Results in nonlocal elliptic equations

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joint work with Luis Caffarelli
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   Stochastic control

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   Integro-differential equations

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   A nonlocal ABP estimate.
   The difference of solutions satisfies an equation
   The nonlocal Evans-Krylov theorem
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Brownian motion $\rightarrow$ Laplace Equation

Let $g : \partial \Omega \rightarrow \mathbb{R}$.
Let $B$ be a Brownian motion.
$B_0 = x$

$B$ hits $\partial \Omega$ at $B_\tau$.

Let $u(x) = \mathbb{E}(g(B_\tau))$

Then

$$\triangle u = 0 \text{ in } \Omega$$
$$u = g \text{ on } \partial \Omega$$
Diffusions $\rightarrow$ Elliptic PDE with coefficients

Let $g : \partial \Omega \to \mathbb{R}$.
\[ X = \sqrt{A}dB \ (A \geq 0) \]
\[ X_0 = x \]

$X$ hits $\partial \Omega$ at $X_{\tau}$.

Let $u(x) = \mathbb{E}(g(X_{\tau}))$

Then
\[ A_{ij} \partial_{ij} u = 0 \text{ in } \Omega \]
\[ u = g \text{ on } \partial \Omega \]
Suppose we can choose the value of the coefficients $a_{ij}$ at every point from a family of choices $a^\alpha_{ij}$ ($\alpha$ is our control). We want to minimize the expected value of $g(X_\tau)$.

The function

$$u(x) = \inf_{\text{all choices of } \alpha \text{ at every point}} \mathbb{E}(g(X_\tau))$$

solves the equation

$$\inf_{\alpha} a^\alpha_{ij} \partial_{ij} u = 0 \text{ in } \Omega$$

(generic $F(D^2 u)$ for $F$ concave)

Evans and Krylov 1982: Solutions of these equations are $C^{2,\alpha}$. 
Optimal stopping $\rightarrow$ obstacle problem

\[ u(x) = \sup_{\tau} E(\varphi(X_{\tau}^x)) \]

where $X_{\tau}^x$ is a Brownian motion starting at $x$ and $\tau$ is any stopping time.

$\nabla u = 0$ \hspace{1cm} where $u > \varphi$ (at the points of no stop)

$\nabla u \leq 0$ everywhere in the domain

$u \geq \varphi$

A similar model is used in financial mathematics for pricing American options

Frehse 1972: Solutions of this equation are $C^{1,1}$.

Caffarelli 1977: The free boundary is smooth except at a small set of singular points.
Processes with jumps

Integro-differential equations (instead of PDEs) arise when we consider processes $X_t$ with discontinuities. Processes that jump from one point to another.

$X_t$ is a purely jump Lévy process.

$u(x) = \mathbb{E}u(X_\tau)$ solves

$$\int_{\mathbb{R}^n} (u(x + y) - u(x))K(y) \, dy = 0 \text{ in } \Omega$$

$u = g$ outside $\Omega$
General kernels

\[ \int_{\mathbb{R}^n} (u(x + y) - u(x))K(y) \, dy = 0 \]

The kernel \( K(y) \) represents the frequency of jumps in every direction \( y \).

- \( K(y) \geq 0 \) for every \( y \).
- \( \int_{\mathbb{R}^n} \min(y^2, 1)K(y) \, dy < +\infty \).

(We will only consider symmetric kernels \( K(y) = K(-y) \))
Fractional Laplacian

The most typical case is

\[ Lu(x) = -(-\Delta)^{\sigma/2}u(x) = c \int_{\mathbb{R}^n} (u(x + y) - u(x)) \frac{1}{|y|^{n+\sigma}} \, dy \]

which is the fractional Laplacian

\[ -(-\Delta)^{\sigma/2}u(\xi) = -|\xi|^{\sigma} \hat{u}(\xi). \]

\(-(-\Delta)^{\sigma/2}\) is for integro-differential equations what \(\Delta\) is for elliptic PDEs.

