DICHOTOMY FOR THE HAUSDORFF DIMENSION OF THE SET OF NONERGODIC DIRECTIONS

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Abstract. Given an irrational $0 < \lambda < 1$, we consider billiards in a $\frac{1}{2} \times 1$ rectangle P_λ with a barrier of length $\alpha = \frac{1-\lambda}{2}$ midway along a vertical side. Let $\mathrm{NE}(P_\lambda)$ be the set of θ such that the flow on P_λ in direction θ is not ergodic. Letting $\{q_k\}$ be the sequence of denominators of the continued fraction expansion of λ we have the following. If $\sum_k \frac{\log\log q_{k+1}}{q_k} = \infty$, then $\mathrm{HDim}\,\mathrm{NE}(P_\lambda) = 0$. If $\sum_k \frac{\log\log q_{k+1}}{q_k} < \infty$, then $\mathrm{HDim}\,\mathrm{NE}(P_\lambda) = \frac{1}{2}$. These extend earlier results of Boshernitzan and Cheung.

1. Introduction

In 1969, ([Ve]) Veech found examples of skew products over a rotation of the circle that are minimal and not uniquely ergodic. These were turned into interval exchange transformations in [KN]. Masur and Smillie gave a geometric interpretation of these examples (see for instance [MT]). Let P_λ denote the billiard in a $\frac{1}{2} \times 1$ rectangle with a horizontal barrier of length α based at the midpoint of a vertical side. There is a standard unfolding procedure which turns billiards in this polygon into flows on a translation surface. In this case the associated translation surface, denoted by (X, ω), is a double cover of a standard flat torus of area one branched over two points z_0 and z_1 a horizontal distance $\lambda = 1 - 2\alpha$ apart on the flat torus. See Figure 1.

The linear flows on this translation surface preserve Lebesgue measure. What Veech showed in these examples is that given θ with unbounded partial quotients in its continued fraction expansion, there is a λ such that the flow on P_λ in direction with slope θ is minimal but not uniquely ergodic.

Let $\mathrm{NE}(P_\lambda)$ denote the set of nonergodic directions, i.e. those directions for which Lebesgue measure is not ergodic. It was shown in [MT] that $\mathrm{NE}(P_\lambda)$ is uncountable if λ is irrational. When λ is rational, a result of Veech ([Ve2]) implies that minimal directions are uniquely

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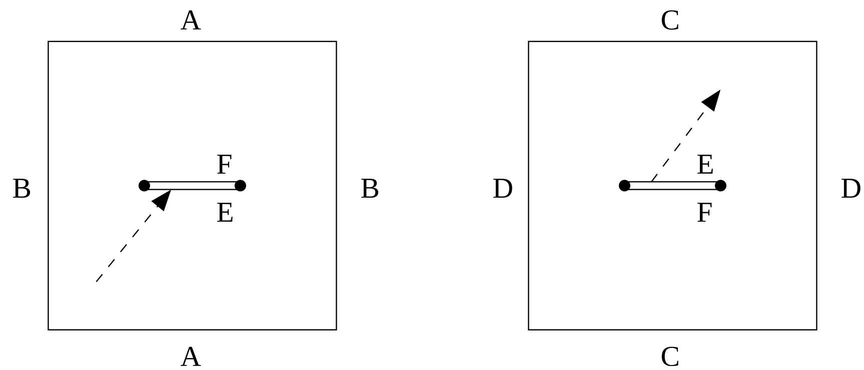

FIGURE 1. The branched double cover (X, ω).

ergodic; thus $\mathrm{NE}(P_\lambda)$ is countable. By a general result of Masur (see [Ma]), the Hausdorff dimension of $\mathrm{NE}(P_\lambda)$ satisfies $\mathrm{HDim}\,\mathrm{NE}(P_\lambda) \leq \frac{1}{2}$. In [Ch1] Cheung proved that this estimate is sharp. He showed that if λ is *Diophantine*, then $\mathrm{HDim}\,\mathrm{NE}(P_\lambda) = \frac{1}{2}$. Recall that λ is Diophantine, if there is an $s > 0$ such that

$$\left|\lambda - \frac{p}{q}\right| < \frac{1}{q^s}$$

does not have any solutions $\frac{p}{q} \in \mathbb{Q}$. This raises the question of the situation when λ is not Diophantine; namely when λ is a Liouville number. Boshernitzan showed that for a residual set of λ, $\mathrm{HDim}\,\mathrm{NE}(P_\lambda) = 0$ (see the Appendix in [Ch1]).

In this paper, we prove the following dichotomy:

Theorem 1.1. *Let (q_k) denote the sequence of denominators in the continued fraction expansion for λ. Then* $\mathrm{HDim}\,\mathrm{NE}(P_\lambda) = 0$ *or* $\frac{1}{2}$, *with the latter case occurring if and only if λ is irrational and*

$$\sum_k \frac{\log\log q_{k+1}}{q_k} < \infty. \tag{1}$$[1]

The proof of Hausdorff dimension 0 will proceed by showing that the set of nonergodic directions can be covered by a union of intervals whose sizes can be estimated. The Hausdorff dimension $\frac{1}{2}$ result is more difficult and proceeds by a construction of a Cantor set of nonergodic directions arising as a limit of directions of slits on the torus. Aspects of this construction were already given in [Ch1] in the case when λ is Diophantine. In the current situation we have to combine

[1]This condition on the denominators of the continued fractions already appeared in complex dynamics in the work of Pérez-Marco ([PM]). Milnor [Mi] provides a very readable account of the long history leading up to this work.

that construction with a new one to deal with the "Liouville" part of λ.

Associated to any translation surface (or more generally a quadratic differential) is a Teichmüller geodesic. For each t the Riemann surface X_t along the geodesic is found by expanding along horizontal lines by a factor of e^t and contracting along vertical lines by e^t. It is known (see [Ma]) that if the vertical foliation of the quadratic differential is nonergodic, then the associated Teichmüller geodesic is *divergent*, i.e. it eventually leaves every compact subset of the stratum.[2] As a by-product of our analysis we obtain:

Theorem 1.2. *Let* $\mathrm{DIV}(P_\lambda)$ *denote the set of divergent directions in* P_λ*, i.e. directions for which the associated Teichmüller geodesic leaves every compact subset of the stratum. Then* $\mathrm{HDim}\,\mathrm{DIV}(P_\lambda) = 0$ *or* $\frac{1}{2}$*, with the latter case occurring if and only if* λ *is irrational.*

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2. Criterion for non-ergodicity

In this section we formulate an explicit, geometric criterion for a direction in (X, ω) to be nonergodic. Sufficiency (Theorem 2.5) follows from a general condition of Masur-Smillie in [MS] while necessity (Theorem 2.6) was obtained in [CE].

Recall z_0, z_1 are the endpoints on the flat torus. We use the same name for the points lying over them on (X, ω).

Definition 2.1. A *slit* is any saddle connection w on (X, ω) joining z_0 and z_1. (Recall that a saddle connection is a straight line in the flat structure of (X, ω) that starts and ends in $\{z_0, z_1\}$ without meeting either point in its interior.) We say w is *separating* if w and its image under the involution of the surface separates X into a pair of tori T_w^1, T_w^2 interchanged by the involution.

A slit w corresponds to a line segment on the flat torus joining z_0 to z_1 and so determines a vector

$$w = (\lambda + m, n).$$

Since λ is irrational, any vector of the above form with $n \neq 0$ is the holonomy of some slit on (X, ω). Such a slit is separating if and only if m and n are even. (See [Ch1] for a proof.)

[2]In [Ma], a stronger assertion was proved, namely the *projection* of the Teichmüller geodesic to the moduli space of Riemann surfaces is also divergent. Theorem 1.2 remains valid with either interpretation of the set $\mathrm{DIV}(P_\lambda)$.

Notation 2.2. For any non horizontal $u, v \in \mathbb{R}^2$, let $\angle uv$ denote the absolute difference between the inverses of their slopes. We have the following analog of the cross-product formula

$$|u \times v| = |u|\ |v| \angle uv$$

where $|\cdot|$ denotes the absolute value of the y-coordinate and $|u \times v|$ is the area of the parallelogram spanned by u and v.

Suppose w, w' are a pair of separating slits. Then we may measure the change in area of the partitions by

$$\chi(w, w') := \text{area}(T^1_w \Delta T^1_{w'}).$$

Lemma 2.3. *Assume w, w' are separating slits satisfying $w' = w + bv$ for some primitive $v \in \mathbb{Z}^2$, i.e.* $\gcd(v) = 1$*, such that $|w \times v| < 1$. Then*

$$\chi(w, w') = |w' \times v| = |w \times v|.$$

Proof. The condition $|w \times v| < 1$ ensures that each waist curve of the cylinder in direction v (in the flat torus) crosses the slit w at most once. The area exchange is a union of $|b|$ parallelograms whose edges have holonomy equal to either $\frac{1}{b}w$ or $\frac{1}{b}w'$. It follows that

$$\chi(w, w') = \frac{1}{|b|}|w \times w'| = |w \times v| = |w' \times v|.$$

□

Definition 2.4. We shall say w, w' are *related by a Dehn twist* about v if the hypothesis of Lemma 2.3 holds. We shall also refer to a primitive vector in $\mathbb{Z}^2$ as a *loop*.

Theorem 2.5. *Suppose $\{w_j\}$ is a sequence of separating slits such that $|w_j|$ increases to infinity, every consecutive pair of slits is related by a Dehn twist, and*

$$\sum_j \chi(w_j, w_{j+1}) < \infty. \tag{2}$$

Then the sequence of slit directions has a limit θ and $\theta \in \text{NE}(P_\lambda)$.

