Description of my current research

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My research concerns partial differential equations. I am mostly interested in regularity for elliptic and parabolic PDEs, with an emphasis on free boundary problems and integro-differential equations. I also work in partial differential equations arising from problems in material sciences and fluid mechanics.

In the next sections I describe some of my current projects.

1 Fully nonlinear integro-differential equations

Integro-differential equations arise naturally in the study of stochastic processes with jumps and have many applications to finance and physics (See for example [16] or [2]). They are also a natural generalization of elliptic partial differential equations. For elliptic PDEs, there is a well established theory of weak solutions and regularity results even in the fully nonlinear case [9]. Many of these results seem to hold for integro-differential equations too. In many cases the generalization is not straight forward but instead presents an interesting mathematical challenge. The study of integro-differential equations is a relatively new area in mathematics full of open problems that is attracting an increasing level of interest.

In a joint work with Luis Caffarelli we are carrying out a program to develop a regularity theory for nonlinear integro-differential equation similar to the regularity theory for fully nonlinear second order differential equations (as in [9]).

The integro-differential equations form a very rich class of equations. Indeed, all second order elliptic PDEs are obtained as limits of integro-differential equations. Integro-differential equations can be understood as elliptic equations of fractional order (which never exceeds two). The simplest example is the fractional Laplace equation, which consists of a function such that $(-\triangle)^s u = 0$ in some domain for $s \in (0,1)$. There are also nonlinear nonloncal equations and a concept of ellipticity of fractional order. There are many general properties of elliptic PDEs that can be generalized in order to obtain a unified theory of existence and uniqueness of their solutions as well as their regularity.

In [4], we prove a nonlocal version of the Alexandroff-Backelman-Pucci estimate and Krylov-Safonov Harnack inequality which leads to Hölder estimates for solutions of integro-differential equations with very rough coefficients. Using that we manage to prove a $C^{1,\alpha}$ regularity result for a general class of nonlinear nonlocal equations. It is important to point out that we obtain estimates that remain uniform if we consider a sequence of integro-differential equations that converges to a second order PDE. In this respect, our estimates recover Krylov-Safonov result as we let the order of the equation approach two.

The $C^{1,\alpha}$ estimates in [4] are for translation invariant equations. In [7], we extended this result to a variable coefficient case and also to other situations, for example when we have lower order terms in the equation.

The most natural equation from stochastic control is the Hamilton-Jacobi-Bellman equation. For purely jump Levy processes it takes the form of an infimum or a supremum of integro-differential equations

$$\inf_i (2-\sigma) \int_{\mathbb{R}^n} (u(x+y) - u(x)) \frac{a_i(y)}{|y|^{n+\sigma}} \, \mathrm{d}y = 0 \qquad \text{for } x \in B_1.$$

We obtain our best result for this particular equation since we can prove that the solutions are classical. In [5], we show that the solution of the integro-differential Hamilton-Jacobi-Bellman equation of order σ is in the class $C^{\sigma+\alpha}$ for some $\alpha > 0$, which means that all the integrals in the left hand side are well defined and Hölder continuous. The functions a_i are assumed to satisfy $0 < \lambda \leq a_i(y) \leq \Lambda$ and $a_i(y) = a_i(-y)$, as well as some extra assumptions on the regularity of their tails.

This result is an integro-differentiable version of the celebrated Evans-Krylov theorem. In order to prove it, we apply all the results from the two previous papers [4] and [7]. Even with those results at hand, the proof is not a direct reproduction of the proof of the second order case, since the proofs available seemed difficult to adapt to the nonlocal case. Instead, the method we developed provides a new proof of the Evans-Krylov theorem even in the second order case, as explained in [6]. The main building blocks on which the proof is based are the same, but they are organized differently.

Relevant publications:

Smooth approximations to solutions of nonconvex fully nonlinear elliptic equations.

L. Caffarelli and L. Silvestre. American Mathematical Society Translations–Series 2 Advances in the Mathematical Sciences 2010; Volume: 229. Nonlinear Partial Differential Equations and Related Topics: Dedicated to Nina N. Uraltseva.