The constant \(c\) depends on \(n\) and \(\sigma\), and \(c \approx (2 - \sigma)\) as \(\sigma \to 2\).
Obstacle problem for the fractional laplacian

\[ u(x) = \sup_{\tau} E(\varphi(X^{x}_{\tau})) \]

where \( X^{x}_{t} \) is an \( \alpha \)-stable Lévy process starting at \( x \) and \( \tau \) is any stopping time.

\[ (-\Delta)^{\alpha/2} u = 0 \quad \text{where} \quad u > \varphi \quad \text{(at the points of no stop)} \]

\[ (-\Delta)^{\alpha/2} u \leq 0 \quad \text{everywhere in} \quad \mathbb{R}^{n} \]

\[ u \geq \varphi \]

A similar model is used in financial mathematics for pricing American options.

Caffarelli, Salsa, S. 2008: Solutions of this equation are \( C^{1,\alpha/2} \).

and the free boundary is smooth except at some singular points
elliptic integro-differential equations

We say

$$\int_{\mathbb{R}^n} (u(x + y) - u(x))K(y) \, dy = 0$$

is uniformly elliptic of order $\sigma \in (0, 2)$ if

1. $K(y) = K(-y)$ for every $y$.
2. $$(2 - \sigma)\frac{\lambda}{|y|^{n+\sigma}} \leq K(y) \leq (2 - \sigma)\frac{\Lambda}{|y|^{n+\sigma}}$$
Nonlinear integro-differential equations

In the same way as for diffusions, we can consider stochastic control problems with jumps to obtain nonlinear equations of the form

\[ 0 = Iu(x) := \sup_{\alpha} \int_{\mathbb{R}^n} (u(x + y) - u(x))K_\alpha(y) \, dy \]
Recovering second order PDEs

Note that the classical PDEs can be recovered from integro-differential equations in several ways. For example:

\[
\triangle u(x) = \lim_{s \to 1} -(-\triangle)^s u(x)
\]

\[
= \lim_{r \to 0} \frac{c}{r^{n+2}} \int_{B_r} u(x + y) - u(x) \, dy
\]
The natural Dirichlet problem is

\[ \mathcal{L}u(x) = 0 \text{ in } \Omega \]
\[ u(x) = g(x) \text{ in } \mathbb{R}^n \setminus \Omega \]

Note that the boundary condition is given in the whole complement of the domain: \( \mathbb{R}^n \setminus \Omega \). This is because of the nonlocal character of the equation.

It can be shown in fairly good generality that this problem admits a unique \emph{viscosity} solution.

\textbf{What about the regularity?}
Uniformly elliptic PDEs

Given $0 < \lambda < \Lambda$, for second order elliptic PDEs $F(D^2 u) = 0$, ellipticity is defined by the following property of the function $F$

$$
\lambda \| Y \| \leq F(X + Y) - F(X) \leq \Lambda \| Y \|
$$

every time $Y$ is a positive matrix.

If $F$ is smooth, this is equivalent to the matrix inequality

$$
\lambda I \leq \frac{\partial F}{X_{ij}} \leq \Lambda I
$$
Pucci’s maximal operator

Uniform ellipticity can also be described by means of the extremal Pucci operators:

\[ M^+(X) = \lambda (\text{sum of negative eigenvalues of } X) \]
\[ + \Lambda (\text{sum of positive eigenvalues of } X) \]
\[ M^-(X) = \Lambda (\text{sum of negative eigenvalues of } X) \]
\[ + \lambda (\text{sum of positive eigenvalues of } X) \]

Now, \( F(D^2u) = 0 \) is a uniformly elliptic equation if

\[ M^-(Y) \leq F(X + Y) - F(X) \leq M^+(Y) \]
Regularity results for fully nonlinear PDEs

  
  If $u$ is a bounded function in $B_1$ such that $M^+ u \geq 0$ and $M^- u \leq 0$ in $B_1$, then $u$ is Hölder continuous in $B_{1/2}$.

- $C^{1,\alpha}$ regularity.
  
  If $u$ is a solution to a uniformly elliptic fully nonlinear equation $F(D^2 u) = 0$ in $B_1$ then $u \in C^{1,\alpha}$ in $B_{1/2}$ for some $\alpha > 0$.