Proof. Let v_j be the loop such that $w_{j+1} = w_j + b_j v_j$ for some (even) $b_j > 0$. Then $\sum |w_j \times v_j| < \infty$, by Lemma 2.3. Since $|w_{j+1}| > |w_j|$, we have $|v_j| \geq 1$ so that

$$\angle w_j w_{j+1} \leq \frac{|w_j \times v_j|}{|w_j||v_j|} + \frac{|v_j \times w_{j+1}|}{|v_j||w_{j+1}|} \leq \frac{2|w_j \times v_j|}{|w_j|}$$

from which the existence of the limit θ follows. It remains to verify that $\lim h_j = 0$ where h_j is the component of w_j orthogonal to θ. (See [MS] or [MT].) This follows easily from

$$\angle w_j\theta \leq \sum_{i\geq j} \angle w_i w_{i+1} \leq \sum_{i\geq j} \frac{2|w_i \times v_i|}{|w_j|}.$$

□

Theorem 2.5 will be used in §4 to obtain the lower bound on Hausdorff dimension of $\mathrm{NE}(P_\lambda)$.

The converse to Thereom 2.5 also holds. That is, to each nonergodic direction θ one can associate a sequence of slits (w_j) whose directions converge to θ and such that all the hypotheses of Theorem 2.5 hold. The construction of the sequence (w_j) may be described as follows.

Let U be the set of vectors of the form $(\lambda + 2m, 2n)$ or of the form (m, n) where $m, n \in \mathbb{Z}$. Assume θ is not horizontal and let it be represented by a vector, denoted by the same letter, whose y-coordinate is one. Let $u_0 \in U$ be a vector with $|u_0| = 1$ and such that $|u_0 \times \theta|$ is smallest possible. Inductively suppose we are given u_j. Let $q > 0$ be the smallest possible integer such that for some $u \in U$ with $|u| = q$ we have $|u \times \theta| < |u_j \times \theta|$. Then let u_{j+1} be any $u \in U$ such that $|u| = q$ and $|u \times \theta|$ is smallest possible. As long as θ is not in the direction of a vector in U, this procedure gives rise to an infinite sequence of vectors in U such that $|u_j|$ is strictly increasing while $|u_j \times \theta|$ is strictly decreasing.

We have the following characterisation of the set of non uniquely ergodic directions in P_λ that implies, in particular, that every ergodic direction is necessarily uniquely ergodic.

Theorem 2.6. *([CE]) Assume θ is not in the direction of a vector in U and let (u_i) be the sequence associated to θ as above. Then θ is a non uniquely ergodic direction if and only if (u_i) is eventually of the form $\dots, w_j, v_j, w_{j+1}, \dots$, alternating between slits and loops, with w_j and w_{j+1} related by a Dehn twist about v_j for all large enough j, and such that (2) holds for the subsequence (w_j) formed by the collection of slits.*

Theorem 2.6 will be used in the next section to obtain the upper bound on Hausdorff dimension of $\mathrm{NE}(P_\lambda)$. The following lemma will also be needed to control the angle that the vectors u_j make with the direction θ.

Lemma 2.7. *For all $j \geq 0$ we have*

$$\frac{|u_j \times u_{j+1}|}{2|u_j||u_{j+1}|} \leq \angle u_j\theta \leq \frac{1}{|u_j||u_{j+1}|}. \tag{3}$$

Proof. Let P_0 be the convex hull of the set $\{\pm u_j, \pm u_{j+1}\}$. Let P be the parallelogram defined as

$$P = \{u \in \mathbb{R}^2 : |u| \leq |u_{j+1}|, \quad |u \times \theta| \leq |u_j \times \theta|\}.$$

Then P contains the vertices of P_0, since $|u_{j+1}| > |u_j|$ and $|u_j \times \theta| > |u_{j+1} \times \theta|$, and since P is convex, it contains all of P_0. The base of P is $2|u_j \times \theta|$ while its height is $2|u_{j+1}|$. By construction, the interior of P contains no nonzero vectors of U so that Minkowski's theorem implies the area of P is at most 4. Since $|u_j \times \theta| = |u_j|\angle u_j\theta$, this proves the right hand inequality in (3). Since

$$\angle u_{j+1}\theta = \frac{|u_{j+1} \times \theta|}{|u_{j+1}|} < \frac{|u_j \times \theta|}{|u_j|} = \angle u_j\theta$$

we have $\angle u_j u_{j+1} < 2\angle u_j\theta$, giving the left hand inequality in (3). □

3. Hausdorff dimension 0

In this section we show $\operatorname{HDim} \operatorname{NE}(P_\lambda) = 0$ under the assumption

$$\sum_k \frac{\log\log q_{k+1}}{q_k} = \infty \tag{4}$$

on the sequence (q_k) of denominators in the continued fraction expansion of λ.

3.1. **Liouville convergents.** We recall two classical results from the theory of continued fractions. The kth convergent of a real number α is a reduced fraction $\frac{m_k}{n_k}$ satisfying

$$\frac{1}{n_k(n_k + n_{k+1})} < \left|\alpha - \frac{m_k}{n_k}\right| < \frac{1}{n_k n_{k+1}}. \tag{5}$$

A partial converse is if a reduced fraction satisfies

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{2q^2} \tag{6}$$

then it is a convergent of α.

Lemma 3.1. *Let $w = (\lambda + m, n)$ be a slit such that*

$$|w| < \frac{q_{k+1}}{2q_k}. \tag{7}$$

for some convergent $\frac{p_k}{q_k}$ of λ. Let $\frac{p}{q}$ denote the fraction $\frac{p_k+mq_k}{nq_k}$ in lowest terms. Then $\frac{p}{q}$ is a convergent of $\frac{\lambda+m}{n}$ satisfying $q_k \le q \le |w|q_k$. Moreover, the height[3] *q' of the next convergent is greater than $\frac{q_{k+1}}{2}$.*

Proof. Using the right hand side of (5) and (7) we get

$$\left|\frac{\lambda+m}{n} - \frac{p_k+mq_k}{nq_k}\right| < \frac{1}{|w|q_kq_{k+1}} < \frac{1}{2n^2q_k^2} \tag{8}$$

which implies that $\frac{p}{q}$ is a convergent of $\frac{\lambda+m}{n}$. Clearly, $q \le |n|q_k = |w|q_k$ and since $\gcd(p_k, q_k) = 1$, n is divisible by $\gcd(p_k + mq_k, nq_k)$ so that $q \ge q_k$. Let q' be the height of the next convergent of $\frac{\lambda+m}{n}$. From the first inequalities in (5) and in (8) we get

$$\frac{1}{2qq'} < \left|\frac{\lambda+m}{n} - \frac{p}{q}\right| < \frac{1}{|w|q_kq_{k+1}}$$

so that

$$q' > \frac{|w|q_kq_{k+1}}{2q} \ge \frac{q_{k+1}}{2}.$$

□

Definition 3.2. When the conclusion of Lemma 3.1 holds, we refer to $\frac{p}{q}$ (or the vector $v = (p, q)$) as the *Liouville convergent* of w indexed by k. (We shall often blur the distinction between the rational $\frac{p}{q}$ and the vector v.)

Lemma 3.3. *Let w, w' be related by a Dehn twist about a loop v satisfying $|w \times v| < \frac{1}{2}$ and $v = (p, q)$ with $q > 0$. Suppose $|w| < |w'| < \frac{q_{k+1}}{2q_k}$ and let u be the Liouville convergent of w indexed by k. Then either (i) $v \ne u$ and*

$$\chi(w, w') > \frac{1}{2q_k}, \tag{9}$$

or (ii) $v = u$ and for any $v' \in \mathbb{Z}^2 \setminus \mathbb{Z}v$ satisfying $|w' \times v'| < \frac{1}{2}$ we have $|v'| > \frac{q_{k+1}}{2}$.

Proof. We have $w' = w + bv$ for some nonzero, even integer b, so that

$$|v| = \frac{|w'-w|}{|b|} \le \frac{|w'|+|w|}{2} < |w'|. \tag{10}$$

Let α' be the inverse slope of w'. Then

$$\left|\alpha' - \frac{p}{q}\right| = \frac{|w' \times v|}{|w'||v|} < \frac{|w \times v|}{|v|^2} < \frac{1}{2q^2}$$

[3]The height of a rational is the smallest positive integer that multiplies it into the integers.

so that $\frac{p}{q}$ is a convergent of α', by (6). Let q' be the height of the next convergent of α'. Then (5) implies

$$\left|\alpha' - \frac{p}{q}\right| > \frac{1}{q(q'+q)} > \frac{1}{2qq'}$$

so that

$$q' > \frac{|w'||v|}{2q|w' \times v|} = \frac{|w'|}{2|w \times v|}.$$

The Liouville convergent $u = (m, n)$ cannot have its height $n < q$ because Lemma 3.1 implies the height n' of the next convergent of α' is greater than $\frac{q_{k+1}}{2} > |w'| > |v| = q$, contradicting the fact that q is the height of a convergent of α', namely $\frac{p}{q}$. Thus, $|u| \geq |v|$.

In case (i), $|u| > |v|$ so that $|u| \geq q' > \frac{|w'|}{2|w \times v|}$. Since $|u| \leq |w'|q_k$, the inequality (9) follows. In case (ii), we have $q' = n' > \frac{q_{k+1}}{2}$, as noted earlier. Given $v' \in \mathbb{Z}^2 \setminus \mathbb{Z}v$, we have

$$\begin{aligned} 1 \leq |u \times v'| &= |u||v'| \left(\angle w'u + \frac{|w' \times v'|}{|w'||v'|} \right) \\ &\leq \frac{|v'|}{2q'} + \frac{|v|}{2|w'|} < \frac{|v'|}{2q'} + \frac{1}{2} \end{aligned}$$

from which it follows that $|v'| > q' > \frac{q_{k+1}}{2}$. □

Let E_r be the set of directions θ for which the associated sequence satisfies $|u_{j+1}| > |u_j|^r$ for infinitely many j.