We show that fully nonlinear elliptic PDEs (which may not have classical solutions) can be approximated with integro-differential equations which have $C^{2,\alpha}$ solutions. This approximation can be used to turn a priori estimates into regularity results. In this paper, we show a $C^{1,\alpha}$ estimate uniform in the approximation. We also study the rate of convergence.

On the Evans-Krylov theorem.

Joint work with Luis Caffarelli. Proceedings of the AMS. 138 (2010), 263-265.

We use the ideas developed for the integro-differential Bellman equation to provide a simplified proof of the Evans-Krylov theorem.

The Evans-Krylov theorem for non local fully non linear equations.

Joint work with Luis Caffarelli. Submitted.

This is the third paper in the series where we develop a regularity theory for fully nonlinear integro-differential equations. We prove that the integro-differential Hamilton-Jacobi-Bellman equation has classical solutions. This is a generalization of the celebrated theorem of Evans and Krylov for convex fully nonlinear elliptic equations. It is interesting to remark that even though there are common ideas, the proof is not an adaptation of the proof in the second order case, but it introduces new ideas even in the local setting.

Regularity results for nonlocal equations by approximation

L. Caffarelli and L. Silvestre. Archive of Rational Mechanics and Analysis. To Appear.

This is the second paper in the series where we develop a regularity theory for fully nonlinear integro-differential equations. We develop the perturbation theory that allows us to obtain regularity results in the variable coefficient case, or in equations that mix either kernels of different orders or drift terms with kernels of order larger than one.

Regularity theory for fully nonlinear integro-differential equations.

Joint work with Luis Caffarelli. Communications on Pure and Applied Mathematics. 62 (2009) Issue 5, 597–638.

This is the first of a series of papers where we develop a regularity theory for fully nonlinear integro-differential equations like the ones that arise from stochastic control problems with purely jump Lvy processes. In this paper we obtain an estimate that plays the role of the Alexander-Backelman-Pucci but in the nonlocal setting. Then we develop a Harnack inequality for equations with discontinuous kernels and we use it to obtain $C^{1,\alpha}$ estimates for fully nonlinear equations. It is important that all estimates are independent of the degree of the equations, so the corresponding results for second order elliptic PDE can be recovered as limit cases.

Hölder estimates for solutions of integro differential eq. like the fractional laplace
Indiana University Mathematical Journal 55 (2006), 1155-1174.

In this paper there is a purely analytical proof of Hölder continuity for harmonic functions with respect to a class of integro differential equations like the ones associated with purely jump processes. There are some previous proofs but they are probabilistic and their assumptions are not as flexible. Our assumptions include the case of an operator with variable order, without any continuity required for that order.

2 Obstacle problem for the fractional Laplacian

The obstacle problem for elliptic partial differential equations appears classically in elasticity as the equation that models the shape of an elastic membrane that is constrained to remain above an obstacle (which pushes the membrane from below). In probability the problem also arises when modeling optimal stopping times. The idea is that one follows a stochastic process and can choose to stop at any time. Whenever we stop, we receive the value of a given payoff function at the point the stochastic process ended. It turns out that the expectation of this payoff function in the best stopping strategy solves an obstacle problem with the payoff function as the obstacle and an elliptic equation related to the properties of the stochastic process that we follow. In financial mathematics, American options are priced using a model that follows this idea.

If we consider an optimal stopping problem with a discontinuous Lévy process, we obtain an obstacle problem for an integro-differential equation. In particular if we consider an α-stable process, we will end up with the obstacle problem for the fractional Laplacian.

Given a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$ which decays at infinity appropriately, the obstacle problem for the fractional Laplacian in $\mathbb{R}^n$ consists in finding the function u satisfying

$$\begin{aligned} u &\geq \varphi && \text{in } \mathbb{R}^n, \\ (-\triangle)^s u &\geq 0 && \text{in } \mathbb{R}^n \qquad (s \in (0,1)), \\ (-\triangle)^s u(x) &= 0 && \text{for those } x \text{ such that } u(x) > \varphi(x), \\ \lim_{|x|\to+\infty} u(x) &= 0. \end{aligned} \tag{1}$$

This is a free boundary problem because the contact set $\{u = \varphi\}$ is not known a priori. The free boundary in this case is precisely $\partial\{u = \varphi\}$.

