Hölder Estimates for Solutions of Integro-differential Equations Like The Fractional Laplace

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ABSTRACT. We provide a purely analytical proof of Hölder continuity for harmonic functions with respect to a class of integro-differential equations like the ones associated with purely jump processes. The assumptions on the operator are more flexible than in previous works. Our assumptions include the case of an operator with variable order, without any continuity assumption in that order.

1. Introduction

Several regularity results in the nonlinear theory of elliptic differential equations are based on Hölder estimates for linear equations with rough coefficients. These results are closely related with Harnack inequalities as it is standard in the subject. In [8], E. De Giorgi proved a $C^\alpha$ estimate for second order uniformly elliptic equations in divergence form with measurable coefficients, and the regularity of minimizers for nonlinear convex functionals followed from there. Other proofs were given by J. Nash [12] and J. Moser [11]. In the non-divergent case, the corresponding result was obtained by Krylov and Safonov [9], and it is an essential tool in proving $C^{1,\alpha}$ regularity for fully nonlinear elliptic equations (See [7]). The regularity of harmonic functions with respect to nonlocal operators was studied in several recent papers like [5] and [6], however their point of view is probabilistic. We obtain a similar result from a purely analytic point of view. Moreover, our assumptions imply many of the previous Hölder estimates in the nonlocal case, and
also include the important case of variable order without assuming any continuity in the order, that was not possible with the previous approach (See [5, Example 2]).

For nonlocal operators, the Harnack inequality does not imply a Hölder estimate as in differential equations, due to the fact that the harmonic function $u$ is assumed to be nonnegative in the whole space $\mathbb{R}^n$. In [13] and [4] a Harnack type inequality is obtained, but it does not lead to a Hölder estimate.

We expect these estimates to be important in a forthcoming nonlinear theory of nonlocal operators.

We will study operators $T$ defined by an integral.

$$ (1.1) \quad Tu(x) = \int_{\mathbb{R}^n} (u(x) - u(x + y) + \chi_{B_r}(y) \nabla u(x) \cdot y)K(x, y) \, dy,$$

where $K$ is a nonnegative function that satisfies

$$ \sup_x \int_{\mathbb{R}^n} (|y|^2 \wedge 1)K(x, y) \, dy < +\infty. $$

With some additional hypotheses we will prove that if $u$ is a bounded function so that $Tu(x) = 0$ for every $x \in B_{2r}$, then for $\alpha$ small enough,

$$ \sup_{x, y \in B_r} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \leq C \frac{1}{r^\alpha} \|u\|_\infty. $$

We are especially interested in the symmetric case where $K(x, y) = K(x, -y)$. However, our method works in a more general setting. If $K(x, y) \, dy$ is replaced by a measure $K(x, dy)$ with a singular component, our result will apply as long as we satisfy (2.2) where $K(rx + x_0, ry) r^n \, dy$ would stand for the pull back of $K(x, dy)$ by the application $x \mapsto rx + x_0$. Moreover, our result could be extended to operators that do not necessarily have an integral representation like (1.1), as Remark 4.4 shows.

In the symmetric case, when $K(x, y) = K(x, -y)$, then the term

$$ \chi_{B_r}(y) \nabla u(x) \cdot y $$

will not have any influence on (1.1) besides making the integral convergent. In this case $\chi_{B_r}(y) \nabla u(x) \cdot y$ could be replaced by $\chi_{B_R}(y) \nabla u(x) \cdot y$, for any $R > 0$, without changing the operator. Moreover,

$$ Tu(x) = \text{PV} \int_{\mathbb{R}^n} (u(x) - u(x+y))K(x, y) \, dy. $$

It would be a little cleaner to write down all the paper in only the symmetric case. We seriously considered doing so. But in order to keep enough generality so as to include most previous results, symmetry is not assumed in this work.
The regularity result is based in Lemma 4.1. It will be proved through point estimates assuming that $u$ is smooth, however the estimates do not depend on the norm of any derivative or modulus of continuity of $u$. Therefore, the result should extend to nonsmooth functions that are merely locally bounded. The problem is that, for the time being, we do not know any good way to make sense of the concept of harmonic or subharmonic function of an operator like (1.1) when $u$ is not smooth. We believe that when there is a good way to define a meaning for (4.1), then it should be possible to apply this result, either adapting the proof or by an approximation process.

The paper is organized as follows. In Section 2, we set up the technical assumptions that we require for the operator $T$ in order for our estimates to apply. As our assumptions may not seem simple or natural, Section 3 is devoted to give several common particular cases where our theory applies. In Section 4 a lemma is proved that is the non-local analogous to De Giorgi’s lemma for subsolutions of divergence form elliptic PDE’s. From this lemma, the main result is derived in Section 5 with a standard iteration technique.

2. NOTATION AND SPECIAL ASSUMPTIONS

We will first define the auxiliary bump function $b(x) = \beta(|x|)$, for $\beta(x) = (1 - x^2)^2$. The only important thing about $b$ is that it is $C^2$, radially symmetric, its support is $B_1$, and it is decreasing in any ray from the origin. Any function with these characteristics would work for us, but we must stick to the same one for the whole paper.

Given a $\delta > 0$, we will make the following assumption for $T$: there are two positive numbers $\kappa < \frac{1}{4}$ and $\eta$ such that for every $x \in \mathbb{R}^n$

$$ (2.1) \quad \kappa Tb(x) + 2 \int_{\mathbb{R}^n - B_{1/4}} (|8y|^\eta - 1)K(x, y) \, dy < \frac{1}{2} \inf_{A \subset B_2, |A| > \delta} \int_A K(x, y) \, dy. $$

Moreover, we will need (2.1) to hold at every scale. That is, if $r > 0$ and $x_0 \in \mathbb{R}^n$, let $u_{r,x_0}(x) = u(rx + x_0)$. By a change of variables we can obtain an operator $T_{r,x_0}$ such that $(T_{r,x_0}u_{r,x_0})(x) = [Tu]_{r,x_0} = Tu(rx + x_0)$. This operator is given by the formula:

$$ T_{r,x_0}v(x) = \int_{\mathbb{R}^n} (v(x) - v(x+y) + \chi_{B_1}(ry) \nabla v(x) \cdot y) K(rx + x_0, ry) r^n \, dy. $$