Lemma 3.4.

$$\operatorname{HDim} E_r \leq \frac{2}{1+r}.$$

Proof. Again let U the set of vectors of the form $(\lambda + m, n)$ or (m, n) where $(m, n) \in \mathbb{Z}^2$ and $n \neq 0$. For any $\theta \in E_r$ and any index j such that $|u_{j+1}| > |u_j|^r$ holds in the associated sequence, Lemma 2.7 implies

$$\angle u_j \theta \leq \frac{1}{|u_j||u_{j+1}|} < \frac{1}{|u_j|^{1+r}}.$$

For each $u \in U$, let $I(u)$ be the interval of length $\frac{2}{|u|^{1+r}}$ centered about the direction of u.

Let $E'_r = E_r \cap [a, a+1]$ for some arbitrary but fixed $a \in \mathbb{R}$. Let $\mathcal{U}(R)$ be the covering of E'_r by intervals $I(u), u \in U$ where $|u| > R$ and $I(u) \cap [a, a+1] \neq \emptyset$. There is a constant $c > 0$ such that for all

$k \in \mathbb{N}$ the number of intervals $I(u)$ with $|u|$ between $2^k R$ and $2^{k+1}R$ is at most $c2^{2k}R^2$. For these intervals we have

$$|I(u)| = \frac{2}{|u|^{1+r}} \leq \frac{2}{2^{k(1+r)}R^{1+r}}.$$

Given $\delta > \frac{2}{1+r}$ we have

$$\sum_{I \in \mathcal{U}(R)} |I|^\delta \leq \sum_{k \geq 0} \frac{c2^{2k+\delta}R^2}{2^{k(1+r)\delta}R^{(1+r)\delta}} = \frac{c2^\delta}{(1-2^{-\nu})R^\nu}$$

where $\nu = (1+r)\delta - 2 > 0$. Given $\varepsilon > 0$ the right hand side above can be made less than ε by choosing R large enough. Therefore, HDim $E_r \leq \delta$ and since $\delta > \frac{2}{1+r}$ was arbitrary, the lemma now follows. □

The Hausdorff dimension 0 result now follows from

Lemma 3.5. *Assume (4) holds. Then any $\theta \in \mathrm{NE}(P_\lambda)$ that is not the direction of a vector in U is contained in $\cap_{r>1} E_r$.*

Proof. Let $n_k > 1$ be the real number defined by $q_{k+1} = q_k^{n_k}$. Note that (4) implies (n_k) is unbounded. Let θ be a direction in $\mathrm{NE}(P_\lambda)$ that is not the direction of a vector in U and suppose $r > 1$ is such that $\theta \notin E_r$. By Theorem 2.6 the sequence associated to θ eventually alternates between slits and loops $\cdots < |w_j| < |v_j| < |w_{j+1}| < \dots$ which, by Lemma 2.3, satisfy $\sum_j |w_j \times v_j| < \infty$. Ignoring a finite number of initial terms, we may further assume that for all j, we have $|w_j \times v_j| < \frac{1}{2}$ and also $|w_{j+1}| \leq |v_j|^r \leq |w_j|^{2r}$, since $\theta \notin E_r$.

Given an index k with n_k large and $|w_0| < q_k$ we consider the number of indices j satisfying

$$q_k \leq |w_j| < |w_{j+1}| < q_k^{n_k - 2}. \tag{11}$$

Since the length of the jth slit with length greater than q_k is at most $q_k^{(2r)^j}$ there are at least $\frac{1}{2}\log_{2r} n_k$ indices in the range (11), assuming n_k is large enough. Let $\ell \subset \mathbb{N}$ be collection of indices k for which n_k is large enough in the foregoing sense and such that

$$q_k^{n_k-2} \leq \min\Big(\frac{q_{k+1}}{2q_k}, \big(\frac{q_{k+1}}{2}\big)^{1/r}\Big).$$

Observe that if $|w_{j+1}| < \frac{q_{k+1}}{2q_k}$ and v_j is the Liouville convergent of w_{j+1} indexed by k then Lemma 3.3(ii) implies $\frac{q_{k+1}}{2} < |v_{j+1}| \leq |w_{j+1}|^r$, which is not possible for j in the given range. Thus, for any $k \in \ell$ and any j in the range (11), v_j is not the Liouville convergent of w_{j+1} indexed by k, so that by Lemma 3.3(i) the area exchanged between the partitions determined by w_j and w_{j+1} is at least $\frac{1}{2q_k}$. By (4) we now have

$$\sum_j \chi(w_j, w_{j+1}) \geq \sum_{k \in \ell} \frac{\log_{2r} n_k}{4q_k} = \sum_{k \in \ell} \frac{\log\log q_{k+1} - \log\log q_k}{4q_k \log 2r} = \infty$$

but this contradicts (2). □

4. Hausdorff dimension $\frac{1}{2}$

In this section we prove $\mathrm{HDim}\,\mathrm{NE}(P_\lambda) = \frac{1}{2}$ for any irrational, Liouville number λ satisfying (1).

We shall construct a set

$$F = F(\lambda, r, M', w_0)$$

of directions depending on a choice of real parameters λ, r, M' and an initial slit w_0, and show that for any $\varepsilon > 0$ the parameters can be chosen so that

$$\mathrm{HDim}\, F > \frac{1}{2} - \varepsilon.$$

The condition (1) will imply that

$$F \subset \mathrm{NE}(P_\lambda).$$

To construct nonergodic directions, we use Theorem 2.5. The general idea is as follows. Starting with an initial slit w_0 we will construct a tree of slits. At level j we will have a collection of slits $\{w_j\}$ of approximately the same length. For each w_j at level j we wish to construct a collection of slits of level $j+1$ each having small cross product with w_j. Depending on the relationship of the length of w_j to the continued fraction expansion of λ, the construction will be one of two types explained below. Either the slits of level $j+1$ will be constructed by a "Liouville construction" or from a "Diophantine construction". The directions of each slit of level j will lie in some interval and the intervals at level j will separated by gaps. The intervals of level $j+1$ will be nested in the intervals of level j. Each nonergodic direction corresponds to a nested intersection of these intervals.

4.1. **Local Hausdorff dimensions.**

To guarantee lower bounds for Hausdorff dimension we will use an estimate of Falconer [Fa] which we explain next. Let

$$F = \cap_{j \geq 0} F_j$$

where each F_j is a finite disjoint union of closed intervals and $F_{j+1} \subset F_j$ for all j. Suppose there are sequences $m_j \geq 2$ and $\varepsilon_j \searrow 0$ such that each interval of F_j contains at least m_j intervals of F_{j+1} and the smallest gap between any two intervals of F_{j+1} is at least ε_j. (Note that $m_j \geq 2$

implies there will always be at least one gap.) Then Falconer's lower bound estimate is

$$\operatorname{HDim} F \geq \liminf_{j} \frac{\log(m_0 \cdots m_j)}{-\log m_{j+1}\varepsilon_{j+1}}.$$

If $\lim_{j\to\infty} m_j\varepsilon_j = 0$, as is necessarily the case if the length of the longest interval in F_j tends to zero as $j \to \infty$, then

$$\operatorname{HDim} F \geq \liminf_{j} d_j$$

where

$$d_j := \frac{\log m_j}{-\log \frac{m_{j+1}\varepsilon_{j+1}}{m_j\varepsilon_j}}. \tag{12}$$

Our goal is that for each $\varepsilon > 0$, we make a construction of a Cantor set of nonergodic directions so that each d_j will satisfy

$$d_j > \frac{1}{2} - \varepsilon.$$

4.2. **Cantor set construction.** Given $r > 1$ and a sequence of positive $\delta_j \to 0$ (which will measure the area interchange defined by consecutive slits), we shall construct a Cantor set F depending on parameters m_j and ε_j that are expressible in terms of r and δ_j. It is based on the assumption, verified later, that we can construct a tree of slits. We start with an initial slit w_0, the unique slit of level 0. Inductively, given a slit w_j of level j we consider slits of the form $w_j + 2v_j$ where $v_j \in \mathbb{Z}^2$ is a primitive vector, i.e. $\gcd(v_j) = 1$, and satisfies

$$|w_j \times v_j| < \delta_j, \quad |w_j|^r \leq |v_j| \leq 2|w_j|^r.$$

We refer to $w_{j+1} = w_j + 2v_j$ of the above form as a *child* of w_j. It satisfies

$$|w_j|^r \leq |w_{j+1}| \leq 5|w_j|^r. \tag{13}$$

The main difficulty in the construction is avoiding slits that have no children at all. To ensure that we can avoid such slits, we shall only use children with "nice Diophantine properties" when we assemble the slits for the next level. However, we shall ensure that at each stage, the number of children used will be at least

$$\rho_j|w_j|^{r-1}\delta_j \tag{14}$$

where ρ_j is to be determined later.

For w a slit, let $I(w)$ denote the interval of length

$$\operatorname{diam} I(w) = \frac{4}{|w|^{r+1}}$$

centered about the direction of w. The following lemma allows us to find estimates for the sizes of intervals and the gaps between them.

Lemma 4.1. *Assume* $|w_0|^{r(r-1)} \geq 64$ *and* $\sup \delta_j < \frac{1}{16}$. *Let* w_{j+1} *be a child of a slit* w_j *of level* j. *Then*

- $I(w_{j+1}) \subset I(w_j)$, *and*
- *if* w'_{j+1} *is another child of* w_j, *then*

$$\operatorname{dist}(I(w_{j+1}), I(w'_{j+1})) \geq \frac{1}{16|w_j|^{2r}}.$$

Proof. Since the distance between the directions of w_j and w_{j+1} is

$$\angle w_j w_{j+1} = \frac{|w_j \times w_{j+1}|}{|w_j||w_{j+1}|} \leq \frac{|w_j \times v_j|}{|w_j||v_j|} < \frac{1}{|w_j|^{r+1}}$$

the first conclusion follows from

$$\frac{1}{|w_j|^{r+1}} + \frac{2}{|w_j|^{r(r+1)}} \leq \frac{2}{|w_j|^{r+1}}$$

which holds easily by the assumption on $|w_0|$.