In [27] and [10], we studied the optimal regularity of the solutions and the free boundary of the obstacle problem for the fractional Laplacian. In [10] we used a characterization of the problem as the *thin* obstacle problem for a degenerate elliptic partial differential equation. This connection was established in [3].

In [3] we show that for every $s \in (0,1)$, the operator $(-\triangle)^s$ corresponds to a Dirichlet to Neumann operator for a degenerate elliptic PDE in the upper half space [3]. In this way we could rewrite the obstacle problem (1) as a thin obstacle problem for a degenerate elliptic equation. We obtained the following equivalent formulation

$$\begin{aligned} \operatorname{div}(r^a \nabla u(x,r)) &= 0 && \text{in } \mathbb{R}^n \times [0,+\infty), \\ u(x,0) &\geq \varphi(x) && \text{in } \mathbb{R}^n, \\ \lim_{r\to0^+} r^a u_r(x,r) &= 0 && \text{for those points } x \in \mathbb{R}^n \text{ where } u(x,0) > \varphi(x), \\ \lim_{r\to0^+} r^a u_r(x,r) &\leq 0 && \text{for every } x \in \mathbb{R}^n. \end{aligned}$$

The most clear advantage is that we now have a local PDE and we can use the more common methods for regularity theory of free boundary problems like monotonicity formulas, blowups, etc.

Other work I have done related to the thin obstacle problem can be found in [25], where we studied the regularity of a solution to the thin obstacle problem for a fully nonlinear uniformly elliptic equation.

For applications, it would be essential to understand the case of general integro-differential operators instead of just the fractional Laplacian. This corresponds to different choices of Lévy processes in the optimal stopping problem. It is natural to expect that the results would extend to general kernels K as long as they behave asymptotically like $c|y|^{-n-\sigma}$ for small y. In [28], the almost optimal regularity of the solution to the problem is extended to the case of an integral differential operator with a kernel $K(y) = c|y|^{-n-\sigma} + g(y)$ for an L^1 function g. This extends the result to, for example, truncated α-stable processes. However, the assumption does not seem to be sharp. One would expect the regularity results to extend to kernels of the form $K(y) = c|y|^{-n-\sigma} + g$ for functions g which may be singular at the origin but $\min(1, |y|^{\sigma})g(y) \in L^1$. A natural kernel that has this form is, for example, $K(y) = c|y|^{-n-\sigma} + a(y)|y|^{-n-r}$ with $r < \sigma$ and any bounded function a.

Another interesting problem, both for the analysis and for the applications, is to understand the optimal stopping problem for Lévy processes with a drift. This corresponds to the obstacle problem for an equation with fractional diffusion and advection. It is to be expected that the optimal regularity result $u \in C^{1,\sigma/2}$ will be lost if there is a drift term in the equation $(b \cdot \nabla u)$ and the power of the Laplacian is less than 1/2. This would be a supercritical case where the fractional Laplacian, which is ultimately what regularizes the solution, has less influence in the low scales than the first derivative. For $\sigma > 1$, th expectation is that a perturbative argument leads to the same regularity as in the case without drift. The critical case $\sigma = 1$ is harder to predict and probably the most interesting.

The parabolic obstacle problem for the fractional heat equation has been analyzed in a very recent paper by Caffarelli and Figalli [11].

Relevant publications:

Reg. estimatres for the sol. and the free bound. of the obst. prob. for the fract. Lap.

Joint work with Luis Caffarelli and Sandro Salsa. Inventiones Mathematicae. 171, Number 2 (2008).

We use a characterization of the fractional Laplacian as a Dirichlet to Neumann operator to rewrite its obstacle problem as a local free boundary problem. We obtain sharp regularity estimates for this free boundary problem and we also prove the regularity of the free boundary at generic points. The techniques used involve studying the blowup limits using a variation of Almgren's monotonicity formula.

Regularity for the nonlinear Signorini problem.

Joint work with Manolis Milakis. Advances in Mathematics. 217, Issue 3 (2008).

We prove that solutions to the thin obstacle problem with arbitrary convex fully nonlinear equations are $C^{1,\alpha}$ for some small $\alpha > 0$.

