MATH 16200 SECTION 30, HOMEWORK 2

DUE DATE TUESDAY, JAN 20

(1) Let F be a field.
  (a) Prove Lemma A, part (2), from the Analysis notes
  (b) Prove Lemma A, part (4), from the Analysis notes
  (c) Show $0 \cdot x = 0$.
  (d) Show $(-1) \cdot x = -x$. (By definition, -1 is the additive inverse of the unit element 1.)

Recall:
Axiom 1: C is simply ordered.
Axiom 2: C has no first or last point.
Axiom 3: The only point sets that are both open and closed are C and $\emptyset$.
Betweenness Proposition: Between any two points, there is another point.

(2) Let C be a set satisfying axioms 1 and 2 and the betweenness proposition, and such that every subset of C which has an upper bound has a least upper bound. Show that C satisfies axiom 3.

(3) Suppose that C' is a set satisfying axioms 1, 2, and 3 (for example, it could be the real numbers). Let a and b be two points of C' and define

$$C = \left(\overline{ab} \times \overline{ab}\right) \setminus \{(a,a),(b,b)\}$$

(Recall that for sets X, Y the notation $X \times Y$ means the set of all pairs (x, y) such that $x \in X, y \in Y$.) If (x, y) and (z, w) are two elements of C, we define $(x, y) < (z, w)$ to hold if and only if **either** of the following two conditions holds:
  (i) $x < z$ in C', **or**
  (ii) $x = z$ and $y < w$ in C'.
(This is called the **lexicographic order** on C.) Show that C satisfies axioms 1, 2, and 3. (Hint: Use Problem 2.)

When C' is taken to be the real numbers $\mathbb{R}$, the C constructed in problem 3 is not isomorphic to $\mathbb{R}$. (That is, there is no one-to-one correspondence between C and $\mathbb{R}$ which preserves order.) *Proving* this, however, is unfortunately beyond the scope of this problem set.