Communications in Partial Differential Equations, 31: 1227–1252, 2006

ISSN 0360-5302 print/1532-4133 online
DOI: 10.1080/03605300600634999

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Regularity for Fully Nonlinear Elliptic Equations with Neumann Boundary Data

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We obtain local C^α, $C^{1,\alpha}$, and $C^{2,\alpha}$ regularity results up to the boundary for viscosity solutions of fully nonlinear uniformly elliptic second order equations with Neumann boundary conditions.

Keywords Fully nonlinear elliptic equations; Neumann boundary conditions; Viscosity solutions.

Mathematics Subject Classification Primary 35J65; Secondary 35B65.

1. Introduction

The theory of viscosity solutions gives a solid framework to study fully nonlinear elliptic equations, and provides a powerful way to prove existence and uniqueness in a very general setting. The question of regularity of the corresponding solutions (that in principle are merely continuous) has been studied extensively in the last decade. There are good results for interior regularity as well as for regularity up to the boundary in the case of the Dirichlet problem. However, for the Neumann problem, there are still not many results. We intend to work in that direction.

On the other hand there are several articles corresponding to uniqueness, comparison theorems, Hölder and Lipschitz continuity for solutions of general fully nonlinear second order elliptic equations with Neumann type boundary conditions. We refer the reader interested in the viscosity solutions approach to Ishii and Lions (1990) and Ishii (1991) where the authors investigate uniqueness results that yield existence via an adaptation of the classical Perron's method. A later article of Barles (1993) gives uniqueness and Lipschitz regularity results for quite general boundary conditions in the case where the boundary is assumed to be smooth enough and the differential operators are basically Lipschitz. Cranny (1996) concerned with C^α regularity of solutions for less regular operators. He achieved these results

Received March 1, 2005; Accepted January 1, 2006
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under some mild geometric conditions upon the domain. We also refer the reader interested in the classical solutions approach to the article of Lions and Trudinger (1986) where the authors proved, using the continuity method, that a problem with an oblique derivative condition at the boundary has a $C^{2,\alpha}$ solution if the equation is convex.

In the present paper we will consider the regularity for viscosity solutions of fully nonlinear uniformly elliptic second order equations with Neumann boundary data. We will always consider the domain to be the upper half ball, and the Neumann boundary data to be given on its base.

We will use the following notation:

$$B_1^+ = \{x \in \mathbb{R}^n : |x| < 1, x_n > 0\}$$
$$\Upsilon = \{x \in \mathbb{R}^n : |x| < 1, x_n = 0\}.$$

The vector $\nu = (0, \ldots, 0, 1)$ is the inner normal to Υ (the base of B_1^+).

The article is organized as follows: The first two sections are the introduction and preliminaries. The third section is devoted to obtaining an extension of the Alexandroff–Backelman–Pucci (ABP) estimate to Neumann boundary conditions. In the fourth section we prove the C^α regularity up to the boundary for the solution of the homogeneous problem

$$\begin{cases} F(D^2u) = 0 & \text{in } B_1^+ \\ u_\nu = 0 & \text{in } \Upsilon. \end{cases} \tag{1.1}$$

In section five we develop some properties for sup- and inf-convolutions and in section six we obtain the $C^{1,\alpha}$ regularity up to the boundary for the solution u of problem (1.1). In the seventh section we get the $C^{2,\alpha}$ regularity up to the boundary for u assuming that F is convex (or concave). In sections eight and nine we extend our results to more general problems corresponding to x dependence on F or to inhomogeneous right hand side. At the end, there is an appendix with the proof of a regularity result for Dirichlet boundary conditions. We expect that our results can be extended to more general (nonflat) domains, but we have not worked in that direction yet. It is our intention to use these results for the upper half ball in a forthcoming article (Milakis and Silvestre, In preparation) associated with Signorini-like obstacle problems.

We believe that it is insightful to think of the Neumann condition as part of the equation and not as boundary data. Our results are local in the sense that we only require a Neumann condition in a piece of the boundary, then obtain regularity there regardless of how the function behaves far from those points. This could be thought as an interior regularity result, if we think of the Neumann condition as part of the equation.

2. Preliminaries

First of all, we make some remarks about our notation. When we say that a function ϕ *touches* another function u from above (or resp. from below) at a point x, we mean that $\phi(x) = u(x)$ and $\phi(y) > u(y)$ (or resp. $\phi(y) < u(y)$) for every y in a neighborhood of x. Strictly speaking, it is not the functions themselves but their graphs that touch each other at the point $(x, u(x)) = (x, \phi(x))$.

When we say that u *solves* $F(D^2u) = 0$ in B_1^+, we always mean in the viscosity sense, and F is always assumed to be uniformly elliptic with constants λ and Λ. A constant is considered universal when it depends only on λ, Λ, and n (the dimension). When we say $u_\nu \geq 0$ on Υ, we also refer to the viscosity sense. By this we mean that for any smooth function ϕ touching u from above at a point x_0 in Υ we have $\phi_\nu(x_0) \geq 0$. Similarly, when we say $u_\nu \leq 0$ in Υ, we mean that if ϕ touches u from below at a point $x_0 \in \Upsilon$, then $\phi_\nu(x_0) \leq 0$.

We also use the notation $S(\lambda, \Lambda, f)$, $\overline{S}(\lambda, \Lambda, f)$, and $S^*(\lambda, \Lambda, f)$ as in Caffarelli and Cabré (1995).

For Dirichlet boundary data, the regularity up to the boundary is fairly well understood. The following propositions are more or less well known for the specialists. However, since we could not find any reference where these propositions are proven, we give the proofs in the appendix of this article.

Proposition 2.1. *Let u be a solution of $F(D^2u) = 0$ in B_1^+ such that $u \in C(\overline{B_1^+})$ and the restriction of u to Υ is C^α for some $\alpha < 1$, then u is $C^\alpha(\overline{B_{1/2}^+})$ up to the boundary. Moreover, we have the estimate*

$$\|u\|_{C^\alpha(\overline{B_{1/2}^+})} \leq C\Big(\|u\|_{C(\overline{B_1^+})} + \|u\|_{C^\alpha(\Upsilon)} + |F(0)|\Big) \tag{2.1}$$

for a constant C depending only on n, λ, Λ, and α ($C \to \infty$ as $\alpha \to 0$).

Proposition 2.2. *Let u be a solution of $F(D^2u) = 0$ in B_1^+ such that $u \in C(\overline{B_1^+})$ and the restriction of u to Υ is $C^{1,\alpha}$ for some $\alpha > 0$, then u is $C^{1,\beta}(\overline{B_{1/2}^+})$ up to the boundary, where $\beta = \min(\alpha, \alpha_0)$ for a universal α_0. Moreover, we have the estimate*

$$\|u\|_{C^{1,\beta}(\overline{B_{1/2}^+})} \leq C\Big(\|u\|_{C(\overline{B_1^+})} + \|u\|_{C^{1,\alpha}(\Upsilon)} + |F(0)|\Big) \tag{2.2}$$

for a constant C depending only on n, λ, Λ, and α ($C \to \infty$ as $\alpha \to 0$).

We are going to develop corresponding results for Neumann boundary data.

In many proofs we use that a C^α (or $C^{1,\alpha}$, or $C^{2,\alpha}$) estimate on Υ plus an interior estimate implies the estimate all the way up to the bottom. This is a standard procedure in the regularity theory that we illustrate in the following propositions.

Proposition 2.3. *Let u be a continuous function in $\overline{B_1^+}$ that satisfies C^α interior estimates. By this we mean that if $B_r(x_0) \subset B_1^+$,*

$$\frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq C\frac{1}{r^\alpha} \operatorname*{osc}_{B_r(x_0)} u \quad \text{for every } x, y \in B_{r/2}(x_0). \tag{2.3}$$

Let us also suppose that u is C^α at the bottom boundary, i.e.,

$$|u(x) - u(y)| \leq C_0|x-y|^\alpha \quad \text{for } x \in \Upsilon \text{ and } y \in B_1^+. \tag{2.4}$$

Then $u \in C^\alpha(B_{1/2}^+)$ and

$$|u(x) - u(y)| \leq CC_0|x-y|^\alpha \tag{2.5}$$

for every $x \in B_{1/2}^+$ and $y \in B_1^+$, where C depends only on the constant of (2.3).

Similar results for $C^{1,\alpha}$ and $C^{2,\alpha}$ estimates are also valid. The statement for $C^{2,\alpha}$ will be needed later in the article and is proven in the appendix for completeness.

Proposition 2.4. *Let $u \in C(B_1^+)$ be a viscosity solution of*

$$F(D^2u) = 0 \quad \text{in } B_1^+$$

for a uniformilly elliptic convex function F. We known from Caffarelli and Cabré (1995) that solutions of such equations have interior $C^{2,\alpha}$ estimates for a universal constant α. Let us also assume that u is $C^{2,\alpha}$ on Υ for the same α. By this we mean that for every $x \in \Upsilon$, there is a second order polynomial P_x such that for any $y \in B_1^+$,

$$|u(y) - P_x(y)| \leq C_0|x - y|^{2+\alpha}. \tag{2.6}$$

Then $u \in C^{2,\alpha}(B_1^+)$ and

$$\|u\|_{C^{2,\alpha}} \leq C \cdot C_0$$

where C is a universal constant.

The proofs of Propositions 2.3 and 2.4 are done in the appendix.

3. An Extension of the ABP Estimate

We obtain an extension to the ABP estimate to Neumann boundary conditions where by $\overline{S}$ class we mean the usual function space dealing with the Pucci's extremal operators (see Section 2.2 of Caffarelli and Cabré, 1995).

Proposition 3.1. *Let $u \in C(\overline{B_1^+})$ be a function that belongs to $\overline{S}(\lambda, \Lambda, f)$ in B_1^+ such that it satisfies $u_\nu = g$ in Υ in the viscosity sense. Then*

$$\inf_{\partial B_r \cap \{x_n > 0\}} u(x) - \inf_{B_r^+} u \leq Cr\left(\int_{\{u=\Gamma_u\}} |f^+(x)|^n \, dx\right)^{1/n} + Cr \sup_{\Upsilon} g \tag{3.1}$$

where Γ_u is the convex envelope of u and C is a universal constant.