An extension problem related to the fractional laplacian.

In collaboration with Luis Caffarelli. Communications in Partial Differential Equations, 32 (2007) 8, 1245.

We characterize the fractional Laplace operator $(-\triangle)^s$ for any power $s \in (0, 1)$ as the Dirichlet to Neumann operator for a degenerate elliptic equation in one more dimension. The elliptic equation can be thought of as a Laplace equation in fractional dimension, and this intuition leads to several useful formulas. The characterization is used to translate nonlocal problems involving fractional diffusion into local problems in one more dimension and therefore it allows us to apply a different type of techniques.

Regularity of the obstacle problem for a fractional power of the laplace operator

Communications on Pure and Applied Mathematics. 60 (2007), no. 1, 67–112.

The solution of the obstacle problem for the operator $(-\triangle)^s$ and an obstacle ϕ is the least function u such that $u \geq \phi$ and $(-\triangle)^s u \geq 0$. In this work it is shown that such solutions are in the space $C^{1,\alpha}$ for every $\alpha < s$. These operators arise in stochastic theory and are related to Levy processes. The obstacle problem is used in financial mathematics to model pricing of american options. When $s = 1/2$, the problem is equivalent to the Signorini problem. It was shown very recently that solutions of the Signorini problem are $C^{1,1/2}$.

3 Equations with Advection and fractional diffusion

I have recently become interested in parabolic equations with advection and fractional diffusion, where there is a subtle interaction between the regularization effect of the diffusion and the possible singularities (shocks) created by the advection. These type of problems concern a Cauchy problem for a scalar function θ and I want to study the regularity of the solution for positive time. The equations considered fall into the following categories.

1. **Linear equations.**

$$\theta_t + b \cdot \nabla\theta + (-\triangle)^s \theta = 0 \tag{2}$$

In this case the interesting question is to obtain a regularity result for θ with minimal assumptions on the vector fields b.

2. **Hamilton-Jacobi equations.**

$$\theta_t + H(\nabla\theta) + (-\triangle)^s \theta = 0 \tag{3}$$

The question for this equation is to understand for what values of s we can guarantee the existence of differentiable solutions.

3. **Conservation laws.**

$$\theta_t + \operatorname{div} F(\theta) + (-\triangle)^s \theta = 0 \tag{4}$$

The question for this equation is to understand what values of s prevent the formation of shocks.

4. **Surface Quasi geostrophic (SQG) equation.**

$$\theta_t + w \cdot \nabla\theta + (-\triangle)^s \theta = 0 \tag{5}$$

Where $w = R^{\perp}\theta$. The question for this case is to understand for what values of s the equation is well posed.

Let us start by discussing the **linear equation**. It is fundamental to have an understanding of the minimal assumptions needed for b in order to obtain regularity for θ, since then we can apply these results for nonlinear equations.

In [8], Caffarelli and Vasseur proved that if b is divergence free and BMO, then θ becomes Hölder continuous for positive time. The proof uses ideas that can be traced back to De Giorgi. It is a parabolic version of the proof of De Giorgi in the De Giorgi-Nash-Moser theorem, adapted to the fractional Laplacian by using its characterization as an extension from [3]. From this result, they immediately obtain the well posedness of the SQG equation in the critical case ($s = 1/2$).

In [33], I proved a very similar result to that of Caffarelli and Vasseur. The result says that if $b \in L^\infty$ then θ becomes immediately Hölder continuous. Note that neither results implies the other: in [8], they assume $b \in BMO\ +\ \operatorname{div} b = 0$, whereas in [33], I assume $b \in L^\infty$. The proofs are completely different. Indeed, without any information on the divergence of b, it seems impossible to apply variational methods (like De Giorgi's) to the equation. The proof in [33] is based on an improvement of oscillation lemma using pointwise estimates that adapt perfectly to the context of viscosity solutions. The estimate immediately implies the differentiability of the solution to the Hamilton-Jacobi equation with critical fractional diffusion (case $s = 1/2$).