The distance between the directions of w_{j+1} and v_j is

$$\angle w_{j+1} v_j = \frac{|w_{j+1} \times v_j|}{|w_{j+1}||v_j|} \leq \frac{|w_j \times v_j|}{|w_j|^{2r}} < \frac{\delta_j}{|w_j|^{2r}}.$$

If $w'_{j+1} = w_j + 2v'_j$ is another child of w_j then

$$\angle v_j v'_j = \frac{|v_j \times v'_j|}{|v_j||v'_j|} \geq \frac{1}{4|w_j|^{2r}}$$

so that by the triangle inequality,

$$\angle w_{j+1} w'_{j+1} \geq \frac{1}{4|w_j|^{2r}} - \frac{\delta_j + \delta_{j+1}}{|w_j|^{2r}} \geq \frac{1}{8|w_j|^{2r}}$$

since $\sup \delta_j < \frac{1}{16}$. Therefore,

$$\operatorname{dist}(I(w_{j+1}), I(w'_{j+1})) \geq \frac{1}{8|w_j|^{2r}} - \frac{4}{|w_j|^{r(r+1)}} \geq \frac{1}{16|w_j|^{2r}}$$

since $|w_0|^{r(r-1)} \geq 64$. □

Let

$$F_j = \cup_w I(w)$$

where the union is taken over all slits of level j. From (13) we have

$$|w_0|^{r^j} \leq |w_j| \leq 5^{\frac{r^j-1}{r-1}} |w_0|^{r^j}, \tag{15}$$

so that the number of children given by (14) is at least

$$m_j := \rho_j \delta_j |w_0|^{r^j(r-1)}$$

while the smallest gap between the associated intervals is at least

$$\varepsilon_j := \frac{1}{16 \cdot 5^{2r\frac{r^j-1}{r-1}} |w_0|^{2r^{j+1}}},$$

by Lemma 4.1.

Now we express the terms in the definition of d_j, given by (12), in terms of r, δ_j and ρ_j. We have

$$m_j \varepsilon_j = \frac{\rho_j \delta_j}{16 \cdot 5^{2r\frac{r^j-1}{r-1}} |w_0|^{r^j(r+1)}}$$

so that

$$\frac{m_{j+1}\varepsilon_{j+1}}{m_j \varepsilon_j} = \frac{\rho_{j+1}\delta_{j+1}/\rho_j\delta_j}{5^{2r^{j+1}} |w_0|^{r^j(r^2-1)}}$$

giving

$$\begin{aligned} d_j &= \frac{r^j(r-1)\log|w_0| + \log(\rho_j\delta_j)}{r^j(r^2-1)\log|w_0| + 2r^{j+1}\log 5 - \log(\rho_{j+1}\delta_{j+1}/\rho_j\delta_j)} \\ &= \frac{1 - \frac{-\log(\rho_j\delta_j)}{r^j(r-1)\log|w_0|}}{1 + r + \frac{2r\log 5}{(r-1)\log|w_0|} + \frac{\log(\rho_j\delta_j/\rho_{j+1}\delta_{j+1})}{r^j(r-1)\log|w_0|}}. \end{aligned}$$

Now making d_j close to $\frac{1}{2}$ will mean making r close to 1 and making the terms

$$\frac{-\log(\rho_j\delta_j)}{r^j(r-1)\log|w_0|} \tag{16}$$

and

$$\frac{2r\log 5}{(r-1)\log|w_0|} + \frac{\log(\rho_j\delta_j/\rho_{j+1}\delta_{j+1})}{r^j(r-1)\log|w_0|} \tag{17}$$

small.

4.3. **Liouville construction.** In this section and the next we describe two constructions of slits. Our first construction is perhaps the main one of the paper and we call it the *Liouville construction.*

Suppose $r > 1$ is given and q_k, q_{k+1} are heights of consecutive convergents of λ. Suppose $w = (\lambda + m, n)$ is a slit satisfying

$$q_k^{\frac{1}{r-1}} \le |w| < \frac{q_{k+1}}{2q_k}.$$

We wish to find children slits w'.

Let $d \in \mathbb{N}$ and $u \in \mathbb{Z}^2$ be determined by

$$(p_k + mq_k, nq_k) = du, \quad \gcd(u) = 1. \tag{18}$$

Note that

$$d = d(w,k) = \gcd(p_k + mq_k, nq_k)$$

and u is the Liouville convergent of w indexed by k. Choose $\tilde{u} \in \mathbb{Z}^2$ so that $|u \times \tilde{u}| = 1$ and so that the y-coordinate of $\tilde{u}$ is nonnegative and less than $|u|$. Note that there are exactly 2 possibilites for $\tilde{u}$. The new slits will be of the form

$$w + 2v \quad \text{where} \quad v = \tilde{u} + au, \quad a \in \mathbb{N}.$$

Let

$$\Lambda(w,k) = \{w + 2v : |w|^r \leq |v| \leq 2|w|^r\}.$$

Since $|u| \leq |w|q_k$ the number of slits in $\Lambda(w,k)$ is at least

$$^\#\Lambda(w,k) \geq 2\left[\frac{|w|^r}{|u|}\right] > \frac{|w|^{r-1}}{q_k} \tag{19}$$

where the last inequality holds since $q_k \leq |w|^{r-1}$ by assumption.

Lemma 4.2. *Suppose $w' = w + 2v \in \Lambda(w,k)$ and $|w'| < \frac{q_{k+1}}{2q_k}$. Then*

$$|w \times v| < \frac{2|w|}{|u|} = \frac{2d(w,k)}{q_k} \tag{20}$$

where u is the Liouville convergent of w indexed by k. Moreover, $d(w',k) \leq 2$ so that the Liouville convergent of w' indexed by k has height is at least $\frac{|w'|q_k}{2}$.

Proof. From (8) we have

$$\angle uw \leq \frac{1}{|w|q_k q_{k+1}}.$$

Since $|v| < |w'| < q_{k+1}$, $|u \times v| = 1$ and $|u| \leq |w|q_k$ we have

$$\angle uv = \frac{|u \times v|}{|u||v|} > \frac{1}{|u|q_{k+1}} \geq \frac{1}{|w|q_k q_{k+1}}$$

so that $\angle vw \leq \angle uv + \angle uw < 2\angle uv$. Therefore,

$$|w \times v| = |w||v|\angle vw < 2|w||v|\angle uv = \frac{2|w|}{|u|}.$$

Since $d(w,k)|u| = |w|q_k$, the first statement follows.

Let $w' = (\lambda + m', n')$ where

$$(m', n') - (m,n) = w' - w = 2v.$$

Now $d' = d(w', k)$ is determined by $d'u' = (p_k + m'q_k, n'q_k)$ for some primitive $u' \in \mathbb{Z}^2$. Note that the Liouville convergent u' of w' indexed by k is well-defined since $|w'| < \frac{q_{k+1}}{2q_k}$. In terms of the basis given by u and $\tilde{u}$ we have

$$d'u' = (p_k + mq_k, nq_k) + 2q_k(\tilde{u} + au) = (2q_k)\tilde{u} + (2aq_k + d)u.$$

Note that $d = \gcd(p_k + mq_k, nq_k)$ is not divisible by any divisor of q_k, since $\gcd(p_k, q_k) = 1$. Therefore,

$$d' = \gcd(2aq_k + d, 2q_k) = \gcd(d, 2q_k) = \gcd(d, 2) \leq 2$$

so that the second statement follows. □

4.4. **Diophantine construction.** Our next general construction is as follows. We call it the *Diophantine construction* and it is accomplished by Lemma 4.4. Many of the ideas in this section were already introduced in [Ch1].

Let us introduce some convenient notation and terminology. Given a slit w, we let

$$\Psi(w)$$

denote the sequence of heights of the convergents in the continued fraction expansion of the direction of w. Given real numbers $a < b$, we say "w has a convergent between a and b" if $\Psi(w) \cap [a, b] \neq \emptyset$.

Definition 4.3. Suppose $1 \leq \alpha < \beta$. We say w is (α, β)-*good* if w has a convergent between $\alpha|w|$ and $\beta|w|$, i.e.

$$\Psi(w) \cap [\alpha|w|, \beta|w|] \neq \emptyset.$$

Let

$$\Delta(w, \alpha, \beta)$$

be the collection of slits of the form $w + 2v$ where $v \in \mathbb{Z} \times \mathbb{Z}_{>0}$ satisfies $\gcd(v) = 1$ and

$$\frac{1}{\beta} < |w \times v| < \frac{1}{\alpha} \quad \text{and} \quad \beta|w| \leq |v| \leq 2\beta|w|. \tag{21}$$

Lemma 4.4. *There is a universal constant $0 < c_0 \leq 1$ such that if $\alpha < c_0\beta$ and w is an (α, β)-good slit then*

$$^{\#}\Delta(w, \alpha, \beta) \geq \frac{c_0\beta}{\alpha}. \tag{22}$$

Moreover, every $w' \in \Delta(w, \alpha, \beta)$ is $(\alpha - \frac{1}{2}, \beta)$-good but not $(1, \alpha - \frac{1}{2})$-good.