Proof. To simplify the notation we can suppose $\inf_{\partial B_r \cap \{x_n > 0\}} u(x) = 0$. From Caffarelli and Cabré (1995, Chapter 3) we know that $\Gamma_u \in C^{1,1}(\overline{B_r^+})$. Let $u(x_0) = \inf_{B_r^+} u$. We will follow the usual procedure of finding a subset of $\nabla\Gamma_u(B_r^+)$. Let us define the following set (see Figure 1)

$$\mathcal{A} := \left\{ A \in \mathbb{R}^n : A \cdot \nu \geq \max g, |A| \leq \frac{-\inf_{B_r^+} u}{2r} \right\}.$$

Take a vector A such that $A \cdot \nu \geq \max g$ and

$$|A| \leq \frac{-\inf_{B_r^+} u}{2r}. \tag{3.2}$$

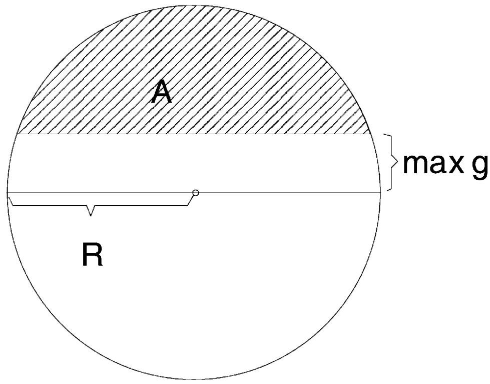

Figure 1. The set $\mathscr{A}$.

Therefore $A \cdot (x - x_0) + u(x_0)$ is a linear function that coincides with u at x_0 and is below u in $\partial B_r \cap \{x_n > 0\}$. Then there is a translation $A \cdot x + b$ such that it touches u from below in a point that is not in $\partial B_r \cap \{x_n > 0\}$. Since $A \cdot v \geq \max g$, then $A \cdot x + b$ cannot touch u at the bottom Υ. Therefore it touches at an interior point and $A \in \nabla\Gamma_u(B_r^+)$. Thus we have:

$$\mathscr{A} \subset \nabla\Gamma_u(B_r^+).$$

The set $\mathscr{A}$ is the upper cap of a ball. Let $R = \frac{-\inf_{B_r^+} u}{2r}$. If $\max g > R$, then $\mathscr{A}$ is empty. If $\max g < R/3$, then $|\mathscr{A}| \geq CR^n$, where C depends only on dimension. To summarize, one of the following two happens:

1. $-\inf_{B_r^+} u < 4r \sup g$;
2. $|\mathscr{A}| \geq CR^n$.

In the second case, we follow the usual proof of the ABP-estimate to obtain

$$\int_{\{u=\Gamma_u\}} |f^+(x)|^n \,\mathrm{d}x \geq CR^n$$

for a universal constant C. And therefore, combining the two cases,

$$-\inf_{B_r^+} u \leq Cr \max\left(\left(\int_{\{u=\Gamma_u\}} |f^+(x)|^n \,\mathrm{d}x\right)^{1/n}, \sup g\right)$$

which is equivalent to what we wanted to prove. □

4. Hölder Regularity

In the present section we intend to prove C^α regularity for the solution up to the boundary. We are going to use the following reflection property and the Hölder regularity for functions in S^* class where by S^* class we mean the usual function space dealing with $\underline{S}$ and $\overline{S}$.

We point out that this regularity has been proven in a much more general situation (see Cranny, 1996). In our case we can provide a simpler proof.

Proposition 4.1 (Reflection Property). *Let $u : B_1^+ \to \mathbb{R}$ be a function that belongs to $S^*(\lambda, \Lambda, f)$ in B_1^+ such that it satisfies $u_\nu = 0$ in Υ in the viscosity sense. Then the reflected function*

$$u^* = \begin{cases} u(x) & \text{when } x_n \geq 0 \\ u(x', -x_n) & \text{when } x_n < 0 \end{cases} \tag{4.1}$$

belongs to the class $S^*(\lambda, \Lambda, f^*)$ in B_1, where f^* is reflected the same way as u^*.

Proof. We will show that u^* belongs to $S^*(\lambda, \Lambda, f^*)$ in B_1, where f is also reflected in the same way as u.

For $\gamma \in \mathbb{R}$, let us consider the function $v_\gamma = u^* + \gamma|x_n|$. It is clear that $v_\gamma \in S^*(\lambda, \Lambda, f^*)$ in B_1^+ as well as $v_\gamma \in S^*(\lambda, \Lambda, f^*)$ in B_1^-, since the Pucci extremal operators depend only on the second derivatives and are invariant under symmetries.

When $\gamma > 0$, then v_γ cannot be touched by any smooth function from above at the points in Υ. Indeed, if ϕ was such a test function, then $\phi(x) - \gamma x_n$ would touch u in the Neumann boundary, and therefore $\phi_\nu \geq \gamma$ at the contact point. But $\phi(x', -x_n) - \gamma x_n$ would also touch u from above at the same point in the boundary, then $\phi_\nu \leq -\gamma$, obtaining a contradiction.

Therefore, since v_γ can never be touched from above at any point in Υ, and in the rest of B_1, v_γ is in the S^* class, then $v_\gamma \in \underline{S}(\lambda, \Lambda, -|f^*|)$ in B_1, when $\gamma > 0$.

Similarly, we obtain $v_\gamma \in \overline{S}(\lambda, \Lambda, |f^*|)$ in B_1, when $\gamma < 0$.

But $v_\gamma \to u^*$ uniformly as $\gamma \to 0$. Since the classes of $\overline{S}$ and $\underline{S}$ are closed under uniform limits, then u^* belongs to both, in other words $u^* \in S^*(\lambda, \Lambda, f^*)$ in B_1. $\square$

Proposition 4.2. *Let $u : B_1^+ \to \mathbb{R}$ be a function that belongs to $S^*(\lambda, \Lambda, f)$ in B_1^+ such that it satisfies $u_\nu = 0$ in the viscosity sense in Υ. Then $u \in C^\alpha(B_{1/2}^+)$ up to the boundary, for a universal $\alpha > 0$. Moreover, we have the estimate*

$$\|u\|_{C^\alpha(\overline{B_{1/2}^+})} \leq C\Big(\|u\|_{L^\infty(B_1^+)} + \|f\|_{L^n(B_1^+)}\Big).$$

Proof. Since the reflected function u^* of Proposition 4.1 is in the class S^* across the boundary. Then u^* is C^α in $B_{1/2}$ by interior estimates (see Caffarelli and Cabré, 1995, Section 4.3). Thus $u \in C^\alpha(\overline{B_{1/2}^+})$ up to the boundary. The estimate follows from the C^α estimates for the S^* class (see Caffarelli and Cabré, 1995, Proposition 4.10). $\square$

Corollary 4.3. *Let u be a solution of a fully nonlinear uniformly elliptic equation $F(D^2u) = 0$ in B_1^+ with Neumann data $u_\nu = 0$ in Υ in the viscosity sense. Then $u \in C^\alpha(\overline{B_{1/2}^+})$ up to the boundary, for a universal $\alpha > 0$. Moreover, we have the estimate*

$$\|u\|_{C^\alpha(\overline{B_{1/2}^+})} \leq C\Big(\|u\|_{L^\infty(B_1^+)} + |F(0)|\Big).$$

5. Sup- and Inf-Convolutions

In Jensen (1988), the author introduced the concept of sup- and inf-convolutions to prove comparison principles for viscosity solutions of second order partial differential equations. We will see in this section, that this concept applies *up to the boundary* in our situation.

Let $u : B_1^+ \to \mathbb{R}$. We will consider the following definition of sup- and inf-convolutions:

$$u^\varepsilon(x) = \sup_{y \in \overline{B_{1-\varepsilon}^+}} \left\{ u(y) - \frac{1}{\varepsilon}|x-y|^2 \right\} \tag{5.1}$$

$$u_\varepsilon(x) = \inf_{y \in \overline{B_{1-\varepsilon}^+}} \left\{ u(y) + \frac{1}{\varepsilon}|x-y|^2 \right\}. \tag{5.2}$$

The following property is standard and its proof can be found in Caffarelli and Cabré (1995, Theorem 5.1).

Proposition 5.1. *The sup-convolution satisfies the following properties*:

1. $u^\varepsilon \in C(\overline{B_{1-\varepsilon}^+})$;
2. $u^\varepsilon \to u$ *uniformly as* $\varepsilon \to 0$;
3. *For any point* $x_0 \in \overline{B_{1-\varepsilon}^+}$, *there is a concave paraboloid of opening* $\frac{2}{\varepsilon}$ *that touches* u^ε *from below. Hence,* $u^\varepsilon \in C^{1,1}$ *by below.*

Lemma 5.2. *Suppose u satisfies* $u_\nu \geq 0$ *in* Υ, *in the viscosity sense. For any* $x \in B_1^+$, *the* sup (*resp.* inf) *in* (5.1) (*resp.* (5.2)) *is achieved for a* $y_0 \in \overline{B_{1-\varepsilon}^+} \setminus \Upsilon$.

Proof. Since we consider u to be continuous and $\overline{B_{1-\varepsilon}^+}$ is a compact set, then the supremum in (5.1) is achieved. We have to check that for the y_0 that achieves this supremum is not in Υ.

Suppose that $y_0 \in \Upsilon$,

$$\begin{aligned} u^\varepsilon(x) &= \sup_{y \in \overline{B_{1-\varepsilon}^+}} \left\{ u(y) - \frac{1}{\varepsilon}|x-y|^2 \right\} \\ &= u(y_0) - \frac{1}{\varepsilon}|x-y_0|^2. \end{aligned} \tag{5.3}$$

Then

$$u(y_0) - \frac{1}{\varepsilon}|x-y_0|^2 + \frac{1}{\varepsilon}|x-y|^2 \geq u(y) \tag{5.4}$$

for every $y \in \overline{B_{1-\varepsilon}^+}$.

Therefore function $v(y) = u(y_0) - \frac{1}{\varepsilon}|x-y_0|^2 + \frac{1}{\varepsilon}|x-y|^2$ touches u from above at the point $y = y_0$. By the Neumann boundary condition in the viscosity sense, we have $v_\nu(y_0) \geq 0$. But $v_\nu(y_0) = \frac{2}{\varepsilon}(y_0 - x) \cdot \nu < 0$ since $x \in B_1^+$ and $y_0 \in \Upsilon$. $\square$

Lemma 5.3. *Let u be a subsolution of the equation* $F(D^2u) \geq 0$ *and* $u_\nu \geq 0$ *in the viscosity sense. Then* u^ε *is also a subsolution of the same equation* (*same conclusion holds with supersolutions if we consider* u_ε *instead of* u^ε).