I am currently preparing a new article [30] where the idea is explored further just for the case of linear equations and all exponents s of the Laplacian. If $s < 1/2$, a Hölder continuity estimate can be obtained for θ if $b \in C^{1-2s}$. This is a strict improvement from what the De Giorgi method can provide in the supercritical case since, from [15], that method requires $\operatorname{div} b = 0$ in addition to the same smoothness assumption $b \in C^{1-2s}$.

I am preparing another article [31] for the case $s < 1/2$ where with a marginally better smoothness assumption on b, we can obtain that the solution θ is $C^{1,\alpha}$ for every $\alpha < 2s$. the assumption for b is

$$\lim_{r \to 0} \sup_{|x-y| \leq r} \frac{|b(x) - b(y)|}{r^{1-2s}} = 0.$$

It may seem surprising that a marginal regularity improvement can provide such dramatic improvement in the regularity of θ. It is not so surprising if one compares with the regularity theory of second order elliptic PDEs. While the result of [30] would be a nonlocal analog of Krylov-Safonov result, the result in [31] is comparable to a nonlocal version of Cordes-Nirenberg. The result would be useful as a regularity criteria for nonlinear equations.

The **Hamilton-Jacobi** equation with subcritical fractional diffusion ($s > 1/2$) is known to have a smooth solution (see [20]). For any $s \in (0, 1)$, there is always a well defined viscosity solution that is Lipschitz in space (see [19]). However, if $s < 1/2$, it is known that even with a smooth initial data, the solution may develop angles. As it was mentioned before, in [33], I proved that the problem is also well posed in the critical case $s = 1/2$. The result in [33] is actually a lot more general, since it shows the differentiability of solutions of equations of the form

$$\theta_t = \inf_i \sup_j \left(c_{ij} + b_{ij} \cdot \nabla \theta + \int_{\mathbb{R}^n} \frac{\theta(x+y) - \theta(x)}{|y|^{n+1}} a_{ij}(y) \, \mathrm{d}y \right)$$

where the a_{ij} are a family of kernels bounded below and above. This type of equations arises from stochastic games with Levy noise. The Hamilton-Jacobi equation with fractional diffusion is a particular case when all a_{ij} are constant.

The case of **conservation laws** with fractional diffusion is similar to the Hamilton-Jacobi equation. It is known that the problem is classically well posed if $s > 1/2$, and shocks can develop if $s < 1/2$ (see [19]). The ideas from [33] can be used to study conservation laws with fractional diffusion. Indeed, the result for the linear equation with $b \in L^\infty$ provides a Hölder a priori estimate for the solution θ of conservations laws with critical fractional diffusion $s = 1/2$. Moreover, the conservation law equation has the extra property that the L^1 norm of the solution does not increase and it is easy to show that the L^∞ norm decays at a certain rate. This allows to obtain large time regularity estimate when s is slightly below $1/2$ as in [12] or [29]. I am planning to work on this in the very near future. There is a very recent preprint [18] where the regularity for long time of the SQG equation is proved for all values of $s \in (0, 1/2)$, thus improving my result from [29]. It would be interesting to see if such result can be achieved with general conservation laws as well.

Let us discuss now the well posedness of the **surface quasi-geostrophic equation.** The SQG models physical phenomena in atmospheric sciences in the case $s = 1/2$. The other values of s still have mathematical interest because the equation serves as a toy model for other problems of fluid mechanics (for example Navier-Stokes). In [13], Constantin, Majda and Tabak present an analogy between the SQG equation and 3D Euler equation.

The SQG equation is considered simpler than Navier-Stokes because the function θ is scalar and the equation has a maximum principle. As a result the solution θ remains bounded in L^∞. However, when $s \geq 1/2$, this bound is not sufficient to obtain further regularity of the solution by considering the nonlinear term as the right hand side of a fractional heat equation.

In the subcritical case ($s > 1/2$) the regularization effect of the fractional Laplacian dominates the equation at small scales, and in [14], Constantin and Wu proved that all solutions become immediately smooth. The case $s = 1/2$ is called critical, because the scaling of the equation coincides with the scaling of the L^∞ bound. The wellposedness of the critical SQG equation was proved recently independently by Kiselev, Nazarov and Volberg in [21] and Caffarelli and Vasseur in [8].

