Then $\overline{\gamma}_m \in \mathcal{X}$ if and only if $\overline{\gamma}'_m \in \mathcal{X}$, and Lemma 4.3 applied to $\overline{\gamma}$ and $\overline{\gamma}_m$ implies

$$\left|\frac{d\mu_{m,N}}{d\mathcal{W}_N}(\overline{\gamma}_m) - \frac{d\mu_{m,N}}{d\mathcal{W}_N}(\overline{\gamma}'_m)\right| \leq 2c'_0 e^{-k/2}\, \frac{d\mu_{m,N}}{d\mathcal{W}_N}(\overline{\gamma}_m)\,. \tag{20}$$

For $\overline{\gamma}_m \notin \mathsf{Nice}_{k,m} \cap \mathsf{Good}_k$, we have by Lemma 4.2,

$$\mu_{m,N}\left[(\mathsf{Good}_k \cap \mathsf{Nice}_{k,m})^c\right] \leq c_0 e^{-k/2}. \tag{21}$$

The coupling rate now follows from putting together (20) and (21):

$$\mathbf{P}\{\overline{\gamma}_m \neq_{k+m} \overline{\gamma}'_m\} = \frac{1}{2}\|\mu_{m,N}(\overline{\gamma}_m) - \mu_{m,N}(\overline{\gamma}'_m)\| \leq (2c'_0 + c_0)e^{-k/2}\,.$$

Take $C_0 = 2c_0' + c_0$ and note that the second inequality in the proposition follows immediately from Lemma 4.3.

$\square$

We now fix an integer K such that

$$C_0 e^{-\frac{K-2}{2}} < \frac{1}{2}, \tag{22}$$

where C_0 is the constant of the previous proposition. We will use the coupling described above for $k \geq K-2$. Otherwise we will use the following.

Proposition 4.5. *There exists $b > 0$, such that if $K \leq N-1$ and $\overline{\gamma}, \overline{\gamma}' \in \mathcal{X}$, then we can find a coupling of $\mu_{K,N}(\overline{\gamma})$ and $\mu_{K,N}(\overline{\gamma}')$ such that with probability at least b,*

$$\overline{\gamma}_K =_{K-2} \overline{\gamma}'_K,$$

and

$$\overline{\gamma}_K \in \mathsf{Good}_{K-2}.$$

Proof. This is proved in the same way as Proposition 3.9. Starting with $\overline{\gamma}$ and $\overline{\gamma}'$, we attach Brownian paths up to first time they hit $\partial\mathcal{B}_K$ in the following way. From the Separation Lemma, with positive probability, by the time the paths reach $\partial\mathcal{B}_1$, they have separated, that is $\overline{\gamma}_1, \overline{\gamma}'_1 \in \mathsf{Sep}$. With positive probability, we can attach paths from $\partial\mathcal{B}_1$ to $\partial\mathcal{B}_2$ so that $\overline{\gamma}_2$ and $\overline{\gamma}'_2$ have the same endpoints and $\overline{\gamma}_2, \overline{\gamma}'_2 \in \mathsf{Sep}$. After this, we can attach the same Brownian paths, which stay very close to the radial lines up to the first time they reach $\partial\mathcal{B}_K$. Thus $\overline{\gamma}_K =_{K-2} \overline{\gamma}'_K$ with positive probability $b(K)$ and the separation ensures $\overline{\gamma}_K \in \mathsf{Good}_K$. The probability depends on K, but we have fixed a particular value of K and we let $b = \min\{b(K), 1/2\}$. $\square$

Proof of Theorem 4.1. Let K be as defined in (22), and let m be the largest integer such that $mK \leq N-1$. We will start by giving a coupling of $\mu_{mK,N}(\overline{\gamma})$ and $\mu_{mK,N}(\overline{\gamma}')$. We will do this one step at a time: first defining $(\overline{\gamma}_K, \overline{\gamma}'_K)$, then $(\overline{\gamma}_{2K}, \overline{\gamma}'_{2K})$, etc. At each stage $n \leq m$, we define the random variable $\sigma(n)$ to be the maximal nonnegative integer j such that, in the coupling,

$$\overline{\gamma}_{nK} =_j \overline{\gamma}'_{nK}$$

and

$$\overline{\gamma}_{nK} \in \mathsf{Good}_j.$$

We define $\sigma(N)$ to be the maximal nonnegative integer j such that in the coupling

$$\overline{\gamma}_N =_j \overline{\gamma}'_N.$$

and do not require the "good" condition if $n = N$. Suppose that we have defined $(\overline{\gamma}_{nK}, \overline{\gamma}'_{nK})$.