Proof. Suppose that $P(x)$ touches u^ε from above at a point x_0.

If $x_0 \in B^+_{1-\varepsilon}$, then $u^\varepsilon(x_0) = u(y_0) - \frac{1}{\varepsilon}|x_0 - y_0|^2$ for some $y_0 \in B^+_{1-\varepsilon}$. Now $u^\varepsilon(x) \geq u(x + y_0 - x_0) - \frac{1}{\varepsilon}|x_0 - y_0|^2$ in a neighborhood of x_0. Therefore $Q(x) = P(x + x_0 - y_0) + \frac{1}{\varepsilon}|x_0 - y_0|^2$ touches u from above at the point y_0. Since u is a subsolution we have $F(D^2P(x_0)) = F(D^2Q(y_0)) \geq 0$.

If $x_0 \in \Upsilon \cap \{|x| < 1 - \varepsilon\}$, then $u^\varepsilon(x_0) = u(y_0) - \frac{1}{\varepsilon}|x_0 - y_0|^2$ for some $y_0 \in \overline{B^+_{1-\varepsilon}}$. Now $u^\varepsilon(x) = u(y_0) - \frac{1}{\varepsilon}|x - y_0|^2$, therefore $P_\nu(x_0) \geq \frac{2}{\varepsilon}(y_0 - x_0) \cdot \nu \geq 0$. □

Proposition 5.4. *Let u be a subsolution of the equation $F(D^2u) \geq 0$ and $u_\nu \geq 0$. Let v be a supersolution of the equation $F(D^2v) \leq 0$ and $v_\nu \leq 0$. Then $u - v \in \underline{S}(\frac{\lambda}{n}, \Lambda)$ in B^+_1 and $(u - v)_\nu \geq 0$ in Υ.*

Proof. The proof uses sup- and inf-convolutions. The proof that $u - v \in \underline{S}\big(\frac{\lambda}{n}, \Lambda\big)$ in B^+_1 can be found in Caffarelli and Cabré (1995, Theorem 5.3). We will concentrate here in the boundary condition. We know that u^ε and u_ε satisfy also the same inequality for the normal derivatives in the boundary Υ.

Let $x_0 \in \Upsilon$. Suppose that $P(x)$ touches $u^\varepsilon - v_\varepsilon$ from above at a point x_0. Let y_0 and y_1 be the point that realize the supremum and infimum respectively:

$$u^\varepsilon(x_0) = u(y_0) - \frac{1}{\varepsilon}|x_0 - y_0|^2 \tag{5.5}$$

$$v_\varepsilon(x_0) = v(y_1) + \frac{1}{\varepsilon}|x_0 - y_1|^2. \tag{5.6}$$

Then $u^\varepsilon(x) = u(y_0) - \frac{1}{\varepsilon}|x - y_0|^2$ and $v_\varepsilon(x) = v(y_1) + \frac{1}{\varepsilon}|x - y_1|^2$ for any x. Therefore,

$$u^\varepsilon(x) - v_\varepsilon(x) = u(y_0) - \frac{1}{\varepsilon}|x - y_0|^2 - v(y_1) - \frac{1}{\varepsilon}|x - y_1|^2 = G(x).$$

Then, $P(x)$ also touches $G(x)$ from above at x_0, thus

$$P_\nu(x_0) \geq \frac{2}{\varepsilon}\nu \cdot (y_0 - x_0) + \frac{2}{\varepsilon}\nu \cdot (y_1 - x_0) \geq 0.$$ □

Remark 5.5. As the referee pointed out, it is also possible to prove Proposition 5.4 using a *doubling variables type* argument, as it is standard in viscosity solutions theory. For a general description of the method see Crandall et al. (1992).

6. Hölder Estimates for the First Derivatives

The main result of this section is the following theorem.

Theorem 6.1. *Let u be a solution of $F(D^2u) = 0$ in B^+_1 and $u_\nu = 0$ in Υ. Then u is $C^{1,\alpha}(\overline{B^+_{1/2}})$ up to the boundary, for a universal $\alpha > 0$. Moreover, we have the estimate*

$$\|u\|_{C^{1,\alpha}(\overline{B^+_{1/2}})} \leq C\Big(\|u\|_{C(\overline{B^+_1})} + |F(0)|\Big) \tag{6.1}$$

for a universal constant C.

Our proof of Theorem 6.1 is an adaptation of the proof of Corollary 5.7 in Caffarelli and Cabré (1995) (the interior $C^{1,\alpha}$ regularity for uniformly elliptic equations). We will use the following lemma, whose proof can be found in Caffarelli and Cabré (1995, Lemma 5.6).

Lemma 6.2. *Let $0 < \alpha < 1$, $0 < \beta \leq 1$, and $K > 0$ be constants. Let $u \in L^\infty([-1, 1])$ satisfy $\|u\|_{L^\infty([-1,1])} \leq K$. Define, for $h \in \mathbb{R}$ with $0 < |h| \leq 1$,*

$$v_{\beta,h}(x) = \frac{u(x+h) - u(x)}{|h|^\beta}, \quad x \in I_h,$$

where $I_h = [-1, 1-h]$ if $h > 0$ and $I_h = [-1-h, 1]$ if $h < 0$. Assume that $v_{\beta,h} \in C^\alpha(I_h)$ and $\|v_{\beta,h}\|_{C^\alpha(I_h)} \leq K$, for any $0 < |h| \leq 1$. We then have:

1. *If $\alpha + \beta < 1$ then $u \in C^{\alpha+\beta}([-1, 1])$ and $\|u\|_{C^{\alpha+\beta}([-1,1])} \leq CK$,*
2. *If $\alpha + \beta > 1$ then $u \in C^{0,1}([-1, 1])$ and $\|u\|_{C^{0,1}} \leq CK$,*

where the constants C in 1. *and* 2. *depend only on $\alpha + \beta$.*

Proof of Theorem 6.1. Let $T^\alpha(B_r^+)$ be the space of functions that are C^α in the horizontal directions,

$$T^\alpha(B_r^+) := \left\{ v \in C(B_r^+) : \sup_{\substack{x,y \in B_r^+ \\ (x-y)\cdot\nu = 0}} \frac{|v(x) - v(y)|}{|x-y|^\alpha} < +\infty \right\}.$$

The norm in this space is given by

$$\|v\|_{T^\alpha(B_r^+)} = \|v\|_{C(B_r^+)} + \sup_{\substack{x,y \in B_r^+ \\ (x-y)\cdot\nu = 0}} \frac{|v(x) - v(y)|}{|x-y|^\alpha}.$$

Let τ be any unit vector parallel to Υ (i.e., $\langle \tau, \nu \rangle = 0$). For any $h < 1/8$, from Proposition 5.4, we have that $v_{\beta,h}(x) = \frac{1}{h^\beta}(u(x+h\tau) - u(x)) \in S\big(\frac{\lambda}{n}, \Lambda\big)$ in $B_{7/8}^+$ and $(v_{\beta,h})_\nu = 0$ in $\Upsilon \cap B_{7/8}$. Hence, by Proposition 4.2 properly rescaled

$$\|v_{\beta,h}\|_{C^\alpha(\overline{B_r^+})} \leq C(r,s)\|v_{\beta,h}\|_{C(\overline{B_{(r+s)/2}^+})} \leq C(r,s)\|u\|_{T^\beta(\overline{B_s^+})}, \tag{6.2}$$

where $0 < r < s \leq \frac{7}{8}$, $0 < h < \frac{s-r}{2}$, α is universal and $C(r, s)$ depends on n, λ, Λ, r, and s.

We can make α slightly smaller if needed so that there is an integer i such that $i\alpha < 1$ and $(i+1)\alpha > 1$. From Corollary 4.3, we know that

$$\|u\|_{T^\alpha(\overline{B_{7/8}^+})} \leq \|u\|_{C^\alpha(\overline{B_{7/8}^+})} \leq C\Big(\|u\|_{C(\overline{B_1^+})} + |F(0)|\Big).$$

Let $K = \big(\|u\|_{C(\overline{B_1^+})} + |F(0)|\big)$ so that $\|u\|_{T^\alpha(\overline{B_{7/8}^+})} \leq CK$.

We can apply now (6.2) with $\beta = \alpha$ and $r = r_1 < s = 7/8$ to get

$$\|v_{\alpha,h}\|_{T^\alpha(\overline{B_{r_1}^+})} \leq C(r_1)\|u\|_{T^\alpha(\overline{B_{7/8}^+})} \leq C(r_1)K.$$

We apply Lemma 6.2 with $\alpha = \beta$. Recall that we can do this for any unit vector τ parallel to Υ. We obtain

$$\|u\|_{T^{2\alpha}(B^+_{r_1})} \leq C(r_1)K.$$

We repeat this process with $\beta = 2\alpha$ to obtain $\|u\|_{T^{3\alpha}(\overline{B^+_{r_2}})} \leq CK$. If we choose $r_{i+1} = 5/8$, at the end we obtain

$$\|u\|_{T^1(\overline{B^+_{3/4}})} \leq CK.$$

Then we apply (6.2) with $\beta = 1$ and get

$$\|v_{1,h}\|_{C^\alpha(\overline{B^+_{5/8}})} \leq C\|u\|_{T^1(\overline{B^+_{3/4}})} \leq CK.$$

Since $v_{1,h}$ is a difference quotient of u for h and τ is any vector parallel to Υ, we obtain $u \in C^{1,\alpha}(\Upsilon \cap \overline{B_{5/8}})$ and $\|u\|_{C^{1,\alpha}(\Upsilon\cap\overline{B_{5/8}})} \leq CK$.

Finally, we apply Proposition 2.2 properly rescaled to obtain $u \in C^{1,\alpha_1}(\overline{B^+_{1/2}})$ and

$$\|u\|_{C^{1,\alpha_1}(\overline{B^+_{1/2}})} \leq C\Big(\|u\|_{C(\overline{B^+_{5/8}})} + \|u\|_{C^{1,\alpha}(\Upsilon\cap\overline{B_{5/8}})} + |F(0)|\Big) \leq CK. \qquad \square$$

7. Hölder Estimates for the Second Derivative

The following lemma was first observed by Krylov (1983). Later a simpler proof was given by Caffarelli (to appear) in the context of viscosity solutions, but unfortunately he did not publish it. His proof can be found in new editions of the book of Gilbarg and Trudinger (2001, Theorem 9.31) stated in a slightly different but equivalent way.