The existence of global in time smooth solutions for the SQG equation when $s < 1/2$ is a major open problem. I proved in [29] that if $s = 1/2 - \varepsilon$ with $\varepsilon \ll 1$, then the solution becomes smooth for large time. The proof uses ideas from [8], but with a more precise analysis of the structure of the nonlinear term of the equation. The point is that, in [8], the relation between w and θ was not used. The vector field w is simply assumed to be some divergence free field in BMO.

Based on ideas that can be traced back to De Giorgi, [8] proves that θ is C^α by proving iteratively a decay of the oscillation of θ in nested dyadic cylinders. The basic idea in [29] is that this iterative procedure can be sharpened by utilizing the improvement of the oscillation of θ in each iteration to update the information on w. The proof depends on the balance between the improvement of oscillation due to the fractional diffusion, and the deterioration of the bounds on w due to scaling. The further away s is from $1/2$, the worse the scaling gets. On the other hand, the longer the parabolic equation evolves, the improvement of oscillation gets better. The appropriate balance is obtained for s sufficiently close to $1/2$ and large times.

There is an additional interesting aspect of the proof in [29]. In each step of the iteration, the oscillation of θ improves in dyadic cylinders. This does not imply that w improves its oscillation in those cylinders, since w is given by an integral transform of θ of order zero and the behavior of θ far away from a cylinder has a significant effect on the value of w inside that cylinder. However, the tails of the integral expression of w are smooth vector fields that define a flow. In each step, a change of variables is performed that follows this flow and cancels out with the nonlocal contribution of θ in the expression for w. This crucial ingredient makes the proof work, and this is important since the proof does not treat the advection term merely as a perturbation of the fractional heat equation.

I proved in [29] that there is a critical time T, that depends on s and converges to zero as $\sigma \to 1/2$, such that the solution θ to the SQG equation becomes smooth after that. The development or not of singularities for the supercritical SQG before this critical time is a difficult open problem. Even though the fact that w is divergence free is used in some estimates in [29], it seems that it should be used in a stronger way in order to solve the problem completely globally in time. How to properly take advantage of the flow of divergence free vector fields for the regularity estimates in advection diffusion equation has been a long standing issue in the mathematics of fluid mechanics. It seems that the fact that $\operatorname{div} u = 0$ is the fundamental difference between the SQG equation and general conservation laws for which shocks do occur. On the other hand, there is no known advantage that can be taken from $\operatorname{div} u = 0$ in the supercritical case. As mentioned above, for the linear equation, the smoothness assumptions on the vector field b in order to obtain a Hölder estimate on θ are the same regardless of whether $\operatorname{div} b = 0$ or not.

For some equations in fluid mechanics, it is possible to prove that weak solutions become smooth for large time simply by combining a global wellposedness result for small data with a decay estimate for weak solutions. This is known for Navier-Stokes since the pioneering work of Leray [22] and it was proved by Cordoba and Cordoba [17] for the critical SQG equation. On the other hand, it is important to point out that this classical idea does not work for either the supercritical SQG equation or supercritical conservation laws, since the type of estimates required for a well posedness result for small data are stronger than the decay provided by the energy estimates.

Relevant publications:

Hölder estimates for advection fractional-diffusion equations

In Preparation.

We analyze conditions for an evolution equation with a drift and fractional diffusion to have a Hölder continuous solution. In case the diffusion is of order one or more, we obtain Hölder estimates for the solution for any bounded drift. In the case when the diffusion is of order less than one, we require the drift to be a Hölder continuous vector field in order to obtain the same type of regularity result.

On the differentiability of the solution to an eq. with drift and fractional diffusion

In preparation.

We analyze conditions for an evolution equation with a drift and fractional diffusion so that its solution has Hölder continuous first derivatives. We obtain that the necessary smoothness assumptions on the drift vector field are only marginally stronger than those needed to obtain only Hölder continuity of the solution.

Holder cont. for int-diff. parabolic eq. with polynomial growth resp. to the gradient.