- If $\sigma(n) \geq K-2$, we define $(\overline{\gamma}_{(n+1)K}, \overline{\gamma}'_{(n+1)K})$ using a coupling as in Proposition 4.4.

- If $\sigma(n) < K-2$, we define $(\overline{\gamma}_{(n+1)K}, \overline{\gamma}'_{(n+1)K})$ using a coupling as in Proposition 4.5.

Let $\mathcal{F}_n$ denote the σ-algebra generated by $(\overline{\gamma}_{nK}, \overline{\gamma}'_{nK})$. Proposition 4.4 implies that if $j \geq K-2$ and $n < m$, then

$$\mathbf{P}\left\{\sigma(n+1) = K + j | \mathcal{F}_n\right\} \geq (1 - C_0 e^{-j/2})\, 1\{\sigma(n) = j\}.$$

Proposition 4.5 along with (22) give

$$\mathbf{P}\left\{\sigma(n+1) \geq K - 2 | \mathcal{F}_n\right\} \geq b.$$

By comparison with a Markov chain (see, e.g., [15]), we can find $c > 0$ and $\beta \leq 1/4$ such that

$$\mathbf{P}\{\sigma(m) \leq mK/2\} \leq c\, e^{-\beta m K}.$$

We have thus produced a coupling of $\mu_{mK,N}(\overline{\gamma})$ and $\mu_{mK,N}(\overline{\gamma}')$ such that, with probability at least $1 - c\, e^{-\beta m K}$, we have $\overline{\gamma}_{mK} =_{mK/2} \overline{\gamma}'_{mK}$ and $\overline{\gamma}_{mK} \in \mathsf{Good}_{mK/2}$.

To complete the proof, use Proposition 4.4 to couple the paths for the last $N - mK$ steps. It is easy to see that there exists C, depending on K, such that, with probability at least $1 - Ce^{-\beta N}$, we have $\overline{\gamma}_N =_{N/2} \overline{\gamma}'_N$, without requiring that $\overline{\gamma}_N \in \mathsf{Good}_j$ for some j in this last step. □

4.3 Some corollaries

Here we establish some straightforward corollaries of the coupling result.

Proposition 4.6. *There exist $c > 0, \beta < \infty$ such that for all $0 \leq m \leq n$ and all $\overline{\gamma}, \overline{\gamma}' \in \mathcal{X}$,*

$$|\mathbf{P}(A_n(\overline{\gamma}) \mid A_m(\overline{\gamma})) - \mathbf{P}(A_n(\overline{\gamma}') \mid A_m(\overline{\gamma}'))| \leq c\, e^{-m\beta}\, e^{-\xi(n-m)}.$$

Proof. Let $\mathcal{F}_m$ denote the σ-algebra generated by $\overline{\gamma}_m, \overline{\gamma}'_m$. Then

$$\mathbf{P}\left(A_n(\overline{\gamma}) \mid \mathcal{F}_m\right) = 1\{\overline{\gamma}_m \in \mathcal{X}\}\, q_{n-m}(\overline{\gamma}_m).$$

Using Theorem 4.1, we can find a coupling of $\overline{\gamma}_m, \overline{\gamma}'_m$ so that, with probability at least $1 - Ce^{-\beta m}$,

$$\overline{\gamma}_m =_{m/2} \overline{\gamma}_m.$$

If $\overline{\gamma}_m =_{m/2} \overline{\gamma}'_m$, then from (19) we have

$$|q_{n-m}(\overline{\gamma}_m) - q_{n-m}(\overline{\gamma}'_m)| \leq c\, e^{-m/2}\, e^{-(n-m)\xi}.$$

If $\overline{\gamma}_m \neq_{m/2} \overline{\gamma}'_m$, we use the fact that for all $\overline{\gamma}^* \in \mathcal{X}$,

$$q_{n-m}(\overline{\gamma}^*) \leq c_*\, e^{-(n-m)\xi}.$$

Now the proposition follows from putting these two estimates together and recalling that $\beta \leq 1/4$.