Lemma 7.1. *Let $u : \overline{B^+_1} \to \mathbb{R}$ be such that $u = 0$ in Υ, and $u \in S(\lambda, \Lambda)$ in B^+_1. Then there is a C^α function $A : \Upsilon \to \mathbb{R}$ such that for every $x \in B^+_{1/2}$,*

$$-C|x_n|^{1+\alpha} \leq u(x) - A(x')x_n \leq C|x_n|^{1+\alpha}$$

where $x = (x', x_n)$, $\alpha > 0$ is universal, C and $\|A\|_{C^\alpha}$ depend on n, λ, Λ, and linearly on $\sup_{B^+_1} u$. The function A is then the normal derivative of u at the boundary Υ.

We can apply this lemma to the normal derivative of a Neumann type problem to find the following estimate.

Corollary 7.2. *Let u be a solution of $F(D^2u) = 0$ in B^+_1 and $u_\nu = 0$ in Υ, then there is a C^α function $A : \Upsilon \to \mathbb{R}$ such that*

$$-C|x_n|^{2+\alpha} \leq u(x) - u(x', 0) - \frac{A(x')}{2}x_n^2 \leq C|x_n|^{2+\alpha}. \tag{7.1}$$

In the previous statements we think of $A(x')$ to be $u_{\nu\nu}(x', 0)$.

Proof. The normal derivative u_ν is well defined since $u \in C^{1,\alpha}$ by Theorem 6.1. We see that u_ν satisfies the hypothesis of Lemma 7.1, then

$$-C|x_n|^{1+\alpha} \leq u_\nu(x) - A(x')x_n \leq C|x_n|^{1+\alpha}.$$

Therefore

$$\begin{aligned} u(x) - u(x', 0) &= \int_0^{x_n} u_\nu(x', y)\mathrm{d}y \\ &\leq \int_0^{x_n} A(x')y + C|y|^{1+\alpha}\,\mathrm{d}y \\ &\leq \frac{A(x')}{2}x_n^2 + C|x_n|^{2+\alpha}. \end{aligned}$$

Similarly, we show $u(x) - u(x', 0) \geq \frac{A(x')}{2}x_n^2 - C|x_n|^{2+\alpha}$. □

Proposition 7.3. *Let u be a solution of $F(D^2u) = 0$ in B_1^+ and $u_\nu = 0$ in Υ. Then if we take its restriction to Υ, $v(x') = u(x', 0)$, then v solves (in the viscosity sense) the equation*

$$F\begin{pmatrix} D^2 v & 0 \\ 0 & A(x') \end{pmatrix} = 0,$$

where $A : \Upsilon \to \mathbb{R}$ is a C^α function, for a universal $\alpha > 0$.

Proof. Let φ be a smooth function on Υ touching v from below in a point in Υ that, for simplicity, we will consider to be the origin. We want to extend φ to B_1 and translate it to turn it into a test function which touches u from below in the interior of B_r^+, for an arbitrarily small r.

Let A be the function of Corollary 7.2. For a small $\varepsilon > 0$, let

$$\tilde{\varphi}(x) = \varphi(x') + \frac{A(0)}{2}x_n^2 - \varepsilon|x|^2.$$

From Corollary 7.2, we know that

$$\begin{aligned} u(x) &\geq u(x', 0) + \frac{A(x')}{2}x_n^2 - C|x_n|^{2+\alpha} \\ &\geq u(x', 0) + \frac{A(0)}{2}x_n^2 - C|x|^{2+\alpha} \quad \text{since } A \text{ is } C^\alpha \\ &\geq u(x', 0) + \frac{A(0)}{2}x_n^2 - \frac{\varepsilon}{2}|x|^2 \quad \text{for } |x| \text{ small enough} \\ &\geq \tilde{\varphi}(x) + \frac{\varepsilon}{2}|x|^2. \end{aligned}$$

Let $r > 0$ be chosen so that the above computation is valid for $|x| < r$.

We will consider two cases: whether $A(0) \leq 0$ or not.

If $A(0) \leq 0$, we translate $\tilde{\varphi}$ in the inner normal direction.

$$\tilde{\varphi}_h(x) = \tilde{\varphi}(x', x_n - h).$$

We choose h such that $\left(-\frac{A(0)}{2}+\varepsilon\right)h^2 \leq \frac{\varepsilon}{2}(r-h)^2$. Therefore $u(0)-\tilde{\varphi}_h(0) \leq u(x)-\tilde{\varphi}_h(x)$ when $|x|=r$. The function $u-\tilde{\varphi}_h$ cannot have a local minimum in $\Upsilon \cap B_r$ because $\partial_\nu u = 0$ and $\partial_\nu \tilde{\varphi}_h > 0$ in there. Therefore $u-\tilde{\varphi}_h$ must have a local minimum at some point $x_1 \in B_r^+$. Since $F(D^2u)=0$ in B_1^+, then $F(D^2\tilde{\varphi}_h(x_1)) \leq 0$. Since we can do all this for r arbitrarily small, and then we can also take $\varepsilon \to 0$, we obtain

$$F\begin{pmatrix} D^2\varphi & 0 \\ 0 & A(0) \end{pmatrix} \leq 0.$$

In the case $A(0) > 0$, the only difference is that we translate $\tilde{\varphi}$ in the outer normal direction

$$\tilde{\varphi}_h(x) = \tilde{\varphi}(x', x_n + h)$$

and then if we choose $\varepsilon < A(0)$, the same reasoning as above applies.

We can do the same thing for test functions touching v from above, therefore we obtain in the viscosity sense

$$F\begin{pmatrix} D^2v & 0 \\ 0 & A(x') \end{pmatrix} = 0. \qquad \square$$

Theorem 7.4. *Assume F to be a convex function and let u be a solution of $F(D^2u)=0$ in B_1^+ and $u_\nu = 0$ in Υ. Then u is $C^{2,\alpha}(\overline{B_{1/2}^+})$ up to the boundary, for a universal $\alpha > 0$. Moreover, we have the estimate*

$$\|u\|_{C^{2,\alpha}(\overline{B_{1/2}^+})} \leq C\left(\|u\|_{C(\overline{B_1^+})} + |F(0)|\right) \tag{7.2}$$

for a universal constant C.

Proof. By Proposition 7.3, the restriction $v(x') = u(x',0)$ satisfies equation

$$F\begin{pmatrix} D^2v & 0 \\ 0 & A(x') \end{pmatrix} = 0$$

where A is C^α. By the $C^{2,\alpha}$ estimates for elliptic equations in Caffarelli and Cabré (1995), we conclude that $v \in C^{2,\alpha}(\Upsilon \cap B_{2/3})$. By Corollary 7.2, then u is $C^{2,\alpha}$ at the boundary $\Upsilon \cap B_{2/3}$, and then from Proposition 2.4 we obtain the desired estimate. $\square$

8. Inhomogeneous Equations

In this section we study the regularity in the case when we have a nonzero right-hand side. The proofs are based on a perturbation of the homogeneous case. For simplicity, in this section we keep the left-hand side independent of x. The proofs of the corresponding results with x dependent left-hand side follow the same spirit but are more complicated. We will outline the general case in the next section.

Theorem 8.1. *Let u be a solution of $F(D^2u) = f(x)$ in B_1^+ and $u_\nu = g$ in Υ, for a bounded function g and $f \in L^n(B_1^+)$. Then u is $C^\alpha(\overline{B_{1/2}^+})$ up to the boundary. For a* *universal* $\alpha > 0$. *Moreover, we have the estimate*

$$\|u\|_{C^\alpha(\overline{B^+_{1/2}})} \le C\Big(\|u\|_{C(\overline{B^+_1})} + \|g\|_{L^\infty} + \|f\|_{L^n} + |F(0)|\Big) \tag{8.1}$$

for a universal constant C.

Proof. If we can show that any function u, continuous in $\overline{B^+_r}$ such that $F(D^2u) = f(x)$ in B^+_r and $u_\nu = g$ in $\Upsilon \cap B_r$ satisfies

$$\operatorname*{osc}_{B^+_{r/2}} u \le (1-\theta) \operatorname*{osc}_{B^+_r} u + C|F(0)|r^2 + Cr(\|g\|_{L^\infty} + \|f\|_{L^n}) \tag{8.2}$$

for universal constants $\theta > 0$ and C. Then applying (8.2) to translations of u we obtain a C^α modulus of continuity for u at the bottom Υ by a standard iterative argument. Then (8.1) follows by interior regularity (or by Proposition 2.1). So we are going to show (8.2).

Let u be as above. Let v be the solution of the problem:

$$\begin{cases} F(D^2v) = 0 & \text{in } B^+_r \\ v_\nu = 0 & \text{in } \Upsilon \cap B_r \\ v = u & \text{in } \partial B_r \cap \{x_n > 0\}. \end{cases}$$

The function v is in the class $S(\lambda/n, \Lambda, F(0))$ in B^+_r. We can reflect it using Proposition 4.1 to get a function $v^* \in S(\lambda/n, \Lambda, F(0))$ in B_r. By simple comparison principle,

$$\max_{\overline{B^+_r}} v = \max_{\overline{B_r}} v^* \le \max_{\partial B_r} v^* + C|F(0)|r^2 \le \max_{\overline{B^+_r}} u + C|F(0)|r^2.$$

Similarly,

$$\min_{\overline{B^+_r}} v \ge \min_{\overline{B^+_r}} u - C|F(0)|r^2.$$

Therefore,

$$\operatorname*{osc}_{\overline{B^+_r}} v \le \operatorname*{osc}_{\overline{B^+_r}} u + C|F(0)|r^2. \tag{8.3}$$

Since $v^* \in S(\lambda/n, \Lambda, F(0))$ in B_r, we can apply Harnack inequality to obtain

$$\operatorname*{osc}_{\overline{B^+_{r/2}}} v = \operatorname*{osc}_{\overline{B_{r/2}}} v^* \le (1-\theta) \operatorname*{osc}_{\overline{B_r}} v^* + C|F(0)|r^2 = (1-\theta) \operatorname*{osc}_{\overline{B^+_r}} v + C|F(0)|r^2. \tag{8.4}$$

Combining (8.3) with (8.4) we get

$$\operatorname*{osc}_{\overline{B^+_{r/2}}} v \le (1-\theta) \operatorname*{osc}_{\overline{B^+_r}} u + C|F(0)|r^2 \tag{8.5}$$

for some universal constant C.