- Evans-Krylov theorem (1982)
  
  If $u$ is a solution to a convex uniformly elliptic fully nonlinear equation $F(D^2 u) = 0$ in $B_1$ then $u \in C^{2,\alpha}$ in $B_{1/2}$ for some $\alpha > 0$. 
Nonlocal extremal operators

The Pucci extremal operators are also given by the formula

$$M^+(D^2u) = \sup_{\lambda I \leq \{a_{ij}\} \leq \Lambda I} a_{ij} \partial_{ij} u$$

$$M^-(D^2u) = \inf_{\lambda I \leq \{a_{ij}\} \leq \Lambda I} a_{ij} \partial_{ij} u$$

An integro-differential analog of order $\sigma$ would be

$$M_{\sigma}^+ u(x) = \sup_{\lambda \leq a(y) \leq \Lambda} \int (u(x + y) - u(x)) \frac{a(y)}{|y|^{n+\sigma}} \, dy$$
Nonlocal extremal operators

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An integro-differential analog of order $\sigma$ would be

$$M^+_\sigma u(x) = \sup_{\lambda \leq a(y) \leq \Lambda} \frac{2 - \sigma}{2} \int \left( u(x + y) + u(x - y) - 2u(x) \right) \frac{a(y)}{|y|^{n+\sigma}} \, dy$$
The Pucci extremal operators are also given by the formula

\[ M^+(D^2 u) = \sup_{\lambda I \leq \{a_{ij}\} \leq \Lambda I} a_{ij} \partial_{ij} u \]

\[ M^-(D^2 u) = \inf_{\lambda I \leq \{a_{ij}\} \leq \Lambda I} a_{ij} \partial_{ij} u \]

An integro-differential analog of order \(\sigma\) would be

\[ M^+_\sigma u(x) = \frac{2 - \sigma}{2} \int_{\mathbb{R}^n} \frac{\Lambda(u(x + y) + u(x - y) - 2u(x))^+ - \lambda(\ldots)^-}{|y|^{n+\sigma}} \, dy \]
Uniform ellipticity for nonlocal equations

We say that a nonlocal operator $I$ is uniformly elliptic of order $\sigma$ if

$$M_\sigma^- v(x) \leq I(u + v)(x) - Iu(x) \leq M_\sigma^+ v(x)$$

($\sigma$ is always in $(0, 2)$)

Examples:

$$Lu(x) = \int (u(x + y) - u(x)) \frac{a(y)}{|y|^{n+\sigma}} \, dy$$

for $\lambda \leq a \leq \Lambda$ and $a(y) = a(-y)$

$Iu(x) = \inf_{\alpha} \sup_{\beta} L_{\alpha\beta} u(x)$

for $L_{\alpha\beta}$ linear as the one above

$Iu(x) = \int \frac{G(u(x + y) + u(x - y) - 2u(x))}{|y|^{n+\sigma}} \, dy$

$G$ monotone Lipschitz and $G(0) = 0$
Theorem (Caffarelli, S.)

Let $u \geq 0$ in $\mathbb{R}^n$, $M^-_\sigma u \leq 0$ and $M^+_\sigma u \geq 0$ in $B_2$.

Then

$$\sup_{B_1} u \leq C \inf_{B_1} u$$

Important: The constant $C$ does not blow up as $\sigma \to 2$.

We can understand the condition $M^-_\sigma u \leq 0$ and $M^+_\sigma u \geq 0$ as that there is some kernel $a(x, y)$ such that

$$\int (u(x + y) + u(x - y) - 2u(x)) \frac{(2 - \sigma)a(x, y)}{|y|^{n+\sigma}} \, dy = 0$$

with $\lambda \leq a(x, y) \leq \Lambda$, and $a$ can be very discontinuous.
Hölder estimates

**Theorem (Caffarelli, S.)**

Let \( u \in L^\infty(\mathbb{R}^n) \), \( M^-_\sigma u \leq 0 \) and \( M^+_\sigma u \geq 0 \) in \( B_2 \).

Then \( u \in C^\alpha(B_1) \) and

\[
u C^\alpha(B_1) \leq C \sup_{\mathbb{R}^n} |u|
\]

Important: The constant \( C \) does not blow up as \( \sigma \to 2 \).