Notice that if $K$ is symmetric, the terms $\chi_{B_1}(ry) \nabla v(x) \cdot y$ and $\chi_{B_1}(y) \nabla v(x) \cdot y$ have the same effect in the integral. If, for example, $T = (-\Delta)^s$ (see Section 3.1), $T_{r,x_0} = r^{-2s}T$. 
To obtain the desired regularity in harmonic functions we will require \((2.1)\) to hold for every \(T_{x_0}^{x_0}\). In other words, we will require that there exist \(\eta\) and \(\kappa < \frac{1}{4}\) such that for every \(r > 0\) and \(x_0 \in \mathbb{R}^n\):

\[
\kappa T_{x_0}^{x_0} b(x) + 2 \int_{\mathbb{R}^n-B_{1/4}} (|y|^\eta - 1) K(r x + x_0, r y) r^n \, dy \quad < \quad \frac{1}{2} \inf_{A \subset \mathbb{R}^n, |A| > \delta} \int_A K(r x + x_0, r y) r^n \, dy.
\]

Normally, we will consider classes of operators with a similar behavior at every scale. So in practice it will be enough to check only \((2.1)\) for that class, and then \((2.2)\) will hold. The formula for \((2.2)\) is not important, what matters is that \((2.2)\) means that \((2.1)\) holds for \(T_{x_0}^{x_0}\) for any value of \(r\) and \(x_0\).

Usually, we are able to find a lower bound for the right-hand side of \((2.1)\), and then we have to prove that we can choose \(\eta\) and \(\kappa\) to make the two terms of the left-hand side as small as desired. The second term controls the long distance behavior in the proof of Lemma 4.1 (probabilists would say “the big jumps”). This is needed since the operators are not local in nature. To control this term (at least at unit scale) it is enough to show that for some \(\eta_0\)

\[
\int_{\mathbb{R}^n-B_{1/4}} |y|^\eta K(x, y) \, dy < +\infty
\]

and then use dominated convergence.

The first term of the left-hand side of \((2.1)\) simply applies \(T\) to a fixed test function \(b\). When \(Tb\) is bounded (which happens for every practical situation), this term can easily be controlled. However, when \(K\) is very non-symmetric it seems really difficult to have a uniform estimate of this term at every scale in \((2.2)\).

In Remark 4.4, we will state a nonlinear analog of these assumptions.

3. PARTICULAR CASES WHERE OUR ASSUMPTIONS APPLY

Having \((2.2)\) as an assumption may seem a little obscure and awkward to check whether it holds. This section is devoted to seeing several examples to illustrate when the theory applies and when it does not.

An observation that can come handy is that the left-hand side of \((2.1)\) (and also \((2.2)\)) is linear in \(K\) and the right-hand side is super linear in the sense that, for a pair \(K_1, K_2\) and \(a_1, a_2 > 0\),

\[
\inf_{A \subset B_2, |A| > \delta} \int_A a_1 K_1(x, y) + a_2 K_2(x, y) \, dy \quad \geq \quad \inf_{A \subset B_2, |A| > \delta} \int_A a_1 K_1(x, y) \, dy + \inf_{A \subset B_2, |A| > \delta} \int_A a_1 K_1(x, y) \, dy.
\]

This immediately implies the next proposition:
Proposition 3.1. If $K_i$ satisfies (2.1) (or (2.2)) for $i = 1 \ldots N$, and $\alpha_i$ is a $N$-uple of positive real numbers, then $\sum \alpha_i K_i$ also satisfies (2.1) (or (2.2)).

Proof. We just have to choose the least value of $\kappa$ and $\eta$ from all the ones that we have for each $K_i$. $lacksquare$

3.1. Fractional powers of the Laplacian

For $s \in (0, 1)$ the operator $(-\Delta)^s$ can be obtained by its singular integral representation:

\[
(-\Delta)^s u(x) = \text{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy = \int_{\mathbb{R}^n} (u(x) - u(x + y) + \chi_{B_1}(y) \nabla u(x) \cdot y) \frac{1}{|y|^{n+2s}} \, dy.
\]

If we take $T = (-\Delta)^s$, let us check that $T$ satisfies (2.1).

Let us first see that $Tb$ is a bounded function. The auxiliary function $b$ is $C^2$ and bounded. Recall that $b$ is a fixed function defined at the beginning of Section 2

\[
(\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy
\]

(3.1) $Tb(x) = \int_{\mathbb{R}^n} (b(x) - b(x + y) + \chi_{B_1}(y) \nabla b(x) \cdot y) \frac{1}{|y|^{n+2s}} \, dy

= I_1 + I_2,$

where we split the domain of integration into the unit ball and its complement,

\[
I_1 = \int_{B_1} (b(x) - b(x + y) + \chi_{B_1}(y) \nabla b(x) \cdot y) \frac{1}{|y|^{n+2s}} \, dy,
\]

(3.2) $I_2 = \int_{\mathbb{R}^n \setminus B_1} (b(x) - b(x + y) + \chi_{B_1}(y) \nabla b(x) \cdot y) \frac{1}{|y|^{n+2s}} \, dy.
\]

(3.3)

To estimate $I_1$, we use that $b$ is a fixed $C^2$ function to get the upper bound

\[
|b(x) - b(x + y) + \chi_{B_1}(y) \nabla b(x) \cdot y| \leq C |y|^2
\]

for a universal constant $C$,

(3.4) $I_1 \leq \int_{B_1} C |y|^2 \frac{1}{|y|^{n+2s}} \leq \frac{C_1}{2 - 2s}$

for a constant $C_1$ depending only on dimension.

To estimate $I_2$, we use that $b$ is bounded, to get the estimate

\[
|b(x) - b(x + y)| \leq 2 \| b \|_{L^\infty}.
\]
The term \( \chi_{B_1}(y) \nabla b(x) \cdot y \) vanishes for \(|y| > 1\), thus we have

\[
I_2 \leq \int_{\mathbb{R}^n \setminus B_1} C \frac{1}{|y|^{n+2s}} = \frac{C_2}{2s}
\]

for a constant \( C_2 \) depending only on dimension.

Therefore, \( Tb = I_1 + I_2 \leq C_1 / (2 - 2s) + C_2 / (2s) \) is a bounded function, with an upper bound depending only on dimension and \( s \). So, choosing \( \kappa \) small, we are able to make \( \kappa Tb \) have an upper bound as small as we wish.