Let $w = u - v$. Then w satisfies the following relations:

$$\begin{cases} w \in S(\lambda/n, \Lambda, f) & \text{in } B_r^+ \\ w_\nu = g & \text{in } \Upsilon \cap B_r \\ w = 0 & \text{in } \partial B_r \cap \{x_n > 0\}. \end{cases}$$

Applying Proposition 3.1 to w and $-w$, we obtain

$$\max_{\overline{B_r^+}} |w| \leq Cr\big(\|f\|_{L^n} + \|g\|_{L^\infty}\big).$$

Therefore,

$$\operatorname*{osc}_{\overline{B_{r/2}^+}} w \leq \operatorname*{osc}_{\overline{B_r^+}} w \leq Cr\big(\|f\|_{L^n} + \|g\|_{L^\infty}\big). \tag{8.6}$$

Adding (8.5) with (8.6), we finally get

$$\begin{aligned} \operatorname*{osc}_{\overline{B_{r/2}^+}} u &\leq \operatorname*{osc}_{\overline{B_{r/2}^+}} v + \operatorname*{osc}_{\overline{B_{r/2}^+}} w \\ &\leq (1-\theta) \operatorname*{osc}_{\overline{B_r^+}} u + C|F(0)|r^2 + Cr(\|f\|_{L^n} + \|g\|_{L^\infty}). \end{aligned} \tag{8.7}$$

□

Theorem 8.2. *Consider $g \in C^\beta(\overline{B_1^+})$ and $f \in L^p(\overline{B_1^+})$ for some $\beta \in (0,1)$ and $p > n$. Let u be a solution of $F(D^2u) = f$ in B_1^+ and $u_\nu = g$ in Υ. Then u is $C^{1,\alpha}(\overline{B_{1/2}^+})$ where $\alpha = \min(\alpha_0, \beta, 1 - n/p)$ and α_0 is a universal constant. Moreover, we have the estimate*

$$\|u\|_{C^{1,\alpha}(\overline{B_{1/2}^+})} \leq C\Big(\|u\|_{C(\overline{B_1^+})} + \|g\|_{C^\beta(\Upsilon)} + \|f\|_{L^p} + |F(0)|\Big), \tag{8.8}$$

where C is a constant depending only on n, λ, Λ, and α.

Proof. By interior estimates (or by Proposition 2.2) it is enough to find a $C^{1,\alpha}$ estimate for the points in Υ. Moreover, for proving (8.8) it is enough to get a universal estimate at the origin, and then apply it to rescaling and translations of u.

Without loss of generality we can assume $u(0) = 0$ and $g(0) = 0$. Let $M = \|u\|_{C(\overline{B_1^+})} + \|g\|_{C^\beta(\Upsilon)} + \|f\|_{L^p} + |F(0)|$. We want to show that there is a universal α_0, and a universal constant $\gamma < 1$, such that for $\alpha = \min(\alpha_0, \beta, 1 - n/p)$, there is a constant C_1, depending only on n, λ, and Λ and a sequence of vectors A_k such that $A_k \cdot \nu = 0$ and

$$\operatorname*{osc}_{B_{\gamma^k}^+}(u(x) - A_k \cdot x) \leq C_1 M \gamma^{k\cdot(1+\alpha)} \tag{8.9}$$

$$|A_{k+1} - A_k| \leq CM\gamma^{k\alpha}. \tag{8.10}$$

We will show this by induction.

We choose a $C_1 > 1$ so that (8.9) holds for $k = 0$ with $A_0 = 0$.

To complete an induction proof, we assume that we already have a sequence of vectors A_k so that (8.9) holds for $k = 0, 1, \ldots, K$; we have to show that there is a vector A_{K+1} such that (8.9) holds for $k = K + 1$.

Let $r = \gamma^K$ and $B = A_K$.

Let v be the solution of the following problem (like in the proof of Theorem 8.1):

$$\begin{cases} F(D^2 v) = 0 & \text{in } B_r^+ \\ v_\nu = 0 & \text{in } \Upsilon \cap B_r \\ v = u - B\cdot x & \text{in } \partial B_r \cap \{x_n > 0\}. \end{cases}$$

From maximum principle (Proposition 3.1) we know

$$\underset{B_r^+}{\operatorname{osc}}\, v \le \underset{B_r^+}{\operatorname{osc}}(u - B\cdot x) + C_2|F(0)|r^2. \tag{8.11}$$

Now we apply the $C^{1,\alpha}$ estimates to v. Theorem 6.1 tells us that ∇v is well defined up to the boundary $\Upsilon \cap B_r$. Let $A = \nabla v(0)$, by the boundary conditions $A\cdot \nu = 0$. Rescaling of the $C^{1,\alpha}$ estimate gives

$$\frac{\operatorname{osc}_{B_{\tilde r}^+}(v(x) - A\cdot x)}{\tilde r^{1+\alpha_1}} \le C_0\left(\frac{1}{r^{1+\alpha_1}}\underset{B_r^+}{\operatorname{osc}}\, v + r^{1-\alpha_1}|F(0)|\right) \tag{8.12}$$

$$|A| \le C\left(\frac{\operatorname{osc}_{B_r^+} v}{r} + r|F(0)|\right) \tag{8.13}$$

for any $\tilde r \le r/2$. Where α_1 is the α of Theorem 6.1.

We choose γ small enough so that $C_0\gamma^{\alpha_1} = (1-\theta) < 1$, for some positive constant θ. Combining (8.12) for $\tilde r = \gamma r$ with (8.11), we get

$$\underset{B_{\gamma r}^+}{\operatorname{osc}}(v(x) - A\cdot x) \le (1-\theta)\gamma\, \underset{B_r^+}{\operatorname{osc}}(u - B\cdot x) + C_3|F(0)|r^2 \tag{8.14}$$

Let $w = u - B\cdot x - v$. Recall $B\cdot\nu = 0$. Like in the proof of Theorem 8.1, we have

$$\begin{cases} w \in S(\lambda/n, \Lambda, f) & \text{in } B_r^+ \\ w_\nu = g & \text{in } \Upsilon \cap B_r \\ w = 0 & \text{in } \partial B_r \cap \{x_n > 0\}. \end{cases}$$

Then, by Proposition 3.1, we have

$$\begin{aligned} \sup_{B_r^+}|w(x)| &\le Cr\|g\|_{L^\infty(\Upsilon\cap B_r)} + Cr\left(\int_{B_r^+}|f|^n\,\mathrm{d}x\right)^{1/n} \\ &\le C\|g\|_{C^\beta(\Upsilon)}r^{1+\beta} + Cr\|f\|_{L^p}r^{1-n/p}. \end{aligned} \tag{8.15}$$

Adding (8.14) with (8.15) we get

$$\begin{aligned} &\underset{B_{\gamma r}^+}{\operatorname{osc}}(u(x) - (A+B)\cdot x) \\ &\quad\le (1-\theta)\gamma\,\underset{B_r^+}{\operatorname{osc}}(u - B\cdot x) + C|F(0)|r^2 + C\|g\|_{C^\beta(\Upsilon)}r^{1+\beta} + C\|f\|_{L^p}r^{2-n/p}. \end{aligned} \tag{8.16}$$

By the inductive hypothesis we have $\operatorname{osc}_{B_r^+}(u - B \cdot x) \le C_1 M \gamma^{K\cdot(1+\alpha)}$, therefore,

$$\operatorname*{osc}_{B_{\gamma r}^+}(u(x) - (A + B) \cdot x) \le M\Big((1-\theta)\gamma C_1 \gamma^{K\cdot(1+\alpha)} + C_2\Big(\gamma^{2K} + \gamma^{K(1+\beta)} + \gamma^{K(2-n/p)}\Big)\Big). \tag{8.17}$$

Now we will choose the right constants α_0 and C_1. We choose α_0 so that $\gamma^{\alpha_0} = (1 - \theta/2)$, $\alpha = \min(\alpha_0, \beta, 1 - n/p))$, and C_1 large enough so that $\frac{\theta\gamma}{2} C_1 \ge 3C_2$. Replacing in (8.17), we get

$$\begin{aligned}
&\operatorname*{osc}_{B_{\gamma r}^+}(u(x) - (A + B) \cdot x) \\
&\quad \le M\Big((1-\theta/2)\gamma C_1 \gamma^{K\cdot(1+\alpha)} + C_2\Big(\gamma^{2K} + \gamma^{K(1+\beta)} + \gamma^{K(2-n/p)}\Big) - \frac{\theta\gamma}{2} C_1 \gamma^{K\cdot(1+\alpha)}\Big) \\
&\quad \le M(1-\theta/2)\gamma C_1 \gamma^{K\cdot(1+\alpha)} \\
&\quad \le M C_1 \gamma^{(K+1)\cdot(1+\alpha)}.
\end{aligned} \tag{8.18}$$

From (8.13), (8.11), (8.9), and that $r = \gamma^K$, we get

$$|A| \le CM\gamma^{\alpha K} \tag{8.19}$$

Taking $A_{K+1} = A + B$, we finish the inductive proof of (8.9) and (8.10).

Let $A_\infty = \lim_{k\to\infty} A_k$. We claim that

$$|u(x) - A_\infty \cdot x| \le CM|x|^{1+\alpha}.$$

Indeed, from (8.9) and (8.10) we get

$$\operatorname*{osc}_{B_{\gamma^k}^+}(u(x) - A_\infty \cdot x) \le \operatorname*{osc}_{B_{\gamma^k}^+}(u(x) - A_k \cdot x) + 2\gamma^k |A_k - A_\infty| \tag{8.20}$$

$$\le C_1 M \gamma^{k\cdot(1+\alpha)} + CM\gamma^k \sum_{j=k}^{\infty} \gamma^{\alpha j} \tag{8.21}$$

$$\le C_1 M \gamma^{k\cdot(1+\alpha)} + CM\gamma^{(1+\alpha)k} \frac{1}{1-\gamma^\alpha} \tag{8.22}$$

$$\le CM\gamma^{k\cdot(1+\alpha)} \tag{8.23}$$

This implies the $C^{1,\alpha}$ estimate at the origin, and the estimate follows by translation and interior estimates. □

Theorem 8.3. *For F a convex, let u be a solution of $F(D^2u) = f$ in $\overline{B_1^+}$ and $u_\nu = g$ in Υ, for a $C^{1,\beta}$ function g, and $f \in C^\beta\big(\overline{B_1^+}\big)$ $(\beta > 0)$. Then u is $C^{2,\alpha}\big(\overline{B_{1/2}^+}\big)$ up to the boundary, for $\alpha = \min(\alpha_0, \beta)$. Where $\alpha_0 > 0$ is a universal constant. Moreover, we have the estimate*

$$\|u\|_{C^{2,\alpha}(\overline{B_{1/2}^+})} \le C\Big(\|u\|_{C(\overline{B_1^+})} + \|g\|_{C^{1,\beta}(\Upsilon)} + \|f\|_{C^\beta(\overline{B_1^+})} + |F(0)|\Big) \tag{8.24}$$

for a constant C depending only on n, λ, Λ, and α.