Proof. By [Ch1,Thm.3], the number of primitive vectors satisfying the last three inequalities in (21) is at least $c_0'\beta/\alpha$ where $c_0' > 0$ is some universal constant. (Remark: To apply [Ch1,Thm.3] one needs to assume $\beta \gg \alpha$, but this hypothesis can be removed, as was shown in [Ch2]. Indeed, by [Ch2,Thm.4] we can take $c_0' = \frac{4}{27\pi}$.) The condition $|w \times v| < 1$ implies that v is a convergent of w' since, writing $w' = (\lambda + m', n')$ and $v = (p, q)$, we have

$$\left|\frac{\lambda + m'}{n'} - \frac{p}{q}\right| = \frac{|w' \times v|}{|w'||v|} < \frac{|w \times v|}{2|v|^2} < \frac{1}{2q^2}$$

and we can use (6). Let q' be the height of the next convergent of w'. Then by (5)

$$\frac{1}{q(q'+q)} < \left|\frac{\lambda + m'}{n'} - \frac{p}{q}\right| < \frac{1}{qq'}$$

so that

$$\frac{1}{q'+q} < \frac{|w \times v|}{|w'|} < \frac{1}{q'}. \tag{23}$$

From the left hand side above, the fact that $|w'| \geq 2|v| = 2q$ and $|w \times v| < \frac{1}{\alpha}$, we have

$$q' > \frac{|w'|}{|w \times v|} - q > (\alpha - \frac{1}{2})|w'|.$$

The angle, by which we mean the distance between inverse slopes, between any two solutions $v, \hat{v}$ to (21) is at least

$$\left|\frac{p}{q} - \frac{\hat{p}}{\hat{q}}\right| \geq \frac{1}{q\hat{q}} \geq \frac{1}{4\beta^2|w|^2}.$$

Take an interval J of length $\frac{2}{\beta^2|w|^2}$ centered at the direction of w and divide it into 8 equal subintervals. The inequality above says that there is at most one solution v whose direction lies in each subinterval. Thus by discarding at most 8 of these solutions, namely those whose directions are in J, we can ensure that the remaining solutions satisfy

$$\frac{|w \times v|}{|w||v|} > \frac{1}{\beta^2|w|^2}.$$

These solutions satisfy all four inequalities in (21) since

$$|w \times v| > \frac{|v|}{\beta^2|w|} \geq \frac{1}{\beta}.$$

We set

$$c_0 = \min(\frac{c_0'}{9}, 1).$$

Then the hypothesis $\alpha < c_0\beta$ implies that $c_0'\beta/\alpha \geq 9$ so that there will be at least 9 solutions to the last three inequalities in (21), and hence, at least one solution to all four in (21). Now, from the right hand side of (23) we have

$$q' < \frac{|w'|}{|w \times v|} < \beta |w'|$$

so that all children w' constructed from the remaining solutions of (21) are $(\alpha - \frac{1}{2}, \beta)$-good. Since q and q' are heights of consecutive convergents of w', none of these children are $(1, \alpha - \frac{1}{2})$-good. The number of children satisfying (21) is at least

$$\frac{c_0'\beta}{\alpha} - 8 \geq \left(9c_0 - 8\frac{\alpha}{\beta}\right)\frac{\beta}{\alpha} \geq \frac{c_0\beta}{\alpha}.$$

□

4.5. **Choice of initial parameters.** Given $\varepsilon > 0$ we choose $r > 1$ such that

$$\frac{1}{1+r} > \frac{1}{2} - \varepsilon.$$

It will be convenient to set

$$M = \frac{1}{r-1}$$

and assume $r < 2$ so that $M > 1$. Choose M' large enough so that

$$M' > 2M$$

and

$$\frac{1 - \frac{M}{M'}}{1 + r + \frac{M}{M'}} > \frac{1}{2} - \varepsilon. \tag{24}$$

We set

$$N = \max(4M'r^5, 3M^2r^3)$$

and also

$$N' = \frac{Nr+1}{r-1}$$

and

$$\phi' = \frac{1}{2r^{N'+1}}.$$

Let

$$\ell_N = \{k : q_{k+1} > q_k^N\}.$$

We shall assume λ is Liouville, so that ℓ_N contains an infinite number of indices.[4] For each $k \in \ell_N$ we define the intervals

$$I_k^B = \left[q_k^{\frac{1}{r}}, q_k^{M'}\right), \quad I_k^C = \left[q_k^{M'}, \frac{q_{k+1}}{2q_k}\right), \quad I_k^D = \left[q_{k+1}^{\frac{1}{N}}, q_{k'}^{\frac{1}{r}}\right)$$

where k' is the next index in ℓ_N immediately after k. Note that since $N > M'r > 2Mr$ we have

$$\frac{q_{k+1}}{2q_k} \geq \frac{q_{k+1}}{q_k^2} > q_{k+1}^{1-\frac{2}{N}} > q_{k+1}^{\frac{1}{r}} \tag{25}$$

and, in particular,

$$\frac{q_{k+1}}{2q_k} > q_k^{\frac{N}{r}} > q_k^{M'}.$$

It follows easily that each of the intervals just defined is nonempty and for any k_0 the union of these intervals for $k \geq k_0$ covers $[q_{k_0}^{\frac{1}{r}}, \infty)$.

Now choose k_0 large enough so that

$$q_{k_0} > \max\left(5^M, \frac{M''}{\phi'}, \exp\left(\frac{2}{c_0\phi'}\right), \frac{100r^{N'+1}}{c_0}\right) \tag{26}$$

where

$$M'' = \max([5 + \log_r M'], M, 64).$$

Then let $k_i, i > 0$ be the ith index in ℓ_N after k_0, so that

$$k_0, k_1, k_2, \ldots \quad \in \ell_N$$

enumerates all the indices of ℓ_N that are at least k_0.

Choose a slit w_{-1} with length $|w_{-1}| \in I_{k_0}^C$ and such that a child $w_0 \in \Lambda(w_{-1}, k_0)$ is such that $|w_0| \in I_{k_0}^C$. (This is possible since $N > M'r^2$.) We call w_0 our initial slit. Applying Lemma 4.2 (first to w_0, then w_{-1}) we get

$$|w_0 \times v| < \frac{2d(w_0, k_0)}{q_{k_0}} \leq \frac{4}{q_{k_0}} < \frac{1}{16}$$

where the last inequality follows by the choice of k_0. The above allows Lemma 4.1 to be applied to the initial slit. We set

$$\delta_0 = \frac{4}{q_{k_0}}, \qquad \rho_0 = \frac{1}{4}$$

so that Lemma 4.2 and (19) imply that w_0 has enough children in $\Lambda(w_0, k_0)$ as required by (14) for $j = 0$.

[4] In the case when λ is not Liouville, i.e. the case dealt with in [Ch1], the set ℓ_N is finite for all sufficiently large N. In this case, our argument essentially reduces to that in [Ch1].

4.6. **Choice of indices.** In this section we define three sequences of indices satisfying

$$0 = j_0^C < j_0^D \leq j_1^B < j_1^C < j_1^D \leq j_2^B < \dots$$

The purpose of these indices is to specify the type of construction that will be used to form the new slits. For $j_i^C \leq j < j_i^D$ we shall use the Liouville construction and for all other j we use the Diophantine construction.

Let $H_0 = \{|w_0|\}$ and for $j > 0$ set

$$H_j = \left[|w_0|^{r^j}, 5^{\frac{r^j-1}{r-1}} |w_0|^{r^j} \right].$$

By (15) the slits of level j will have their lengths in this set.

Set $j_0^C = 0$, $j_0^B = -1$, and for $i > 0$ set

$$j_i^B = \max\{j \geq 0 : \sup H_j < q_{k_i}^{\frac{1}{r}}\}$$

and

$$j_i^C = \min\{j \geq 0 : \inf H_j \geq q_{k_i}^{M'r}\}.$$

For $i \geq 0$, set

$$j_i^D = \max\{j \geq 0 : \sup H_j < q_{k_i+1}^{\frac{1}{r}}\}.$$

Lemma 4.5. *For any $i \geq 0$,*

(a) $j_i^C \leq j \leq j_i^D$ *implies* $H_j \subset I_{k_i}^C$,
(b) $j_i^D \leq j \leq j_{i+1}^B$ *implies* $H_j \subset I_{k_i}^D$,
(c) $j_i^B < j_i^C < j_i^D \leq j_{i+1}^B$,
(d) $j_i^C - j_i^B \leq [5 + \log_r M']$.

Proof. First, note that

$$\inf H_{j+1} = (\inf H_j)^r$$

and since $|w_0| \in I_{k_0}^C$ we have $|w_0| \geq q_{k_0}^{M'} > 5^{M^2}$ so that

$$\sup H_j < (5^M)^{r^j} \inf H_j < (\inf H_j)^r.$$

To prove (a) we note that the definition of j_i^C implies

$$\inf H_{j_i^C} \geq q_{k_i}^{M'r} > q_{k_i}^{M'}$$

while the definition of j_i^D and (25) imply

$$\sup H_{j_i^D} < q_{k_i+1}^{\frac{1}{r}} < \frac{q_{k_i+1}}{2q_{k_i}}.$$

To prove (b) we note that the definition of j^B_{i+1} implies

$$\sup H_{j^B_{i+1}} < q_{k_{i+1}}^{\frac{1}{r}}$$

while the definition of j^D_i implies

$$(\inf H_{j^D_i})^{r^2} > \sup H_{j^D_i+1} \geq q_{k_i+1}^{\frac{1}{r}} \tag{27}$$

so that, since $N > r^3$, we have $\inf H_{j^D_i} \geq q_{k_i}^{1/N}$.

Using the definition of j^B_i and $M' > 1 > \frac{1}{r}$ we have

$$\sup H_{j^B_i} < q_{k_i}^{\frac{1}{r}} < q_{k_i}^{M'r}$$

so that the first inequality in (c) follows from the definition of j^C_i. Using the definition of j^C_i and $N > M'r^5$ we have

$$\sup H_{j^C_i+1} < (\inf H_{j^C_i-1})^{r^3} < q_{k_i}^{M'r^4} < q_{k_i+1}^{\frac{1}{r}}$$

so that the second inequality in (c) follows from the definition of j^D_i. The third inequality in (c) follows by comparing the definitions of j^D_i and j^B_{i+1} and noting that $q_{k_{i+1}} \geq q_{k_i+1}$.