□

Proposition 4.7. *Let $Q_n(\overline{\gamma}) = e^{n\xi}\, q_n(\overline{\gamma})$. There exist a bounded function $Q : \mathcal{X} \to (0, \infty)$ and $c > 0, \beta < \infty$ such that if $\overline{\gamma} \in \mathcal{X}$, then the following hold:*

$$\lim_{n\to\infty} Q_n(\overline{\gamma}) = Q(\overline{\gamma}),$$

$$|Q(\overline{\gamma}) - Q_n(\overline{\gamma})| \leq c\, Q(\overline{\gamma})\, e^{-n\beta},$$

$$\frac{1}{c} \leq \frac{Q(\overline{\gamma})}{q_1(\overline{\gamma})} \leq c.$$

Proof. Note that

$$\frac{q_{n+1}(\overline{\gamma})}{q_n(\overline{\gamma})} = \mathbf{E}_n[q_1(\overline{\gamma}^*)],$$

where the expectation on the right denotes the expectation with respect to the probability measure $\mu_n(\overline{\gamma})$. Using the separation lemma, and more specifically Corollary 3.4, we see that there exists a constant $c > 0$ such that for $n \geq 1$,

$$c \leq \frac{q_{n+1}(\overline{\gamma})}{q_n(\overline{\gamma})} \leq 1.$$

Consider two initial configurations $\overline{\gamma}, \overline{\gamma}' \in \mathcal{X}$. By (19), if $\overline{\gamma}_n =_{n/2} \overline{\gamma}'_n$, then

$$|q_1(\overline{\gamma}_n) - q_1(\overline{\gamma}'_n)| \leq c\, e^{-n/2}.$$

But by Theorem 4.1, we have $\overline{\gamma}_n \neq_{n/2} \overline{\gamma}'_n$ with probability at most $Ce^{-\beta n}$. Using this and the bound $\beta \leq 1/4$,

$$\left| \frac{q_{n+1}(\overline{\gamma})}{q_n(\overline{\gamma})} - \frac{q_{n+1}(\overline{\gamma}')}{q_n(\overline{\gamma}')} \right| \leq c\, e^{-\beta n}.$$

A similar argument shows that for $m \leq n$, and all $\overline{\gamma}, \overline{\gamma}' \in \mathcal{X}$,

$$\left| \frac{q_{n+1}(\overline{\gamma})}{q_n(\overline{\gamma})} - \frac{q_{m+1}(\overline{\gamma}')}{q_m(\overline{\gamma}')} \right| \leq c\, e^{-\beta m}.$$

In particular, the limit

$$\lim_{n\to\infty} \frac{q_{n+1}(\overline{\gamma})}{q_n(\overline{\gamma})}$$

exists and is independent of the initial configuration $\overline{\gamma}$. Since $q_n(\overline{\gamma}) \asymp q_1(\overline{\gamma})\, e^{-n\xi}$, the limit must equal $e^{-\xi}$. Therefore,

$$|\, Q_{n+1}(\overline{\gamma}) - Q_n(\overline{\gamma})\,| \leq c\, e^{-n\beta}\, Q_n(\overline{\gamma}),$$

and by iterating this, we see for all positive integers m,

$$|\, Q_{n+m}(\overline{\gamma}) - Q_n(\overline{\gamma})\,| \leq c\, e^{-n\beta}\, Q_n(\overline{\gamma}),$$

with a different constant c. In particular, the sequence $\{Q_n(\overline{\gamma})\}$ is a Cauchy sequence in n and has a limit $Q(\overline{\gamma})$ satisfying

$$| \, Q_n(\overline{\gamma}) - Q(\overline{\gamma}) \, | \leq c \, e^{-n\beta} \, Q(\overline{\gamma}).$$

This establishes the result for integer n, but it is easy to extend it to non-integer $n \geq 1$. Recalling that for all $n \geq 1$ and all $\overline{\gamma} \in \mathcal{X}$, we have $Q_n(\overline{\gamma}) \leq c_*$, this result also proves the first claim in Theorem 2.8.