Proof. The proof will follow the same ideas as Theorem 8.2, but instead of approximating with planes, we must use paraboloids, and we have to use Theorem 7.4 instead of 6.1.

Let $M = \|u\|_{C(\overline{B_1^+})} + \|g\|_{C^{1,\beta}(\Upsilon)} + \|f\|_{C^\beta} + |F(0)|$. We want to show that there is an $\alpha > 0$, and a universal constant $\gamma < 1$, such that there is a constant C_1, depending only on n, λ, and Λ, and a sequence of paraboloids $P_k(x) = \frac{1}{2}x^T Q_k x + A_k \cdot x + u(0)$ such that $F(Q_k) = f(0)$, $A_k \cdot v = 0$ and $(Q_k)_{jn} = \partial_{jn} P_k(x) = \partial_j g(0)$ for every $j \neq n$, and

$$\operatorname*{osc}_{B_{\gamma^k}}(u(x) - P_k(x)) \leq C_1 M \gamma^{k \cdot (2+\alpha)} \tag{8.25}$$

$$|A_{k+1} - A_k| \leq CM\gamma^{k(1+\alpha)} \tag{8.26}$$

$$|Q_{k+1} - Q_k| \leq CM\gamma^{k\alpha}. \tag{8.27}$$

As before, this will show that estimate (8.24) holds punctually at $x = 0$; and then this implies the full estimate (8.24) by translations and interior estimates. We can subtract a suitable plane to u such that $u(0) = 0$ and $g(0) = u_v(0) = 0$. So we suppose $u(0) = 0$ and $g(0) = 0$.

We will show (8.25) by induction.

We choose $C_1 > 1$ so that (8.25) holds $k = 0$ with $A_0 = 0$ and Q_0 the symmetric matrix such that $\partial_j g(0) = Q_{jn}$ for every $j \neq n$, $Q_{ij} = 0$ for $i, j \neq n$ and Q_{nn} chosen so that $F(Q) = f(0)$. Note that $|Q| \leq C(|F(0)| + |f(0)|) \leq CM$ for a universal constant C. This is the only part where the term $|F(0)|$ in the definition of M matters.

To complete an induction proof, we assume that we already have such a sequence of paraboloids $P_k = \frac{1}{2}x^T Q_k x + A_k \cdot x$ so that (8.25), (8.26), and (8.27) hold for $k = 0, 1, \ldots, K$; we have to show that there is another paraboloid P_{K+1} such that (8.25)–(8.27) hold for $k = K + 1$.

Let $r = \gamma^{-K}$.

Let v be the solution of the following problem (like in the proof of Theorem 8.2):

$$\begin{cases} F(D^2 v + Q_K) = 0 & \text{in } B_r^+ \\ v_v = 0 & \text{in } \Upsilon \cap B_r \\ v = u - P_k(x) & \text{in } \partial B_r \cap \{x_n > 0\}. \end{cases}$$

From maximum principle (Proposition 3.1) and that $F(Q_K) = 0$, we know that

$$\operatorname*{osc}_{\overline{B_r^+}} v \leq \operatorname*{osc}_{\overline{B_r^+}}(u - P_k(x)). \tag{8.28}$$

Now we apply the $C^{2,\alpha}$ estimates to v. Theorem 7.4 tells us that ∇v and $D^2 u$ are well defined up to the boundary $\Upsilon \cap B_r$. Let $B = \nabla v(0)$ and $R = D^2 v(0)$, and let $\overline{P}$ be the paraboloid $x^T Rx + B \cdot x$. By the boundary conditions $\partial_n \overline{P} = 0$ and by the equation and the fact that $D^2 v$ is continuous up to the boundary $F(Q_K + R) = 0$. Since $F(Q_K) = 0$, rescaling of the $C^{2,\alpha}$ estimate gives

$$\frac{\operatorname{osc}_{B_{\tilde{r}}}(v(x) - \overline{P}(x))}{\tilde{r}^{2+\alpha_1}} \leq C_0 \frac{1}{r^{2+\alpha_1}} \operatorname*{osc}_{\overline{B_r^+}} v \tag{8.29}$$

$$|B| \le C \frac{\operatorname{osc}_{B_r^+} v}{r} \tag{8.30}$$

$$|R| \le C \frac{\operatorname{osc}_{B_r^+} v}{r^2} \tag{8.31}$$

for any $\tilde{r} \le r/2$, where α_1 is the α of Theorem 7.4.

We choose γ small enough so that $C_0 \gamma^{\alpha_1} = (1-\theta) < 1$, for some positive constant θ. Combining (8.29) for $\tilde{r} = \gamma r$ with (8.28), we get

$$\operatorname*{osc}_{B_{\gamma r}^+}(v(x) - \overline{P}(x)) \le (1-\theta)\gamma^2 \operatorname*{osc}_{B_r^+}(u - P_K(x)). \tag{8.32}$$

Let $w = u - P_K(x) - v$. Like in the proof of Theorem 8.2, we have

$$\begin{cases} w \in S(\lambda/n, \Lambda, f - f(0)) & \text{in } B_r^+ \\ w_\nu = g(x) - \partial_n P_K(x) & \text{in } \Upsilon \cap B_r \\ w = 0 & \text{in } \partial B_r \cap \{x_n > 0\}. \end{cases}$$

Recall that $\partial_j g = (Q_K)_{jn} = \partial_{jn} P_K(0)$ and $g(0) = 0$, so $g - \partial_n P_K$ is of order $|x|^{1+\beta}$ around $x = 0$. Then, by Proposition 3.1, we have

$$\begin{aligned} \sup_{B_r^+} |w(x)| &\le Cr \|g - \partial_n P_K\|_{L^\infty(\Upsilon \cap B_r)} + Cr\left(\int_{B_r^+} |f - f(0)|^n \,\mathrm{d}x\right)^{1/n} \\ &\le C\|g\|_{C^{1,\beta}(\Upsilon)} r^{2+\beta} + C\|f\|_{C^\beta} r^{2+\beta}. \end{aligned} \tag{8.33}$$

Adding (8.32) with (8.33) we get

$$\operatorname*{osc}_{B_{\gamma r}}(u(x) - P_K(x) - \overline{P}(x)) \le (1-\theta)\gamma^2 \operatorname*{osc}_{B_r^+}(u - P_K(x)) + C\|g\|_{C^{1,\beta}(\Upsilon)} r^{2+\beta} + C\|f\|_{C^\beta} r^{2+\beta}. \tag{8.34}$$

By the inductive hypothesis, $\operatorname{osc}_{B_r^+}(u - P_K(x)) \le C_1 M \gamma^{K\cdot(2+\alpha)}$. Thus we get

$$\operatorname*{osc}_{B_{\gamma r}}(u(x) - P_K(x) - \overline{P}(x)) \le M\big((1-\theta)\gamma^2 C_1 \gamma^{K\cdot(2+\alpha)} + C_2 \gamma^{K(2+\beta)}\big). \tag{8.35}$$

Now we will choose the right constants α_0 and C_1. We choose α_0 so that $\gamma^{\alpha_0} = (1-\theta/2)$, $\alpha = \min(\alpha_0, \beta)$, and C_1 large enough so that $\frac{\theta\gamma^2}{2} C_1 \ge C_2$. Replacing in (8.35), we get

$$\begin{aligned} \operatorname*{osc}_{B_{\gamma r}}(u(x) - P_K(x) - \overline{P}(x)) &\le M\left((1-\theta/2)\gamma^2 C_1 \gamma^{K\cdot(2+\alpha)} + C_2\gamma^{K(2+\beta)} - \frac{\theta\gamma^2}{2} C_1 \gamma^{K\cdot(2+\alpha)}\right) \\ &\le M(1-\theta/2)\gamma^2 C_1 \gamma^{K\cdot(2+\alpha)} \\ &\le M C_1 \gamma^{(K+1)\cdot(2+\alpha)}. \end{aligned} \tag{8.36}$$

From (8.30), (8.28), and that $r = \gamma^K$, we get

$$|B| \le C\gamma^{(1+\alpha)K}. \tag{8.37}$$

And from (8.31) and (8.28), we get

$$|R| \le C\gamma^{\alpha K}. \tag{8.38}$$

Taking $P_{K+1} = P_K + \overline{P}$, we finish the inductive proof of (8.25), (8.26), and (8.27).

Like in the proof of Theorem 8.2, this implies a $C^{2,\alpha}$ estimate punctually at the origin. Thus the estimate follows. □

9. General Equations

This section is concerned with the Hölder regularity for the first derivatives of solutions of

$$F(D^2u, x) = f(x) \tag{9.1}$$

for $x \in B_1^+$ and $u_\nu = g$ on Υ. In Caffarelli and Cabré (1995, Chapter 8), interior regularity results are obtained for an equation like (9.1) by a perturbation argument of the homogeneous case. With the results we have so far in this article, we can extend the proof in Caffarelli and Cabré (1995) for $C^{1,\alpha}$ regularity up to the boundary in the Neumann problem. It is a key idea to think of the Neumann condition as part of the equation and not the boundary data. After all, if the Neumann condition is part of the equation, then we are actually talking about *interior* regularity.

The results so far in this article provide us with good estimates for the equation that we obtain when we "freeze" the value of x. We assume that the oscillation of function $F(M, x)$ in x is sufficiently "small" so that the Neumann condition $u_\nu = g$ has a C^α right-hand side, and that the corresponding homogeneous equation "with constant coefficients"

$$F(D^2u, x_0) = 0 \quad \text{in } B_r^+$$
$$u_\nu = g(y_0) \quad \text{in } \Upsilon$$

has $C^{1,\bar{\alpha}}$ estimates for any fixed x_0, where y_0 is the projection of x_0 into Υ.

We intend to follow proof in Caffarelli and Cabré (1995, Theorem 8.3) for the interior regularity. We realize everything follows the same way as long as we modify two lemmas. We are just going to outline the required modifications.

The proof of Lemma 9.1 (Proposition 4.14 in Caffarelli and Cabré, 1995) is based on the ABP estimate and C^α regularity of the solution. Since we have a corresponding result by Propositions 3.1 and 4.2, we can extend the lemma to our case.