Now, let us estimate the second term in (2.1), which in this case reads

\[
2 \int_{\mathbb{R}^n - B_1/4} (|8y||n - 1|) \frac{1}{|y|^{n+2s}} \, dy.
\]

We notice that the integral is finite if \( \eta < 2s \). Moreover, the integrand decreases when \( \eta \) decreases, and tends to zero as \( \eta \to 0 \). We can apply the dominated convergence theorem to see that the integral goes to zero as \( \eta \to 0 \). Thus, this term can be as small as we wish if we choose \( \eta \) small. For a given \( \varepsilon > 0 \), the value of \( \eta \) for which this term is less than \( \varepsilon \) depends on dimension and on \( s \).

For the right-hand side of (2.1), we observe that

\[
\frac{1}{2} \int_A \frac{1}{|y|^{n+2s}} \, dy \geq \frac{1}{2} \delta \frac{1}{2^{n+2s}},
\]

then

\[
\frac{1}{2} \inf_{A \subset B_1, |A| > \delta} \int_A K(x, y) \, dy
\]

is strictly positive. Therefore, if we pick \( \kappa \) and \( \eta \) small enough, the left-hand side of (2.1) will be smaller than the right-hand side. Notice that the values of \( \kappa \) and \( \eta \) for which this happens depend only on \( s \) and dimension.

Since \( T_{r, x_0} = r^{-2s} T \) and every term in (2.1) is linear in \( T \), then every \( T_{r, x_0} \) satisfies (2.1). Thus \( T \) also satisfies (2.2).

### 3.2. Sum of powers of the Laplacian

If we consider

\[
T = \sum_{i=1}^N a_i (-\Delta)^{s_i}
\]

for \( a_i > 0 \) and \( s_i \in (0, 1) \), then \( T \) satisfies (2.2) because of Proposition 3.1.

### 3.3. Kernels comparable to those of \((-\Delta)^s\)

In [6], Bass and Levin consider operators like (1.1) with kernels satisfying

\[
\frac{a}{|y|^{n+2s}} \leq K(x, y) \leq \frac{A}{|y|^{n+2s}},
\]

(3.6)

\[
K(x, y) = K(x, -y),
\]

(3.7)
for \(0 < a \leq A\) and \(s \in (0, 1)\).

If we follow closely the estimates for the fractional laplacian, we see that the
only thing we are using is their growth estimates at zero and infinity. The same
proof, line by line, works for these operators too.

Now, if \(T\) is an operator whose kernel satisfies (3.6) and (3.7), then so is
\(r^{2s} T_{r,x_0}\). We do not have to worry about the term \(X_{B_1}(r\gamma)\nabla v(x) \cdot \gamma\) in \(T_{r,x_0}\)
since \(K\) is symmetric. Since (2.1) is linear in \(T\), then \(T_{r,x_0}\) satisfies (2.1) uniformly
for every \(r > 0\) and \(x_0 \in \mathbb{R}^n\).

It is important to notice that the choice of \(\eta\) and \(\kappa\) goes to zero as \(s\) goes
either to 0 or to 1. This means that this method would not allow us to obtain a
similar estimate for second order operators. This suggests that the estimates are
not completely sharp, since second order uniformly elliptic equations in nondiver-
gence form can be obtained as a limit of operators like the ones considered here.
But from the result of Krylov and Safonov [9] we know that the Hölder estimates
do hold for second order equations.

### 3.4. Operators of variable order

Taking a closer look at (2.1), we can notice that the condition must be satisfied for each single value of \(x \in \mathbb{R}^n\), but
there is no interaction with the neighboring points of \(x\). Therefore, if we have
a family of kernels \(K_\alpha\) so that they satisfy the assumption (2.2) with a uniform
choice of \(\kappa\) and \(\eta\), then a \(K\) such that \(K(x, \gamma) = K_\alpha(x, \gamma)\) would also satisfy
the hypothesis (2.2). This kernel \(K\) would produce an operator \(T\) that applies a
different operator \(T_\alpha\) (corresponding to each \(K_\alpha\)) depending on the point where
it is evaluated.

We can apply this observation to kernels satisfying (3.6) and (3.7) as long as
we keep uniform bounds on \(\kappa\) and \(\eta\). But no continuity whatsoever is required in \(x\). Therefore we can consider kernels satisfying the following conditions:

\begin{align}
\frac{a}{|y|^{n+2s(x)}} \leq K(x, \gamma) \leq \frac{A}{|y|^{n+2s(x)}},
\end{align}

\begin{align}
K(x, \gamma) = K(x, -\gamma),
\end{align}

for \(0 < a \leq A\) and \(s \in (0, 1)\) as long as \(0 < s_1 \leq s(x) \leq s_2 < 1\) so that we can
keep a uniform choice for \(\kappa\) and \(\eta\).

We could alternatively prove this case by direct computation as we did for the
fractional Laplace operators.

To obtain an upper bound for \(Tb(x)\), we split the integral as in Section 3.1:
\(Tb(x) = I_1 + I_2\). This time we have the inequalities

\begin{align}
I_1 \leq \int_{B_1} C|y|^2 \frac{A}{|y|^{n+2s_1}} \leq \frac{C_1 A}{2 - 2s_2}
\end{align}
for a constant $C_1$ depending only on dimension.

$$I_2 \leq \int_{\mathbb{R}^n \setminus B_1} C \frac{A}{|y|^{n+2s_1}} = \frac{C_2 A}{2s_1}$$

for a constant $C_2$ depending only on dimension.

Therefore, $Tb(x) = I_1 + I_2 \leq C_1 A / (2 - 2s_2) + C_2 A / (2s_1)$ is bounded independently of $x$, with an upper bound depending only on $s_1$, $s_2$, $A$, and dimension. So, as before, we can choose $\kappa$ small, in order to make $\kappa Tb(x)$ as small as we wish.

For the second term in (2.1),

$$2 \int_{\mathbb{R}^n \setminus B_{1/4}} (|8y|^\eta - 1)K(x, y) \, dy \leq 2 \int_{\mathbb{R}^n \setminus B_{1/4}} (|8y|^\eta - 1) \frac{4^{2s_2 - 2s_1} A}{|y|^{n+2s_1}} \, dy$$

and, as before, we can make it as small as we wish as $\eta$ goes to zero.

For the right-hand side of (2.1), we observe that

$$\frac{1}{2} \int_A K(x, y) \, dy \geq \frac{1}{2} \delta \frac{a}{2n+2s_2};$$

then

$$\frac{1}{2} \inf_{A \subset B_2, |A| > \delta} \int_A K(x, y) \, dy$$

is strictly positive. Therefore, as in Section 3.1, if we pick $\kappa$ and $\eta$ small enough, the left hand side of (2.1) will be smaller than the right-hand side. The values of $\eta$ and $\kappa$ for which this happens depend on $s_1$, $s_2$, $a$, $A$, and dimension.