We can understand the condition \( M^-_\sigma u \leq 0 \) and \( M^+_\sigma u \geq 0 \) as that there is some kernel \( a(x, y) \) such that

\[
\int (u(x + y) + u(x - y) - 2u(x)) \frac{(2 - \sigma)a(x, y)}{|y|^{n+\sigma}} \, dy = 0
\]

with \( \lambda \leq a(x, y) \leq \Lambda \), and \( a \) can be very discontinuous.
Differentiability of solutions

**Theorem (Caffarelli, S.)**

If $I$ is a nonlocal elliptic operator of order $\sigma$ and $u$ is a bounded function such that $Iu = 0$ in $B_1$, then $u \in C^{1+\alpha}(B_{1/2})$ and

$$u_{C^{1+\alpha}(B_{1/2})} \leq C \left( \sup_{\mathbb{R}^n} |u| + |I0| \right)$$

Important: The constant $C$ does not blow up as $\sigma \to 2$. 
More regular solutions for concave problems

Theorem (Caffarelli, S.)

If $I$ is a concave nonlocal elliptic operator of order $\sigma$ and $u$ is a bounded function such that $Iu = 0$ in $B_1$, then $u \in C^{\sigma + \alpha}(B_{1/2})$ and

$$u_{C^{\sigma + \alpha}(B_{1/2})} \leq C \left( \sup_{\mathbb{R}^n} |u| + |I0| \right)$$

Important: The constant $C$ does not blow up as $\sigma \to 2$. $\alpha$ can also be chosen independently of $\sigma$. 
Alexandroff-Bakelman-Pucci estimate

The proof of Harnack inequality for elliptic PDEs of second order is based on the ABP estimate: if $M^+ u \geq -f$ in $B_1$, $u \leq 0$ on $\partial B_1$, and $\Gamma$ is the concave envelope of $u$ in $B_2$ then

$$c(\max_{B_1} u)^n \leq |\nabla \Gamma(B_1)| = \int_{\{u=\Gamma\}} \det(-D^2\Gamma) \, dx \leq C \int_{\{u=\Gamma\}} f^n \, dx$$

For integro differential equations, we need some alternative way to measure $\{u = \Gamma\}$.
Alexandroff-Bakelman-Pucci estimate

The proof of Harnack inequality for elliptic PDEs of second order is based on the ABP estimate: if $M^+ u \geq -f$ in $B_1$, $u \leq 0$ on $\partial B_1$, and $\Gamma$ is the concave envelope of $u$ in $B_2$ then

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For integro differential equations, we need some alternative way to measure $\{u = \Gamma\}$
No cancellations in the integral

Let \( x \in \{ u = \Gamma \} \), \(-f(x) \leq M^+ u(x)\)

\[
M_\sigma^+ u(x) = \int \frac{\Lambda(u(x + y) + u(x - y) - 2u(x))^+ - \lambda(u(x + y) + u(x - y) - 2u(x))^+}{|y|^{n+\sigma}} \, dy
\]
Let \( x \in \{ u = \Gamma \} \), \( -f(x) \leq M^+ u(x) \)

\[
M^+_\sigma u(x) = \int \frac{\Lambda(u(x + y) + u(x - y) - 2u(x))^+ - \lambda(u(x + y) + u(x - y) - 2u(x))^+}{|y|^{n+\sigma}} \, dy
\]
We compare $u(y) - u(x) - y \cdot \nabla \Gamma(x)$ with $A|y|^2$. 

\[ \int (2 - \sigma) \frac{\lambda(u(x+y)+u(x-y)-2u(x))^-}{|y|^{n+\sigma}} \, dy \leq c \frac{f(x)}{A} \int_{B_r} \frac{(2-\sigma)A|y|^2}{|y|^{n+\sigma}} \, dy \]

$\left| \{ u(y) - u(x) - y \cdot \nabla \Gamma(x) \leq A|y|^2 \} \right| \leq C \frac{f(x)}{A} r^n$
Catching up with the integrals

