162 Homework 4, Due Tuesday February 3

1. Let $A \subset \mathbf{R}$ and $f : A \to \mathbf{R}$ a function. Show that the following statements are equivalent.

   (a) For all $x_0 \in A$ and for all $x_n \in A$ with $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

   (b) f is continuous.

2. Suppose $f : A \to \mathbf{R}$ is a function and a is a limit point of A. We have an alternate definition of limits which says $\lim_{x\to a} f(x) = L$ if for all $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |x - a| < \delta$ and $x \in A$ then $|f(x) - L| < \epsilon$.

   Prove this definition is equivalent to the definition in the notes.

3. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous such that for all $x, y \in \mathbf{R}$ we have $f(x+y) = f(x)+f(y)$. Show

   (a) $f(n) = nf(1)$ for all $n \in \mathbf{N}$.

   (b) $f(n) = nf(1)$ for all $n \in \mathbf{Z}$

   (c) $f(x) = xf(1)$ for all $x \in \mathbf{Q}$

   (d) $f(x) = xf(1)$ for all $x \in \mathbf{R}$.

   For the last part you can use that $\overline{Q} = \mathbf{R}$.

4. For each of the following three functions defined on $\{x \in \mathbf{R} : x > 0\}$ can the function be extended to the point $x = 0$ in such a way that the new function is continuous at $x = 0$?

   (a) $f(x) = \frac{1}{x}$

   (b) $f(x) = \frac{\sin(x)}{x}$.

   (c) $f(x) = \sin(\frac{1}{x})$.