It is very important to observe that the operator $r^{2s_r} \mathfrak{r}_r \mathfrak{x}_0 \mathfrak{t} \mathfrak{r} \mathfrak{x}_0 \mathfrak{t}$ satisfies also conditions (3.8) and (3.9) for $s_{r \mathfrak{r}, \mathfrak{x}_0} (\mathfrak{x}) = s (\mathfrak{r} \mathfrak{x} + \mathfrak{x}_0)$ instead of $s(\mathfrak{x})$. This implies that it also satisfies (2.1) for $\kappa$ and $\eta$ depending only on $\delta$, $a$, $A$, $s_1$, and $s_2$. But we see that assumption (2.1) remains invariant if we multiply $T$ by a function of $x$. Then $T_{r \mathfrak{r}, \mathfrak{x}_0} \mathfrak{t}$ satisfies (2.1) for any $r$ and $\mathfrak{x}_0$ with a uniform choice of $\kappa$ and $\eta$, or in other words, $T$ satisfies (2.2).

This lack of continuity in $x$ cannot be achieved with the hypothesis of [5], as Example 2 in that paper points out.

Since it is known ([9]) that for purely diffusive operators the solutions are Hölder continuous, then it would not be unreasonable to expect the same to be true for operators like the above but without the condition $s_2 < 1$. At the present time we are unable to prove or disprove this.

### 3.5. The assumptions of Bass and Kassmann

In [5], Bass and Kassmann study the operator (1.1) having a structure with a (not necessarily absolutely continuous) measure $K(x, \, dy)$ (they actually use the negative of (1.1), but that does...
not make any difference). For the assumptions on $K$ they define:

\begin{align*}
(3.12) & \quad S(x, r) = \int_{|y| \geq r} K(x, dy), \\
(3.13) & \quad L(x, r) = S(x, r) + r^{-1} \int_{1 \leq |y| \leq r} |yK(x, dy)| \\
& \quad \quad \quad + r^{-2} \int_{|y| < r} |y|^2 K(x, dy), \\
(3.14) & \quad N(x, r) = \inf \left\{ K(x, A - x) : A \subset B(x, 2r), \right. \\
& \quad \quad \quad \left. |A| \geq \frac{1}{3 \cdot 2^d |B(x, r)|} \right\}.
\end{align*}

Then they assume:

$$\sup_x L(x, 1) < \infty$$

and also the following.

(a) There exist $\kappa_1 > 0$ and $\sigma > 0$ such that

$$\frac{S(x, \lambda r)}{S(x, r)} \leq \kappa_1 \lambda^{-\sigma}, \quad x \in \mathbb{R}^d, 1 < \lambda < 1/r, \ r < 1.$$  \hfill (3.15)

(b) There exist $\kappa_2 > 0$ such that if $x \in \mathbb{R}^d, r < 1, r/2 \leq s \leq 2r$, and $|x_1 - x_2| \leq 2r$, then

$$N(x_1, r) \geq \kappa_2 L(x_2, s).$$  \hfill (3.16)

These assumptions are designed to apply to Hölder estimates in balls of radius less than 1, and they are sharpened in that respect. Assumption (a) would be the same as the bound in the second term of (2.2) if we did not have $\lambda < 1/r$. The assumption (b) is almost the same as our bound for the first term of (2.2), as $L$ is used in [5] to estimate the operator in a smooth test function. However, our estimate uses the same point in both sides of the inequality; the fact that we have to consider two different points $x_1$ and $x_2$ in (b) prevents us from applying the result of [5] to very irregular cases as those in Section 3.4.

### 3.6. Minimizers of functionals in $H^s$

Given $s \in (0, 1)$, one of the possible norms for the Sobolev space $H^s(\mathbb{R}^n)$ is given by

$$\|u\|^2_{H^s} = \int |u(x)|^2 \, dx + \iint \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy.$$  \hfill (3.17)

Given $g \in H^s(\mathbb{R}^n)$ and $\lambda \leq h(x, y) \leq \Lambda$, a boundary value problem can be stated in this space as considering the minimizer of a functional like

$$J(u) = \iint \frac{h(x, y)}{|x - y|^{n+2s}} |u(x) - u(y)|^2 \, dx \, dy.$$
over all $u \in H^s(\mathbb{R}^n)$ such that $u(x) = g(x)$ for $x \in \mathbb{R}^n \setminus B_1$.

The Euler-Lagrange equation for $J$ gives an equation for the minimizer $u$. Let us compute the Frechet derivative of $J$.

$$J(u + tv) = \iint \frac{h(x, y)}{|x - y|^{n+2s}} |u(x) + tv(x) - u(y) - tv(y)|^2 \, dx \, dy$$

$$= J(u) + 2t \iint \frac{h(x, y)}{|x - y|^{n+2s}} (v(x) - v(y)) \cdot (u(x) - u(y)) \, dx \, dy$$

$$+ O(t^2).$$

Then

$$\langle DJ(u), v \rangle = 2t \iint \frac{h(x, y)}{|x - y|^{n+2s}} (v(x) - v(y)) \cdot (u(x) - u(y)) \, dx \, dy.$$ 

At this point, one is tempted to split the integral in two terms and exchange $x$ with $y$ in one of the terms to obtain

$$\langle DJ(u), v \rangle = 2t \int v(x) \left( \int \frac{h(x, y) + h(y, x)}{|x - y|^{n+2s}} \cdot (u(x) - u(y)) \, dy \right) \, dx.$$

However, we cannot always do that. The inner integral in (3.18) is not absolutely convergent. If further symmetry is assumed, it can be defined as a principal value, and $DJ(u)$ takes a form as in (1.1) with

$$K(x, y-x) = \frac{h(x, y) + h(y, x)}{|x - y|^{n+2s}}.$$

When this $K$ satisfies $K(x, y-x) = K(x, -y)$, then a simple calculation shows that

$$\langle DJ(u), v \rangle =$$

$$= 2t \int_{\mathbb{R}^n} v(x) \left( \lim_{r \to 0} \int_{\mathbb{R}^n \setminus B_r(x)} \frac{h(x, y) + h(y, x)}{|x - y|^{n+2s}} \cdot (u(x) - u(y)) \, dy \right) \, dx$$

$$= 2t \int_{\mathbb{R}^n} v(x) \left( \text{PV} \int_{\mathbb{R}^n} (u(x) - u(x+y))K(x, y) \, dy \right) \, dx$$

$$= 2t \langle Tu, v \rangle.$$

A kernel $K$ obtained by this means would always satisfy $K(x, y-x) = K(y, x-y)$. In other words, not any functional like (3.17) gives rise to an
operator like (1.1) nor the other way round. This essential difference has to be understood as the difference between divergence and nondivergence second order equations. Our result does not apply in general to minimizers of (3.17). In this respect, the result in this paper is flavored more along the lines of the Krylov-Safonov Harnack inequality than the De Giorgi-Nash-Moser Harnack inequality.