To prove (d), let $n = j^C_i - j^B_i$ so that $\inf H_{j^C_i} = (\inf H_{j^B_i})^{r^n}$. From

$$\inf H_{j^B_i} > (\sup H_{j^B_i+1})^{\frac{1}{r^2}} \geq (q_{k_i})^{\frac{1}{r^3}} \tag{28}$$

and

$$\inf H_{j^C_i} = (\inf H_{j^C_i-1})^r < q_{k_i}^{M'r^2}$$

we obtain $M'r^2 > r^{n-3}$ so that

$$j^C_i - j^B_i = n < 5 + \log_r M',$$

giving (d). $\square$

4.7. **Enough children.** In this section we specify how, in building our Cantor set F, the slits of level $j+1$ are obtained from the slits of level j. We need to specify the values of δ_j and ρ_j, then verify that the number of children we construct for each slit w of level j is at least (14). We also need to check that $\delta_j < \frac{1}{16}$ so that Lemma 4.1 can be applied to all slits of level j. We will do this for each interval of indices and the existence of enough solutions as required by (14) will be stated as a proposition in each case.

Let $W_0 = \{w_0\}$. Since $|w_0| \in I^C_{k_0}$, we consider first the case

$$j^C_i \leq j < j^D_i.$$

In this case, the slits of level $j+1$ are given by the Liouville construction:

$$W_{j+1} = \bigcup_{w \in W_j} \Lambda(w, k_i).$$

If $j_i^C < j < j_i^D$ Lemma 4.2 implies $d(w, k_i) \leq 2$ for every slit of level j, leading us to define

$$\delta_j = \frac{4}{q_{k_i}}, \qquad \rho_j = \frac{1}{4}. \tag{29}$$

The choice of k_0, Lemma 4.2 and the estimate (19) imply

Proposition 4.6. *For any $j_i^C < j < j_i^D$ we have $\delta_j < \frac{1}{16}$. Moreover, every slit w of level j has at least (14) children in $\Lambda(w, k_i)$.*

We postpone for the time being the discussion of $j = j_i^C$ for $i \geq 1$. To continue with the construction we need the following lemma.

Lemma 4.7. *Every slit w of level j_i^D is $(\frac{q_{k_i}}{2}, |w|^{r-1})$-good.*

Proof. Since j_i^C is *strictly* less than j_i^D, every slit of level j_i^D is constructed as above by the Liouville construction. By the second statement of Lemma 4.2 it follows that w has a convergent between $\frac{q_{k_i}|w|}{2}$ and $q_{k_i}|w|$. It remains to verify that $q_{k_i} \leq |w|^{r-1}$, but this follows from $|w| \in H_j \subset I_j^C$, since it implies $|w| \geq q_{k_i}^{M'} > q_{k_i}^M$. □

Now we consider the case

$$j_i^D \leq j < j_{i+1}^B.$$

Definition 4.8. We say w is N'-*normal* if for all $1 \leq t \leq N'+1$,

$$\Psi(w) \cap [\phi' r^{t+j-j_i^D} q_{k_i}|w|, |w|^{1+(r-1)t}] \neq \emptyset. \tag{30}$$

Here, the index i is determined by j, while the index j is determined by the condition $|w| \in H_j$.

It will be convenient to also introduce the following.

Definition 4.9. We say w is n-*good* if

$$\Psi(w) \cap [\phi' r^{n+j-j_i^D} q_{k_i}|w|, |w|^r] \neq \emptyset$$

or equivalently, if it is $(\phi' r^{n+j-j_i^D} q_{k_i}, |w|^{r-1})$-good.

From the definitions it follows immediately that

$$(N'+1) - \text{good} \quad \Rightarrow \quad N' - \text{normal} \quad \Rightarrow \quad 1 - \text{good}.$$

Lemma 4.10. *Every slit of level j_i^D is N'-normal.*

Proof. By Lemma 4.7, every slit w of level j_i^D is $(\frac{q_{k_i}}{2}, |w|^{r-1})$-good. By definition of ϕ', w is $(N'+1)$-good, hence N'-normal. □

For $j_i^D \le j < j_{i+1}^B$ we set

$$\delta_j = \frac{1}{\phi' r^{j-j_i^D} q_{k_i}}, \qquad \rho_j = c_0\phi'. \tag{31}$$

The following proposition is basic for the rest of this section.

Proposition 4.11. *Let w be an N'-normal slit with $|w| \in H_j$ where $j_i^D \le j < j_{i+1}^B$. Then w has*

$$\frac{c_0|w|^{r-1}}{q_{k_i} r^{j-j_i^D}}$$

N'-normal children w' of the form $w' = w + 2v$ where $|w \times v| < \delta_j$.

We defer the proof until the next section and assume it holds for now. The slits of level $j+1$ are given by

$$W_{j+1} = \bigcup_{w \in W_j} \Delta'_j(w)$$

where $\Delta'_j(w)$ denotes the set of N'-normal children of w. Then with the choices of δ_j and ρ_j in (31), Lemma 4.10 and Proposition 4.11 imply

Proposition 4.12. *For any $j_i^D \le j < j_{i+1}^B$ we have $\delta_j < \frac{1}{16}$. Moreover, every slit w of level j has at least (14) children in $\Delta'_j(w)$.*

Next, consider levels

$$j_{i+1}^B \le j < j_{i+1}^C.$$

Lemma 4.13. *Each slit w of level j_{i+1}^B is $(\phi' q_{k_i}, |w|^{r-1})$-good.*

Proof. Proposition 4.11 implies every slit of level j_{i+1}^B is N'-normal and, in particular, 1-good. Therefore, each slit w of level j_{i+1}^B has a convergent between $\phi' r^{1+j_{i+1}^B - j_i^D} q_{k_i}|w| \ge \phi' q_{k_i}|w|$ and $|w|^r$. □

We now apply the Diophantine construction. For $j_{i+1}^B \le j < j_{i+1}^C$ the slits of level $j+1$ are given by

$$W_{j+1} = \bigcup_{w \in W_j} \Delta(w, \alpha_j, |w|^{r-1}),$$

where

$$\alpha_j = \phi' q_{k_i} - \frac{j - j_{i+1}^B}{2}.$$

Lemmas 4.13 and 4.4 imply that every slit w of level j is $(\alpha_j, |w|^{r-1})$-good; this holds even for $j = j_{i+1}^C$. By Lemma 4.5(d) and the choice of k_0 in (26), we have $j_{i+1}^C - j_{j+1}^B \le [5 + \log_r M'] \le \phi' q_{k_0}$, so that

$$\alpha_j \ge \frac{\phi' q_{k_i}}{2}.$$

This leads us to define, for $j_{i+1}^B \le j < j_{i+1}^C$,

$$\delta_j = \frac{2}{\phi' q_{k_i}}, \qquad \rho_j = \frac{c_0}{2}. \tag{32}$$

Lemma 4.4 now implies

Proposition 4.14. *For any $j_{i+1}^B \le j < j_{i+1}^C$ we have $\delta_j < \frac{1}{16}$. Moreover, every slit w of level j has at least (14) children in $\Delta(w, \alpha_j, |w|^{r-1})$.*

Finally we consider the level $j = j_{i+1}^C$. Every slit w of this level is $(\alpha_j, |w|^{r-1})$-good, as noted earlier; similarly, none of them are $(1, \alpha_j)$-good. It follows that for each slit w in this level the Liouville convergent indexed by k_i has height at least

$$|u| \ge \alpha_j |w| \ge \frac{\phi' q_{k_i} |w|}{2}.$$

By Lemma 4.2, for each child $w + 2v \in \Lambda(w, k_i)$ we have

$$|w \times v| < \frac{2|w|}{|u|} \le \frac{4}{\phi' q_{k_i}}.$$

Hence, for $j = j_{i+1}^C$ we set

$$\delta_j = \frac{4}{\phi' q_{k_i}}, \qquad \rho_j = \frac{\phi' q_{k_i}}{4 q_{k_{i+1}}}. \tag{33}$$

Now the choice of k_0 in (26) and (19) give

Proposition 4.15. *For $j = j_{i+1}^C$, we have $\delta_j < \frac{1}{16}$ and every slit w of this level has at least (14) children in $\Lambda(w, k_i)$.*

4.8. **Proof of Proposition 4.11.** In this section we prove Proposition 4.11. The idea is to be able to apply Lemma 4.4 in the appropriate setting. We begin with

Lemma 4.16. *Let w be a slit with $|w| \in H_j$ where $j_i^D \le j \le j_{i+1}^B$. Then $100 r^{N'+1} (r^{j - j_i^D} q_{k_i})^2 < c_0 |w|^{(r-1)^2}$.*

Proof. By definition of j_i^D, $\sup H_{j_i^D + 1} \ge q_{k_i+1}^{1/r}$ so that $\inf H_{j_i^D} > q_{k_i}^{N/r^3}$. Write $t = j - j_i^D$ so that

$$|w| \ge (\inf H_{j_i^D})^{r^t} > q_{k_i}^{N r^{t-3}}.$$

It is enough to show that

$$(N' + 1 + t) \log r + \log q_{k_i} + \log 100 - \log c_0 < N r^{t-3} (r-1)^2 \log q_{k_i}.$$

Since $N \ge 3M^2 r^3$, this reduces to showing that for all $t \ge 0$,

$$(N' + 1 + t) \log r + \log 100 - \log c_0 < (3r^t - 1) \log q_{k_i}.$$

Since $t\log r \le r^t$ and $r^t \ge 1$, this further reduces to showing

$$(N'+1)\log r + \log 100 - \log c_0 < \log q_{k_i},$$

which holds by the choice of k_0 in (26). □

Lemma 4.17. *Suppose that $q_{k_i+1}^{1/N} < |w| < q_{k_{i+1}}^{1/r}$. Then, if w is N'-good then it is also N'-normal.*

Proof. To say that w is N'-good means (the direction of) w has a convergent p/q whose height satisfies

$$\phi' r^{N'+j-j_i^D} q_{k_i}|w| \le q \le |w|^r.$$

If w is $(N'+1)$-good, it would be N'-normal and we would be done; hence, we may reduce to the situation where

$$q < \phi' r^{N'+1+j-j_i^D} q_{k_i}|w|.$$

Let q' be the height of the next convergent of w. If $q' \le |w|^{1+(r-1)N'}$ then (30) is satisfied by q for all $1 \le t \le N'$, and by q' for all $N' \le t \le N'+1$. Thus, w would be N'-normal and we are again done.