The last assertion follows from a direct application of Corollary 3.4 $\square$

Definition If $K_1, K_2 \subset \mathbb{R}^3$ are compact subsets of $\mathbb{R}^3$ with $K_1 \cap K_2$ finite, and $\mathbf{w} = (w_1, w_2) \in \mathbb{R}^3 \times \mathbb{R}^3$, let

$$Q_n(\mathbf{K}; \mathbf{w}) = e^{n\xi} \, \mathbf{P}^{\mathbf{w}}[A_n(K_1, K_2)].$$

$$Q(\mathbf{K}; \mathbf{w}) = \lim_{n \to \infty} A_n(\mathbf{K}; \mathbf{w}) = \lim_{n \to \infty} e^{n\xi} \mathbf{P}^{\mathbf{w}}[A_n(K_1, K_2)]. \tag{23}$$

If $K_1 \cap K_2$ is infinite, we define $Q(\mathbf{K}; \mathbf{w}) = 0$

Proposition 4.8. *The limit* (23) *exists. If $K_1, K_2 \subset \overline{\mathcal{B}}$ are disjoint and $w_1, w_2 \in \overline{\mathcal{B}}$, and $n \geq 1$,*

$$| \, Q(\mathbf{K}; \mathbf{w}) - Q_n(\mathbf{K}; \mathbf{w}) \, | \leq C \, e^{-n\beta} \, Q(\mathbf{K}; \mathbf{w}). \tag{24}$$

It is translation invariant

$$Q(\mathbf{K} + \mathbf{z}; \mathbf{w} + \mathbf{z}) = Q(\mathbf{K}; \mathbf{w})$$

and satisfies the scaling rule

$$Q(e^r \mathbf{K}; e^r \mathbf{w}) = e^{r\xi} \, Q(\mathbf{K}; \mathbf{w}), \tag{25}$$

Proof. The proof of (24) is essentially the same as that of Proposition 4.7. Brownian scaling implies that

$$\mathbf{P}^{e^r \mathbf{w}} \left[A_{r+n}(e^r K_1, e^r K_2) \right] = \mathbf{P}^{\mathbf{w}} \left[A_n(K_1, K_2) \right],$$

from which (25) follows immediately. Also, if $|z| = 1$, the closed disk of radius e^n about z contains $\mathcal{B}_{\log(e^n - 1)}$ and is contained in $\mathcal{B}_{\log(e^n+1)}$. Hence, if $\mathbf{z} = (z_1, z_2)$,

$$\mathbf{P}^{\mathbf{w}} \left[A_{\log(e^n+1)}(K_1, K_2) \right] \leq \mathbf{P}^{\mathbf{w}+\mathbf{z}} \left[A_n(K_1 + z_1, K_2 + z_2) \right] \leq \mathbf{P}^{\mathbf{w}} \left[A_{\log(e^n-1)}(K_1, K_2) \right].$$

$\square$

5 Invariant measure

With the coupling result, the proof of the existence of the measure ν proceeds as in [6, 12, 15]. We start by defining $\pi_k \nu$ for positive integers k. The coupling result implies that for any $\overline{\gamma} \in \mathcal{X}$, the collection of measures $\{\pi_k \mu_n(\overline{\gamma}) : n = 1, 2, \ldots\}$ is a Cauchy sequence of measures. Indeed, if $n \geq m \geq 2k$,

$$\|\pi_k \mu_n(\overline{\gamma}) - \pi_k \mu_m(\overline{\gamma})\| \leq c\, e^{-\beta m}.$$