**Lemma:** Assume $M_{\sigma}^+ u \geq -f$ in $B_1$ (where $M_{\sigma}^+$ is now the maximal operator of order $\sigma$), $u \leq 0$ in $\mathbb{R}^n \setminus B_1$ and $\Gamma$ is the concave envelope of $u$ in $B_3$. If $u(x) = \Gamma(x)$, for every $A > 0$ there is ring $R_r(x)$ such that

$$|R_r \cap \{ u(y) \leq u(x) - (y-x) \cdot \nabla \Gamma(x) - Ar^2 \}| \leq C \frac{f(x)}{A} r^n$$

$$\Gamma(y) \leq u(x) - (y - x) \cdot \nabla \Gamma(x) - Ar^2 \text{ for all } y \in B_{r/2}$$
Catching up with the integrals

**Lemma:** Assume \( M_{\sigma}^+ u \geq -f \) in \( B_1 \) (where \( M_{\sigma}^+ \) is now the maximal operator of order \( \sigma \)). \( u \leq 0 \) in \( \mathbb{R}^n \setminus B_1 \) and \( \Gamma \) is the concave envelope of \( u \) in \( B_3 \). If \( u(x) = \Gamma(x) \), for every \( A > 0 \) there is ring \( R_r(x) \) such that

\[
| R_r \cap \{ u(y) \leq u(x) - (y-x) \cdot \nabla \Gamma(x) - Ar^2 \} | \leq C \frac{f(x)}{A} r^n 
\]

\( \Gamma(y) \leq u(x) - (y - x) \cdot \nabla \Gamma(x) - Ar^2 \) for all \( y \) in \( B_{r/2} \)
Consequences of the lemma

Around each point \( x \in \{ u = \Gamma \} \) there is a (small) ball \( B_r(x) \) such that

- \( u \geq \Gamma - Cf(x)r^2 \) in a large proportion of \( B_r(x) \).
- \( |\nabla \Gamma(B_r(x))| \leq Cf(x)^n |B_r| \).

By covering the whole contact set \( \{ u = \Gamma \} \) with a subfamily of such balls with finite overlapping we find

\[
|\nabla \Gamma(B_1)| \leq C \left| \left\{ u(x) \geq \Gamma(x) - Cr_0^2 \right\} \right|
\]

\( (r_0 \text{ is the maximum possible value of } r, \text{ which depends on } \sigma) \)
Thus we obtain
\[
c(\max u)^n \leq |\nabla \Gamma(B_1)| \leq C \left| \{u(x) \geq \Gamma(x) - Cr_0^2\} \right|
\]
which is good enough to carry out the rest of the proof of Harnack inequality and Hölder estimates.
Difference of solutions

If $u$ and $v$ are solutions to the same equation $Iu = Iv = 0$, then their difference solves

$$M^-(u - v) \leq 0 \leq M^+(u - v)$$

One can understand this as a linear equation with a priori discontinuous coefficients.

$$\int_{\mathbb{R}^n} ((u - v)(x + y) + (u - v)(x - y) - 2(u - v)(x)) K(x, y) \, dy = 0$$

where $(2 - \sigma) \frac{\lambda}{|y|^{n+\sigma}} \leq K(x, y) \leq (2 - \sigma) \frac{\Lambda}{|y|^{n+\sigma}}$ with no continuity a priori in $x$. 
More on difference of solutions

If $u$ and $v$ are solutions to the same equation $Iu = Iv = 0$, then

$$M^-(u - v) \leq 0 \leq M^+(u - v)$$

which also implies that the integrals of positive and negative incremental quotients

$$\int_{\mathbb{R}^n} ((u - v)(x + y) + (u - v)(x - y) - 2(u - v)(x)) \frac{1}{|y|^{n+\sigma}} \, dy$$

are comparable.
More on difference of solutions

If \( u \) and \( v \) are solutions to the same equation \( Iu = Iv = 0 \), then

\[
M^-(u - v) \leq 0 \leq M^+(u - v)
\]

which also implies that the integrals of positive and negative incremental quotients

\[
\int_{\mathbb{R}^n} \delta_y(u - v)(x) \pm \frac{1}{|y|^{n+\sigma}} \, dy
\]

are comparable.
\textbf{$C^{1,\alpha}$ estimates}

The differential quotient $w_h = \frac{u(x+he) - u(x)}{h}$ satisfies an equation

$$M^- w_h \leq 0 \leq M^+ w_h$$

\implies $w_h$ is $C^\alpha$ independently of $h$, and $u \in C^{1,\alpha}$.