4. The Main Lemma

The proof of the main theorem of this paper is based on the following lemma. The idea of proving a result like this was taken from the paper by De Giorgi [8], where he obtained Hölder regularity for weak solutions to divergence form elliptic equations using a lemma that looks very similar to this one. Lemmas of this type are sometimes called growth lemmas. They became a common tool in regularity theory for elliptic equations since the work of Landis [10]. To adapt it to nonlocal equations, it was necessary to add Condition (4.3) to control the behavior of \( u \) away from the origin.

**Lemma 4.1.** Suppose our operator satisfies \( (2.1) \), and \( u \) is a function that satisfies the following assumptions (where \( \delta \) and \( \eta \) are the same as in (2.2)):

\[
\begin{align*}
(4.1) & \quad Tu(x) \leq 0 \quad \text{when } x \in B_1, \\
(4.2) & \quad u(x) \leq 1 \quad \text{when } x \in B_1, \\
(4.3) & \quad u(x) \leq 2|2x|^\eta - 1 \quad \text{when } x \in \mathbb{R}^n \setminus B_1, \\
(4.4) & \quad \delta < |\{x \in B_1 : u(x) \leq 0\}|.
\end{align*}
\]

Then \( u \leq 1 - \gamma \) in \( B_{1/2} \) for some \( \gamma > 0 \) depending only on \( \kappa \).

**Remark 4.2.** We are assuming that (2.1) holds for a triplet of values \( \delta, \eta \) and \( \kappa \).

For a given kernel \( K \), the values of \( \kappa \) and \( \eta \) for which Assumption (2.1) holds will depend on \( \delta \). The value of \( \gamma \) in the last lemma depends only on \( \kappa \) \( (\gamma = \kappa(\beta(\frac{1}{2}) - \beta(\frac{3}{4}) \text{, where } \beta \text{ is the fixed function } \beta(x) = (1 - x^2)^2) \). But since \( \kappa \) depends on \( \delta \), then it could be said that \( \gamma \) depends on \( \delta \) too.

The only value of \( \delta \) for which we are actually going to apply the lemma is \( \delta = \frac{1}{2} |B_1| \).

**Proof.** Let \( \gamma = \kappa(\beta(\frac{1}{2}) - \beta(\frac{3}{4}) \text{ (recall } b(x) = \beta(|x|)) \). Suppose there is a point \( x_0 \in B_{1/2} \) such that \( u(x_0) > 1 - \gamma = 1 - \kappa \beta(\frac{1}{2}) + \kappa \beta(\frac{3}{4}) \). Then \( u(x_0) + \kappa b(x_0) > 1 + \kappa \beta(\frac{3}{4}) \), and for every \( y \in B_1 \setminus B_{3/4} \), \( u(x_0) + \kappa b(x_0) > u(y) + \kappa b(y) \). That means that the supremum of \( u(x) + \kappa b(x) \) for \( x \in B_1 \) is greater than 1 and is achieved in an interior point of \( B_{3/4} \). Let us call that point \( x_1 \). Now we will evaluate \( T(u + \kappa b)(x_1) \).

On one hand, \( T(u + \kappa b)(x_1) = Tu(x_1) + \kappa Tb(x_1) \leq \kappa Tb(x_1) \).
On the other hand, we have

\[
T(u + \kappa b)(x_1) = \int_{\mathbb{R}^n} \left( (u + \kappa b)(x_1) - (u + \kappa b)(x_1 + y) + \nabla (u + \kappa b)(x_1) \cdot y \right) K(x_1, y) \, dy.
\]

Since \( u + \kappa b \) has a local maximum at \( x_1 \), \( \nabla (u + \kappa b)(x_1) = 0 \). Besides, for any point \( z \in B_1 \), we know \( (u + \kappa b)(x_1) \geq (u + \kappa b)(z) \). Let \( A_0 = \{ y : x_1 + y \in B_1 \land u(x_1 + y) = 0 \} \). We use (4.3) and that \( \kappa < \frac{1}{4} \) to obtain the lower bound:

\[
T(u + \kappa b)(x_1) \geq \int_{(x_1 + y) \in \mathbb{R}^n \setminus B_1} \left( (u + \kappa b)(x_1) - (u + \kappa b)(x_1 + y) \right) K(x_1, y) \, dy
\]

\[
+ \int_{(x_1 + y) \in B_1} (u + \kappa b)(x_1) - (u + \kappa b)(x_1 + y))K(x_1, y) \, dy
\]

\[
\geq \int_{\mathbb{R}^n \setminus B_{1/4}} \left( 2 - 2 \left( |y| + \frac{3}{4} \right) \right) K(x_1, y) \, dy + \int_{A_0} (1 - 2\kappa)K(x_1, y) \, dy
\]

\[
\geq \int_{\mathbb{R}^n \setminus B_{1/4}} (2 - 2|8y|^\eta)K(x_1, y) \, dy + \int_{A_0} \frac{1}{2}K(x_1, y) \, dy.
\]

Therefore

\[
\kappa Tb(x_1) \geq 2 \int_{\mathbb{R}^n \setminus B_{1/4}} (1 - |8y|^\eta)K(x_1, y) \, dy + \int_{A_0} \frac{1}{2}K(x_1, y) \, dy
\]

\[
\geq 2 \int_{\mathbb{R}^n \setminus B_{1/4}} (1 - |8y|^\eta)K(x_1, y) \, dy + \frac{1}{2} \inf_{A \subset B_1, |A| > \delta} \int_A K(x_1, y) \, dy.
\]

But this is a contradiction with (2.1).