Here $\|\cdot\|$ denotes variation distance, but since measures for fixed k are absolutely continuous with respect to an appropriate Wiener measure, we can also consider it as an L^1-metric on the density with respect to Wiener measure. Hence, there exists a limit which we denote by $\pi_k \nu$ which is also absolutely continuous with respect to Wiener measure. The same coupling argument shows that for any $\overline{\gamma} \in \mathcal{X}$ and $n \geq 2k$,

$$\|\pi_k \mu_n(\overline{\gamma}) - \pi_k \nu\| \leq c\, e^{-\beta n}.$$

Using this we can see that the $\{\pi_k \nu\}$ satisfy the appropriate consistency condition so we can combine them to give the measure ν.

There is a minor technical detail to show that the paths under measure ν have finite time duration. Let $T_k(\overline{\gamma})$ denote the sum of the time durations of γ^1 and γ^2 between the times of the first visit to $\partial \mathcal{B}_{-k}$ to the first visit to $\partial \mathcal{B}_{1-k}$. Using standard estimates for Brownian motion, one can easily show that there exist c, α such that

$$\nu \{\overline{\gamma} : T_1(\overline{\gamma}) \geq r\} \leq c\, e^{-\alpha r}.$$

Using this, Brownian scaling, and (26) below we see that there exists c' such that for all $r > 0$,

$$\nu \left\{\overline{\gamma} : T_k(\overline{\gamma}) \geq r\, e^{-2k}\right\} \leq c'\, e^{-\alpha r}.$$

Using a Borel-Cantelli argument, we can see that this implies that

$$\nu \left\{\overline{\gamma} : \sum_{k=1}^{\infty} T_k(\overline{\gamma}) = \infty \right\} = 0.$$

This completes the proof of Theorem 2.8.

If Y is a function on $\mathcal{X}$, we write $\nu[Y] = \int Y\, d\nu$. We omit the easy proof of the next proposition which gives some properties of the measure ν.

Proposition 5.1. *For all $n > 0$,*

$$\mu_n[\nu] = \nu,$$

$$\nu[q_n] = e^{-n\xi}, \quad \nu[Q_n] = 1.$$

$$\nu[\mathsf{Sep}] \geq \rho_1,$$

$$\frac{d\pi_n \nu}{d\nu}(\overline{\gamma}) = Q_n(\overline{\gamma}). \tag{26}$$

Let us define the measure $\overline{\nu}$ by

$$\frac{d\overline{\nu}}{d\nu}(\overline{\gamma}) = Q(\overline{\gamma}).$$

Remark We have defined analogues of measures that are sometimes called quasi-invariant measures for subMarkov chains. As a simple analogue, suppose that X_n is an aperiodic, irreducible Markov chain on a finite state space that we write as $A \cup \{z\}$. Let T denote the first visit to z and let P denote the subMarkov transition matrix on A with entries

$$p(x,y) = \mathbf{P}\{X_n = y, T > n | X_0 = x\}.$$

Let $p_n(x,y)$ denote entries of P^n. Then the analogue of $e^{-\xi}$ is the largest eigenvalue of P and the analogues of $Q, \nu, \overline{\nu}$ in this case are given by

$$Q(x) = \lim_{n\to\infty} e^{\xi n}\mathbf{P}\{T > n | X_0 = x\},$$
$$\nu(y) = \lim_{n\to\infty} \mathbf{P}\{X_n = y | X_0 = x, T > n\},$$
$$\overline{\nu}(z) = \lim_{n\to\infty} \mathbf{P}\{X_n = y | X_0 = x, T > 2n\}.$$

6 Future directions

We plan on extending these coupling results to more general intersection exponents. Briefly, let $B_t^1, ..., B_t^{m+n}$ be independent 3-dimensional Brownian motions, started uniformly on $\partial\mathcal{B}$. As before, for $1 \le j \le m+n$, let $T_k^j = \inf\{t : B_t^j \in \partial\mathcal{B}_k\}$ and let

$$\Gamma_k^1 = B^1[0, T_k^1] \cup \cdots \cup B^m[0, T_k^m], \qquad \Gamma_k^2 = B^{m+1}[0, T_k^{m+1}] \cup \cdots \cup B^{m+n}[0, T_k^{m+n}].$$