Lemma 9.1. *Let u be continuous in $\overline{B_1^+}$ that belongs to $S(\lambda, \Lambda, f)$ in B_1^+ such that it satisfies $u_\nu = g$ on Υ and f is a continuous function. Denote by φ the restriction of u on $\partial B_1 \cap \{x_n > 0\}$ and let $\rho(|x-y|)$ be a modulus of continuity of φ; that is ρ is a nondecreasing function with $\lim_{\delta\to 0}\rho(\delta) = 0$ such that*

$$|\varphi(x) - \varphi(y)| \le \rho(|x-y|)$$

for all $x, y \in \partial B_1 \cap \{x_n > 0\}$. *In addition let* K *be a positive constant such that* $\|\varphi\|_{L^\infty} \le K$, $\|g\|_{L^\infty} \le K$ *and* $\|f\|_{L^n(B_1^+)} \le K$. *Then there exists a modulus of continuity* ρ^* *of* u *in* $\overline{B_1^+}$, *i.e.,* ρ^* *is nondecreasing,* $\lim_{\delta\to 0}\rho^*(\delta) = 0$ *and*

$$|u(x) - u(y)| \le \rho^*(|x-y|)$$

for all x, y *in* $\overline{B_1^+}$. *Where* ρ^* *depends on* n, λ, Λ, K, *and* ρ.

Next we adapt Lemma 8.2 in Caffarelli and Cabré (1995) (see also Theorem 2.1 in Swiech, 1997) to our case. We consider the function

$$\beta(x) = \beta_F(x) = \sup_{\mathcal{S}\setminus\{O\}} \frac{|F(M,x) - F(M,0)|}{1 + \|M\|}$$

which measures the oscillation of F in x near the origin (recall that $\mathcal{S}$ denotes the space of symmetric matrices).

Lemma 9.2. *Suppose that* F *is continuous in* x, $F(0,x) \equiv 0$, *and* $\beta = \beta_F$ *is Hölder continuous in* B_1^+ *(for some exponent in* $(0,1)$*). Let* u_0 *be a continuous function on* $\partial B_1 \cap \{x_n > 0\}$, *has* $\rho = \rho(s)$ *as modulus of continuity on* $\partial B_1 \cap \{x_n > 0\}$ *and satisfy* $\|u_0\|_{L^\infty(\partial B_1 \cap \{x_n > 0\})} \le K$, *for some positive constant* K.

Then, given $\varepsilon > 0$, *there exists* $\delta > 0$ *depending only on* $\varepsilon, n, \lambda, \Lambda, \rho, K$ *such that if* f *is Hölder continuous in* B_1^+, g *is Hölder continuous on* Υ,

$$\|\beta\|_{L^n(B_1^+)} \le \delta, \quad \|f\|_{L^n(B_1^+)} \le \delta, \quad \text{and} \quad \|g - g_0\|_{L^\infty(\Upsilon)} \le \delta \text{ for } g_0 \le K$$

then any two viscosity solutions v *and* w *of, respectively,*

$$\begin{cases} F(D^2 v, x) = f(x) & \text{in } B_1^+ \\ v = u_0 & \text{on } \partial B_1 \cap \{x_n > 0\} \\ v_\nu = g & \text{on } \Upsilon \end{cases}$$

and

$$\begin{cases} F(D^2 w, 0) = 0 & \text{in } B_1^+ \\ w = u_0 & \text{on } \partial B_1 \cap \{x_n > 0\} \\ w_\nu = g_0 & \text{on } \Upsilon \end{cases}$$

satisfy

$$\|v - w\|_{L^\infty(B_1^+)} \le \varepsilon.$$

Note that it is no restriction to assume $F(0,x) \equiv 0$, since $F(D^2u, x) = f(x)$ may be written as $F(D^2u, x) - F(0,x) = f(x) - F(0,x)$. For the proof of the previous lemma we refer the reader to Caffarelli and Cabré (1995, Lemma 8.2) (or Theorem 2.1 in Swiech, 1997) the proof of which is still valid in our case due to Lemma 9.1.

As we mentioned before, in order to prove $C^{1,\alpha}$ estimates for our solution, we would like to use the "freezing" argument as in Theorem 8.3 of Caffarelli and Cabré

(1995). In fact if we follow the lines of the proof, we realize that all the arguments are applied perfectly in our case, except the fact that we use domains B_r^+, instead of balls B_r, with the Neumann condition on the bottom Υ. But this is not a problem since we treat the Neumann conditions as a part of our equation and thus we also apply for them the compactness method (see Lemma 9.2). To conclude we state the main theorem.

Theorem 9.3. *Assume that $F(0, x) \equiv 0$, F is continuous in $x \in B_1^+$ and both $\beta = \beta_F$ and f are Hölder continuous in B_1^+ and g is Hölder continuous on Υ. Suppose that there is a constant $0 \leq \bar{\alpha} < 1$ such that for any $u_0 \in C(\partial B_1 \cap \{x_n > 0\})$ and $g_0 \leq K$, there exists a $w \in C(\overline{B_1^+}) \cap C^{1,\bar{\alpha}}(B_{1/2}^+)$ which is viscosity solution of*

$$\begin{cases} F(D^2 w, 0) = 0 & \text{in } B_1^+ \\ w = u_0 & \text{on } \partial B_1 \cap \{x_n > 0\} \\ w_\nu = g_0 & \text{on } \Upsilon. \end{cases}$$

Assume that $0 < \alpha < \bar{\alpha}$, $r_0 > 0$, and $C_1 > 0$. Then there exists $\theta > 0$ depending on $n, \lambda, \Lambda, \alpha, \bar{\alpha}$, and the $C^{1,\bar{\alpha}}$ norm such that if

$$\left(⨍_{B_r^+(0)} \beta^n \right)^{1/n} \leq \theta \quad \text{and} \quad \left(⨍_{B_r^+(0)} |f|^n \right)^{1/n} \leq C_1 r^{\alpha - 1}$$

for all $r \leq r_0$, then any viscosity solution u of

$$\begin{cases} F(D^2 u, x) = f(x) & \text{in } B_{r_0}^+(0) \\ w_\nu = g & \text{on } \Upsilon \cap B_{r_0} \end{cases}$$

is $C^{1,\alpha}$ in the sense that there is an affine function l such that

$$\|u - l\|_{L^\infty(B_r^+(0))} \leq C_2 r^{1+\alpha}$$
$$r_0^{-\alpha} |\nabla l| \leq C_2$$

for all $r \leq r_0$ and

$$C_2 \leq C\left(r_0^{-(1+\alpha)} \|u\|_{L^\infty(B_{r_0}(0))} + \|g\|_{C^\alpha} + C_1 \right),$$

where $C > 0$ depends only on $n, \lambda, \Lambda, \alpha, \bar{\alpha}$, and the $C^{1,\bar{\alpha}}$ norm of w.

10. Appendix

In this appendix, we will give a proof of Propositions 2.1, 2.2, 2.3, and 2.4.

For functions vanishing in the boundary, the $C^{1,\alpha}$ regularity follows from Lemma 7.1.

Lemma 10.1. *Let u be a solution of $F(D^2 u) = 0$ in B_1^+ such that $u \in C(\overline{B_1^+})$ and $u = 0$ in Υ, then u is $C^{1,\alpha}(\overline{B_{1/2}^+})$ up to the boundary, for a universal α. Moreover, we have the* *estimate*

$$\|u\|_{C^{1,\alpha}(\overline{B^+_{1/2}})} \leq C\Big(\|u\|_{C(\overline{B^+_1})} + |F(0)|\Big) \tag{10.1}$$

for a universal constant C.

Proof. The function u belongs to $S^*\big(\frac{\lambda}{n}, \Lambda, F(0)\big)$ in B^+_1, and it vanishes in Υ, therefore we can apply Lemma 7.1 to obtain

$$|u(x) - A(x')x_n| \leq C|x_n|^{1+\alpha}$$

for all $x \in B^+_{1/2}$, where $C = C_0\|u\|_{L^\infty(B^+_1)}$ and $\|A\|_{C^\alpha} \leq C_1\|u\|_{L^\infty(B^+_1)}$, for universal constants C_0 and C_1. Therefore

$$\begin{aligned} |u(x) - A(0)x_n| &\leq |u(x) - A(x')x_n| + |x_n||A(x') - A(0)| \\ &\leq C_0\|u\|_{L^\infty(B^+_1)}|x_n|^{1+\alpha} + |x_n|C_1\|u\|_{L^\infty(B^+_1)}|x'|^\alpha \\ &\leq C\|u\|_{L^\infty(B^+_1)}|x|^{1+\alpha} \end{aligned}$$

for a universal C.

Then u is punctually $C^{1,\alpha}$ at the origin. In the same way, we can show it is punctually $C^{1,\alpha}$ at every point in $\Upsilon \cap B_{1/2}$ with a uniform bound. Now the lemma follows using the interior $C^{1,\alpha}$ estimates for the equation $F(D^2u) = 0$. □

Propositions 2.1 and 2.2 follow from Lemma 10.1 in the same way Theorem 8.2 follows from Theorem 6.1. We are going to give a detailed proof of Proposition 2.2, that is the one that we actually use in this article. The proof of Proposition 2.1 is very similar. The proof is written almost with the same words as the proof of Theorem 8.2 to stress the similarity.

Proof of Proposition 2.2. Set $g(x') = u(x', 0)$. We are going to use an iteration process similar to the proof of Theorem 6.1. By subtracting a suitable plane at the origin, we can suppose that $u(0) = g(0) = 0$ and $\nabla_{n-1}g(0) = 0$. We are going to prove a right decay for the function at the origin, and from there the estimate follows.

Let $M = \|u\|_{L^\infty(B^+_1)} + |F(0)| + \|g\|_{C^{1,\alpha}(\Upsilon)}$.

We want to show that there is a universal α_0, and a universal constant $\gamma < 1$, such that for $\beta = \min(\alpha_0, \alpha)$, there is a constant C_1, depending only on n, λ, and Λ and a sequence of real numbers a_k such that

$$\operatorname*{osc}_{B_{\gamma^k}}(u(x) - a_k \cdot x_n) \leq C_1 M\gamma^{k\cdot(1+\beta)} \tag{10.2}$$

$$|a_{k+1} - a_k| \leq CM\gamma^{k\beta}. \tag{10.3}$$

We will show this by induction.