**Remark 4.3.** It is to be noticed that the condition \( Tu(x) \leq 0 \) is used only at one (carefully chosen) point \( x_1 \). Another important observation is that the condition \( Tu(x) \leq 0 \) could be replaced by \( Tu(x) \leq \varepsilon \) for small enough \( \varepsilon \) (depending also on \( \kappa \)). If we take \( \varepsilon < (\kappa/2) \sup Tb \), then we get the result with \( y = (\kappa/2)(\beta(\frac{1}{2}) - \beta(\frac{1}{2})) \).

**Remark 4.4.** The integral representation of the operator \( T \) has little to do with the proof. Instead, the proof is based on the behavior of \( T \) with respect to a few test functions and some sort of ellipticity. Linearity is only used for the inequality \( T(u + \kappa b) \leq Tu + \kappa Tb \). If we could get this inequality in some way, then the lemma would apply to any operator \( T \) that applies to \( C^2 \) functions \( u \) with the property that \( |u(x)| \leq (1 + |x|^2)^{\eta_0} (0 < \eta_0 < 1) \), such that the following hold:
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- (ellipticity) If for a pair of functions $u$ and $v$
  \[ u(x_0) = v(x_0), \quad u(x) \leq v(x), \quad \text{for every } x \in \mathbb{R}^n, \]
  then $Tu(x_0) \geq Tv(x_0)$.
- There is $\eta > 0$ and $\frac{1}{4} > \kappa > 0$ such that:

\[ \kappa Tb(x) \leq \leq \inf \{ Tu(y) : u(y) \leq (1 + |x - y|^2)^\eta \wedge \{|y \in B_2(x) : u(y) = 0\}| \geq \delta \}. \]

The property is therefore nonlinear, in the sense that Lemma 4.1 holds for any operator satisfying these assumptions.

The minimum or the maximum of two operators satisfying these assumptions will also satisfy them.

It seems interesting to try to find an explicit (probably nonlinear) operator that is a maximal for a class of operators satisfying these hypotheses. This operator would play the role of the maximal Pucci operator in the theory of uniformly elliptic equations. This maximal operator takes a particularly simple form for operators of constant order like in Section 3.3. For $s \in (0, 1)$ and $0 < a \leq A$, we can define

\begin{equation}
M^+ u(x) = \int_{\mathbb{R}^n} A \left( \frac{u(x) - (u(x + y) + u(x - y))/2^+}{|y|^{n+2s}} \right) \, dy,
\end{equation}

where $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$. It is simple to check that $M^+(a, A)$ is the supremum of all the operators as in Section 3.3. Similarly, the infimum can be computed by

\begin{equation}
M^- u(x) = \int_{\mathbb{R}^n} a \left( \frac{u(x) - (u(x + y) + u(x - y))/2^+}{|y|^{n+2s}} \right) \, dy.
\end{equation}

Notice that these operators satisfy

\[ M^- h(x) \leq M^- (u+h)(x) - M^- u(x) \leq M^+ h(x), \]
\[ M^- h(x) \leq M^+ (u+h)(x) - M^+ u(x) \leq M^+ h(x). \]

By scaling Lemma 4.1, we can obtain the following extension:
Corollary 4.5. Suppose our operator satisfies (2.2), and \( u \) is a function that satisfies the following assumptions in a ball \( B_r(x_0) \), for a given \( \delta > 0 \):

\[
\begin{align*}
(4.7) & \quad Tu(x) \leq 0 \quad \text{when } x \in B_r(x_0), \\
(4.8) & \quad u(x) \leq A \quad \text{when } x \in B_r(x_0), \\
(4.9) & \quad u(x) \leq A \left( 2 \left| \frac{x - x_0}{r} \right|^{\eta} - 1 \right) \quad \text{when } x \in \mathbb{R}^n \setminus B_r(x_0), \\
(4.10) & \quad \delta < \frac{|\{x \in B_r(x_0) : u(x) \leq 0\}|}{|\mathbb{R}^n|}.
\end{align*}
\]

Then \( u \leq (1 - \gamma)A \) in \( B_{r/2}(x_0) \) for some \( \gamma > 0 \) depending on \( \kappa, \eta \) and \( \delta \).

Proof. We consider \( v(x) = (1/A)u(rx + x_0) \). By definition of \( T_{r,x_0} \), \( T_{r,x_0}v(x) = (1/A)[Tu](rx + x_0) \). Then \( v \) satisfies

\[
\begin{align*}
(4.11) & \quad T_{r,x_0}v(x) \leq 0 \quad \text{when } x \in B_1, \\
(4.12) & \quad v(x) \leq 1 \quad \text{when } x \in B_1, \\
(4.13) & \quad v(x) \leq 2|x|^\eta - 1 \quad \text{when } x \in \mathbb{R}^n \setminus B_1, \\
(4.14) & \quad \delta < \frac{|\{x \in B_1 : v(x) \leq 0\}|}{|\mathbb{R}^n|}.
\end{align*}
\]

Since \( T \) satisfies (2.2), then \( T_{r,x_0} \) satisfies (2.1). So we can apply Lemma 4.1 to obtain \( v \leq 1 - \gamma \) in \( B_{1/2} \). Recalling that \( u(x) = Av((x - x_0)/r) \), we obtain the result of the corollary.

5. Hölder Regularity

Theorem 5.1. Let \( u \) be a bounded function so that \( Tu(x) = 0 \) for every \( x \in B_{2r} \). Suppose that \( K \) satisfies (2.2) for \( \delta = |B_1|/2 \). Then for \( \alpha \) small enough, \( u \in C^\alpha(B_r) \) and

\[
\sup_{x,y \in B_r} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C \frac{1}{r^\alpha} \|u\|_\infty,
\]

where the values of \( \alpha \) and \( C \) depend only on the constants \( \kappa \) and \( \eta \) of (2.2).

The proof of this theorem follows from Lemma 4.1 or Corollary 4.5 by a standard iteration technique. The only thing we must be careful is to keep a control of the nonlocal part of the estimate to make sure that in every step we satisfy Condition (4.3) of Lemma 4.1. The details follow.

Proof. We can reduce the problem to the case \( r = 1 \) and \( \text{osc}_{B_1} u = 1 \) by considering the rescaled function \( \tilde{u}(x) = u(rx)/(2\|u\|_\infty) \). So, we will assume this case.
Let $x_0 \in B_1$. We want to show that there exist a $C > 0$ so that
\begin{equation}
|u(x_0) - u(y)| \leq C|x_0 - y|^\alpha
\end{equation}
for any $y \in \mathbb{R}^n$.

