MATH 16200 SECTION 30, HOMEWORK 7

DUE DATE TUESDAY, FEB 24

(1) (a) Show that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous bijective function, then its inverse f^{-1} is also continuous.

(b) By definition the **integer part** of a real number x, denoted by $[x]$, is the unique integer n such that $n \le x < n + 1$. The **fractional part** of x is by definition

$$\{x\} := x - [x] \in [0, 1).$$

Note that $\{x\}$ is a *number*, and should not be confused with the set containing only the point x. Now for a fixed nonzero irrational number α, define a function $f : \mathbb{Q} \setminus \{0\} \to (0, \alpha)$ by $f(x) = \alpha\{\frac{x}{\alpha}\}$.

(i) Show that f is one to one.

(ii) Show that f is continuous. (Hint: It may help to show that if x is sufficiently close to a, for x and a in the domain, then $[\frac{x}{\alpha}] = [\frac{a}{\alpha}]$)

(iii) If we consider f as a function from $\mathbb{Q} \setminus \{0\}$ to $f(\mathbb{Q} \setminus \{0\})$ then f is bijective. Show, by proving the following steps, that the inverse function of f is nowhere continuous on $f(\mathbb{Q} \setminus \{0\})$:

(A) Set $J := f(\mathbb{Q} \setminus \{0\})$ and take $y \in J$. Show that for all $n \in \mathbb{N} \setminus \{0\}$ there exists a rational number x_n in the region $(y + \alpha n - \frac{1}{n}, y + \alpha n + \frac{1}{n})$. Find the limit of the sequence x_n, or show that the limit does not exist.

(B) Show that there is $N \in \mathbb{N} \setminus \{0\}$ such that for all $n \ge N$ one has $y_n := f(x_n) \in (y - \frac{1}{n}, y + \frac{1}{n})$. Find the limit of the sequence y_n, or show that the limit does not exist.

(C) Conclude that f^{-1} is not continuous at any point in its domain. This shows that part (a) does not work for all continuous bijective functions $f : A \to B$.