To complete the proof, we argue by contradiction to show that

$$q' > |w|^{1+(r-1)N'}$$

cannot hold. Indeed, suppose it does. Then $|w| > 2$ and $q \le |w|^r$ imply

$$q' > |w|^{2+Nr} > 2|w|^{1+Nr} \ge 2|w|q^N.$$

Writing $w = (\lambda + m, n)$ we have

$$\left|\frac{\lambda+m}{n} - \frac{p}{q}\right| < \frac{1}{qq'}$$

so that

$$\left|\lambda + m - \frac{np}{q}\right| < \frac{|w|}{qq'} < \frac{1}{2q^{1+N}}$$

from which it follows, by (6), that $\frac{\tilde p}{\tilde q} = m - \frac{np}{q}$ is a convergent of λ. Let $\tilde q'$ denote the height of the next convergent of λ. Since, by (5),

$$\frac{1}{2\tilde q \tilde q'} < \left|\lambda - \frac{\tilde p}{\tilde q}\right| < \frac{|w|}{qq'}$$

we have

$$\tilde q' > \frac{qq'}{2|w|\tilde q} > q^N \ge \tilde q^N,$$

from which it follows that $\tilde q \in \ell_N$. Since $\tilde q \le q < |w|^r < q_{k_{i+1}}$ we must have $\tilde q \le q_{k_i}$. Hence $\tilde q' \le q_{k_i+1}$ so that

$$\frac{1}{2q_{k_i}q_{k_i+1}} \le \frac{1}{2\tilde q\tilde q'} < \frac{|w|}{qq'}.$$

Since $q \geq \phi' r^{N'+j-j_i^D} q_{k_i}|w| \geq \frac{q_{k_i}|w|}{2r}$ and $q' > 2|w|^{1+Nr}$, it follows that

$$q_{k_i+1} > \frac{qq'}{2|w|q_{k_i}} \geq \frac{|w|^{1+Nr}}{2r} > |w|^N$$

giving us the desired contradiction. □

Proof of Proposition 4.11. Recall from (31) that for $j_i^D \leq j < j_{i+1}^B$

$$\delta_j = \frac{1}{\phi' r^{j-j_i^D} q_{k_i}}.$$

Let u be the convergent of w of maximal height $|u| \leq |w|^r$. Since w is N'-normal, it is, in particular, 1-good. Therefore,

$$|u| = \phi' r^{t_1+j-j_i^D} q_{k_i}|w| \quad \text{for some} \quad t_1 \geq 1.$$

If $t_1 \geq N'+1$ then w is $(N'+1)$-good so that we may apply Lemma 4.4 with $\alpha = \phi' r^{N'+1+j-j_i^D} q_{k_i}$ and $\beta = |w|^{r-1}$ to obtain at least

$$\frac{c_0|w|^{r-1}}{\phi' r^{N'+1+j-j_i^D} q_{k_i}} = \frac{2c_0|w|^{r-1}}{q_{k_i} r^{j-j_i^D}}$$

children (twice as many as we need) of the form $w + 2v$ with

$$|w \times v| < \frac{\delta_j}{r^{N'+1}} < \delta_j.$$

Again by Lemma 4.4 all children are N'-good because

$$\alpha - \frac{1}{2} > \frac{\alpha}{r} \tag{34}$$

or equivalently, $2(r-1)\phi' r^{N'+j-j_i^D} q_{k_i} > 1$, which holds by the choice of k_0 in (26). By Lemma 4.17 and Lemma 4.5(b) it follows that all children are N'-normal. Hence, we may reduce to the case where

$$t_1 < N' + 1.$$

Let q be the height of the next convergent of w after u. Since w is N'-normal, the above implies that

$$q = |w|^{1+(r-1)t_2} \quad \text{for some} \quad t_2 \leq t_1.$$

(Otherwise, (30) fails for $t_1 < t < t_2$.) Apply Lemma 4.4 with $\alpha = \phi' r^{t_1+j-j_i^D} q_{k_i}$ and $\beta = |w|^{r-1}$ to obtain at least

$$\frac{c_0|w|^{r-1}}{\phi' r^{t_1+j-j_i^D} q_{k_i}} > \frac{2c_0|w|^{r-1}}{q_{k_i} r^{j-j_i^D}} \tag{35}$$

children (again, twice as many we need) of the form $w + 2v$ with

$$|w \times v| < \frac{\delta_j}{r^{t_1}} < \delta_j.$$

Since w is t_1-good, (34) and Lemma 4.4 imply all children constructed are $(t_1 - 1)$-good.

We wish to show that the number of children that are *not* N'-normal is at most half of the total number of children. We will divide the children that are not N'-normal into strips. Within each strip we will divide the children into clusters. We will estimate the number of strips, the number of clusters inside each strip and then the number of children inside each cluster. The product of these three numbers will be an upper bound for the number of children that are not N'-normal.

To begin, suppose w' is one of these children that is not N'-normal. Let u' be the convergent of w' of maximal height $|u'| \le |w'|^r$. Since w' is $t_1 - 1$-good and not N'-normal, we have

$$|u'| = \phi' r^{t_3 + j - j_i^D} q_{k_i} |w'| \quad \text{for some} \quad t_3 \ge t_1 - 1. \tag{36}$$

Since w' is not N'-normal, it is not $(N'+1)$-good so that

$$t_3 < N' + 1.$$

Let q' be the height of the next convergent of w' after u'. The definition of u' implies

$$q' = |w'|^{1+(r-1)t_4} \quad \text{for some} \quad t_4 > 1.$$

If $t_4 \le t_3$ then (30) for w' holds for $t \le t_3$ because $|u'| \in \Psi(w')$ while it holds for $t > t_3$ since $q' \in \Psi(w')$. Hence, w' would be N'-normal, contrary to assumption. Therefore, $t_4 > t_3$ so that

$$t_4 > \bar{t}_3 := \max(t_3, 1).$$

We record

$$t_1 - r\bar{t}_3 \le 2 - r. \tag{37}$$

To see this note that if $t_3 > 1$ then since $t_3 \ge t_1 - 1$,

$$t_1 - rt_3 = (t_1 - t_3) - (r-1)t_3 < 2 - r,$$

whereas if $t_3 \le 1$ then

$$t_1 - r \le t_3 + 1 - r \le 2 - r.$$

Lemma 4.18. *Let u' the convergent of w' as above. Then u' determines a (nonzero) integer $a \in \mathbb{Z}$ with $|a| < 2r^{N'+1}$ such that*

$$|w \times u' + 2a| < \frac{1}{|w|^{r(r-1)\bar{t}_3}}.$$

Proof. Write $w' = w + 2v$ and recall that since $|w' \times v| = |w \times v| < \delta_j < 1$ (as in the proof of Lemma 4.4) v is a convergent of w'. Let v' be the next convergent of w' after v. Then $|v'| > \frac{|w'|}{2\delta_j}$, by first inequality in (23), so that

$$\frac{|u'|}{|v'|} < 2\delta_j \phi' r^{t_3+j-j_i^D} q_{k_i} = 2r^{t_3} < 2r^{N'+1}.$$

Since $|u'| > |v|$ we either have $u' = v'$ or u' comes after v' in the continued fraction expansion of w'. In any case, we have $u' = av' + bv$ for some *nonnegative* integers $a \geq b \geq 0$ with $\gcd(a, b) = 1$. The above implies $0 < a < 2r^{N'+1}$. Since $v \times v' = \pm 1$ we have

$$|w' \times u'| = |w \times u' + 2v \times u'| = |w \times u' \pm 2a|.$$

On the other hand,

$$|w' \times u'| < \frac{|w'|}{q'} = \frac{1}{|w'|^{(r-1)t_4}} < \frac{1}{|w|^{r(r-1)\bar{t}_3}}.$$

These pair of inequalities give the lemma. □

We continue with the proof of Proposition 4.11. Suppose w'' is another child constructed that is not N'-normal. Let u'' be the convergent of w'' of maximal height $|u''| \leq |w''|^r$. Suppose further that it determines the same integer a determined by u' as in Lemma 4.18. Then we say u' and u'' belong to the same *strip*. The number of strips is bounded by the number of possible values of a. Thus by Lemma 4.18 the number of strips is bounded by

$$\left\lceil 4r^{N'+1} \right\rceil.$$

Now suppose u', u'' belong to the same strip. We say u' and u'' lie in the same *cluster* if they differ by a multiple of u.

Then we claim that if u', u'' belong to the same strip then either

$$|u'' - u'| > \frac{|w|^r}{4} \tag{38}$$

or they belong to the same cluster.