Let $\alpha = \min(\eta, -\log(1-\gamma) / \log 2)$, where $\gamma$ is the constant of Lemma 4.1 and $\eta$ is the one from (2.2). We choose $C = 2^\alpha$. Since $\alpha$ depends only on $\eta$ and the $\gamma$ from Lemma 4.1 and $C$ is computed from $\alpha$, then $\alpha$ and $C$ depend only on the constants $\kappa$ and $\eta$ of (2.2).

We will show by induction, that for any integer number $k$,
\begin{equation}
\text{osc}_{B_{2^{-k}}(x_0)} u \leq 2^{-k\alpha}.
\end{equation}
More precisely, we will construct a nondecreasing sequence $b_k$ and a nonincreasing sequence $a_k$ such that
\begin{equation}
b_k \leq u(x) \leq a_k \quad \text{when } x \in B_{2^{-k}}(x_0) \text{ and } a_k - b_k = 2^{-k\alpha}.
\end{equation}

Suppose that $2^{-k} \leq |x_0 - y| < 2^{-k+1}$, then by (5.2) we would conclude
\begin{equation}
|u(y) - u(x_0)| \leq 2^{(-k+1)\alpha} \leq 2^\alpha |x_0 - y|^\alpha,
\end{equation}
obtaining (5.1).

Let us construct $a_k$ and $b_k$ by induction. For $k = 0$, we can take $b_k = \inf u$ and $a_k = b_k + 2^{-k\alpha}$ because $\text{osc}_{\mathbb{R}^n} u = 1$.

Let us assume we already have $a_j$ and $b_j$ for any $j \leq k$; we have to find suitable $a_{k+1}$ and $b_{k+1}$.

Let $m = (a_k + b_k)/2$; then, by (5.3), $|u - m| \leq \frac{1}{2} 2^{-k\alpha}$ in $B_{2^{-k}}(x_0)$.

Consider $\tilde{u}(x) = 2.2^{\alpha k}(u(2^{-k}x + x_0) - m)$. Then $|\tilde{u}(x)| \leq 1$ for $x \in B_1$ and $T_{2^{-k}x_0} \tilde{u} = 0$ in $B_1$. Let us suppose that $|x \in B_1 : \tilde{u}(x) \leq 0| \geq \frac{1}{2} |B_1|$.

When $|\gamma| > 1$, let $j \geq 0$ such that $2^j \leq |\gamma| < 2^{j+1}$, then $2^{-k+j} \leq |2^{-k} \gamma| < 2^{-k+j+1}$, then by the inductive hypothesis
\begin{align*}
\tilde{u}(\gamma) &= 2.2^{\alpha k}(u(2^{-k} \gamma + x_0) - m) \\
&\leq 2.2^{\alpha k}(a_{k-j-1} - b_{k-j-1} + b_k - m) \\
&\leq 2.2^{\alpha k} \left(2^{-j-1}\alpha - \frac{1}{2} 2^{-k\alpha}\right) \leq 2.2^{(j+1)\alpha} - 1 \leq 2|2^j \gamma|^\alpha - 1.
\end{align*}

The function $\tilde{u}$ is harmonic in $B_1$ with respect to the operator $T_{2^{-k}x_0}$ that also satisfies (2.1), since $K$ satisfies (2.2). We have all what we need to apply Lemma 4.1 and obtain $\tilde{u}(x) \leq 1 - \gamma$ for every $x \in B_{1/2}$. We then scale back to $u$ to see that $u \leq b_k + ((2-\gamma)/2) 2^{-k\alpha}$ in $B_{2^{-k+1}}$. Then we can define $b_{k+1} = b_k$ and
\[ a_{k+1} = 2^{-k-1}a + b_k, \] and we still satisfy \( u \leq a_{k+1} \) in \( B_{2^{-k-1}} \) because \( \alpha \) was chosen so that \( (2-\gamma)/2 \leq 2^{-\alpha} \).

In the case \( |x| \in B_1 : \bar{u}(x) = 0 \leq \frac{1}{2}|B_1| \), we do the same reasoning with \( -\bar{u} \) instead of \( \bar{u} \) to obtain that \( u \geq a_k - ((2-\gamma)/2)2^{-k\alpha} \). So we define \( a_{k+1} = a_k \) and \( b_{k+1} = a_k - 2^{-(k-1)\alpha} \).

**Remark 5.2.** Theorem 5.1 holds for any operator such that if \( Tu = 0 \), then \( T(Au + C) = 0 \), and \( T_{r,x} \) satisfies the assumptions of Remark 4.4 for every \( r > 0 \) and \( x \in \mathbb{R}^n \).

If we were interested only in the constant order case, we could state Theorem 5.1 in the following fashion.

**Theorem 5.3.** Given \( s \in (0,1) \) and \( 0 < a \leq A \), let \( M^+ \) and \( M^- \) be the operators defined in (4.5) and (4.6). Let \( u \) be a bounded function so that \( M^+ u \geq 0 \) and \( M^- u \leq 0 \) in \( B_{2r} \). Then for a small \( \alpha \), \( u \in C^\alpha(B_r) \) and

\[
\sup_{x,y \in B_r} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \leq C \frac{1}{r^\alpha} \|u\|_\infty.
\]

Moreover, if \( u \) is merely continuous, the conditions \( M^+ u \geq 0 \) and \( M^- u \leq 0 \) in \( B_{2r} \) could be taken in the viscosity sense. By this we mean that for every smooth function \( \varphi \) touching \( u \) from above at a point \( x_0 \), i.e.,

\[ \varphi(x_0) = u(x_0), \quad \varphi(x) \geq u(x), \quad \text{for every } x \in \mathbb{R}^n, \]

\( M^- \varphi(x_0) \leq 0 \), and for every smooth function \( \varphi \) touching \( u \) from below at \( x_0 \), \( M^+ \varphi(x_0) \geq 0 \).

The theory of viscosity solutions for nonlocal operators seems to be at an early stage of development at the time this paper is written. Some references can be found in [1], [2] or [3].

If the operator \( T \) has a structure like in Section 3.4, not only is Theorem 5.1 valid, but we can also have a right-hand side using Remark 4.3 and the scaling properties of the operators of Section 3.4.