(there is a technical difficulty because $u$ may not be $C^1$ outside of the domain)
Concavity

If $I$ is concave and $u$ is a solution of $Iu = 0$ then a mollification is a subsolution.

$$I(u \ast \eta) \geq 0$$

In particular for

$$\delta_y u(x) := (u(x + y) + u(x - y) - 2u(x))$$

we have

$$M^+ \delta_y u(x) \geq 0$$
If \( I \) is concave and \( u \) is a solution of \( Iu = 0 \) then a mollification is a subsolution.

\[
I(u * \eta) \geq 0
\]

In particular for

\[
\int_{B_h} \delta_y u(x) K(y) \, dy \approx u * K - \left( \int K \, dy \right) u
\]

we have

\[
M^+ \int_{B_h} \delta_y u(x) K(y) \, dy \geq 0
\]
For the proof of Evans-Krylov theorem, it is not enough to have

\[ M^+ \delta_y u(x) \geq 0 \]

to get \( u \in C^{2,\alpha} \).

The equation has to be used further. In particular that

\[ \| D^2 u^+ \| \approx \| D^2 u^- \| \]
Evans-Krylov theorem

For the proof of Evans-Krylov theorem, it is not enough to have

\[ M^+ \delta_y u(x) \geq 0 \]

to get \( u \in C^{2,\alpha} \).

The equation has to be used further. In particular that

\[ \| (D^2 u(x) - D^2 u(y))^+ \| \approx \| (D^2 u(x) - D^2 u(y))^- \| \]
What is used

Positive and negative parts of the integral control each other.

\[
\int (\delta_y u(x) - \delta_y u(0))^+ \frac{(2 - \sigma)}{|y|^{n+\sigma}} \, dy \approx \int (\delta_y u(x) - \delta_y u(0))^- \frac{(2 - \sigma)}{|y|^{n+\sigma}} \, dy
\]

Linear integral operators are subsolutions

\[
M^+ \int (\delta_y u(x) - \delta_y u(0))K(y) \, dy \geq 0
\]

for any \( K \geq 0 \).
Steps in the proof

Step 1. Prove that the integrals converge absolutely:

\[ \int |\delta_y u(x)| \frac{(2 - \sigma)}{|y|^{n+\sigma}} \, dy \leq C \]

Step 2. Prove that the function is \( C^{\sigma + \alpha} \).
Scheme of step 2.

We prove that

\[ P(x) := \int (\delta_y u(x) - \delta_y u(0))^+ \frac{(2 - \sigma)}{|y|^{n+\sigma}} \, dy \leq C|x|^\alpha \]

This implies that

\[ \int |\delta_y u(x) - \delta_y u(0)| \frac{(2 - \sigma)}{|y|^{n+\sigma}} \, dy \leq C|x|^\alpha \]

which immediately implies that \( u \in C^{\sigma+\alpha} \).
Inductive argument

We show that for every $r \in (0, 1)$,

$$\sup_{B_{r/2}} P(x) \leq (1 - \theta) \sup_{B_r} P(x)$$

for some $\theta > 0$

Thus we get $P(x) \leq C|x|^\alpha$
Inductive argument

We show that for every $r \in (0, 1)$,

$$\sup_{B_{r/2}} P(x) \leq (1 - \theta) \sup_{B_r} P(x) \quad \text{for some } \theta > 0$$

Thus we get $P(x) \leq C|x|^\alpha$.
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Inductive argument

We show that for every $r \in (0, 1),$ 

$$\sup_{B_{r/2}} P(x) \leq (1 - \theta) \sup_{B_r} P(x)$$  for some $\theta > 0$

Thus we get $P(x) \leq C|x|^\alpha$
Inductive argument

We show that for every $r \in (0, 1)$,

$$\sup_{B_{r/2}} P(x) \leq (1 - \theta) \sup_{B_r} P(x)$$

for some $\theta > 0$

Thus we get $P(x) \leq C|x|^\alpha$
The inductive step

Let $P(x_0) = \max_{B_1/2} P$. We want to show $P(x_0) \leq (1 - \theta)$ for some $\theta > 0$. 

\[ P \leq 1 \text{ in } B_1 \]
The inductive step

Let $P(x_0) = \max_{\overline{B}_{1/2}} P$. We want to show $P(x_0) \leq (1 - \theta)$ for some $\theta > 0$.