Discrete and Continuous Dynamical Systems Volume: 28, Number: 3, November 2010. A special issue Dedicated to Louis Nirenberg on the Occasion of his 85th Birthday Part II

We obtain Hölder estimates for an equation of the form

$$u_t + H(x, \nabla u) + (-\triangle)^s u.$$

where $s \in [1/2, 1)$ and H is allowed to have a polynomial growth respect to ∇u of order $2s$. No regularity of H is assumed respect to x.

Eventual regularization of the slightly supercritical fractional Burgers equation.

Joint work with Chi Hin Chan and Magdalena Czubak. Discrete and Continuous Dynamical Systems Volume: 27, Number: 2, June 2010. A special issue Trends and Developments in DE/Dynamics Part I.

We show that even though the weak solutions to the Burgers equation with supercritical fractional diffusion can develop singularities in finite time, the entropy solutions become smooth again after a certain period of time if the exponent of the diffusion is within certain range.

On the differentiability of the sol. to the Hamilton-Jacobi eq. with critical fract. diff.

Advances in Mathematics. To appear.

We study the Hamilto-Jacobi equation with fractional diffusion

$$u_t + H(\nabla u) + (-\triangle)^s u = 0.$$

It was already known that the Caucky problem for this equation has a Lipschitz viscosity solution for any s in the range $s \in (0, 1)$ and a classical solution in the subcritical case $s \in (1/2, 1)$. In the supercritical case $s \in (0, 1/2)$ the solution may become non differentiable regardless of the smoothness of the initial data. In this article we show that the solution is $C^{1,\alpha}$ for the critical case $s = 1/2$. In this case, all terms in the equation are of order one, so no perturbative techniques can be applied.

Eventual regularization in the slightly supercritical quasi-geostrophic equation

Annales de l'Institut Henri Poincare (C) Non Linear Analysis 27 (2010), Issue 2, Pages 693-704.

We study weak solutions of the surface quasi-geostrophic equations for supercritical diffusion. We prove that in some range of diffusion exponents the weak solutions become smooth for large time. The proof uses an iterative improvement of oscillation inspired by the work of Caffarelli and Vasseur. In each step of the iteration the obtained improvement of oscillation is used to improve our understanding of the drift term. We have to partially follow the flow of the drift term in order to obtain an extra cancellation that allows us to study a slightly supercritical regime. In that way, more information about the nonlinearity is used than in the work of Caffarelli and Vasseur for the critical case.

4 Liouville theorems and regularity for advection diffusion equations

In a joint work with G. Seregin, V. Šverák and A. Zlatos, we are studying Liuville theorems and regularity properties for advection-diffusion equations of the form

$$u_t + b \cdot \nabla u - \triangle u = 0.$$

We assume that $\operatorname{div} b = 0$ and we are interested in exploring the minimal assumptions on b for which the Liuville theorem would apply and also there would be a priori estimates for a modulus of continuity.

The key is to understand the role played by the divergence free condition $\operatorname{div} b = 0$. Can we relax the regularity assumptions on b using that $\operatorname{div} b = 0$? It turns out that the assumptions on b can be relaxed only marginally compared to the non divergence free case. In the non divergence free case, one would need $b \in L^p_t L^p_x$ with $2/p + n/q = 1$ with $p < \infty$ to prove regularity of the solution. On the other hand, for the divergence free case, we can prove that the solution u is Hölder continuous if $b \in L^\infty_t BMO^{-1}_x$ through a modification of De Giorgi-Nash-Moser Harnack inequality. This Harnack inequality also provides a Liouville theorem under the same hypothesis on the vector field b (the hypothesis is scale invariant).

In order to show how sharp the hypothesis on b is, we construct a counterexample in two dimensions of the Liouville theorem for the stationary problem where $b = \nabla^\perp h$ and the stream function h grows like $\log|x| \log\log|x|$ for large x. Thus, in some sense, h is very close to BMO.

The local regularity of solutions has some subtleties. For the stationary problem in two dimensions we can obtain a logarithmic modulus of continuity just from the energy estimate (from assuming that $u \in H^1$ locally) and the maximum principle (that for every circle, the essential supremum of u is achieved on the boundary). There is no further assumption on b and the equation is not used except for obtaining those two properties. This may suggest that the divergence free assumption on b has a strong and unexpected effect on the regularity of the solution u. On the other hand, the two dimensional case is quite special. The Sobolev norm H^1 is borderline Hölder continuous to begin with. In fact, in three dimensions, we have an example of a divergence free vector field for which the solution of the stationary problem does not have a modulus of continuity.