Then the intersection exponent $\xi(m,n)$ is defined as

$$\mathbf{P}\{\Gamma_k^1 \cap \Gamma_k^2 = \emptyset\} \approx e^{-\xi(m,n)k}.$$

Note that $\xi = \xi(1,1)$ and that $\xi(m,n)$ measures the probability that a set of m independent paths avoids a set of n independent paths. These exponents can be extended in a natural way for all $\lambda \ge 0$ to $\xi(k,\lambda)$. They were first introduced in [13] and their existence follows, as before, from a subadditivity argument.

While in 2 dimensions all these exponents have been computed (see [9] and [10]), not much is known of their 3-dimensional counterparts. The only known values are $\xi(k,0) = 0$ and $\xi(2,1) = \xi(1,2) = 1$. Looking at $\xi(k,\lambda)$ as functions of λ, it was proved in [7] that they are strictly concave. One question of interest is whether these functions are also analytic. In [11], an exponential coupling of weighted Brownian paths was used to prove that 2-dimensional intersection exponents are analytic. While the coupling from [11] relies on conformal invariance of planar Brownian motion and cannot be generalized to three dimensions, we believe that our coupling argument carries over from $\xi(1,1)$ to $\xi(k,\lambda)$, hence providing a fast convergence to an invariant measure in the general case. This in turn should be sufficient to prove analyticity of 3-dimensional exponents.

A long range goal is to give an effective way to study the multifractal nature of the Brownian path.

7 Simulations for ξ

The value of the intersection exponent ξ is not known, and it is possible that it will never be known exactly. However, one can do simulations, and we report the results of our recent trials. In [2], it was proved that Brownian exponents and simple random walk exponents are the same. That is to say, if S^1 and S^2 are simple random walks started at the origin, then

$$\mathbf{P}\{S^1(0,n] \cap S^2(0,n] = \emptyset\} \approx n^{-\zeta},$$

where $\zeta = \xi/2$. It is believed that this probability is asymptotic to $cn^{-\zeta}$ for some c, and this is what we assume here.

Therefore, as in [3], we do simulations of the random walk exponent. Suppose we run M pairs of independent simple random walks, started at the origin. If $M(n)$ denotes the number of (pairs of) paths that have no intersections in the time interval $(0, n]$, then the probability of no intersection by time n is estimated by $M(n)/M$. Let

$$k(n) = \frac{\log M - \log M(n)}{\log n}.$$

This quantity should converge to ζ as $n \to \infty$.

We ran one million pairs of 3-dimensional random walks of length $100,000$, started at the origin. We use the same number of walks as in [3], but our walks are much longer. Our simulation results are included in Table 1.Our simulations suggest $\xi = 2\zeta$ is around .57, which is consistent with simulations in [3].

Similar to the simulation analysis in [3], one can estimate ζ using the sequence

$$h(n) = \frac{\log M(n) - \log M(n+m)}{\log(m+n) - \log n},$$

which should also converge to ζ as $n \to \infty$. Let $m = 10,000$. We observe that our simulations lead to more variation in the value of $h(n)$ than in the value of $k(n)$, as it can be seen in Table 1, but again suggests ξ is around .57.

n	$M(n)$	k(n)	h(n)
10,000	74,629	0.2818	0.2874
20,000	61,151	0.2822	0.2948
30,000	54,262	0.2827	0.2857
40,000	49,981	0.2827	0.2838
50,000	46,914	0.2828	0.2953
60,000	44,455	0.2830	0.2895
70,000	42,515	0.2831	0.2787
80,000	40,962	0.2830	0.2822
90,000	39,623	0.2830	0.2746
100,000	38,493	0.2829	–

Table 1: Simulations using $1,000,000$ pairs of $100,000$ step walks.

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