Recall

$$P(x_0) = \int (\delta_y u(x_0) - \delta_y u(0))^+ \frac{(2 - \sigma)}{|y|^{n+\sigma}} \, dy$$
The inductive step

Let $P(x_0) = \max_{\bar{B}_{1/2}} P$. We want to show $P(x_0) \leq (1 - \theta)$ for some $\theta > 0$.

Recall

$$P(x_0) = \int \left( \delta_y u(x_0) - \delta_y u(0) \right) \frac{(2 - \sigma)}{|y|^{n+\sigma}} \chi_A \, dy$$

where $A = \{ y : \delta_y u(x) - \delta_y u(0) > 0 \}$
A tool: weak Harnack inequality

The following versions of the weak Harnack inequality are available for sub and super-solutions.

**Theorem**

Let $u \geq 0$ in $\mathbb{R}^n$ and $M^- u \leq 0$ in $B_1$ (supersolution).

$$|\{u > t\} \cap B_1| \leq Ct^{-\varepsilon} \inf_{B_{1/2}} u \quad \text{for every } t > 0.$$ 

**Theorem**

If $M^+ u \geq 0$ in $B_1$ (subsolution) then

$$u(x) \leq C \int_{\mathbb{R}^n} \frac{|u(y)|}{1 + |y|^{n+\sigma}} \, dy \quad \text{in } B_{1/2}.$$
First possibility

Since $M^+ \left( \int (\delta_y u(x) - \delta_y u(0)) \frac{(2-\sigma)}{|y|^{n+\sigma}} \chi_A \, dy \right) \geq 0$, if we had

$$\int (\delta_y u(x) - \delta_y u(0)) \frac{(2-\sigma)}{|y|^{n+\sigma}} \chi_A \, dy \leq (1 - C\theta)$$

in a fraction of $B_1$, we would obtain

$$P(x_0) = \int (\delta_y u(x_0) - \delta_y u(0)) \frac{(2-\sigma)}{|y|^{n+\sigma}} \chi_A \, dy \leq 1 - \theta$$

by weak Harnack inequality.

But what if the opposite inequality holds in most of $B_1$?
Second posibility

If

\[ \int (\delta_y u(x) - \delta_y u(0)) \frac{(2 - \sigma)}{|y|^{n+\sigma}} \chi_A \, dy \geq (1 - C\theta) \]

in most of \( B_1 \), that means that the same choice of set \( A \) is approximately optimal to compute \( P(x) \) in most of \( B_1 \).

\[ \int (\delta_y u(x) - \delta_y u(0))^+ \frac{(2 - \sigma)}{|y|^{n+\sigma}} \, dy \approx \int (\delta_y u(x) - \delta_y u(0)) \frac{(2 - \sigma)}{|y|^{n+\sigma}} \chi_A \, dy \]

with the same set \( A \) for most \( x \in B_1 \).
Second possibility

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\]

with the same set \( A \) for most \( x \in B_1 \).
The punchline

If we have \( \int (\delta_y u(x) - \delta_y u(0))^+ \frac{(2-\sigma)}{|y|^{n+\sigma}} \, dy \) very positive in most of \( B_1 \)

But then we can apply weak Harnack and obtain that \( \int (\delta_y u(x) - \delta_y u(0))^+ \frac{(2-\sigma)}{|y|^{n+\sigma}} \chi_{A^c} \, dy \) is strictly negative for all \( x \in B_{1/2} \)

This is a contradiction at \( x = 0 \)!

This finishes the proof of the inductive step \( \Rightarrow P(x) \leq C|x|^\alpha \Rightarrow u \in C^{\sigma+\alpha} \).
The punchline

If we have \( \int (\delta_y u(x) - \delta_y u(0)) \frac{(2-\sigma)}{|y|^{n+\sigma}} \, dy \) very negative in most of \( B_1 \)

But then we can apply weak Harnack and obtain that
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If we have \[ \int (\delta_y u(x) - \delta_y u(0)) \left( \frac{2-\sigma}{|y|^{n+\sigma}} \right) \chi_{A^c} \, dy \] very negative in most of \( B_1 \)

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