We choose $C_1 > 1$ so that (10.2) holds for $k = 0$ with $a_0 = 0$.

To complete an induction proof, we assume that we already have a sequence a_k so that (10.2) holds for $k = 0, 1, \ldots, K$; we have to show that there is a real number a_{K+1} such that (10.2) holds for $k = K + 1$.

Let $r = \gamma^K$ and $b = a_K$.

Let v be the solution of the following problem

$$\begin{cases} F(D^2v) = 0 & \text{in } B_r^+ \\ v = 0 & \text{in } \Upsilon \cap B_r \\ v = u - b \cdot x_n & \text{in } \partial B_r \cap \{x_n > 0\}. \end{cases}$$

From maximum principle, we know

$$\operatorname*{osc}_{B_r^+} v \le \operatorname*{osc}_{B_r^+}(u - b \cdot x) + C_2|F(0)|r^2 \tag{10.4}$$

Now we apply the $C^{1,\alpha}$ estimates to v. Lemma 10.1 tells us that ∇v is well defined up to the boundary $\Upsilon \cap B_r$. Let $a = \partial_n v(0)$, by the boundary conditions $\nabla v(0) = (0, \ldots, 0, a)$. Rescaling of the $C^{1,\alpha}$ estimate gives

$$\frac{\operatorname{osc}_{B_{\tilde r}}(v(x) - a \cdot x_n)}{\tilde r^{1+\alpha_1}} \le C_0\left(\frac{1}{r^{1+\alpha_1}} \operatorname*{osc}_{B_r^+} v + r^{1-\alpha_1}|F(0)|\right) \tag{10.5}$$

$$|a| \le C\left(\frac{\operatorname{osc}_{B_r^+} v}{r} + r|F(0)|\right) \tag{10.6}$$

for any $\tilde r \le r/2$. Where α_1 is the α of Lemma 10.1.

We choose γ small enough so that $C_0\gamma^{\alpha_1} = (1-\theta) < 1$, for some positive constant θ. Combining (10.5) for $\tilde r = \gamma r$ with (10.4), we get

$$\operatorname*{osc}_{B_{\gamma r}^+}(v(x) - a \cdot x) \le (1-\theta)\gamma \operatorname*{osc}_{B_r^+}(u - b \cdot x) + C_3|F(0)|r^2 \tag{10.7}$$

Let $w = u - b \cdot x_n - v$. We have

$$\begin{cases} w \in S(\lambda/n, \Lambda) & \text{in } B_r^+ \\ w = g & \text{in } \Upsilon \cap B_r \\ w = 0 & \text{in } \partial B_r \cap \{x_n > 0\}. \end{cases}$$

Then, by the maximum principle,

$$\sup_{B_r^+} |w(x)| \le C\|g\|_{L^\infty(\Upsilon \cap B_r)} \le C\|g\|_{C^{1+\beta}(\Upsilon)} r^{1+\alpha}. \tag{10.8}$$

Adding (10.7) with (10.8) we get

$$\operatorname*{osc}_{B_{\gamma r}}(u(x) - (a+b) \cdot x_n) \le (1-\theta)\gamma \operatorname*{osc}_{B_r^+}(u - b \cdot x_n) + C|F(0)|r^2 + C\|g\|_{C^\beta(\Upsilon)} r^{1+\alpha}. \tag{10.9}$$

By the inductive hypothesis $\operatorname{osc}_{B_r^+}(u - b \cdot x_n) \le C_1 M \gamma^{K\cdot(1+\beta)}$. Replacing,

$$\operatorname*{osc}_{B_{\gamma r}}(u(x) - (a+b) \cdot x_n) \le M\Big((1-\theta)\gamma C_1 \gamma^{K\cdot(1+\beta)} + C_2\big(\gamma^{2K} + \gamma^{K(1+\alpha)}\big)\Big). \tag{10.10}$$

Now we will choose the right constants α_0 and C_1. We choose α_0 so that $\gamma^{\alpha_0} = (1-\theta/2)$, $\alpha = \min(\alpha_0, \beta)$, and C_1 large enough so that $\frac{\theta\gamma}{2} C_1 \geq 3C_2$. Replacing in (10.10), we get

$$\begin{aligned}
\operatorname*{osc}_{B_{\gamma r}}(u(x) - (a+b)\cdot x_n) &\leq M\left((1-\theta/2)\gamma C_1 \gamma^{K\cdot(1+\beta)} + C_2\left(\gamma^{2K} + \gamma^{K(1+\alpha)}\right) - \frac{\theta\gamma}{2} C_1 \gamma^{K\cdot(1+\alpha)}\right) \\
&\leq M(1-\theta/2)\gamma C_1 \gamma^{K\cdot(1+\beta)} \\
&\leq MC_1 \gamma^{(K+1)\cdot(1+\beta)}. \qquad (10.11)
\end{aligned}$$

From (10.6), (10.4), and that $r = \gamma^K$, we get

$$|a| \leq CM\gamma^{\beta K}. \qquad (10.12)$$

Taking $a_{K+1} = a + b$, we finish the inductive proof of (10.2) and (10.3).

Let $a_\infty = \lim_{k\to\infty} a_k$. We claim that

$$|u(x) - a_\infty \cdot x_n| \leq CM|x|^{1+\alpha};$$

indeed, from (10.2) and (10.3) we get

$$\begin{aligned}
\operatorname*{osc}_{B_{\gamma^k}}(u(x) - a_\infty \cdot x_n) &\leq \operatorname*{osc}_{B_{\gamma^k}}(u(x) - a_\infty \cdot x_n) + \gamma^k |a_k - a_\infty| \qquad (10.13) \\
&\leq C_1 M \gamma^{k\cdot(1+\alpha)} + CM\gamma^k \sum_{j=k}^{\infty} \gamma^{\alpha j} \qquad (10.14) \\
&\leq C_1 M \gamma^{k\cdot(1+\alpha)} + CM\gamma^{(1+\alpha)k} \frac{1}{1-\gamma^\alpha} \qquad (10.15) \\
&\leq CM\gamma^{k\cdot(1+\alpha)}. \qquad (10.16)
\end{aligned}$$

This implies the $C^{1,\alpha}$ estimate at the origin, and the estimate follows by translation and interior estimates. □

Now let us prove Propositions 2.3 and 2.4.

Proof of Proposition 2.3. Let $x \in B^+_{1/2}$ and $y \in B^+_1$. Let $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. We consider two cases whether $|x - y| \leq |x_n|/2$ or not.

If $|x-y| \leq |x_n|/2$, we apply (2.3) for $r = |x_n|$ and $x_0 = x$, then

$$\frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq C\frac{1}{r^\alpha} \operatorname*{osc}_{B_r(x)} u$$

but $\operatorname{osc}_{B_r(x)} u \leq 2C_0 |2x_n|^\alpha$ by (2.4), since $r = |x_n|$

$$\frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq C2^\alpha C_0 \leq CC_0$$

for a constant C depending only on the constant of (2.3).

If $|x-y| > |x_n|/2$, we apply (2.4) to obtain

$$\begin{aligned}|u(x)-u(y)| &\le |u(x)-u(x',0)| + |u(y)-u(x',0)| \\ &\le C_0(|x_n|^\alpha + |y-(x',0)|^\alpha) \\ &\le C_0(2^\alpha + 3^\alpha)|x-y|^\alpha \le 5C_0|x-y|^\alpha,\end{aligned}$$

where for the last inequality we used that $|x_n| < 2|x-y|$, and since $|y-(x',0)| \le |x-y| + |x_n|$, then $|y_n| \le 3|x-y|$.

Putting the two cases together, we obtain (2.5). □

Proof of Proposition 2.4. We are going to show that for any $x \in B^+_{1/2}$ there is a second order polynomial P_x such that for any $y \in B_1$,

$$|u(y) - P_x(y)| \le C \cdot C_0|x-y|^{2+\alpha} \tag{10.17}$$

The statement of the theorem clearly follows from this.

Let us write $x = (x', x_n)$ and $\bar{x} = (x', 0)$ be the projection of x on the boundary Υ. We know from the assumptions that there is a polynomial $P_{\bar{x}}$ such that

$$|u(y) - P_{\bar{x}}(y)| \le C_0|y-\bar{x}|^{2+\alpha}. \tag{10.18}$$

The function $v = u - P_{\bar{x}}$ solves

$$F(D^2 v + A) = 0 \quad \text{in } B_1^+$$

where A is the constant matrix $A = D^2 P_{\bar{x}}$. From Caffarelli and Cabré (1995), this equation has a $C^{2,\alpha}$ interior estimate that does not depend on A. Applying it in the ball $B_{x_n/2}(x)$ and recalling (10.18) we obtain that there is a polynomial R such that

$$\begin{aligned}|R(x)| &= |v(x)| \le C_0|x_n|^{2+\alpha} \\ |\nabla R(x)| &\le C \cdot C_0|x_n|^{1+\alpha} \\ |D^2 R(x)| &\le C \cdot C_0|x_n|^\alpha\end{aligned}$$

and

$$|v(y) - R(y)| \le C\Big(\sup_{B_{x_n/2}(x)} |v|\Big)\frac{1}{|x_n|^{2+\alpha}}|y-x|^{2+\alpha} \le C \cdot C_0|y-x|^{2+\alpha}. \tag{10.19}$$

Let us define

$$P_x = P_{\bar{x}} + R.$$

Now, if $|y-x| < x_n/2$ we have

$$|u(y) - P_x(y)| = |v(y) - R(y)| \le C \cdot C_0|y-x|^{2+\alpha}.$$

We are only left to prove (10.17) for the case $|y - x| > x_n/2$. In that case,

$$\begin{aligned}
|u(y) - P_x(y)| &\le |u(y) - P_{\bar{x}}(y)| + |R(y)| \\
&\le C_0|y - \bar{x}|^{2+\alpha} + |R(x) + \nabla R(x) \cdot (y - x) + (y - x)^t D^2 R(y - x)| \\
&\le C_0|y - \bar{x}|^{2+\alpha} + C \cdot C_0(x_n^{2+\alpha} + |y - x|x_n^{1+\alpha} + |y - x|^2 x_n^{\alpha}) \\
&\le C \cdot C_0|y - x|^{2+\alpha}
\end{aligned}$$

which finishes the proof. □

Remark 10.2. The proof of a $C^{1,\alpha}$ estimate up to the boundary works the same way replacing the second order polynomials by first order ones.

Acknowledgment

We would like to thank Prof. Luis Caffarelli for the fruitful discussion of ideas regarding this work.

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