We prove the claim. Since u', u'' determine the same a, Lemma 4.18 implies

$$|w \times (u'' - u')| < \frac{2}{|w|^{r(r-1)\bar{t}_3}} < \frac{2}{|w|^{r(r-1)}}. \tag{39}$$

since $\bar{t}_3 \geq 1$. Write $u'' - u' = d\bar{u}$ where $d = \gcd(u'' - u')$. Then

$$\angle w\bar{u} = \frac{|w \times (u'' - u')|}{|u'' - u'||w|} < \frac{2}{|u'' - u'||w|^{1+r(r-1)}} < \frac{2}{|u'' - u'||w|^r}.$$

Suppose now that the first possibility in the claim does not hold, i.e.

$$|w|^r \geq 4|u'' - u'|.$$

Then

$$\angle w\bar{u} \leq \frac{1}{2|u'' - u'|^2},$$

so that $\bar{u}$ is a convergent of w. Since $t_3 < N' + 1$, (36) implies

$$|u'' - u'| < 2\phi' r^{N'+1+j-j_i^D} q_{k_i}|w| = r^{j-j_i^D} q_{k_i}|w| \leq |w|^r$$

by Lemma 4.16 and $r < 2$. Since u is the convergent of maximal height at most $|w|^r$ and $|\bar{u}| \leq |u'' - u'| \leq |w|^r$, we have $|\bar{u}| \leq |u|$. Now suppose that $|\bar{u}| < |u|$. We will arrive at a contradiction. Since u is a convergent of w coming after $\bar{u}$,

$$|w \times \bar{u}| > \frac{|w|}{2|u|}$$

which together with (39) implies

$$\frac{d|w|}{2|u|} < \frac{2}{|w|^{r(r-1)}}$$

which, since $|w_0|^{(r-1)^2} \geq q_{k_0}^{M'(r-1)^2} \geq 4$, gives

$$|u| > \frac{d|w|^{r+(r-1)^2}}{4} \geq |w|^r$$

contradicting the definition of u. We conclude that $\bar{u} = u$, so that u', u'' differ by a multiple of u. That is, they belong to the same cluster. This proves the claim.

The above claim says that the difference in the y-coordinates between u' and u'' in different clusters is greater than $\frac{|w|^r}{4}$. On the other hand, we saw that $|u'' - u'| \leq |w|^r$. Hence, there are at most 4 clusters.

Finally, we find an upper bound for the number of u' in each cluster. Since q is the next convergent of w after u we have

$$|w \times u| > \frac{|w|}{2q} = \frac{1}{2|w|^{(r-1)t_2}} \geq \frac{1}{2|w|^{(r-1)t_1}}$$

and so by (39) and (37)

$$\frac{|w \times (u'' - u')|}{|w \times u|} < 4|w|^{(r-1)(t_1 - r\bar{t}_3)} \leq 4|w|^{(r-1)-(r-1)^2}.$$

Now since by definition, for u', u'' in the same cluster, $u' - u''$ is a multiple of u, it follows that the number of possible multiples of u and hence the number in each cluster is bounded by

$$\left\lceil 4|w|^{(r-1)-(r-1)^2} \right\rceil.$$

Now we multiply the number of strips by the number of clusters and the number in each cluster to find the total number of children that are not N'-normal to be bounded above by

$$100r^{N'+1}|w|^{(r-1)-(r-1)^2} \leq \frac{c_0|w|^{r-1}}{q_{k_i} r^{j-j_i^D}},$$

by Lemma 4.16. This bound is at most half the total number of children obtained in (35). Hence, we conclude there are at least as many N'-normal children as there are children that are not N'-normal. This completes the proof of Proposition 4.11. □

The construction of the set $F = F(\lambda, r, M', w_0)$ is now complete.

4.9. **Calculations of cross products.** We first check that

$$F \subset \mathrm{NE}(P_\lambda)$$

which follows by the next lemma.

Lemma 4.19. *For any sequence (w_j) determining a point of F we have*

$$\sum_j \chi(w_j, w_{j+1}) < \infty.$$

Proof. Recall the definition of δ_j given by (29), (31) (32), and (33). By construction, we have $\chi(w_j, w_{j+1}) < \delta_j$ for all j. We break the sum into the intervals $j_i^B \leq j \leq j_i^C$, $j_i^C < j < j_i^D$, and $j_i^D \leq j < j_{i+1}^B$.

From the definitions, we easily have

$$(j_i^D - j_i^C)\log r \leq \log\log q_{k_i+1}$$

so that, by (29),

$$\sum_{k_i \in \ell_N} \sum_{j_i^C < j < j_i^D} \chi(w_j, w_{j+1}) \leq \frac{4}{\log r} \sum_{k_i \in \ell_N} \frac{\log\log q_{k_i+1}}{q_{k_i}} < \infty.$$

Since $j_i^C - j_i^B \leq M''$ we have, by (32) and (33),

$$\sum_{k_i \in \ell_N} \sum_{j_i^B \leq j \leq j_i^C} \chi(w_j, w_{j+1}) \leq \sum_{k_i \in \ell_N} \frac{4(M''+1)}{\phi' q_{k_{i-1}}} < \infty.$$

Let $R = \sum_{j \geq 0} r^{-j}$. Then, by (31), we have

$$\sum_{k_i \in \ell_N} \sum_{j_i^D \leq j < j_{i+1}^B} \chi(w_j, w_{j+1}) \leq \sum_{k_i \in \ell_N} \frac{R}{\phi' q_{k_i}} < \infty.$$

□

4.10. **Calculations of local Hausdorff Dimensions.** The proof of Theorem 1.1 will be complete with the proof of the following lemma.

Lemma 4.20. *For all j we have $d_j > \frac{1}{2} - \varepsilon$.*

Proof. By (24) it is enough to show that both of the terms (16) and (17) are bounded by $\frac{M}{M'}$. We consider the expression (17) first.

From (32), (29), (33), and (31) we have

$$\rho_j\delta_j = \begin{cases} \dfrac{c_0}{\phi' q_{k_{i-1}}} & : \quad j_i^B \le j < j_i^C \\[2ex] \dfrac{1}{q_{k_i}} & : \quad j_i^C \le j < j_i^D \\[2ex] \dfrac{c_0}{r^{j-j_i^D} q_{k_i}} & : \quad j_i^D \le j < j_{i+1}^B \end{cases}$$

so that for $j \neq j_i^C - 1$ we have

$$\frac{\rho_j\delta_j}{\rho_{j+1}\delta_{j+1}} \in \left\{1, r, \frac{1}{c_0}, \frac{\phi'}{c_0}, \frac{\phi'}{r^{j_{i+1}^B - j_i^D}}\right\}$$

while for $j = j_i^C - 1$ we have

$$\frac{\rho_j\delta_j}{\rho_{j+1}\delta_{j+1}} = \frac{c_0 q_{k_i}}{\phi' q_{k_{i-1}}}.$$

In the second case we have

$$\frac{\log(\rho_j\delta_j/\rho_{j+1}\delta_{j+1})}{r^j(r-1)\log|w_0|} \le M\left(\frac{\log q_{k_i}}{r^{j_i^C-1}\log|w_0|}\right) \le \frac{M}{M'}$$

by the definition of j_i^C, while in the first case the same inequality holds since $\log|w_0| \ge M' \max(r, c_0^{-1})$, by the choice of k_0 in (26).

We now turn to the expression (16). For $j_i^C \le j < j_i^D$ we have

$$\frac{-\log(\rho_j\delta_j)}{r^j(r-1)\log|w_0|} \le M\left(\frac{\log q_{k_i}}{r^{j_i^C}\log|w_0|}\right) \le \frac{M}{M'},$$

by the definition of j_i^C.

Now we turn to the possibility that $j_i^B \le j < j_i^C$ $(i \ge 1)$. Using $\phi' < 1$, $c_0^{-1} < q_{k_0}$, $\log q_{k_i} \ge \log q_{k_{i-1}+1} \ge N \log q_{k_{i-1}}$, (28), and $N \ge 2M'r^3$ we have

$$\begin{aligned} \frac{-\log(\rho_j\delta_j)}{r^j(r-1)\log|w_0|} &\le M\left(\frac{\log q_{k_{i-1}} + \log(\phi'/c_0)}{r^{j_i^B}\log|w_0|}\right) \\ &\le \frac{2M\log q_{k_{i-1}}}{r^{j_i^B}\log|w_0|} \le \frac{2Mr^3}{N} \le \frac{M}{M'}. \end{aligned}$$

Next consider $j_i^D \leq j < j_{i+1}^B$. Using $jr^{-j}\log r \leq 1$, $q_{k_0} > \max(c_0^{-1}, e^2)$, $\log q_{k_i} \leq N^{-1}\log q_{k_i+1}$, (27), and $N \geq 4M'r^3$ we have

$$\frac{-\log(\rho_j\delta_j)}{r^j(r-1)\log|w_0|} \leq M\left(\frac{\log q_{k_i} + (j-j_i^D)\log r + \log(1/c_0)}{r^j\log|w_0|}\right)$$
$$\leq \frac{2M\log q_{k_i}}{r^{j_i^D}\log|w_0|} + \frac{M}{\log|w_0|} \leq \frac{2Mr^3}{N} + \frac{M}{2M'} \leq \frac{M}{M'},$$

and the lemma follows. □

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. The construction of the set F as well as the lower bound $\frac{1}{2}$ estimate on its Hausdorff dimension remains valid for any irrational λ. (Note that Lemma 4.19 fails when (4) holds and F cannot be a subset of $\mathrm{NE}(P_\lambda)$, since the latter has Hausdorff dimension zero.) On the other hand, Proposition 3.6 of [Ch2] says that $\lim_j \delta_j = 0$ implies $F \subset \mathrm{DIV}(P_\lambda)$, so that $\mathrm{HDim}\,\mathrm{DIV}(P_\lambda) \geq \frac{1}{2}$ for all irrational λ. The opposite inequality follows from a more general result in [Ma]. Lastly, when $\lambda \in \mathbb{Q}$, the set $\mathrm{DIV}(P_\lambda)$ is countable, so that its Hausdorff dimension vanishes. □

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