Relevant publications:

On divergence free drifts
In collaboration with G. Seregin, V. Šverák and A. Zlatos. In Preparation.

5 Optimal design of composites

It is an important issue in engineering to design composite materials with advantageus effective properties. Consider for example composites made of two or more specific materials with certain thermal conductivities. If the mixing of the two materials is done at the microscopic scale, we may want to know the conductivity properties of the composite at the macroscopic scale. Moreover, we may want to find the best possible way to arrange the two materials microscopically in order to maximize or minimize the macroscopic conductivity. The same problem can be stated for the other properties of the materials, like electric resistivity or linear elasticity.

In mathematics, this kind of problems are studied within the framework of homogenization. During my postdoc at the Courant Institute, Bob Kohn got me interested in the problem of designing a composite that would maximize the sum of two conductivities (say electrical and thermal). There was strong numerical evidence (see [34]) that such design would be achieved by separating the two phases of the composite by a periodic minimal surface.

In [27] I found a connection between optimal design of composites and free boundary problems that characterizes the optimal designs. This was achieved by obtaining a new method to prove upper and lower bounds for the effective conductivity of a composite. In particular it can be used to show that the well known Hashin-Stickman bounds are achieved when one phase corresponds to the contact set in an obstacle problem in a box with neumann boundary conditions. This had previously been observed in [24].

For the case of the sum of two conductivities, in [27] I showed that the optimal microstructures correspond to the solutions of a free boundary problem with a somewhat nonstandard boundary condition. If we assume that the free boundary had zero mean curvature, the boundary conditions simplify and lead to more well known free boundary problems. However, a simple computation shows that these problems cannot have as the free boundary the periodic minimal surfaces that were previously suggested.

The most important idea in [27] is the observation that in the optimal configurations the solutions of the cell problem agree with the directional derivatives of a potential function. This idea simplifies the proof of upper bounds greatly, and it is used to reprove Bergman's cross property bounds. In [23], Liping Liu used that idea to provide a new proof for the full Hashin-Shtrickman bounds for multiphase composite and study their attainability conditions.

The classical problem of finding bounds for a single conductivity is completely solved in the case of composites of two components. The Hashin-Shtrickman bounds can always be achieved by some two phase composite for any value of the parameters. When the number of phases is larger than two the situation is different. In some range of parameters, the well known Hashin-Shtrickman bounds are achievable, but not in all. What is most dramatic is that as the volume ratio of the most conductive phase goes to zero, the limit of the upper bound still depends on the largest conductivity, even though that material disappeared from the composite. In two dimensions, there are other bounds obtained by Nesi and Astala ([26] and [1]). In [32], I am working on new bounds for multiphase materials in any dimension that refine the Hashin-Shtrickman bounds and actually converge to the optimal two-phase bounds when the volume ratio of all phases but any two go to zero. The idea is to apply estimates from harmonic analysis to the potential function found using the "gradient of potential" idea from [27].

Relevant publications:

Upper bounds for multiphase composites in any dimension

In Preparation

We prove a rigorous upper bound for the effective conductivity of a composite made of several isotropic components. This upper bound coincides with the Hashin Shtrikman bound when the volume ratio of all phases but any two vanish.

A characterization of optimal two-phase multifunctional composite designs.

Proc. of the Royal Soc. of London A 463, Number 2086 (2007).

We study the problem of optimal design of a two-phase composite material to maximize the sum of thermal and electrical conductivity. We obtain a characterization of an optimal configuration in terms of a free boundary problem. We use our characterization to argue that the optimal interface is not, as has been suggested, a periodic minimal surface. In the process of obtaining this result, we provide a new proof of the relevant case of Bergmans cross property bound. The proof introduces the key idea in the optimal configurations the solutions of the cell problem agree with the directional derivatives of a potential function. This same idea can be used to obtain a very elementary proof of the Hashin-Shtrickman bounds.

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