**Theorem 5.4.** Let \( u \) be a bounded function so that \( Tu(x) = f(x) \) for every \( x \in B_{2r} \), where \( f \) is a bounded function and \( T \) is of the form (1.1) with \( K \) satisfying

\[
\frac{a}{|y|^{n+2s(x)}} \leq K(x,y) \leq \frac{A}{|y|^{n+2s(x)}},
\]

\[
K(x,y) = K(x,-y),
\]

for \( 0 < a \leq A \) and \( s \in (0,1) \) and \( 0 < \inf s(x) \leq s(x) < 1 \).

Then for \( \alpha \) small enough, \( u \in C^\alpha(B_r) \), and we have the estimate

\[
\frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \leq C \frac{1}{r^\alpha} \left( \|u\|_\infty + \max\{r^{\inf 2s}, r^{\sup 2s}\}\|f\|_\infty \right)
\]
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for any $x \in B_r$, where $\alpha$ and $C$ depend only on $a$, $A$, $\inf s$, and $\sup s$.

**Proof.** First of all, we normalize $u$ so that $r = 1$, $\text{osc} \, u \leq 1$ and $\|f\|_{\infty} \leq \varepsilon$, where $\varepsilon$ is the constant of Remark 4.3. To achieve this, we consider the following $\tilde{u}$ instead of $u$:

$$\tilde{u}(x) = \frac{u(rx)}{2\|u\|_{\infty} + \max(r^{2\inf s}, r^{\sup s})\|f\|_{\infty}/\varepsilon}$$

so that

$$r^{2s(rx)}T_{r,0}\tilde{u}(x) = \frac{r^{2s(rx)}f(rx)}{\|u\|_{\infty} + \max(r^{2\inf s}, r^{\sup s})\|f\|_{\infty}/\varepsilon} \leq \tilde{f}(x) \leq \varepsilon$$

and the operators $r^{2s(rx)}T_{r,0}$ satisfy (5.4) and (5.5) with $s(rx)$ instead of $s(x)$.

Then we can continue in the same way as in the proof of Theorem 5.1 but using the observation in Remark 4.3 instead of Lemma 4.1. We are able to obtain an improvement of the oscillation in the first iteration step because we are considering a small enough right-hand side.

For any $x_0 \in B_1$ we construct, as in the proof of Theorem 5.1, a pair of sequences $a_k$ and $b_k$ bounding $\tilde{u}$ in $B_{2^{-k}}(x_0)$ from above and from below, respectively, so that $a_k - b_k = 2^{-k\alpha}$.

In each iteration step, we rescale $\tilde{u}$ by considering

$$v(x) = 2^{\alpha k}(\tilde{u}(2^{-k}x + x_0) - m),$$

where $m = (a_k + b_k)/2$. This function satisfies the equation:

$$\left(2^{-k}\right)^{2s_{2^{-k}x_0}(x)}T_{2^{-k},x_0}v(x) = \left(2^{-k}\right)^{-\alpha + 2s_{2^{-k}x_0}(x)}\tilde{f}(2^{-k}x + x_0).$$

That means that for $\alpha < 2\inf s$, the right-hand side is less then or equal to $\tilde{f}(2^{-k}x - x_0) < \varepsilon$. Moreover, the operators $\left(2^{-k}\right)^{2s_{2^{-k}x_0}(x)}T_{2^{-k},x_0}$ satisfy (5.4) and (5.5) uniformly. Thus, we can apply Remark 4.3 to either $v$ or $-v$ to get an improvement of the oscillation of $\tilde{u}$ in $B_{2^{-k-1}}$ and all the iteration steps can be carried out as in the proof of Theorem 5.1. We obtain the estimate

$$\frac{|\tilde{u}(x_0) - \tilde{u}(y)|}{|x_0 - y|^{\alpha}} \leq 2^\alpha$$

for any $y \in \mathbb{R}^n$.

Thus, replacing $u$ by (5.6),

$$\frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \leq C \frac{1}{r^{\alpha}} \left(\|u\|_{\infty} + \max(r^{\inf 2s}, r^{\sup 2s})\|f\|_{\infty}\right)$$

for any $x \in B_r$, where $C$ and $\alpha$ depend on $\kappa$ and $\eta$ from (2.2) and $\varepsilon$ from Remark 4.3, but all those quantities can be computed from $a$, $A$, $\inf s$, and $\sup s$.  □
6. Applications

6.1. A Liouville property

Corollary 6.1. Let $T$ be an operator that satisfies (2.2); if $u$ is a bounded global solution of $Tu(x) = 0$, then $u$ is constant.

Proof. Given $x, y \in \mathbb{R}^n$, take any $r > 0$ so that $x, y \in B_r$; by Theorem 5.1,

\[ \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \leq C \frac{1}{r^{\alpha}} \|u\|_{\infty}. \]

Taking $r$ large enough, the right-hand side converges to zero. Therefore $u(x) = u(y)$ for any $x$ and $y$, and $u$ is constant.

6.2. Nonlinear equations

As an example we will consider two kernels $K_1$ and $K_2$ depending only on $y$ so that the operators:

\[ Tiu(x) = PV \int (u(x) - u(x+y)) + \nabla u(x) \cdot y \, K_i(y) \, dy \]

satisfy (2.2).

We consider kernels depending only on $y$ so that the operators $Ti$ commute with translations and therefore also with differentiation.

Now, let $F : \mathbb{R}^2 \to \mathbb{R}$ be a function that is strictly increasing in each coordinate in the sense that $\partial_i F \geq C$ for $i = 1, 2$. The estimate will not depend on its smoothness.

We can obtain an interior $C^{1,\alpha}$ estimate for bounded solutions $u$ to the nonlinear equation

(6.1) $F(T_1u(x), T_2u(x)) = 0$.

We have all the necessary ingredients to perform a proof almost identical to the one of Section 5.3 in [7]. The idea is that the difference between $u$ and a translation of $u$ solves an equation with an operator that satisfies (2.2), then a telescopic sum iteration method shows that $u$ is Lipschitz, and finally the derivative of $u$ also solves an equation for which we can conclude that $\nabla u \in C^{\alpha}$.

The same property could be obtained for a nonlinear operator $F$ that maps $C^2$ functions $u$ such that $|u(x)| \leq (1 + |x|^2)^{\eta_0}$ into continuous functions such that there is a uniform choice of $\eta$ and $\kappa$ so that for each such function $u$, the operator:

\[ Tv = F(u + v) - F(u) \]

satisfies the assumptions of Remark 5.2.

In the constant order case, the right assumption would be

\[ M^- v \leq F(u + v) - F(u) \leq M^+ v, \]

where $M^+$ and $M^-$ are the operators defined in (4.5) and (4.6).
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